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Research article

Solvability of product of n-quadratic Hadamard-type fractional integral equations in Orlicz spaces

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Abstract: The current study demonstrated and studied the existence of monotonic solutions, as well as the uniqueness of the solutions for a general and abstract form of a product of n-quadratic fractional integral equations of Hadamard-type in Orlicz spaces L_φ. We utilized the analysis of the measure of non-compactness associated with Darbo's fixed-point theorem and fractional calculus to obtain the results.

Keywords: Hadamard fractional integral operator; n-product of quadratic integral equation; measure of non-compactness (MNC); Orlicz spaces L_φ
Mathematics Subject Classification: 45G10, 46E30, 47H30, 47N20

1. Introduction

The theory of fractional integral and differential equations has a fundamental role in several branches of science, such as economics, biology, engineering, physics, electrical circuits, electrochemistry, earthquakes, fluid dynamics, traffic models, and viscoelasticity (cf. [1–3]).

Hadamard fractional integral operators were defined by Hadamard in 1892 [4]. These operators have a kernel of logarithmic function of arbitrary order, which is not of convolution type. Consequently, they should be examined separately from the more well-known Caputo and Riemann-Liouville fractional operators. These types of operators have been studied by several researchers in numerous function spaces. (cf. [5–7]).

The present work investigates and establishes the existence theorem as well as the uniqueness of the solution to a general and abstract form of a product of n-quadratic fractional integral equations of

Hadamard-type in Orlicz spaces L_φ, which has the form

$$y(s) \ = \prod_{i=1}^{n} \Big(h_i(s) + G_{2_i}(y)(s) + \frac{G_{1_i}(y)(s)}{\Gamma(\alpha_i)} \cdot \int_1^s \Big(\log \frac{s}{\tau} \Big)^{\alpha_i - 1} \frac{G_{3_i}(y)(\tau)}{\tau} \, d\tau \Big), \ \ s \in [1, e], \ 0 < \alpha_i < 1, \quad (1.1)$$

in arbitrary Orlicz spaces L_φ, where G_{j_i}, $j = 1, 2, 3$ are general operators.

The theory of fractional calculus in Orlicz spaces was studied by O'Neill in 1965 [8], and, subsequently, several interesting articles were published on this topic (see, for example, [9–11]).

Orlicz spaces L_φ are suitable spaces for studying operators with strong nonlinearities (e.g., exponential growth) rather than polynomial growth in Lebesgue spaces L_p, $p \geq 1$, (see [12, 13]). These are motivated by some problems in statistical physics and mathematical physics (see [14, 15]). In particular, the thermodynamics problem

$$y(s) + \int_I a(s, u) \cdot e^{y(u)} \, du = 0,$$

contains exponential nonlinearity (cf. [16]).

Moreover, quadratic integral equations have been applied in astrophysics, radiative transfer theory, or neutron transport [17–19]. It should be noted that several kinds of quadratic integral equations have been investigated in L_p spaces [20–22] and in L_φ-spaces [12, 13, 23] using the measure of non-compactness analysis associated with Darbo's fixed-point hypothesis via different sets of assumptions.

It is useful to study the product of two or more than two operators, as mentioned by Medveď and Brestovanská in [24, 25]; however, they consider the Banach algebras of continuous functions, which have a different technique in the proof. Since Orlicz spaces are not Banach algebras, we use the methods given in [26, 27] to obtain our results.

In [26], the author proved some fixed point theorems and employed them in examining the solution of the equation

$$y(s) = \prod_{i=1}^{n} \Big(g_i(s) + \int_a^s K_i(s, \tau, y(\tau)) \, d\tau \Big),$$

in some types of ideal spaces like $L_p, p > 1$ and Orlicz spaces $L_\varphi(I)$, $I = [a, b]$, where φ verifies the Δ_2-condition.

In [27], the existence theorems for the product of n-integral equations operating on n-distinct Orlicz spaces

$$y(s) = \prod_{i=1}^{n} \Big(g_i(s) + \lambda_i \cdot h_i(s, y(s)) \cdot \int_a^b K_i(s, \tau) f_i(\tau, y(\tau)) \, d\tau \Big),$$

were discussed in Orlicz spaces $L_\varphi([a, b])$, for $n \geq 2$, when the function φ verifies the so-called Δ', Δ_3, and Δ_2-conditions.

The author in [28] demonstrated and proved some basic theorems for the Riemann-Liouville fractional integral operator and investigated the existence theorems in L_φ-spaces for the equation

$$y(s) \ = \ y(s) + G(y)(s) \int_0^s \frac{(s - \tau)^{\alpha - 1}}{\Gamma(\alpha)} f(\tau, y(\tau)) \, d\tau, \ \ 0 < \alpha < 1, \ s \in [0, d].$$

In [29], some basic theorems were demonstrated and proved for the Hadamard fractional order integral operator, and the existence theorems were also investigated for the equation:

$$y(s) \; = \; G_3(y)(s) + \frac{G_1(y)(s)}{\Gamma(\alpha)} \int_1^s \Big(\log \frac{s}{\tau}\Big)^{\alpha-1} \frac{G_2(y)(\tau)}{\tau} \, d\tau, \quad 0 < \alpha < 1, \; s \in [1, e],$$

in Orlicz spaces L_φ.

Basic theorems for the Erdélyi-Kober fractional order integral operator can be found, both demonstrated and proved, in [30], where the existence theorems were also investigated for the following equation:

$$y(s) \; = \; g(s) + \; f_1(s, y(s)) + f_2\Big(s, \; \frac{\beta h_1(s, y(s))}{\Gamma(\alpha)} \cdot \int_0^s \frac{\tau^{\beta-1} h_2(\tau, y(\tau))}{(s^\beta - \tau^\beta)^{1-\alpha}} \, d\tau\Big), \quad s \in [0, d],$$

where $0 < \alpha < 1$ and $\beta > 0$ in both L_p and L_φ spaces.

This paper is motivated by studying monotonic solutions for a general and abstract form of a product of n-quadratic fractional integral equations of Hadamard-type in Orlicz spaces L_φ. We provide two existence theorems, namely (the existence and the uniqueness of) the solutions for Eq (1.1). The measure of non-compactness and Darbo's fixed point theorem are our main tools for examining the obtained results.

2. Preliminaries

Let $\mathbb{R}^+ = [0, \infty) \subset \mathbb{R} = (-\infty, \infty)$ and $I = [1, e], \; e \approx 2.718$. A function $M : [0, \infty) \to [0, \infty)$ points to a Young function if

$$M(\tau) = \int_0^\tau u(s) dt, \quad \text{for} \;\; \tau \geq 0,$$

where $u : [0, \infty) \to [0, \infty)$ is a left-continuous-increasing function and is neither equal to infinite, nor zero on $\mathbb{R}^+$. The functions N and M are referred to the complementary Young functions, if $M(y) = \sup_{z \geq 0}(yz - N(y))$. Furthermore, if M is finite-valued with $\lim_{\tau \to 0} \frac{M(\tau)}{\tau} = 0$, $\lim_{\tau \to \infty} \frac{M(\tau)}{\tau} = \infty$, and $M(\tau) > 0$ if $\tau > 0$ ($M(\tau) = 0 \iff \tau = 0$), then M is said to be an N-function.

The Orlicz space $L_M = L_M(I)$ is the space of all measurable functions $y : I \to \mathbb{R}$ with the Luxemburg norm

$$\|y\|_M = \inf_{\epsilon > 0} \left\{ \int_I M\left(\frac{y(\tau)}{\epsilon}\right) \, d\tau \leq 1 \right\}.$$

Let $E_M = E_M(I)$ contain the set of all bounded functions of L_M and have absolutely continuous norms.

Definition 2.1. *[31] The Hadamard-type fractional integral of an integrable function y of order $\alpha > 0$ is given by*

$$J^\alpha y(s) = \frac{1}{\Gamma(\alpha)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\alpha-1} \frac{y(\tau)}{\tau} \, d\tau, \quad s > 1, \;\; \alpha > 0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} \, ds$.

Proposition 2.1. *[5] The operator J^α maps a.e. nondecreasing and nonnegative functions to functions of similar types.*

Lemma 2.1. *[29] Assume, that M and N are complementary N-functions with* $\int_0^s M(\tau^{\alpha-1})\,d\tau < \infty,\ \alpha \in (0,1)$. *Moreover, suppose that φ is N-function, where*

$$k(s) = \frac{1}{\epsilon^{\frac{1}{1-\alpha}}} \int_0^{s\epsilon^{\frac{1}{1-\alpha}}} M(\tau^{\alpha-1})\,d\tau \in E_\varphi$$

for a.e. $\tau \in I$ and $\epsilon > 0$, then the operator $J^\alpha : L_N \to L_\varphi$ is continuous and verifying

$$\|J^\alpha y\|_\varphi \leq \frac{2}{\Gamma(\alpha)} \|k\|_\varphi \|y\|_N.$$

The following lemma characterizes the product of the operators in L_φ:

Lemma 2.2. ([32, Theorem 1]) *Let $n \geq 2$. If φ and $\varphi_i,\ i = 1, \cdots n$ are arbitrary N-functions, then the following conditions are equivalent:*

(1) For every $u_i \in L_{\varphi_i}$, $\prod_{i=1}^n u_i \in L_\varphi$.
(2) There exists a constant $K > 0$ s.t.

$$\Big\| \prod_{i=1}^n u_i \Big\|_\varphi \leq K \prod_{i=1}^n \|u_i\|_{\varphi_i},$$

for every $u_i \in L_{\varphi_i}, i = 1, 2, \cdots n$.
(3) There exists a constant $C > 0$ s.t.

$$\prod_{i=1}^n \varphi_i^{-1}(s) \leq C\varphi^{-1}(s)$$

for every $s \geq 0$.
(4) There exists a constant $C > 0$ s.t. $\forall\ s_i \geq 0, i = 1, \cdots n$,

$$\varphi\Big(\frac{\prod_{i=1}^n s_i}{C}\Big) \leq \sum_{i=1}^n \varphi_i(s_i).$$

Let $S = S(I)$ refer to all Lebesgue measurable functions on the interval I. The set S concerning the metric

$$d(y, z) = \inf_{\epsilon>0}[\epsilon + meas\{\tau : |y(\tau) - z(\tau)| \geq \epsilon\}]$$

becomes a complete space, where "*meas*" points to the Lebesgue measure in $\mathbb{R}$. The convergence w.r. to d is identical to the convergence in measure on I (cf. Proposition 2.14 in [34]). We call the compactness in S by "compactness in measure".

Lemma 2.3. *[23] Let $Y \subset L_M$ be a bounded set, and there is a family $(\Omega_c)_{0\leq c\leq e-1} \subset I$ s.t. meas $\Omega_c = c$ for every $c \in [1, e]$, and for every $y \in Y$,*

$$y(s_1) \geq y(s_2), \quad (s_1 \in \Omega_c,\ s_2 \notin \Omega_c).$$

Thus, Y represents a compact in measure set in L_M.

Definition 2.2. *[23] Let $\emptyset \neq Y \subset L_M$ be bounded, then*

$$\beta_H(Y) = \inf\{r > 0 : \ \exists \text{ a finite subset } Z \text{ of } L_M \text{ s.t. } Y \subset \ Z \ + \ B_r \ \},$$

is called the Hausdorff measure of non-compactness (MNC), where $B_r = \{m \in L_M : \ \|m\|_M \leq r\}$.

The measure of equi-integrability c of the set $Y \in L_M$ is given by

$$c(Y) \ = \ \lim_{\epsilon \to 0} \ \sup_{mesD \leq \epsilon} \ \sup_{y \in Y} \|y \cdot \chi_D\|_{L_M},$$

where $\epsilon \ > \ 0$ and χ_D is the characteristic function of $D \subset I$ (cf. [33] or [34]).

Lemma 2.4. *[23, 33] Let $\emptyset \neq Y \subset E_M$ provide a bounded and compact in measure set, then we have*

$$\beta_H(Y) = c(Y).$$

3. Main results

Rewrite Eq (1.1) as

$$y = B(y) = \prod_{i=1}^{n} B_i(y) = \prod_{i=1}^{n} \Big(h_i + G_{2_i}(y) + \ U_i(y)\Big),$$

where

$$U_i(y) = G_{1_i}(y) \cdot A_i(y), \ \ A_i(y) = J_i^{\alpha_i} G_{3_i}(y),$$

s.t. $J_i^{\alpha_i}$ is as in Definition 2.1 and $G_{j_i}(y)$ are general operators that act on some different Orlicz spaces for $j = 1, 2, 3$ and $i = 1, \cdots, n$.

Next, we discuss the existence of L_φ solutions for Eq (1.1).

3.1. The existence of L_φ-solutions

For $i \ = \ 1, \cdots, n$, suppose that $\varphi, \varphi_i, \varphi_{1_i}, \varphi_{2_i}$ are N-functions and that N_i, M_i are complementary N-functions with $\int_0^s M_i(\tau^{\alpha_i - 1})\, d\tau < \infty$, $\alpha_i \in (0, 1)$, and consider the assumptions:

(N1) There exists a constant $K > 0$ s.t. for every $u_i \in L_{\varphi_i}$, and we have $\| \prod_{i=1}^n u_i\|_\varphi \leq K \prod_{i=1}^n \|u_i\|_{\varphi_i}$.

(N2) There exists a constant $k_{1_i} > 0$ such that for every $u_1 \in L_{\varphi_{1_i}}$ and $u_2 \in L_{\varphi_{2_i}}$, we get $\|u_1 u_2\|_{\varphi_i} \leq k_{1_i}\|u_1\|_{\varphi_{1_i}}\|u_2\|_{\varphi_{2_i}}$.

(N3) The functions $h_i \in E_{\varphi_i}$ are a.e. nondecreasing on the interval I.

(N4) $G_{1_i} : L_\varphi \to L_{\varphi_{1_i}}$ take continuously $E_\varphi \to E_{\varphi_{1_i}}$, the operators $G_{2_i} : L_\varphi \to L_{\varphi_i}$ take continuously $E_\varphi \to E_{\varphi_i}$, and the operators $G_{3_i} : L_\varphi \to L_{N_i}$ take continuously $E_\varphi \to E_{N_i}$.

(N5) There exist positive functions $g_{1_i} \in L_{\varphi_{1_i}}$, $g_{2_i} \in L_{\varphi_i}$, $g_{3_i} \in L_{N_i}$ s.t. for $s \in I$, $|G_{j_i}(y)(s)| \leq g_{j_i}(s)\|y\|_\varphi$; and G_{j_i}, $j = 1, 2, 3$, takes the set of all a.e. nondecreasing functions to functions of similar properties. Moreover, suppose that for any $y \in E_\varphi$, we have $G_{1_i}(y) \in E_{\varphi_{1_i}}$, $G_{2_i}(y) \in E_{\varphi_i}$, and $G_{3_i}(y) \in E_{N_i}$.

(N6) Assume that $k_i(s) = \frac{1}{\epsilon^{\frac{1}{1-\alpha_i}}} \int_0^{s\epsilon^{\frac{1}{1-\alpha_i}}} M_i(\tau^{\alpha_i - 1})\, d\tau \in E_{\varphi_{2_i}}$ for $\epsilon > 0$ and $s \in I$.

(N7) Suppose that $\exists\, r > 0$ and $L_i > 0$ verify

$$\prod_{i=1}^{m} L_i = K \prod_{i=1}^{n} \left(\|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \cdot r + \frac{2k_{1_i}\|k_i\|_{\varphi_{2_i}}}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \cdot r^2 \right) \leq r \tag{3.1}$$

and

$$\prod_{i=1}^{n} \left(\|g_{2_i}\|_{\varphi_i} + \frac{2k_{1_i}\|k_i\|_{\varphi_{2_i}} \cdot r}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \right) < \frac{1}{r^n K}.$$

Theorem 3.1. *Let the assumptions (N1)–(N7) be verified, then there exists a solution $y \in E_\varphi$ of (1.1) that is a.e. nondecreasing on I.*

Proof. **I.** In what follows, put $i = 1, \cdots, n$. First, Lemma 2.1 implies that each $J_i^\alpha : L_{N_i} \to L_{\varphi_{2_i}}$ is continuous. By assumption (N4), we have that the operators $G_{1_i} : E_\varphi \to E_{\varphi_{1_i}}$, $G_{2_i} : E_\varphi \to E_{\varphi_i}$, and $G_{3_i} : E_\varphi \to E_{N_i}$ are continuous, then $A_i = J_i^{\alpha_i} G_{3_i} : E_\varphi \to E_{\varphi_{2_i}}$ are continuous. By assumption (N2) and the Hölder inequality, we get that $U_i = G_{1_i} \cdot A_i : E_\varphi \to E_{\varphi_i}$, and they are continuous. By using assumptions (N3), we have the operators $B_i : E_\varphi \to E_{\varphi_i}$. Finally, assumption (N1) and the Hölder inequality give us that $B = \prod_{i=1}^n B_i : E_\varphi \to E_\varphi$ is continuous.

II. We shall establish the ball $B_r(E_\varphi) = \{y \in L_\varphi : \|y\|_\varphi \leq r\}$, where r is defined in assumption (N7). Let $y \in B_r(E_\varphi)$, and by recalling Lemma 2.1, we have

$$\begin{aligned}
\|B_i(y)\|_{\varphi_i} &\leq \|h_i\|_{\varphi_i} + \|G_{2_i}(y)\|_{\varphi_i} + \|U_i y\|_{\varphi_i} \\
&\leq \|h_i\|_{\varphi_i} + \left\|g_{2_i} \cdot \|y\|_\varphi\right\|_{\varphi_i} + \|G_{1_i}(y) \cdot A_i(y)\|_{\varphi_i} \\
&\leq \|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \|y\|_\varphi + k_{1_i} \|G_{1_i}(y)\|_{\varphi_{1_i}} \cdot \|A_i(y)\|_{\varphi_{2_i}} \\
&\leq \|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \|y\|_\varphi + k_{1_i} \left\|g_{1_i} \cdot \|y\|_\varphi\right\|_{\varphi_{1_i}} \cdot \|J_i^{\alpha_i} G_{3_i}(y)\|_{\varphi_{2_i}} \\
&\leq \|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \|y\|_\varphi + k_{1_i} \left\|g_{1_i}\right\|_{\varphi_{1_i}} \|y\|_\varphi \frac{2}{\Gamma(\alpha_i)} \|k_i\|_{\varphi_{2_i}} \left\|g_{3_i} \cdot \|y\|_\varphi\right\|_{N_i} \\
&\leq \|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \|y\|_\varphi + k_{1_i} \left\|g_{1_i}\right\|_{\varphi_{1_i}} \|y\|_\varphi \frac{2}{\Gamma(\alpha_i)} \|k_i\|_{\varphi_{2_i}} \left\|g_{3_i}\right\|_{N_i} \|y\|_\varphi \\
&\leq \|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \|y\|_\varphi + \frac{2k_{1_i}\|k_i\|_{\varphi_{2_i}}}{\Gamma(\alpha_i)} \left\|g_{1_i}\right\|_{\varphi_{1_i}} \left\|g_{3_i}\right\|_{N_i} \|y\|_\varphi^2 \\
&\leq \|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \cdot r + \frac{2k_{1_i}\|k_i\|_{\varphi_{2_i}}}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \cdot r^2.
\end{aligned}$$

Therefore, utilizing assumption (N1), we have

$$\begin{aligned}
\|B(y)\|_\varphi &\leq K \prod_{i=1}^{n} \|B_i(y)\|_{\varphi_i} \\
&\leq K \prod_{i=1}^{n} \left(\|h_i\|_{\varphi_i} + \left\|g_{2_i}\right\|_{\varphi_i} \cdot r + \frac{2k_{1_i}\|k_i\|_{\varphi_{2_i}}}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \cdot r^2 \right) \leq r.
\end{aligned}$$

By using assumption (N7), we have that $B : B_r(E_\varphi) \to E_\varphi$ is continuous.

III. Let $Q_r \subset B_r(E_\varphi)$ contain the a.e. nondecreasing functions of I. The set Q_r is a closed, nonempty, bounded, and convex set in L_φ; see [23]. Furthermore, Q_r is compact in measure (thanks to Lemma 2.3).

IV. Next, we discuss the monotonicity for the operator B. Take $y \in Q_r$, then y is a.e. nondecreasing on I. By assumption (N5), the operators $G_{j_i}(y)$, $j = 1, 2, 3$ are a.e. nondecreasing on I, by Proposition, 2.1 the operator A_i is of the same type, then the operators $U_i(y) = G_{1_i}(y) \cdot A_i(y)$ are a.e. nondecreasing on I, and by using assumption (N3), we have that $B : Q_r \to Q_r$ is continuous.

V. We will demonstrate that B is a contraction w.r. to the MNC. Suppose that $\emptyset \neq Y \subset Q_r$. For $y \in Y$ and for a set $D \subset I$, $\epsilon > 0$, meas$D \leq \epsilon$. By assumption (N4), we have

$$\|G_{1_i}(y) \cdot \chi_D\|_{\varphi_{1_i}} \leq \|G_{1_i}(y \cdot \chi_D)\|_{\varphi_{1_i}} \leq \left\|g_{1_i} \cdot \|y \cdot \chi_D\|_{\varphi}\right\|_{\varphi_{1_i}} \leq \left\|g_{1_i}\right\|_{\varphi_{1_i}} \|y \cdot \chi_D\|_{\varphi}$$

and, similarly,

$$\|G_{2_i}(y) \cdot \chi_D\|_{\varphi_i} \leq \left\|g_{2_i}\right\|_{\varphi_i} \|y \cdot \chi_D\|_{\varphi},$$

then we have

$$\begin{aligned}
\|B_i(y) \cdot \chi_D\|_{\varphi_i} &\leq \|h_i \cdot \chi_D\|_{\varphi_i} + \|G_{2_i}(y) \cdot \chi_D\|_{\varphi_i} + \|U_i(y) \cdot \chi_D\|_{\varphi_i} \\
&\leq \|h_i \cdot \chi_D\|_{\varphi_i} + \|G_{2_i}(y \cdot \chi_D)\|_{\varphi_i} + \|G_{1_i}(y) \cdot A_i(y) \cdot \chi_D\|_{\varphi_i} \\
&\leq \|h_i \cdot \chi_D\|_{\varphi_i} + \|g_{2_i}\|_{\varphi_i} \|y \cdot \chi_D\|_{\varphi} + k_{1_i} \|G_{1_i}(y) \cdot \chi_D\|_{\varphi_{1_i}} \cdot \|A_i(y) \cdot \chi_D\|_{\varphi_{2_i}} \\
&\leq \|h_i \cdot \chi_D\|_{\varphi_i} + \|g_{2_i}\|_{\varphi_i} \|y \cdot \chi_D\|_{\varphi} + k_{1_i} \|G_{1_i}(y \cdot \chi_D)\|_{\varphi_{1_i}} \cdot \|A_i(y)\|_{\varphi_{2_i}} \\
&\leq \|h_i \cdot \chi_D\|_{\varphi_i} + \|g_{2_i}\|_{\varphi_i} \|y \cdot \chi_D\|_{\varphi} + \frac{2k_{1_i}}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|y \cdot \chi_D\|_{\varphi} \|k_i\|_{\varphi_{2_i}} \|G_{3_i}(y)\|_{N_i} \\
&\leq \|h_i \cdot \chi_D\|_{\varphi_i} + \|g_{2_i}\|_{\varphi_i} \|y \cdot \chi_D\|_{\varphi} + \frac{2k_{1_i}}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|y \cdot \chi_D\|_{\varphi} \|k_i\|_{\varphi_{2_i}} \|g_{3_i}\|_{N_i} \|y\|_{\varphi} \\
&\leq \|h_i \cdot \chi_D\|_{\varphi_i} + \|g_{2_i}\|_{\varphi_i} \|y \cdot \chi_D\|_{\varphi} + \frac{2k_{1_i} \|k_i\|_{\varphi_{2_i}} \cdot r}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \|y \cdot \chi_D\|_{\varphi}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|B(y) \cdot \chi_D\|_{\varphi} &\leq K \prod_{i=1}^{n} \|B_i(y) \cdot \chi_D\|_{\varphi_i} \\
&\leq K \prod_{i=1}^{n} \left(\|h_i \cdot \chi_D\|_{\varphi_i} + \|g_{2_i}\|_{\varphi_i} \|y \cdot \chi_D\|_{\varphi} + \frac{2k_{1_i} \|k_i\|_{\varphi_{2_i}} \cdot r}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \|y \cdot \chi_D\|_{\varphi} \right).
\end{aligned}$$

Since $h_i \in E_{\varphi_i}$, we obtain

$$\lim_{\varepsilon \to 0} \{ \sup_{meas\ D \leq \varepsilon} [\sup_{y \in Y} \{\|h_i \cdot \chi_D\|_{\varphi_i}\}] \} = 0.$$

From the definition of $c(y)$, we have

$$c(B(Y)) \leq r^n K \prod_{i=1}^{n} \left(\|g_{2_i}\|_{\varphi_i} + \frac{2k_{1_i} \|k_i\|_{\varphi_{2_i}} \cdot r}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \right) c(Y),$$

where $\|y \cdot \chi_D\|_{\varphi}^n = \|y \cdot \chi_D\|_{\varphi}^{n-1} \|y \cdot \chi_D\|_{\varphi} \leq r^n \|y \cdot \chi_D\|_{\varphi}$.

Since $\emptyset \neq Y \subset Q_r$ is a bounded and compact in measure subset of E_{φ}, we can employ Lemma 2.4 to get

$$\beta_H(B(Y)) \leq r^n K \prod_{i=1}^{n} \left(\|g_{2_i}\|_{\varphi_i} + \frac{2k_{1_i} \|k_i\|_{\varphi_{2_i}} \cdot r}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \right) \cdot \beta_H(Y).$$

Since $\prod_{i=1}^{n} \left(\|g_{2_i}\|_{\varphi_i} + \frac{2k_{1_i} \|k_i\|_{\varphi_{2_i}} \cdot r}{\Gamma(\alpha_i)} \|g_{1_i}\|_{\varphi_{1_i}} \|g_{3_i}\|_{N_i} \right) < \frac{1}{r^n K}$, we have finished (cf. [26]). □

Remark 3.1. *If the N-functions $N_i, i = 1, \cdots, n$ verify the Δ'-condition, then Theorem 3.1 is valid on the unite balls $B_1(E_\varphi) = \{y \in L_\varphi : \|y\|_\varphi \leq 1\}$. Furthermore, if they verify the Δ_3 or Δ_2-conditions, then Theorem 3.1 is valid on the whole E_φ (cf. [13, 23]).*

3.1.1. Uniqueness of the solution

Now, we discuss the uniqueness of Eq (1.1).

Theorem 3.2. *Let assumption (N1)–(N7) be verified. If*

$$C = \sum_{j=1}^{n} \left[K\left(\|g_{2_j}\|_{\varphi_j} + \frac{4k_{1_j} \cdot r\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)} \|g_{1_j}\|_{\varphi_{i_j}} \left\|g_{3_j}\right\|_{N_j} \right) \cdot \prod_{i=1, i\neq j}^{n} L_i \right] < 1,$$

where r and L_i are defined in assumption (N7), then Eq (1.1) has a unique solution $y \in L_\varphi$ in Q_r.

Proof. Let y and z be any two different solutions of Eq (1.1), then we obtain

$$\begin{aligned}
|y - z| &= \left| \prod_{i=1}^{n} B_i(y) - \prod_{i=1}^{n} B_i(z) \right| \\
&\leq \left| \prod_{i=1}^{n} B_i(y) - B_1(z) \prod_{i=2}^{n} B_i(y) \right| + \left| B_1(z) \prod_{i=2}^{n} B_i(y) - B_1(z) B_2(z) \prod_{i=3}^{n} B_i(y) \right| \\
&\quad + \cdots + \left| B_n(y) \prod_{i=1}^{n-1} B_i(z) - \prod_{i=1}^{n} B_i(z) \right| \\
&\leq |B_1(y) - B_1(z)| \cdot \prod_{i=2}^{n} |B_i(y)| + |B_1(z)| \cdot |B_2(y) - B_2(z)| \cdot \prod_{i=3}^{n} |B_i(y)| \\
&\quad + \cdots + |B_n(y) - B_n(z)| \cdot \prod_{i=1}^{n-1} |B_i(z)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|y - z\|_\varphi &\leq K\left\|B_1(y) - B_1(z)\right\|_{\varphi_1} \prod_{i=2}^{n} \|B_i(y)\|_{\varphi_i} + K\|B_1(z)\|_{\varphi_1} \left\|B_2(y) - B_2(z)\right\|_{\varphi_2} \prod_{i=3}^{n} \|B_i(y)\|_{\varphi_i} \\
&\quad + \ldots + K\left\|B_n(y) - B_n(z)\right\|_{\varphi_n} \prod_{i=1}^{n-1} \|B_i(z)\|_{\varphi_i}. \qquad (3.2)
\end{aligned}$$

To calculate the above inequality, we need the following estimation. For $j = 1, \cdots, n$, and by using Lemma 2.1, we have

$$\begin{aligned}
&\left\|B_j(y) - B_j(z)\right\|_{\varphi_j} \leq \left\|G_{2_j}(y) - G_{2_j}(z)\right\|_{\varphi_j} + \left\|G_{1_j}(y)A_j(y) - G_{1_j}(z)A_j(z)\right\|_{\varphi_j} \\
&\leq \left\|g_{2_j} \cdot \|y\|_\varphi - g_{2_j} \cdot \|z\|_\varphi\right\|_{\varphi_j} + \left\|G_{1_j}(y)A_j(y) - G_{1_j}(z)A_j(y)\right\|_{\varphi_j} + \left\|G_{1_j}(z)A_j(y) - G_{1_j}(z)A_j(z)\right\|_{\varphi_j} \\
&\leq \left\|g_{2_j} \cdot \left|\|y\|_\varphi - \|z\|_\varphi\right|\right\|_{\varphi_j} + k_{1_j}\left\|G_{1_j}(y) - G_{1_j}(z)\right\|_{\varphi_{1_j}} \left\|A_j(y)\right\|_{\varphi_{2_j}} + k_{1_j}\left\|G_{1_j}(z)\right\|_{\varphi_{1_j}} \left\|A_j(y) - A_j(z)\right\|_{\varphi_{2_j}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \ \|g_{2_j}\|_{\varphi_j}\|y-z\|_\varphi + k_{1_j}\Big\|g_{1_j}\cdot\big|\|y\|_\varphi-\|z\|_\varphi\big|\Big\|_{\varphi_{1_j}}\big\|J_j^{\alpha_j}G_{3_j}(y)\big\|_{\varphi_{2_j}}\\
&\quad +k_{1_j}\Big\|g_{1_j}\cdot\|z\|_\varphi\Big\|_{\varphi_{1_j}}\big\|J_j^{\alpha_j}G_{3_j}(y)-J_j^{\alpha_j}G_{3_j}(z)\big\|_{\varphi_{2_j}}\\
&\leq \ \|g_{2_j}\|_{\varphi_j}\|y-z\|_\varphi + k_{1_j}\|g_{1_j}\|_{\varphi_{1_j}}\|y-z\|_\varphi\frac{2}{\Gamma(\alpha_i)}\|k_j\|_{\varphi_{2_j}}\big\|g_{3_j}\big\|_{N_j}\|y\|_\varphi\\
&\quad +k_{1_j}\|g_{1_j}\|_{\varphi_{1_j}}\cdot\|z\|_\varphi\frac{2}{\Gamma(\alpha_j)}\|k_j\|_{\varphi_{2_j}}\big\|g_{3_j}\big\|_{N_j}\|y-z\|_\varphi\\
&\leq \ \left(\|g_{2_j}\|_{\varphi_j}+\frac{4k_{1_j}\cdot r\|k_j\|_{\varphi_{2_j}}}{\Gamma(\alpha_j)}\|g_{1_j}\|_{\varphi_{1_j}}\big\|g_{3_j}\big\|_{N_j}\right)\|y-z\|_\varphi. \qquad (3.3)
\end{aligned}
$$

By substituting from (3.1) and (3.3) in (3.2), we obtain

$$
\begin{aligned}
\|y-z\|_\varphi \ &\leq \ \Bigg[K\left(\|g_{2_1}\|_{\varphi_1}+\frac{4k_{1_1}\cdot r\|k_1\|_{\varphi_{2_1}}}{\Gamma(\alpha_1)}\|g_{1_1}\|_{\varphi_{1_1}}\big\|g_{3_1}\big\|_{N_1}\right)\prod_{i=2}^{n}L_i\\
&\quad +KL_1\left(\|g_{2_2}\|_{\varphi_2}+\frac{4k_{1_2}\cdot r\|k_2\|_{\varphi_{2_2}}}{\Gamma(\alpha_2)}\|g_{1_2}\|_{\varphi_{1_2}}\big\|g_{3_2}\big\|_{N_2}\right)\prod_{i=3}^{n}L_i\\
&\quad +...+K\left(\|g_{2_n}\|_{\varphi_n}+\frac{4k_{1_n}\cdot r\|k_n\|_{\varphi_{2n}}}{\Gamma(\alpha_n)}\|g_{1_n}\|_{\varphi_{1n}}\big\|g_{3_n}\big\|_{N_n}\right)\prod_{i=1}^{n-1}L_i\Bigg]\|y-z\|_\varphi\\
&= \ C\cdot\|y-z\|_\varphi.
\end{aligned}
$$

Since $C<1$, we get $y=z$ (a.e.), and we have finished. □

4. Examples

We need to provide some examples to demonstrate our results.

Example 4.1. *Put the N-functions $M_i(u)=N_i(u)=u^2$ and $\varphi_{2_i}(u)=\exp|u|-|u|-1$. We shall show that $J_i^{\alpha_i}: L_{N_i}\to L_{\varphi_{2_i}},\ i=1,\cdots,n$ are continuous, and Lemma 2.1 is verified.*
Indeed: For $s\in[1,e]$ and any $\alpha_i\in(0,1)$, we have

$$k_i(s)=\int_0^s M_i(\tau^{\alpha_i-1})\,d\tau=\int_0^s\tau^{2\alpha_i-2}\,d\tau=\frac{s^{2\alpha_i-1}}{2\alpha_i-1}.$$

Moreover,

$$\int_1^e\varphi_{2_i}(k_i(s))\,d\tau=\int_1^e\left(e^{\frac{s^{2\alpha_i-1}}{2\alpha_i-1}}-\frac{s^{2\alpha_i-1}}{2\alpha_i-1}-1\right)ds<\infty.$$

Thus for $y\in L_{N_i}$, we get that $J_i^{\alpha_i}: L_{N_i}\to L_{\varphi_{2_i}}$ is continuous.

Remark 4.1. *For more details and information about the acting and continuity assumptions of $G_i(y)=g_i(s)\cdot y(s)$, (see our assumption (N5) and [15, Theorem 18.2]).*

Example 4.2. *Let $G_{j_i}(y)(s)=g_i(s)\cdot y(s),\ j=1,2,3,$ and $i=1,\cdots n$, then we have*

$$y(s)=\prod_{i=1}^{n}\left(h_i(s)+g_{2_i}(s)\cdot y(s)+g_{1_i}(s)\cdot y(s)\int_1^s\left(\log\frac{s}{\tau}\right)^{\alpha_i-1}\frac{g_{3_i}(\tau)\cdot y(\tau)}{\tau}\,d\tau\right),\ \alpha_i\in(0,1),\ s\in[1,e],$$

which provides a special case of Eq (1.1).

5. Conclusions

The current study demonstrates and studies two existence theorems, namely, (the existence and the uniqueness) the monotonic solutions for a general and abstract form of a product of n-quadratic Hadamard-type fractional integral equations in Orlicz spaces L_φ. The measure of non-compactness associated with Darbo's fixed-point theorem and fractional calculus are the main tools used to obtain our results in L_φ-spaces. For the upcoming work in this direction, we will look for some numerical solutions for similar problems in different function spaces.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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