Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 10997–11006.
DOI:10.3934/math.2024538
Received: 09 January 2024
Revised: 11 March 2024
Accepted: 13 March 2024
Published: 20 March 2024

Research article

Strongly convex functions and extensions of related inequalities with applications to entropy

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Abstract: We extended the Mercer inequlaity, Fejér-Hermite-Hadamard, and Jensen inequalities for strongly convex functions. Moreover, we obtained several results in information theory and mathematical analysis using obtained inequalities.

Keywords: Jensens inequality; entropy; strongly convex; Mercer inequality
Mathematics Subject Classification: 26B25, 26D20

1. Introduction

For $a, b \in \mathbb{R}, a < b$, and $I := [a, b]$, the function $\varphi : I \to \mathbb{R}$ is convex if

$$\varphi\left(tx_1 + (1-t)x_2\right) \le t\varphi(x_1) + (1-t)\varphi(x_2)$$

and φ is named strongly convex (S-C) with modulus k if

$$\varphi\left(tx_1 + (1-t)x_2\right) \le t\varphi(x_1) + (1-t)\varphi(x_2) - kt(1-t)\left(x_1 - x_2\right)^2,$$

for all $x_1, x_2 \in I$ and all $t \in [0, 1]$

Mercer inequality [8], Hermite-Hadamard inequality (H-H inequality) [6], and Jensen's inequality [7] are some types of important inequalities in different fields of mathematical analysis and optimization.

In [4], Fejér investigated an extension of the H-H inequality. Azócar et al. [1] obtained the following Fejér inequality for the S-C function.

Because of the comprehensive results of the Jensen's, Mercer, and H-H inequalities, some researchers exteneded their studies via mappings of various types (see [2, 3, 10, 17, 18]).

The theory of convex function is significant in entropy estimation and optimization [11–16].

An equivalent condition for the convexity of a continuous function is given in [21]. Also, in [22], S. Zlobec generalized some of the basic integral properties of convex functions.

Throughout this article, suppose that $\mathbf{x} = \{x_i\} \subseteq [a, b]$ and $\mathbf{t} = \{t_i\}, 0 \leq t_i$ with $\sum_{i=1}^{n} t_i = 1$.

Theorem 1.1. *[9] Suppose that $\varphi : I \to \mathbb{R}$ is an S-C function with modulus k, then*

$$\varphi\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i \varphi(x_i) - kV(\mathbf{x}),$$

for all $x_1, x_2, \ldots, x_n \in [a, b], t_i \geq 0 (i = 1, \ldots, n)$ with $\sum_{i=1}^{n} t_i = 1, \bar{\mathbf{x}} = \sum_{i=1}^{n} t_i x_i$, and $V(\mathbf{x}) := \sum_{i=1}^{n} t_i (x_i - \bar{\mathbf{x}})^2$.

Definition 1.1. *The Shannon entropy of a probability distribution* **t** *is defined by*

$$H(\mathbf{t}) := -\sum_{i=1}^{n} t_i \log(t_i).$$

Proposition 1.1. *[19] Let $\eta := \min\{t_i : i = 1, \ldots, n\}$ and $\vartheta := \max\{t_i : i = 1, \ldots, n\}$, then*

$$\mathbf{m}(\eta, \vartheta) := \eta \log\left(\frac{2\eta}{\eta + \vartheta}\right) + \vartheta \log\left(\frac{2\vartheta}{\eta + \vartheta}\right) \leq \log n - H(\mathbf{t}) \leq \log\left(\frac{(\eta + \vartheta)^2}{4\eta\vartheta}\right) := \mathbf{M}(\eta, \vartheta).$$

Our main goal is to obtain some intersting Jensen, Mercer, and Fejér-Hermite-Hadamard inequalities for S-C functions. Furthermore, we use those inequalities in mathematical analysis and the Shannon entropy to obtain a strong bound for entropy of a probability distribution.

2. Main results

In this section, we generelize the Mercer, Jensen type, and Fejér-Hermite-Hadamard inequalities for S-C functions.

Lemma 2.1. *Suppose that $\varphi : I \to \mathbb{R}$ is an S-C function with modulus k. If $w_1, w_2, w_3 \in I$, and $w_1 < w_2 < w_3$, then*

(i) $\frac{\varphi(w_2)-\varphi(w_1)}{2} \leq \varphi\left(\frac{w_2+w_3}{2}\right) - \varphi\left(\frac{w_1+w_3}{2}\right) - \frac{k}{4}(w_2 - w_1)(2w_3 - w_2 - w_1)$,

(ii) $\frac{\varphi(w_3)-\varphi(w_2)}{2} \geq \varphi\left(\frac{w_1+w_3}{2}\right) - \varphi\left(\frac{w_1+w_2}{2}\right) + \frac{k}{4}(w_3 + w_2 - 2w_1)(w_3 - w_2)$.

Proof. Since $w_1 < w_2 < \frac{w_2+w_3}{2} < w_3$, there are $s, t \in [0, 1]$, $s + t = 1$ such that $w_2 = s\left(\frac{w_2+w_3}{2}\right) + tw_1$. Therefore,

$$\begin{aligned}
&\frac{\varphi(w_1) - \varphi(w_2)}{2} + \varphi\left(\frac{w_2 + w_3}{2}\right) \\
&= \frac{1}{2}\left[\varphi(w_1) - \varphi\left(tw_1 + s\left(\frac{w_2 + w_3}{2}\right)\right)\right] + \varphi\left(\frac{w_2 + w_3}{2}\right) \\
&\geq \frac{1}{2}\left[\varphi(w_1) - \left(t\varphi(w_1) + s\varphi\left(\frac{w_2 + w_3}{2}\right) - kts\left(\frac{w_2 + w_3}{2} - w_1\right)^2\right)\right] + \varphi\left(\frac{w_2 + w_3}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s}{2}\varphi(w_1) + \frac{2-s}{2}\varphi\left(\frac{w_2+w_3}{2}\right) + \frac{k}{2}ts\left(\frac{w_2+w_3}{2} - w_1\right)^2 \\
&\geq \varphi\left(\frac{s}{2}w_1 + \frac{2-s}{2}\left(\frac{w_2+w_3}{2}\right)\right) + \left(k\frac{s(2-s)}{4} + \frac{k}{2}ts\right)\left(\frac{w_2+w_3}{2} - w_1\right)^2 \\
&= \varphi\left(\frac{s}{2}w_1 + \left(\frac{w_2+w_3}{2}\right) - \frac{1}{2}(w_2 - tw_1)\right) + \frac{k}{4}\left(4s - 3s^2\right)\left(\frac{w_2+w_3-2w_1}{2}\right)^2 \\
&= \varphi\left(\frac{w_1+w_3}{2}\right) + \frac{k}{4}\left(\frac{4(w_2-w_1)(2w_3-w_2-w_1)}{(w_2+w_3-2w_1)^2}\right)\left(\frac{w_2+w_3-2w_1}{2}\right)^2 \\
&= \varphi\left(\frac{w_1+w_3}{2}\right) + \frac{k}{4}(w_2-w_1)(2w_3-w_2-w_1).
\end{aligned}
$$

Similarly, we obtain (ii) by putting $w_2 = t\left(\frac{w_1+w_2}{2}\right) + sw_3$, where $w_1 < \frac{w_1+w_2}{2} < w_2 < w_3$. □

Theorem 2.1. *Assume that $\varphi : [a,b] \to \mathbb{R}$ is an S-C function with modulus k and*

$$
\Delta_k(p,q) = \varphi(p) + \varphi(q) - 2\varphi\left(\frac{p+q}{2}\right) - \frac{k}{2}(p-q)^2,
$$

where $a \leq p, q \leq b$, then

$$
\max_{p,q} \Delta_k(p,q) = \Delta_k(a,b). \tag{2.1}
$$

Proof. Taking $w_1 = a, w_2 = p$, and $w_3 = b$ in the part (i) of Lemma 2.1 and $w_1 = p, w_2 = q$, and $w_3 = b$ in the part (ii) of Lemma 2.1, we gain

$$
\frac{\varphi(p) - \varphi(a)}{2} \leq \varphi\left(\frac{p+b}{2}\right) - \varphi\left(\frac{a+b}{2}\right) - \frac{k}{4}(p-a)(2b-p-a), \tag{2.2}
$$

$$
\frac{\varphi(b) - \varphi(q)}{2} \geq \varphi\left(\frac{p+b}{2}\right) - \varphi\left(\frac{p+q}{2}\right) + \frac{k}{4}(b-q)(b+q-2p), \tag{2.3}
$$

respectively.

By (2.3), we get

$$
\frac{\varphi(q) - \varphi(b)}{2} \leq \varphi\left(\frac{p+q}{2}\right) - \varphi\left(\frac{p+b}{2}\right) - \frac{k}{4}(b-q)(b+q-2p). \tag{2.4}
$$

Now, from (2.2) and (2.4), we conclude (2.1). □

Corollary 2.1. *Assume that $\varphi : [a,b] \to \mathbb{R}$ is an S-C function with modulus k and $x \in [a,b]$, then*

$$
\varphi(a+b-x) - \frac{k}{2}(a+b-2x)^2 \leq \varphi(a) + \varphi(b) - \varphi(x) - \frac{k}{2}(b-a)^2. \tag{2.5}
$$

Proof. Replacing p by x and q by $a+b-x$ in Theorem 2.1 gives the desired result. □

Lemma 2.2. *Let $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$ and let $\varphi : [a,b] \to \mathbb{R}$ be an S-C function with modulus k, then*

$$
\varphi\left(\lambda(a+b) + (\mu - \lambda)\sum_{i=1}^{n} t_i x_i\right) \leq \lambda\varphi(a) + \lambda\varphi(b) + (\mu - \lambda)\sum_{i=1}^{n} t_i \varphi(x_i) \tag{2.6}
$$

$$-k\left[\sum_{i=1}^{n} t_i(x_i-\bar{\mathbf{x}})^2+\frac{\lambda}{2}(b-a)^2+\lambda\mu\,(a+b-2\bar{\mathbf{x}})^2-\frac{\lambda}{2}\sum_{i=1}^{n} t_i\,(a+b-2x_i)^2\right],$$

for all $x_1, x_2, \ldots, x_n \in [a,b], t_i \geq 0$ $(1 \leq i \leq n)$ *with* $\sum_{i=1}^{n} t_i = 1$ *and* $\bar{\mathbf{x}} = \sum_{i=1}^{n} t_i x_i$.

Proof. By applying Theorem 1.1 and (2.5), we obtain

$$\begin{aligned}
&\varphi\left(\lambda(a+b)+(\mu-\lambda)\sum_{i=1}^{n} t_i x_i\right)=\varphi\left(\lambda\sum_{i=1}^{n} t_i(a+b-x_i)+\mu\sum_{i=1}^{n} t_i x_i\right)\\
&\leq \lambda\varphi\left(\sum_{i=1}^{n} t_i(a+b-x_i)\right)+\mu\varphi\left(\sum_{i=1}^{n} t_i x_i\right)-k\lambda\mu\left(a+b-2\sum_{i=1}^{n} t_i x_i\right)^2\\
&\leq \lambda\left(\sum_{i=1}^{n} t_i\varphi(a+b-x_i)-k\sum_{i=1}^{n} t_i(x_i-\bar{\mathbf{x}})^2\right)+\mu\varphi\left(\sum_{i=1}^{n} t_i x_i\right)-k\lambda\mu\left(a+b-2\sum_{i=1}^{n} t_i x_i\right)^2\\
&\leq \lambda\left(\varphi(a)+\varphi(b)-\sum_{i=1}^{n} t_i\varphi(x_i)-\frac{k}{2}(b-a)^2+\frac{k}{2}\sum_{i=1}^{n} t_i\,(a+b-2x_i)^2\right)\\
&\quad -k\lambda\sum_{i=1}^{n} t_i(x_i-\bar{\mathbf{x}})^2+\mu\sum_{i=1}^{n} t_i\varphi(x_i)-k\mu\sum_{i=1}^{n} t_i(x_i-\bar{\mathbf{x}})^2-k\lambda\mu\,(a+b-2\bar{\mathbf{x}})^2\\
&=\lambda\varphi(a)+\lambda\varphi(b)+(\mu-\lambda)\sum_{i=1}^{n} t_i\varphi(x_i)+\frac{k\lambda}{2}\sum_{i=1}^{n} t_i\,(a+b-2x_i)^2\\
&\quad -k\left[\sum_{i=1}^{n} t_i(x_i-\bar{\mathbf{x}})^2+\frac{\lambda}{2}(b-a)^2+\lambda\mu\,(a+b-2\bar{\mathbf{x}})^2\right].
\end{aligned}$$

□

Corollary 2.2. *Let* $\lambda, \mu \in [0,1]$ *with* $\lambda+\mu=1$*, and let* φ *be an S-C function with modulus* k *on* $[a,b]$ *and* $x \in [a,b]$*, then*

$$\begin{aligned}
&\varphi(\lambda(a+b)+(\mu-\lambda)x)-\frac{k\lambda}{2}(a+b-2x)^2\\
&\leq \lambda\varphi(a)+\lambda\varphi(b)+(\mu-\lambda)\varphi(x)-k\lambda\left[\frac{1}{2}(b-a)^2+\mu(a+b-2x)^2\right].
\end{aligned}$$

Corollary 2.3. *Let* φ *be an S-C function with modulus* k *on* $[a,b]$ *and* $s,t,\lambda,\mu \in [0,1]$ *with* $s+t=1$ *and* $\lambda+\mu=1$*, then*

$$\begin{aligned}
&\varphi(\lambda(a+b)+(\mu-\lambda)(sa+tb))+(\lambda-\mu)\varphi(sa+tb)-\frac{k\lambda}{2}(s-t)^2(b-a)^2\\
&\leq \lambda\varphi(a)+\lambda\varphi(b)-k\lambda\left[\frac{1}{2}(b-a)^2+\lambda\mu(a+b-2sa-2tb)^2\right].
\end{aligned}$$

Proof. Substitute x by $sa+tb$ in Corollary 2.2 to get the inequality. □

Corollary 2.4. *Let* φ *be an S-C function with modulus* k *on* $[a,b]$ *and* $s,t \in [0,1]$, $s+t=1$*, then*

$$2\varphi\left(\frac{a+b}{2}\right)\leq\varphi(sa+tb)+\varphi(ta+sb)-\frac{k}{2}(s-t)^2(b-a)^2\leq\varphi(a)+\varphi(b)-\frac{k}{2}(b-a)^2.$$

Proof. Set $\lambda = 1$ in Corollary 2.3 to get the righthand side of the inequality. On the other hand, by definition, we have

$$\varphi\left(\frac{a+b}{2}\right) = \varphi\left(\frac{sa+tb}{2} + \frac{ta+sb}{2}\right) \le \frac{1}{2}\varphi(sa+tb) + \frac{1}{2}\varphi(ta+sb) - \frac{k}{4}(s-t)^2(a-b)^2.$$

□

The Corollary 2.4 shows a kind of pre-H-H inequalities.

Next, we prove a generalization of the H-H type inequality for S-C functions.

Theorem 2.2. *Assume that $\varphi : [a,b] \to \mathbb{R}$ is an S-C function with modulus k and ω is a nonnegative function on $[a,b]$, then*

$$\begin{aligned} 2\varphi\left(\frac{a+b}{2}\right)\int_a^b \omega(u)du &\le \int_a^b (\omega(u) + \omega(a+b-u))\varphi(u)du - \frac{k}{2}\int_a^b (a+b-2u)^2\omega(u)du \\ &\le \left(\varphi(a) + \varphi(b) - \frac{k}{2}(b-a)^2\right)\int_a^b \omega(u)du. \end{aligned}$$

Proof. By multiplying the two-sided inequality in Corollary 2.4 with $\omega(a + t(b-a))$, integrating with respect to variable t from 0 to 1, and by interchanging $u = a + t(b-a)$, we get

$$\begin{aligned} 2\varphi\left(\frac{a+b}{2}\right)\int_a^b \omega(u)du &\le \int_a^b (\varphi(u) + \varphi(a+b-u))\omega(u)du - \frac{k}{2}\int_a^b (a+b-2u)^2\omega(u)du \\ &\le \left(\varphi(a) + \varphi(b) - \frac{k}{2}(b-a)^2\right)\int_a^b \omega(u)du, \end{aligned}$$

since

$$\int_a^b (\omega(u) + \omega(a+b-u))\varphi(u)du = \int_a^b (\varphi(u) + \varphi(a+b-u))\omega(u)du.$$

□

Corollary 2.5. *Let $\varphi : [a,b] \to \mathbb{R}$ be an S-C function with modulus k, then*

$$\varphi\left(\frac{a+b}{2}\right) + \frac{k}{12}(b-a)^2 \le \frac{1}{b-a}\int_a^b \varphi(u)du \le \frac{\varphi(a)+\varphi(b)}{2} - \frac{k}{6}(b-a)^2.$$

Proof. Putting $\omega \equiv 1$ in Theorem 2.2 and after some calculations, the desired inequality follows. □

If we consider $\omega(u) = u^{\alpha-1}$ in Theorem 2.2, then we have the following result.

Corollary 2.6. *Let $\alpha > 0$, $0 < a < b$, and $\varphi : [a,b] \to \mathbb{R}$ be an S-C function with modulus k, then*

$$\begin{aligned} 2\varphi\left(\frac{a+b}{2}\right) \le& \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b (u^{\alpha-1} + (a+b-u)^{\alpha-1})\varphi(u)du - \frac{k\alpha}{2\,(b^\alpha - a^\alpha)}\int_a^b (a+b-2u)^2 u^{\alpha-1}du \\ \le& \varphi(a) + \varphi(b) - \frac{k}{2}(b-a)^2. \end{aligned}$$

If $\alpha \to 0^+$ in Corollary 2.6, then we get the next corollary.

Corollary 2.7. *Let* $0 < a < b$ *and* $\varphi : [a, b] \to \mathbb{R}$ *be an S-C function with modulus k, then*

$$2\varphi\left(\frac{a+b}{2}\right)\frac{\log\left(\frac{b}{a}\right)}{a+b} \leq \int_a^b \frac{\varphi(u)}{u(a+b-u)}du - \frac{k}{2}\left((a+b)^2\log\left(\frac{b}{a}\right) - 2b^2 + 2a^2\right)$$
$$\leq \left(\varphi(a) + \varphi(b) - \frac{k}{2}(b-a)^2\right)\frac{\log\left(\frac{b}{a}\right)}{a+b}.$$

Let $\varphi : [a, b] \to \mathbb{R}$ be an S-C function with modulus k and $0 \leq s, t \leq 1$ such that $s + t = 1$. We define

$$\Delta_k^*(s, t, x, y) = s\varphi(x) + t\varphi(y) - \varphi(sx + ty) - kst(x - y)^2,$$

for all $x, y \in [a, b]$.

Theorem 2.3. *Let* $\varphi : [a, b] \to \mathbb{R}$ *be an S-C function with modulus k, then*

$$\max_{s,t\in\mathbf{t};x,y\in[a,b]} \Delta_k^*(s, t, x, y) \leq \Delta_k(a, b).$$

Proof. First, we prove that

$$\Delta_k^*(s, t, x, y) \leq \Delta_k(x, y),$$

for all $s, t \in \mathbf{t}$ and $x, y \in [a, b]$.

Now,

$$\begin{aligned}
&\Delta_k(x, y) - \Delta_k^*(s, t, x, y)\\
&= t\varphi(x) + s\varphi(y) + \varphi(sx + ty) + kst(x - y)^2 - 2\varphi\left(\frac{x+y}{2}\right) - \frac{k}{2}(x - y)^2\\
&\geq \varphi(tx + sy) + \varphi(sx + ty) + 2kst(x - y)^2 - \frac{k}{2}(x - y)^2 - 2\varphi\left(\frac{x+y}{2}\right)\\
&= \varphi(tx + sy) + \varphi(sx + ty) - \frac{k}{2}(s - t)^2(x - y)^2 - 2\varphi\left(\frac{x+y}{2}\right)\\
&\geq 2\varphi\left(\frac{(tx + sy) + (sx + ty)}{2}\right) - 2\varphi\left(\frac{x+y}{2}\right)\\
&= 0.
\end{aligned}$$

The remains of the proof is an application of Theorem 2.1. □

Theorem 2.4. *Suppose that* $\varphi : [a, b] \to \mathbb{R}$ *is an S-C function with modulus k, then*

$$\begin{aligned}
\mathcal{J}_\varphi(\mathbf{t}, \mathbf{x}) :&= \sum_{i=1}^n t_i\varphi(x_i) - \varphi\left(\sum_{i=1}^n t_i x_i\right)\\
&\leq \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right) - \frac{k}{4}(b - a)^2 - k(b - x_r)(x_r - a),
\end{aligned}$$

where $(b - x_r)(x_r - a) = \min_i \{(b - x_i)(x_i - a)\}$.

Proof. Since $x_i \in [a,b]$, there is a sequence $\{\lambda_i\}, \lambda_i \in [0,1]$ with $x_i = \lambda_i a + (1-\lambda_i)b$ for $i = 1,2,\ldots$. Thus,

$$\sum_{i=1}^{n} t_i\varphi(x_i) - \varphi\left(\sum_{i=1}^{n} t_i x_i\right) = \sum_{i=1}^{n} t_i\varphi(\lambda_i a + (1-\lambda_i)b) - \varphi\left(\sum_{i=1}^{n} t_i\left(\lambda_i a + (1-\lambda_i)b\right)\right)$$
$$\leq \sum_{i=1}^{n} t_i\left(\lambda_i\varphi(a) + (1-\lambda_i)\varphi(b) - k\lambda_i(1-\lambda_i)(b-a)^2\right) - \varphi\left(a\sum_{i=1}^{n} t_i\lambda_i + b\sum_{i=1}^{n} t_i(1-\lambda_i)\right).$$

Setting $\lambda := \sum_{i=1}^{n} t_i\lambda_i$ and $\mu := 1 - \sum_{i=1}^{n} t_i\lambda_i$, by the use of Theorem 2.3, we gain

$$\mathcal{J}_\varphi(\mathbf{t},\mathbf{x}) \leq \lambda\varphi(a) + \mu\varphi(b) - \varphi(\lambda a + \mu b) - k(b-a)^2\sum_{i=1}^{n} t_i\lambda_i(1-\lambda_i)$$
$$\leq \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right) - \frac{k}{2}(b-a)^2 + k\lambda\mu(b-a)^2 - k\lambda_r(1-\lambda_r)(b-a)^2,$$

because

$$\lambda_r(1-\lambda_r) = \frac{(b-x_r)(x_r-a)}{(b-a)^2} \leq \frac{(b-x_i)(x_i-a)}{(b-a)^2} = \lambda_i(1-\lambda_i),$$

for all $i = 1,2,\ldots,n$. Hence,

$$\mathcal{J}_\varphi(\mathbf{t},\mathbf{x}) \leq \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right) - \frac{k}{4}(b-a)^2 - k(b-x_r)(x_r-a).$$

□

3. Applications

3.1. Applications in analysis

In the following, we obtain an application of Lemma 2.2, which improves the results from [8].

Proposition 3.1. *Let* $0 < a \leq x_i \leq b$, $t_i \geq 0$ $(i = 1,2,\ldots,n)$ *with* $\sum_{i=1}^{n} t_i = 1, \lambda, \mu \geq 0$, *and* $\lambda + \mu = 1$, *then*

$$\tilde{G}_\lambda \leq \tilde{A}_\lambda e^{-\frac{1}{2b^2}\left(V(\mathbf{x})+\frac{1}{2}(b-a)^2-\frac{\lambda}{2}\bar{\mathbf{y}}\right)} \leq \tilde{A}_\lambda, \tag{3.1}$$

where $\tilde{A}_\lambda := \lambda(a+b) + (\mu-\lambda)\bar{\mathbf{x}}, \tilde{G}_\lambda := \frac{(ab)^\lambda}{\prod_{i=1}^{n} x_i^{t_i(\lambda-\mu)}}, y_i := (a+b-2x_i)^2$, *and* $\mathbf{y} := \{y_i\}$.

Proof. Letting $\varphi(x) := -\log(x)$ and $k = \frac{1}{2b^2}$ in Lemma 2.2, the desired inequality follows. □

Remark 3.1. *Putting* $\lambda = 1$ *in* (3.1), *we obtain*

$$\tilde{G} \leq \tilde{A}e^{-\frac{1}{2b^2}\left(V(\mathbf{x})+\frac{1}{2}(b-a)^2-\frac{1}{2}\bar{\mathbf{y}}\right)} \leq \tilde{A},$$

where $\tilde{A} = a + b - \bar{\mathbf{x}}$ *and* $\tilde{G} = \frac{ab}{\prod_{i=1}^{n} x_i^{t_i}}$ *(see [8]).*

Example 3.1. *Assume that $\beta \geq 2$ and $0 < a < b$, then $\varphi(x) = x^\beta$ is an S-C function with modulus $k := \frac{\beta(\beta-1)}{2}a^{\beta-2}$ on $[a, b]$. Further, if $\beta = 2n$ $(n = 1, 2, \dots)$, then $\varphi(x) = x^\beta$ is an S-C function with modulus k on arbitrary interval $[a, b]$.*

Proof. Let $x, y \in [a, b]$. Define the following functions on $[0, 1]$:

$$g(t) := (tx + (1-t)y)^\beta + \frac{\beta(\beta-1)}{2}a^{\beta-2}t(1-t)(x-y)^2$$

and

$$h(t) := tx^\beta + (1-t)y^\beta.$$

Since $g(0) = h(0)$, $g(1) = h(1)$, $h'' \equiv 0$, and

$$g''(t) = \beta(\beta-1)(x-y)^2\left((tx + (1-t)y)^{\beta-2} - a^{\beta-2}\right) \geq 0,$$

$g(t) \leq h(t)$ for every $t \in [0, 1]$. Therefore, x^β is an S-C function with modulus $\frac{\beta(\beta-1)}{2}a^{\beta-2}$ on $[a, b]$. □

In the next proposition, we give an extension of the pre-Grüss inequality (see [5, 20]):

$$\sum_{i=1}^{n} t_i x_i^2 - \left(\sum_{i=1}^{n} t_i x_i\right)^2 \leq \frac{1}{4}(b-a)^2.$$

Proposition 3.2. *Assume that $x_i \in [a, b]$ and $\{t_i\} \in \mathbf{t}$, then*

$$\sum_{i=1}^{n} t_i x_i^2 - \left(\sum_{i=1}^{n} t_i x_i\right)^2 \leq \frac{1}{4}(b-a)^2 - (b-x_r)(x_r-a),$$

where $(b-x_r)(x_r-a) = \min_i \{(b-x_i)(x_i-a)\}$.

Proof. It follows from Example 3.1 and Theorem 2.4 with $\varphi(x) = x^2$. □

3.2. Applications to entropy

In this subsection, new Shannon entropy bounds are found, that improve the entropy bounds from [19].

Proposition 3.3. *Assume that $\eta := \min\{t_i : i = 1, \dots, n\}$ and $\vartheta := \max\{t_i : i = 1, \dots, n\}$, then*

$$0 \leq \log n - H(\mathbf{t}) \leq \log\left(\frac{(\eta+\vartheta)^2}{4\eta\vartheta}\right) - \frac{(\vartheta-\eta)^2}{8\vartheta^2} := \Gamma(\eta, \vartheta) \leq \mathbf{M}(\eta, \vartheta).$$

Proof. Replace $\varphi(x)$ by $-\log(x)$ in Theorem 2.4 and consider $k = \frac{1}{2b^2}$, $x_i := \frac{1}{t_i}$ for all $i = 1, \dots, n$. □

Proposition 3.4. *Assume that $\varphi : [a, b] \to \mathbb{R}$ is an S-C function with modulus k, then*

$$\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i) - \varphi\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) \leq \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right) - \frac{k}{4}(b-a)^2 - k(b-x_r)(x_r-a),$$

where $(b-x_r)(x_r-a) = \min_i \{(b-x_i)(x_i-a)\}$.

Proof. Set $t_i = \frac{1}{n}$ for every $i = 1, \ldots, n$ in Theorem 2.4. □

Proposition 3.5. *Assume that* $\eta := \min\{t_i : i = 1, \ldots, n\}$ *and* $\vartheta := \max\{t_i : i = 1, \ldots, n\}$, *then*

$$0 \leq \log n - H(\mathbf{t}) \leq n \left\{ \eta \log \left(\frac{2\eta}{\eta + \vartheta} \right) + \vartheta \log \left(\frac{2\vartheta}{\eta + \vartheta} \right) - \frac{1}{8\vartheta} (\vartheta - \eta)^2 \right\} := \Lambda(\eta, \vartheta).$$

Proof. Replace $\varphi(x)$ by $x \log(x)$ in Proposition 3.4 and modulus $k = \frac{1}{2b}$, $x_i := t_i$ for all $i = 1, \ldots, n$. □

Example 3.2. *Let* $j \geq 2$ *be an integer,* $n = 10^j$, $\eta = 10^{-j-1}$, *and* $\vartheta = 10^{-j+1}$, *then*

$$\mathbf{M}(\eta, \vartheta) - \Gamma(\eta, \vartheta) \simeq 0.1225$$

and

$$n.\mathbf{m}(\eta, \vartheta) - \Lambda(\eta, \vartheta) \simeq 12.25.$$

4. Conclusions

In this work, we have stablished some new inequalities such as the Mercer, Fejér-Hermite-Hadamard, and Jensen inequalities for strongly convex functions. Next, using these inequalities, we get some applications in analysis and entropy of probability distributions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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