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Research article

Study tsunamis through approximate solution of damped geophysical Korteweg-de Vries equation

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Abstract: The article studied tsunami waves with consideration of important wave properties such as velocity, width, and collision through finding an approximate solution to the damped geophysical Korteweg-de Vries (dGKdV). The addition of the damping term in the GKdV is a result of studying the nonlinear waves in bounded nonplanar geometry. The properties of the wave in bounded nonplanar geometry are different than the unbounded planar geometry, as many experiments approved. Thus, this work reported for the first time the analytical solution for the dGKdV equation using the Ansatz method. The used method assumed a suitable hypothesis and the initial condition of the GKdV. The GKdV is an integrable equation and the solution can be found by several known methods either analytically or numerically. On the other hand, the dGKdV is a nonintegrable equation and does not have an initial exact solution, and this is the challenge. In this work, the novel Ansatz method proved its ability to reach the approximate solution of dGKdV and presented the effect of the damping term as well as the Coriolis effect term in the amplitude of the wave. The advantage of the Ansatz method was that the obtained solution was in a general solution form depending on the exact solution of GKdV. This means the variety of nonlinear wave structures like solitons, lumps, or cnoidal can be easily investigated by the obtained solution. We realized that the amplitude of a tsunami wave decreases if the Coriolis term or damping term increases, while it increases if wave speed increases.

Keywords: analytical solution; ansatz method; geophysical Korteweg-de Vries; damped equation
Mathematics Subject Classification: 34K25, 34C27, 34D20, 92D25

1. Introduction

One of the serious problems on the earth is tsunami waves. Tsunami is a type of wave that appears in the oceans and occurs by large earthquakes, submarine landslides, or volcanic eruptions. The life of tsunami waves simply includes three parts: Generation, propagation, and run-up. In this paper, we are concerned about the propagation part [1]. The main features of tsunamis are the height of tsunami waves increasing once the waves approach the mainland, the speed of the tsunami waves depending on the depth of the ocean [2] and, tsunami waves are very fast in deep water, and slow down in shallow water [3].

Studying the mechanism of the waves mathematically can be done either by building a mathematical model or using an existing differential equation as a tool to analyze and investigate. The mathematical model for tsunami waves was introduced in reference [4]. Tsunamis in shallow water can be studied by nonlinear shallow water equations, Korteweg-de Vries (KdV), or the geophysical Korteweg-de Vries (GKdV). In this article, the tsunami wave is analyzed and discussed in the framework of GKdV. In addition, we will study the effect of damping terms in the GKdV which means we will study the damped geophysical Korteweg-de Vries (dGKdV) equation.

This article is organized as follows: We introduce briefly the importance of GKdV in Section 1. In Section 2, we will present the tsunami wave model as an application for the GKdV equation in the ocean. Section 3 will describe the Ansatz method through the dGKdV. Section 4 is a discussion of the results and the final section is the conclusion of the work.

2. Model of tsunami wave

The tsunami wave has four stages. Stage one starts when the wave is generated via an earthquake in deep water, which can be very devastating waves. In the second stage, the tsunami wave takes a shape of depression, elevation, or a combination of both of them in the deep water where the tsunami wave becomes a long wave with a small amplitude and unchangeable speed [5–7]. Assume the speed of the tsunami wave is denoted by c, which is also dependent on the ocean depth. In the third stage, the tsunami wave propagates from deep water to shallow water; through this stage, the wavelength decreases and wave amplitude increases. This stage indicates wave dispersion and nonlinearity. The fourth stage is when the tsunami wave reaches the coast carrying debris. The third stage is very important because of the effect of nonlinearity when the wave transverses the deep water.

We note that the theoretical studies of the tsunami wave gave more attention to the depression wave than the evaluation despite both types of waves having the same damping [8–11]. Most of these studies investigated the tsunami wave near shallow water and used the nonlinear shallow water equation to study the effect of the wave mass, amplitude, and polarity. The results of the previous studies showed that the nonlinear steepness effect is more when the type of wave in the third stage is depression. It is very important to capture the effects of wave number dispersion when the tsunami propagates to the coast. Consider the surface water waves as an elevation case. The KdV equation is a very well-known and significant equation that describes the propagation of waves in oceans and shallow water. Also, it used to study the weakly linear and nonlinear waves in several media, such as optical fiber, plasma, liquids composing of gas bubbles, and turbulence [12–17]. The general form of the KdV is

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{2.1}$$

where α and β denote the coefficients of the nonlinear and dispersion terms, respectively. The sign of α represents the polarity of the waves. The Coriolis effect term is important to study the effect of Earth's turning on the flow of tsunamis. The KdV with additional term for the Coriolis effect is known as the nonlinear GKdV equation [18] and is governed by

$$u_t - \omega u_x + \alpha u u_x + \beta u_{xxx} = 0, \tag{2.2}$$

where u is the free surface elevation and ω is the constant related to the Coriolis effect.

The Eq (2.2) has been solved by several numerical and analytical methods. In ref. [19], the authors construct the analytical solutions using the extended direct algebraic method (EDAM), the (G'/G)-expansion method, and the extended simple equation method (ESEM). The semi-analytical solution for GKdV is computed by the homotopy perturbation method (HPM), the Adomain decomposition method (ADM), and the ADM- Pade approximation technique [20]. The numerical solutions for Eq (2.2) have been found by the finite element method [21]. Karunakar and Chakraverty [22] found the exact solitary wave solution of the Eq (2.2) as

$$u(x,t) = (2 + c + \omega_0) sech^2\left(\sqrt{\frac{3}{2}(c + \omega_0}(x - ct)\right), \tag{2.3}$$

where c is the wave speed. All the attempts in literature for solving the GKdV are in the unbounded planar geometry. The experimental applications in plasma found that the properties of propagation on bounded nonplanar geometry such as velocity, width, and density are different than on unbounded planar geometry [23]. The tsunami waves propagate through the entire depth of the ocean and their speed is different in shallow water than in deep ocean. Thus, the velocity and width of tsunami waves are changeable. Therefore, it is important to add the damping term in the GKdV to study the effect of the velocity and width of the tsunami waves. The dGKdV equation is governed by

$$v_t - \omega v_x + \alpha v v_x + \beta v_{xxx} + R(t) v = 0, \tag{2.4}$$

where $R(t)$ is the damping term, $\alpha = \frac{3}{2}$, and $\beta = \frac{1}{6}$. In the case of planar geometry, $R(t) \equiv 0$, while in the case of nonplanar geometry such as cylindrical and spherical, $R(t) \not\equiv 0$. The damping term makes Eq (2.2) a non-integrable equation and most of the known methods fail to reach the solution. This article is devoted to finding the analytical solution for the dGKdV equation by the novel Ansatz method [24].

3. Analytical solution

The Eq (2.4) is a non-integrable equation which means the energy of the solution is not conserved and decays along t. The analytical solution of Eq (2.4) will be found by the Ansatz method. The following are the steps of the computation.

- Suppose that the solution of Eq (2.4) is in the following Ansatz:

$$v(x,t) = f_1(t) u(f_2(t)x, f_3(t)), \tag{3.1}$$

  where $f_i(t),\ i = 1,2,3$ are functions and are determined later and $u(x,t)$ is the exact solution of undamped GKdV Eq (2.2). Thus,

$$u_t|_{u\leftarrow\hat{u}} = \omega \hat{u}_x - \frac{3}{2}\hat{u}\hat{u}_x - \frac{1}{6}\hat{u}_{xxx}. \tag{3.2}$$

- Substituting (3.1) into Eq (2.4) gives:

$$f_1'\hat{u} + f_1 f_2' x\hat{u}_t + f_1 f_3'\hat{u}_t - \omega f_1 f_2 \hat{u}_x + \frac{3}{2} f_1^2 f_2 \hat{u}\hat{u}_x + \frac{1}{6} f_1 f_2^3 \hat{u}_{xxx} + R(t) f_1 \hat{u} \tag{3.3}$$

where $\hat{u} = f_1 u(f_2 x, f_3)$.

- Rearranging Eq (3.3) and using Eq (3.2) to obtain the following system of $f_i,\ i = 1, 2, 3,$

$$f_1' + R(t) f_1 = 0, \tag{3.4}$$
$$f_3' - f_2 = 0, \tag{3.5}$$
$$f_3' - f_1 f_2 = 0, \tag{3.6}$$
$$f_3' - f_2^3 = 0, \tag{3.7}$$

and note that t$f_1 f_2' x\hat{u}_t$ is a residual error. The function $R(t)$ is defined based on the geometry of the structure in ref. [25] as follows:

$$R(t) = \begin{cases} 0 & \text{the planar geometry} \\ m/2t & \text{the nonplanar geometry.} \end{cases} \tag{3.8}$$

- Solving $f_1' + R(t) f_1 = 0$ with the initial condition $f_1(t_0) = 1$ in a nonplanar case implies

$$f_1(t) = (t_0/t)^{m/2}, \tag{3.9}$$

where t_0 is the initial point in the t domain.

- It is obvious that $f_3' = f_2$ or $f_3' = f_2^3$. By inserting $f_3' = f_2$ from Eq (3.5) and the value of f_1 in Eq (3.9), we obtain: $f_2 = 1$, which implies $f_1 = 1$ and $m = 0$. This is the planar case. On the other hand, using Eq (3.7), ($f_3' = f_2^3$), and f_1 in Eq (3.9), we get $f_2 = 1$ or $f_2 = \left(\frac{t_0}{t}\right)^{m/4}$. If $f_2 = 1$, we have the planar case. Therefore, $f_2(t) = \left(\frac{t_0}{t}\right)^{m/4}$.
- Solving Eq (3.7) for f_3 leads to

$$f_3(t) = \frac{4\left(t_0 - t\left(\frac{t_0}{t}\right)^{\frac{3m}{4}}\right)}{3m-4}.$$

Therefore, the solution for Eq (2.4) is

$$u(x,t) = 2(\alpha + c)\left(\frac{t_0}{t}\right)^{m/2} \operatorname{sech}^2\left(\sqrt{\frac{3}{2}}\sqrt{\alpha + c}\left(x\left(\frac{t_0}{t}\right)^{m/4} - \frac{4c\left(t_0 - t\left(\frac{t_0}{t}\right)^{\frac{3m}{4}}\right)}{3m-4}\right)\right).$$

4. Discussion of results

Figure 1 shows the solution when $m = 0$, which is the solution of the integrable GKdV equation. Figures 2 and 3 present the solution for dGKdV $m = 1$ and $m = 2$, respectively. We noticed the behavior of the solution in the planer case is different than the nonplanar case. We found that the amplitude of the wave does not change along t in the planer case while it decreases along t in the nonplanar case. This reflects the importance of adding the effect of the damping term in the governing equation. Figure 4 shows the plot of the solution with $m = 0, 1, 2$. The method can present the effect of the damping term. The amplitude of the wave decreases as m increases.

Figure 5 presents the effect of the Coriolis effect term α. We noticed that as the value ω increases, the wave height and wavelength decrease. Thus, we realized by the solution of the dGKdV equation, the earth's rotation causes the shrink of the tsunami wave amplitude. Figure 6 presents the solution of dGKdV and shows that the amplitude of the wave changes based on the speed of the wave. Therefore, we obtained that the amplitude as well as the speed of tsunami waves depend on the depth of the water that it is traveling through. If the tsunami wave is fast, the amplitude of the wave increases.

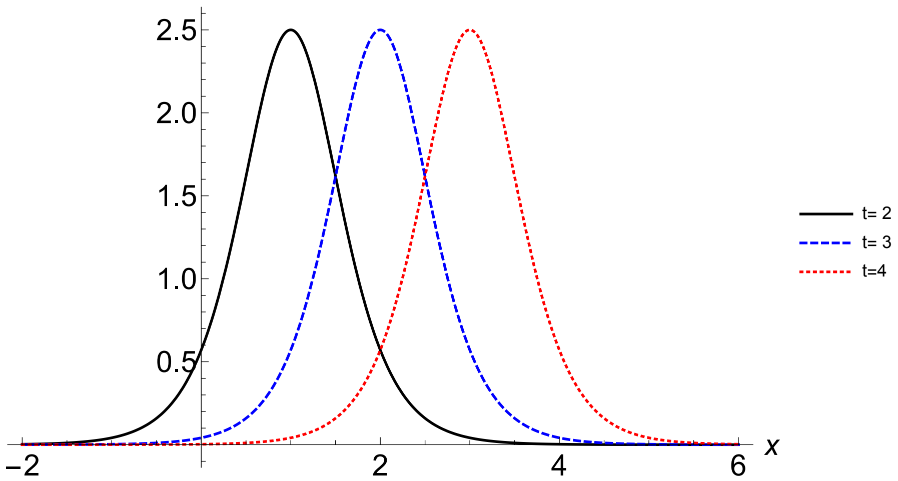

Figure 1. The analytical solution for the GKdV $m = 0, t_0 = 1, \alpha = 0.25, c = 1$ for different t.

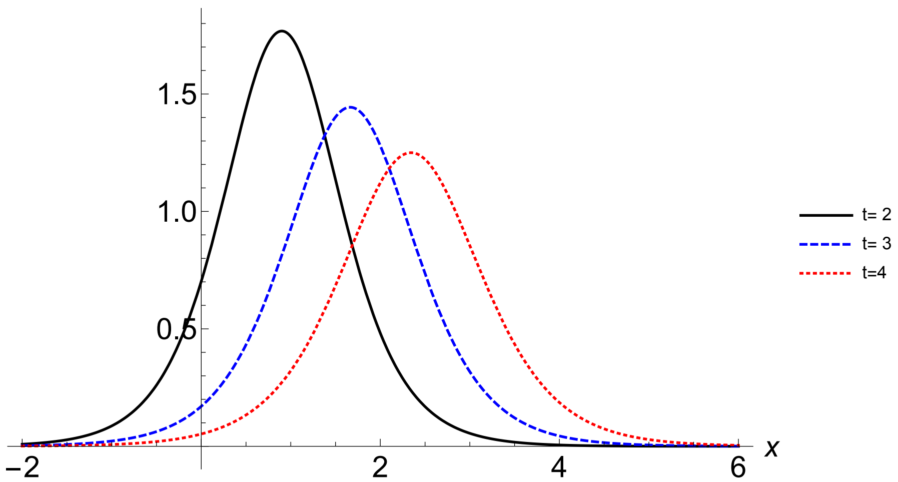

Figure 2. The analytical solution for the dGKdV $m = 1, t_0 = 1, \alpha = 0.25, c = 1$ for different t.

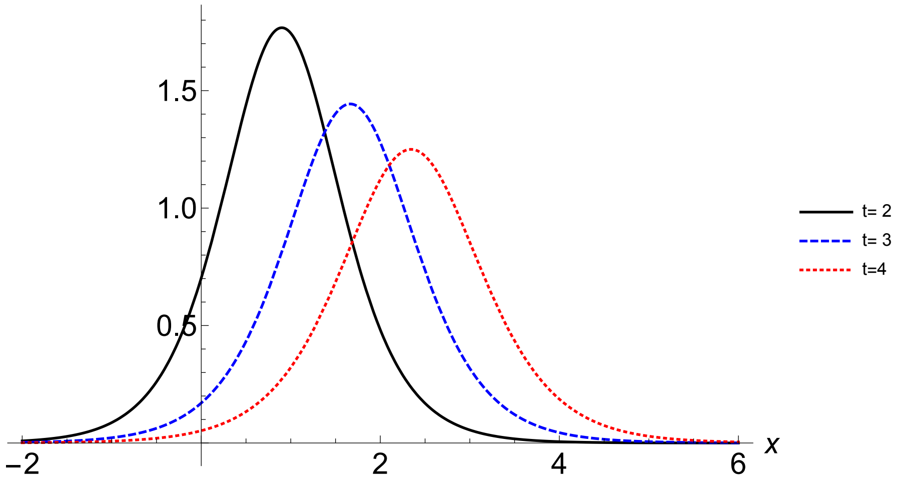

Figure 3. The analytical solution for the dGKdV $m = 2, t_0 = 1, \alpha = 0.25, c = 1$ for different t.

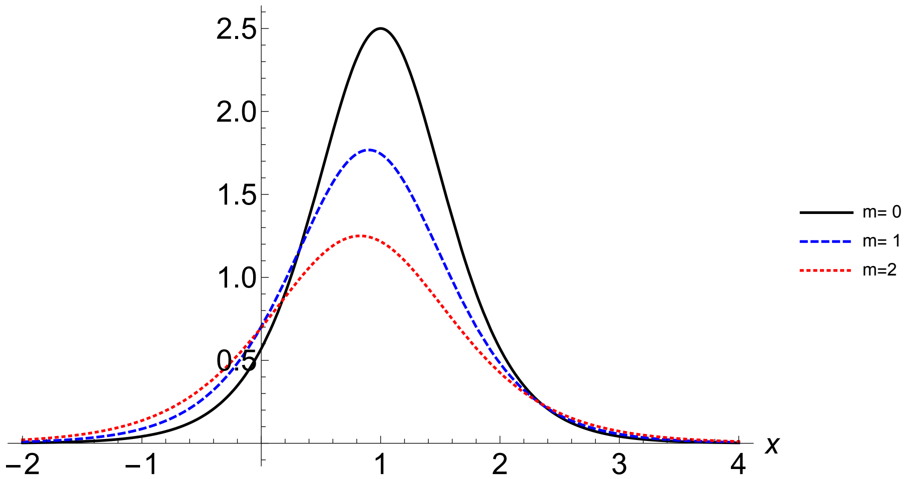

Figure 4. The analytical solution for the dGKdV $t = 2, t_0 = 1, \alpha = 0.25, c = 1$ for different m.

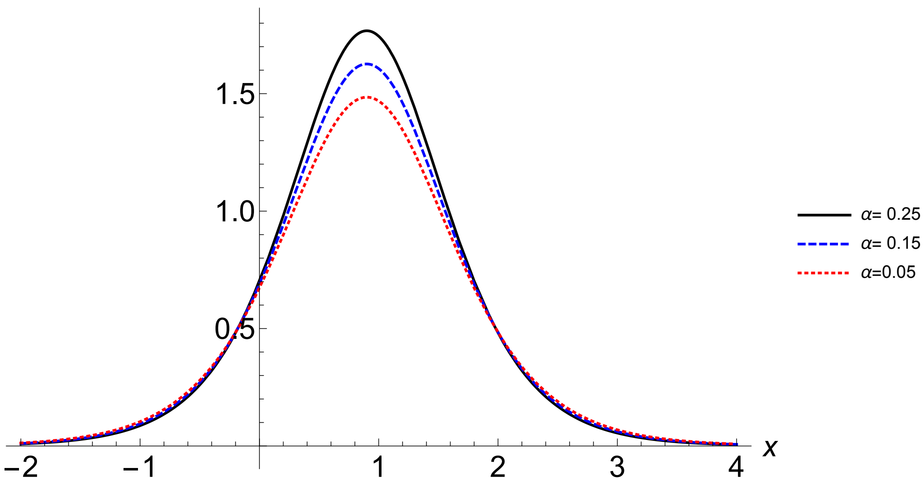

Figure 5. The analytical solution for the dGKdV $m = 1, c = 1, t = 2, t_0 = 1$ for different α.

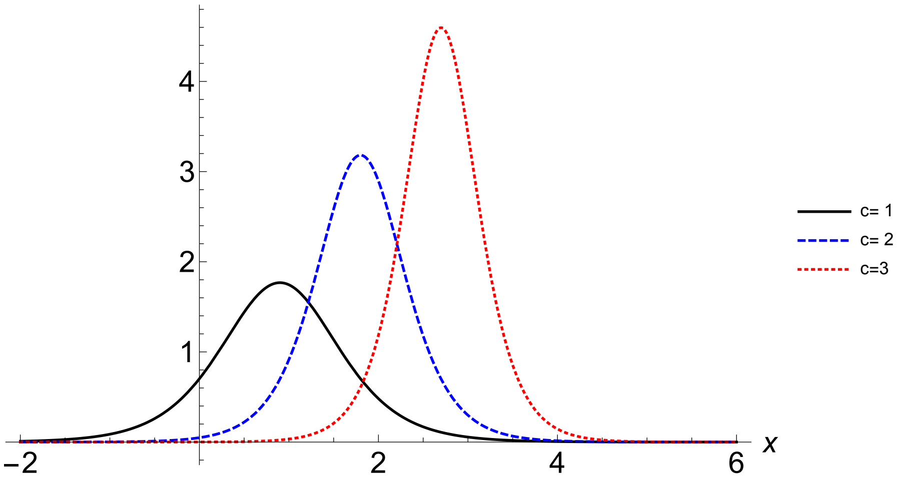

Figure 6. The analytical solution for the dGKdV $m = 1$ for different c.

5. Conclusions

The analytical solution for dGKdV is introduced in this article. The general solution was found by the novel Ansatz method. The solution was used to study the effect of the damping term and the Coriolis effect term in the GKdV. Tsunami waves are one of the applications of the dGKdV equation in the ocean. The obtained solution depicted that the amplitude of the wave increases with the increase of wave speed, while it decreases when the Coriolis effect term or damping term increases. In future work, the Ansatz method will be used to study different phenomena on Earth that are described by the non-integrable equation.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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