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Research article

Positive solutions for a Riemann-Liouville-type impulsive fractional integral boundary value problem

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Abstract: In this work, we investigate a Riemann-Liouville-type impulsive fractional integral boundary value problem. Using the fixed point index, we obtain two existence theorems on positive solutions under some conditions concerning the spectral radius of the relevant linear operator. Our method improves and generalizes some results in the literature.

Keywords: fractional-order differential equations; integral boundary value problems; impulse; positive solutions; fixed point index
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1. Introduction

In this work, we study the following Riemann-Liouville-type impulsive fractional integral boundary value problem

$$\begin{cases} {}_{t_k}D_t^{\beta} z(t) = -f\left(t, z(t)\right), \ t \neq t_k, \\ \Delta D^{\beta-1} z\left(t_k\right) = -I_k\left(z\left(t_k\right)\right), \ k = 1, \ldots, m, \\ z(0) = z'(0) = 0, \ z'(1) = \displaystyle\int_0^1 g(s, z(s)) d\alpha(s), \end{cases} \tag{1.1}$$

where $2 < \beta \leq 3$ is a real number, ${}_{t_k}D_t^{\beta}$ is the Riemann-Liouville fractional derivative, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, ${}_{t_k}D_t^{\beta-1} z\left(t_k^+\right) = \lim_{h\to 0^+} {}_{t_k}D_t^{\beta-1} z\left(t_k + h\right)$ and ${}_{t_k}D_t^{\beta-1} z\left(t_k^-\right) = \lim_{h\to 0^-} {}_{t_k}D_t^{\beta-1} z\left(t_k + h\right)$

represent the right and left limits of ${}_{t_k}D_t^{\beta-1}z(t)$ at $t = t_k$, respectively, ${}_{t_k}D_t^{\beta-1}z\left(t_k^-\right) = {}_{t_k}D_t^{\beta-1}z(t_k)$, and $\Delta D^{\beta-1}z(t_k) = {}_{t_k}D_t^{\beta-1}z\left(t_k^+\right) - {}_{t_{k-1}}D_t^{\beta-1}z\left(t_k^-\right)$. In addition, the functions f, g, α, I_k satisfy the conditions:

(H0) $f, g \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $k = 1, 2, ..., m$, $\mathbb{R}^+ := [0, +\infty)$,

(H1) α is a function of bounded variation with $\alpha(t) \geq 0$, and $\alpha(t) \not\equiv 0$, $t \in [0,1]$.

In comparison to integer calculus when describing natural phenomena and objective laws, fractional calculus is more accurate and applicable in physics, chemistry, and engineering. Many scholars have applied the methods of nonlinear analysis to study fractional boundary value problems, and a large number of results have been obtained; see for example [1–31] and the references therein. In [1], the authors used some fixed-point techniques to study the existence, uniqueness, and multiplicity of positive solutions for the fractional integral boundary value problem

$$\begin{cases} {}_0D_t^{\alpha}x(t) + q(t)f(t, x(t)) = 0, \ 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \ {}_0D_t^{\beta}x(1) = \int_0^1 h(s, x(s))dA(s), \end{cases}$$

where ${}_0D_t^{\alpha}$, ${}_0D_t^{\beta}$ are Riemann-Liouville fractional derivatives. In [2], the authors studied the following p-Laplacian fractional boundary value problem involving the Riemann-Stieltjes integral:

$$\begin{cases} -{}_0D_t^{\beta}(\varphi_p(-{}_0D_t^{\alpha}z(t) - g(t, z(t), {}_0D_t^{\gamma}z(t)))) = f(t, z(t), {}_0D_t^{\gamma}z(t)), \ 0 < t < 1, \\ {}_0D_t^{\alpha}z(0) = {}_0D_t^{\alpha+1}z(0) = {}_0D_t^{\gamma}z(0) = 0, \\ {}_0D_t^{\alpha}z(1) = 0, \ {}_0D_t^{\gamma}z(1) = \int_0^1 {}_0D_t^{\gamma}z(s)dA(s), \end{cases}$$

where ${}_0D_t^{\alpha}$, ${}_0D_t^{\beta}$, ${}_0D_t^{\gamma}$ are Riemann-Liouville fractional derivatives. The authors used fixed point theorems on a sum operator in partial ordering Banach spaces to investigate the existence and uniqueness of positive solutions for their problem.

In [3], the authors studied the impulsive fractional integral boundary value problem

$$\begin{cases} {}_{t_k}D_t^{\alpha}u(t) = f\left(t, u(t), u'(t), {}_{t_k}D_t^{\alpha-1}u(t)\right), \ t \neq t_k, \\ \Delta D^{\beta-1}u(t_k) = I_k(u(t_k)), \ k = 1, \ldots, m, \\ u(0) = u'(0) = 0, \ u'(1) = \int_0^{\eta} g(s, u(s))ds, \end{cases}$$

and they adopted the contraction mapping principle and the fixed point theorem to establish the existence and uniqueness of nontrivial solutions when the nonlinearities f, g, I_k satisfy some Lipschitz conditions. In [4], the authors studied positive solutions for the fractional integral boundary value problem

$$\begin{cases} D_{0+}^{\alpha}\chi(t) + h(t)f(t, \chi(t)) = 0, \quad 0 < t < 1, \\ \chi(0) = \chi'(0) = \chi''(0) = 0, \\ \chi(1) = \lambda \int_0^{\eta} \chi(s)ds, \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies the conditions

(HZ1) $\liminf_{\chi\to 0^+} \frac{f(t,\chi)}{\chi} > \lambda_1$, $\limsup_{\chi\to+\infty} \frac{f(t,\chi)}{\chi} < \lambda_1$ uniformly with respect to $t \in [0,1]$,

(HZ2) $\limsup_{\chi\to 0^+} \frac{f(t,\chi)}{\chi} < \lambda_1$, $\liminf_{\chi\to+\infty} \frac{f(t,\chi)}{\chi} > \lambda_1$ uniformly with respect to $t \in [0,1]$,

where λ_1 is the first eigenvalue of the operator $(L_{Z1}\chi)(t) = \int_0^1 G_Z(t,s)h(s)\chi(s)ds$ and G_Z is the Green's function.

Motivated by the aforementioned works, in this paper we use the fixed point index to study positive solutions for (1.1) under some conditions concerning the spectral radius of the relevant linear operator. Note that the considered linear operator can include the Riemann-Stieltjes integral condition in (1.1) and the approach is quite different from previous works in the literature. Moreover, we also consider the effect of the impulsive term and our conditions are more general than (HZ1)–(HZ2).

2. Preliminaries

In this section, we first present the definitions of the Riemann-Liouville-type fractional integral and derivative. For the other necessary definitions and notations, we refer the reader to the books [8,13,17].

Definition 2.1. The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $z : (a, +\infty) \to \mathbb{R}$ is given by

$$ {}_aI_t^{\beta} z(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} z(s)ds, \quad a > 0, $$

provided that the right-hand side is point-wise defined on $(a, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\beta > 0$ of a continuous function $z : (a, +\infty) \to \mathbb{R}$ is given by

$$ {}_aD_t^{\beta} z(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\beta-1} z(s)ds, $$

where $a > 0, n-1 < \beta \leq n$, provided that the right-hand side is point-wise defined on $(a, +\infty)$.

Let $C([0,1],\mathbb{R})$ be the Banach space of continuous functions from $[0,1]$ to $\mathbb{R}$ with the norm $\|z\| = \sup_{0\leq t\leq 1} |z(t)|$. Define the Banach space $PC^1([0,1],\mathbb{R})$ as follows

$$ PC^1([0,1],\mathbb{R}) = \Big\{ z \in C([0,1],\mathbb{R}) : {}_{t_k}D_t^{\beta-1} z\,(t_k^+) \text{ and } {}_{t_k}D_t^{\beta-1} z\,(t_k^-) \text{ exist with } \\ {}_{t_k}D_t^{\beta-1} z\,(t_k) = {}_{t_k}D_t^{\beta-1} z\,(t_k^-), k = 0, 1, \ldots, m \Big\} $$

with the norm $\|z\|_{PC^1} = \max\left\{\|z\|, \left\|{}_{t_k}D_t^{\beta-1} z\right\|\right\}$. Let $P = \{z \in C([0,1],\mathbb{R}) : z(t) \geq 0, t \in [0,1]\}$ and $P_0 = \{z \in P : z(t) \geq t^{\beta-1}\|z\|, t \in [0,1]\}$. Then P, P_0 are cones on $C([0,1],\mathbb{R})$.

Lemma 2.3. (see [3, Lemma 2.4]) Let $h, V \in C([0,1],\mathbb{R})$ and $v_k \in \mathbb{R}$, $k = 1,2,...,m$. Then, the boundary value problem

$$ \begin{cases} {}_{t_k}D_t^{\beta} z(t) = -h\,(t), & t \neq t_k, \\ \Delta D^{\beta-1} z\,(t_k) = -v_k, & k = 1, \ldots, m, \\ z(0) = z'(0) = 0, & z'(1) = \int_0^1 V(s)d\alpha(s) \end{cases} \tag{2.1} $$

has a solution of the form

$$ z(t) = \int_0^1 G(t,s)h(s)ds + \frac{t^{\beta-1}}{\beta-1} \int_0^1 V(s)d\alpha(s) + \sum_{k=1}^m H(t,t_k)v_k,\ 0 \leq t \leq 1, $$

where

$$G(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}t^{\beta-1}(1-s)^{\beta-2}-(t-s)^{\beta-1},\ 0\le s\le t\le 1,\\ t^{\beta-1}(1-s)^{\beta-2},\ 0\le t\le s\le 1,\end{cases}$$

and

$$H(t,t_k)=\frac{1}{\Gamma(\beta)}\begin{cases}t^{\beta-1},\ 0\le t\le t_k<1,\\ 0,\ 0<t_k<t\le 1.\end{cases}$$

Lemma 2.4. (see [9]) The function G has the following properties:

(C1) $G(t,s)\ge 0$ for $t,s\in[0,1]$;

(C2) $t^{\beta-1}G(1,s)\le G(t,s)\le G(1,s)$ for $t,s\in[0,1]$.

From Lemma 2.3 and (H0)–(H1), we define an operator $\mathcal{T}:P\to P$ as follows:

$$(\mathcal{T}z)(t)=\int_0^1 G(t,s)f(s,z(s))\,ds+\frac{t^{\beta-1}}{\beta-1}\int_0^1 g(s,z(s))\,d\alpha(s)+\sum_{k=1}^{m}H(t,t_k)I_k(z(t_k)),\ 0\le t\le 1. \quad (2.2)$$

From Lemma 2.3 we see that if there exists $z^*\in P\backslash\{0\}$ such that $\mathcal{T}z^*=z^*$, then this z^* is the positive solution for (1.1). Hence, in what follows we study the existence of positive fixed points of the operator $\mathcal{T}$.

Lemma 2.5. Suppose that (H0)–(H1) hold. Then, $\mathcal{T}(P)\subset P_0$.

By Lemma 2.4 and the method of [21, Lemma 2.6], we obtain the conclusion, so, we omit its proof.

Lemma 2.6. Let

$$(\mathcal{L}_{\mu,\nu}z)(t)=\mu\int_0^1 G(t,s)z(s)ds+\nu\frac{t^{\beta-1}}{\beta-1}\int_0^1 z(s)d\alpha(s)$$

with $\mu,\nu\ge 0$ and $\mu^2+\nu^2\ne 0$. Then $\mathcal{L}_{\mu,\nu}(P)\subset P_0$ and the spectral radius of $\mathcal{L}_{\mu,\nu}$, denoted by $r(\mathcal{L}_{\mu,\nu})$, which satisfies the inequality

$$\mu\int_0^1 G(1,s)s^{\beta-1}ds+\frac{\nu}{\beta-1}\int_0^1 s^{\beta-1}d\alpha(s)\le r(\mathcal{L}_{\mu,\nu})\le\mu\int_0^1 G(1,s)ds+\frac{\nu}{\beta-1}\int_0^1 d\alpha(s). \quad (2.3)$$

Proof. If $z\in P$, then from Lemma 2.4(C2) we have

$$(\mathcal{L}_{\mu,\nu}z)(t)\le\mu\int_0^1 G(1,s)z(s)ds+\nu\frac{1}{\beta-1}\int_0^1 z(s)d\alpha(s),$$

and

$$\begin{aligned}(\mathcal{L}_{\mu,\nu}z)(t)&\ge t^{\beta-1}\mu\int_0^1 G(1,s)z(s)ds+\nu\frac{t^{\beta-1}}{\beta-1}\int_0^1 z(s)d\alpha(s)\\&\ge t^{\beta-1}\|\mathcal{L}_{\mu,\nu}z\|,\ t\in[0,1].\end{aligned}$$

Hence, $\mathcal{L}_{\mu,\nu}(P)\in P_0$, as required.

Let $(L_\mu z)(t) = \mu \int_0^1 G(t,s)z(s)ds$ and $(L_\nu z)(t) = \nu \frac{t^{\beta-1}}{\beta-1} \int_0^1 z(s)d\alpha(s),\ t \in [0,1]$. Then for all $n \in \mathbb{N}^+$ we have

$$(L_\mu^n z)(t) = \mu^n \underbrace{\int_0^1 \cdots \int_0^1}_{n} G(t,s_1)G(s_1,s_2)\cdots G(s_{n-1},s_n)z(s_n)ds_1\cdots ds_n$$
$$\geq \mu^n \underbrace{\int_0^1 \cdots \int_0^1}_{n} t^{\beta-1}G(1,s_1)s_1^{\beta-1}G(1,s_2)\cdots s_{n-1}^{\beta-1}G(1,s_n)z(s_n)ds_1\cdots ds_n,$$

and

$$(L_\nu^n z)(t) = \left(\frac{\nu}{\beta-1}\right)^n t^{\beta-1}\left[\int_0^1 s^{\beta-1}d\alpha(s)\right]^{n-1}\int_0^1 z(s)d\alpha(s),\ t\in[0,1].$$

Consequently, we have

$$\|L_\mu^n\| \geq \max_{t\in[0,1]}(L_\mu^n \mathbf{1})(t) \geq \mu^n\left[\int_0^1 G(1,s)s^{\beta-1}ds\right]^{n-1}\int_0^1 G(1,s)ds,$$

and

$$\|L_\nu^n\| \geq \max_{t\in[0,1]}(L_\nu^n \mathbf{1})(t) \geq \left(\frac{\nu}{\beta-1}\right)^n\left[\int_0^1 s^{\beta-1}d\alpha(s)\right]^{n-1}\int_0^1 d\alpha(s),$$

where $\mathbf{1}(t)\equiv 1,\ t\in[0,1]$. Therefore, Gelfand's theorem implies that

$$r(L_\mu) = \liminf_{n\to\infty}\sqrt[n]{\|L_\mu^n\|} \geq \mu\int_0^1 G(1,s)s^{\beta-1}ds,$$

and

$$r(L_\nu) = \liminf_{n\to\infty}\sqrt[n]{\|L_\nu^n\|} \geq \frac{\nu}{\beta-1}\int_0^1 s^{\beta-1}d\alpha(s).$$

Combining the two inequalities, we get

$$r(\mathcal{L}_{\mu,\nu}) \geq \mu\int_0^1 G(1,s)s^{\beta-1}ds + \frac{\nu}{\beta-1}\int_0^1 s^{\beta-1}d\alpha(s).$$

On the other hand, we note that

$$r(L_\mu) \leq \mu\int_0^1 G(1,s)ds,\ \text{and}\ r(L_\nu) \leq \frac{\nu}{\beta-1}\int_0^1 d\alpha(s),$$

and then

$$r(\mathcal{L}_{\mu,\nu}) \leq \mu\int_0^1 G(1,s)ds + \frac{\nu}{\beta-1}\int_0^1 d\alpha(s).$$

Therefore, we obtain (2.3). This completes the proof. □

From Lemma 2.6, we find $r(\mathcal{L}_{\mu,\nu}) > 0$. Consequently, the Krein-Rutman theorem [32] implies that there exists $\zeta_{\mu,\nu} \in P\backslash\{0\}$ such that

$$\mathcal{L}_{\mu,\nu}\zeta_{\mu,\nu} = r(\mathcal{L}_{\mu,\nu})\zeta_{\mu,\nu}. \tag{2.4}$$

From [19, 33], the conjugate space of $C([0,1],\mathbb{R})$ is $E^* := \{\gamma : \gamma$ has bounded variation on $[0,1]\}$. Moreover, the dual cone of P and the bounded linear functional on $C([0,1],\mathbb{R})$ can be expressed by

$$P^* := \{\gamma \in E^* : \gamma \text{ is non-decreasing on } [0,1]\} \text{ and } \gamma(z) = \int_0^1 z(t)d\gamma(t), z \in C([0,1],\mathbb{R}), \gamma \in E^*.$$

Note that $r(\mathcal{L}_{\mu,\nu}) > 0$ in Lemma 2.6, and there exists $\psi_{\mu,\nu} \in P^*\backslash\{0\}$ such that

$$\mathcal{L}^*_{\mu,\nu}\psi_{\mu,\nu} = r\left(\mathcal{L}_{\mu,\nu}\right)\psi_{\mu,\nu}, \tag{2.5}$$

where $\mathcal{L}^*_{\mu,\nu} : E^* \to E^*$ is the conjugate operator of $\mathcal{L}_{\mu,\nu}$, denoted by

$$\left(\mathcal{L}^*_{\mu,\nu}\gamma\right)(t) := \mu\int_0^t ds\int_0^1 G(\tau,s)d\gamma(\tau) + \nu\alpha(t)\int_0^1 \frac{\tau^{\beta-1}}{\beta-1}d\gamma(\tau), \gamma \in E^*.$$

Lemma 2.7. (see [34]) Let E be a Banach space, $\Omega \subset E$ a bounded open set, and $A : \overline{\Omega}\cap P \to P$ a completely continuous operator. If there exists $z_0 \in P\backslash\{0\}$ such that $z - Az \neq \lambda z_0,$ for all $z \in \partial\Omega\cap P, \lambda \geq 0$, then the fixed point index $i(A, \Omega\cap P, P) = 0$.

Lemma 2.8. (see [34]) Let E be a Banach space, $\Omega \subset E$ a bounded open set with $0 \in \Omega$, and $A : \overline{\Omega}\cap P \to P$ a completely continuous operator. If $z \neq \lambda Az,$ for all $z \in \partial\Omega\cap P, 0 \leq \lambda \leq 1$, then the fixed point index $i(A, \Omega\cap P, P) = 1$.

3. Main results

Consider the coefficients $\mu_i, \nu_i \geq 0$ with $\mu_i^2 + \nu_i^2 \neq 0,\ i = 1,2,3,4$. From Lemma 2.6, $r(\mathcal{L}_{\mu_i,\nu_i}) > 0$. Then there exists $\psi_{\mu_i,\nu_i} \in P^*\backslash\{0\}$ such that

$$\mathcal{L}^*_{\mu_i,\nu_i}\psi_{\mu_i,\nu_i} = r\left(\mathcal{L}_{\mu_i,\nu_i}\right)\psi_{\mu_i,\nu_i},\ i = 1,2,3,4. \tag{3.1}$$

Remark 3.1. Let $z \in P$. Then we have

$$\int_0^1 z(t)d\psi_{\mu_i,\nu_i}(t) \geq 0,\ \int_0^1 d\psi_{\mu_i,\nu_i}(t) > 0,\ \int_0^1 t^{\beta-1}d\psi_{\mu_i,\nu_i}(t) > 0,\ \int_0^1 H(t,t_k)d\psi_{\mu_i,\nu_i}(t) > 0.$$

To see this note that $\psi_{\mu_i,\nu_i} \in P^*\backslash\{0\}$, and from the definition of the Riemann-Stieltjes integral we have

$$\int_0^1 z(t)d\psi_{\mu_i,\nu_i}(t) = \lim_{\rho\to 0}\sum_{j=1}^n z(\xi_j)\left[\psi_{\mu_i,\nu_i}\left(t_j\right) - \psi_{\mu_i,\nu_i}\left(t_{j-1}\right)\right] \geq 0,$$

and

$$\begin{aligned}\int_0^1 d\psi_{\mu_i,\nu_i}(t) &= \lim_{\rho\to 0}\sum_{j=1}^n \left[\psi_{\mu_i,\nu_i}\left(t_j\right) - \psi_{\mu_i,\nu_i}\left(t_{j-1}\right)\right] \\ &\geq [\psi_{\mu_i,\nu_i}(1) - \psi_{\mu_i,\nu_i}(0)] \\ &> 0,\end{aligned}$$

for all divisions t_j: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < t_{n+1} = 1$, $\rho = \max_{1\le j\le n}\left(t_j - t_{j-1}\right)$, $\xi_j \in \left[t_{j-1}, t_j\right]$, $j = 1, 2, \cdots, n$. The other two inequalities can be similarly proven.

Now, we list our assumptions for the nonlinearities $f, g, I_k (k = 1, 2, ..., m)$:

(H2) There exist $\mu_1, \nu_1 \ge 0$ $(\mu_1^2 + \nu_1^2 \neq 0)$ and $l_k \ge 0$ $(\sum\limits_{k=1}^{m} l_k^2 \neq 0), k = 1, 2, ..., m$ such that

$$\text{if } r\left(\mathcal{L}_{\mu_1,\nu_1}\right) < 1 \Rightarrow \sum_{k=1}^{m} l_k t_k^{\beta-1} \int_0^1 H(t, t_k) d\psi_{\mu_1,\nu_1}(t) > [1 - r\left(\mathcal{L}_{\mu_1,\nu_1}\right)] \int_0^1 d\psi_{\mu_1,\nu_1}(t),$$

$$\liminf_{z\to+\infty} \frac{f(t,z)}{z} \ge \mu_1, \ \liminf_{z\to+\infty} \frac{g(t,z)}{z} \ge \nu_1 \text{ uniformly on } t \in [0,1], \text{ and } \liminf_{z\to+\infty} \frac{I_k(z)}{z} \ge l_k, k = 1, 2, ..., m.$$

(H3) There exist $\mu_2, \nu_2 \ge 0$ $(\mu_2^2 + \nu_2^2 \neq 0)$ and $\widetilde{l_k} \ge 0$ $(\sum\limits_{k=1}^{m} \widetilde{l}_k^2 \neq 0), k = 1, 2, ..., m$ such that

$$r\left(\mathcal{L}_{\mu_2,\nu_2}\right) < 1 \Rightarrow [1 - r\left(\mathcal{L}_{\mu_2,\nu_2}\right)] \int_0^1 t^{\beta-1} d\psi_{\mu_2,\nu_2}(t) > \sum_{k=1}^{m} \widetilde{l_k} \int_0^1 H(t, t_k) d\psi_{\mu_2,\nu_2}(t),$$

$$\limsup_{z\to0^+} \frac{f(t,z)}{z} \le \mu_2, \ \limsup_{z\to0^+} \frac{g(t,z)}{z} \le \nu_2 \text{ uniformly on } t \in [0,1], \text{ and } \liminf_{z\to0^+} \frac{I_k(z)}{z} \le \widetilde{l_k}, k = 1, 2, ..., m.$$

(H4) There exist $\mu_3, \nu_3 \ge 0$ $(\mu_3^2 + \nu_3^2 \neq 0)$ and $\bar{l}_k \ge 0$ $(\sum\limits_{k=1}^{m} \bar{l}_k^2 \neq 0), k = 1, 2, ..., m$ such that

$$\text{if } r\left(\mathcal{L}_{\mu_3,\nu_3}\right) < 1 \Rightarrow \sum_{k=1}^{m} \bar{l}_k t_k^{\beta-1} \int_0^1 H(t, t_k) d\psi_{\mu_3,\nu_3}(t) > [1 - r\left(\mathcal{L}_{\mu_3,\nu_3}\right)] \int_0^1 d\psi_{\mu_3,\nu_3}(t),$$

$$\liminf_{z\to0^+} \frac{f(t,z)}{z} \ge \mu_3, \ \liminf_{z\to0^+} \frac{g(t,z)}{z} \ge \nu_3 \text{ uniformly on } t \in [0,1], \text{ and } \liminf_{z\to0^+} \frac{I_k(z)}{z} \ge \bar{l}_k, k = 1, 2, ..., m.$$

(H5) There exist $\mu_4, \nu_4 \ge 0$ $(\mu_4^2 + \nu_4^2 \neq 0)$ and $\widehat{l_k} \ge 0$ $(\sum\limits_{k=1}^{m} \widehat{l}_k^2 \neq 0), k = 1, 2, ..., m$ such that

$$r\left(\mathcal{L}_{\mu_4,\nu_4}\right) < 1 \Rightarrow (1 - r\left(\mathcal{L}_{\mu_4,\nu_4}\right)) \int_0^1 t^{\beta-1} d\psi_{\mu_4,\nu_4}(t) > \sum_{k=1}^{m} \widehat{l_k} \int_0^1 H(t, t_k) d\psi_{\mu_4,\nu_4}(t),$$

$$\limsup_{z\to+\infty} \frac{f(t,z)}{z} \le \mu_4, \ \limsup_{z\to+\infty} \frac{g(t,z)}{z} \le \nu_4 \text{ uniformly on } t \in [0,1], \text{ and } \liminf_{z\to+\infty} \frac{I_k(z)}{z} \le \widehat{l_k}, k = 1, 2, ..., m.$$

Theorem 3.2. Suppose that (H0)–(H3) hold. Then, (1.1) has at least one positive solution.

Proof. Let $S_1 = \{z \in P : z - \mathcal{T}z = \lambda\widetilde{z}, \ \lambda \ge 0\}$, where $\widetilde{z} \in P_0$ is a fixed element. We first prove that S_1 is a bounded set in P. Note that $z \in S_1$, and from Lemma 2.5 we have

$$z \in P_0, \text{ i.e., } z(t) \ge t^{\beta-1}\|z\|, t \in [0,1] \text{ and } z(t_k) \ge t_k^{\beta-1}\|z\|, k = 1, 2, ..., m. \tag{3.2}$$

By (H2) there exist $\widetilde{c}, \widetilde{c}_k > 0 (k = 1, 2, ..., m)$ such that

$$f(t,z) \ge \mu_1(z - \widetilde{c}), \ g(t,z) \ge \nu_1(z - \widetilde{c}), \ I_k(z) \ge l_k z - \widetilde{c}_k, \ z \in \mathbb{R}^+, t \in [0,1], k = 1, 2, ..., m.$$

Consequently, if $z \in S_1$, we have

$$\begin{aligned} z(t) &\geq (\mathcal{T} z)(t) \\ &\geq \mu_1 \int_0^1 G(t,s)(z(s)-\widetilde{c})ds + \nu_1 \frac{t^{\beta-1}}{\beta-1}\int_0^1 (z(s)-\widetilde{c})d\alpha(s) + \sum_{k=1}^{m} H(t,t_k)\left(l_k z(t_k) - \widetilde{c}_k\right). \end{aligned} \tag{3.3}$$

Multiplying by $d\psi_{\mu_1,\nu_1}(t)$ on both sides of (3.3) and integrating over $[0, 1]$, from (3.1) we have

$$\begin{aligned} \int_0^1 z(t)d\psi_{\mu_1,\nu_1}(t) \geq& \int_0^1 \left[\mu_1 \int_0^1 G(t,s)(z(s)-\widetilde{c})ds + \nu_1 \frac{t^{\beta-1}}{\beta-1}\int_0^1 (z(s)-\widetilde{c})d\alpha(s)\right] d\psi_{\mu_1,\nu_1}(t) \\ &+ \sum_{k=1}^{m}\int_0^1 H(t,t_k)\left(l_k z(t_k)-\widetilde{c}_k\right) d\psi_{\mu_1,\nu_1}(t) \\ =& \int_0^1 (z(s)-\widetilde{c}) d\left(\mu_1 \int_0^s d\tau \int_0^1 G(t,\tau)d\psi_{\mu_1,\nu_1}(t) + \nu_1 \alpha(s)\int_0^1 \frac{t^{\beta-1}}{\beta-1} d\psi_{\mu_1,\nu_1}(t)\right) \\ &+ \sum_{k=1}^{m}\int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t)\left(l_k z(t_k)-\widetilde{c}_k\right) \\ =& \int_0^1 (z(s)-\widetilde{c}) d(\mathcal{L}^*_{\mu_1,\nu_1}\psi_{\mu_1,\nu_1})(s) + \sum_{k=1}^{m}\int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t)\left(l_k z(t_k)-\widetilde{c}_k\right) \\ =& \int_0^1 (z(s)-\widetilde{c}) d(r\left(\mathcal{L}_{\mu_1,\nu_1}\right)\psi_{\mu_1,\nu_1})(s) + \sum_{k=1}^{m}\int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t)\left(l_k z(t_k)-\widetilde{c}_k\right). \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^1 z(t)d\psi_{\mu_1,\nu_1}(t) + \widetilde{c} r\left(\mathcal{L}_{\mu_1,\nu_1}\right)\int_0^1 d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m}\widetilde{c}_k \int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t) \\ \geq& r\left(\mathcal{L}_{\mu_1,\nu_1}\right)\int_0^1 z(t)d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m} l_k \int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t) z(t_k). \end{aligned} \tag{3.4}$$

There are two cases to consider.

Case 1. $r\left(\mathcal{L}_{\mu_1,\nu_1}\right) \geq 1$. From (3.2) and (3.4) we have

$$\begin{aligned} &[r\left(\mathcal{L}_{\mu_1,\nu_1}\right)-1]\|z\|\int_0^1 t^{\beta-1}d\psi_{\mu_1,\nu_1}(t) + \|z\|\sum_{k=1}^{m} l_k t_k^{\beta-1}\int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t) \\ \leq& \widetilde{c} r\left(\mathcal{L}_{\mu_1,\nu_1}\right)\int_0^1 d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m}\widetilde{c}_k \int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t), \end{aligned}$$

and thus

$$\|z\| \leq \frac{\widetilde{c} r\left(\mathcal{L}_{\mu_1,\nu_1}\right)\int_0^1 d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m}\widetilde{c}_k \int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t)}{[r\left(\mathcal{L}_{\mu_1,\nu_1}\right)-1]\int_0^1 t^{\beta-1}d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m} l_k t_k^{\beta-1}\int_0^1 H(t,t_k)d\psi_{\mu_1,\nu_1}(t)}.$$

Case 2. Now $r\left(\mathcal{L}_{\mu_1,\nu_1}\right) < 1$. (H2), (3.2), and (3.4) imply that

$$[r\left(\mathcal{L}_{\mu_1,\nu_1}\right) - 1]\|z\| \int_0^1 d\psi_{\mu_1,\nu_1}(t) + \|z\| \sum_{k=1}^{m} l_k t_k^{\beta-1} \int_0^1 H(t,t_k) d\psi_{\mu_1,\nu_1}(t)$$
$$\leq \widetilde{c} r\left(\mathcal{L}_{\mu_1,\nu_1}\right) \int_0^1 d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m} \widetilde{c}_k \int_0^1 H(t,t_k) d\psi_{\mu_1,\nu_1}(t),$$

and then

$$\|z\| \leq \frac{\widetilde{c} r\left(\mathcal{L}_{\mu_1,\nu_1}\right) \int_0^1 d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m} \widetilde{c}_k \int_0^1 H(t,t_k) d\psi_{\mu_1,\nu_1}(t)}{[r\left(\mathcal{L}_{\mu_1,\nu_1}\right) - 1] \int_0^1 d\psi_{\mu_1,\nu_1}(t) + \sum_{k=1}^{m} l_k t_k^{\beta-1} \int_0^1 H(t,t_k) d\psi_{\mu_1,\nu_1}(t)}.$$

Combining the two cases, we have proved that S_1 is a bounded set, as required. Now, we choose a sufficiently large $R_1 > \sup S_1$ such that

$$z - \mathcal{T}z \neq \lambda \widetilde{z},\ z \in \partial B_{R_1} \cap P,\ \lambda \geq 0, \tag{3.5}$$

where $B_{R_1} = \{z \in P : \|z\| < R_1\}$. Therefore, Lemma 2.7 implies that

$$i(\mathcal{T}, B_{R_1} \cap P, P) = 0. \tag{3.6}$$

By (H3) there exists $r_1 > 0$ such that

$$f(t,z) \leq \mu_2 z,\ g(t,z) \leq \nu_2 z,\ I_k(z) \leq \widetilde{l}_k z,\ z \in [0,r_1], t \in [0,1], k = 1,2,...,m. \tag{3.7}$$

Now, we prove that

$$z \neq \lambda \mathcal{T} z,\ z \in \partial B_{r_1} \cap P,\ \lambda \in [0,1], \tag{3.8}$$

where $B_{r_1} = \{z \in P : \|z\| < r_1\}$. If the claim is false, then there exist a $z_1 \in \partial B_{r_1} \cap P,\ \lambda_1 \in [0,1]$ such that

$$z_1 = \lambda_1 \mathcal{T} z_1.$$

By Lemma 2.5, z_1 satisfies (3.2), and from (3.7) we have

$$z_1(t) \leq (\mathcal{T} z_1)(t) \leq \mu_2 \int_0^1 G(t,s) z_1(s) ds + \nu_2 \frac{t^{\beta-1}}{\beta - 1} \int_0^1 z_1(s) d\alpha(s) + \sum_{k=1}^{m} H(t,t_k) \widetilde{l}_k z_1(t_k). \tag{3.9}$$

Multiplying by $d\psi_{\mu_2,\nu_2}(t)$ on both sides of (3.9) and integrating over [0, 1], from (3.1) we obtain

$$\begin{aligned}\int_0^1 z_1(t)d\psi_{\mu_2,\nu_2}(t) &\le \mu_2\int_0^1\int_0^1 G(t,s)z_1(s)dsd\psi_{\mu_2,\nu_2}(t) + \nu_2\int_0^1\frac{t^{\beta-1}}{\beta-1}\int_0^1 z_1(s)d\alpha(s)d\psi_{\mu_2,\nu_2}(t)\\ &+ \sum_{k=1}^m \widetilde{l}_k\int_0^1 H(t,t_k)d\psi_{\mu_2,\nu_2}(t)z_1(t_k)\\ &= \int_0^1 z_1(s)d\left(\mu_2\int_0^s d\tau\int_0^1 G(t,\tau)d\psi_{\mu_2,\nu_2}(t) + \nu_2\alpha(s)\int_0^1\frac{t^{\beta-1}}{\beta-1}d\psi_{\mu_2,\nu_2}(t)\right)\\ &+ \sum_{k=1}^m \widetilde{l}_k\int_0^1 H(t,t_k)d\psi_{\mu_2,\nu_2}(t)z_1(t_k)\\ &= \int_0^1 z_1(s)d(\mathcal{L}^*_{\mu_2,\nu_2}\psi_{\mu_2,\nu_2})(s) + \sum_{k=1}^m \widetilde{l}_k\int_0^1 H(t,t_k)d\psi_{\mu_2,\nu_2}(t)z_1(t_k)\\ &= \int_0^1 z_1(s)d(r\left(\mathcal{L}_{\mu_2,\nu_2}\right)\psi_{\mu_2,\nu_2})(s) + \sum_{k=1}^m \widetilde{l}_k\int_0^1 H(t,t_k)d\psi_{\mu_2,\nu_2}(t)z_1(t_k).\end{aligned}$$

This, together with (3.2), implies that

$$[1-r\left(\mathcal{L}_{\mu_2,\nu_2}\right)]\|z_1\|\int_0^1 t^{\beta-1}d\psi_{\mu_2,\nu_2}(t) \le \|z_1\|\sum_{k=1}^m \widetilde{l}_k\int_0^1 H(t,t_k)d\psi_{\mu_2,\nu_2}(t).$$

This contradicts (H3) unless $\|z_1\| = 0$. Note that $\|z_1\| = 0$ also contradicts $z_1 \in \partial B_{r_1}\cap P,\ r_1 > 0$. Therefore, we obtain that (3.8) holds, as required. From Lemma 2.8 we have

$$i(\mathcal{T}, B_{r_1}\cap P, P) = 1. \tag{3.10}$$

Note that R_1 can be chosen large enough such that $R_1 > \sup S_1$ and $R_1 > r_1$. Therefore, from (3.6) and (3.10) we have

$$i(\mathcal{T}, (B_{R_1}\backslash\overline{B}_{r_1})\cap P, P) = i(\mathcal{T}, B_{R_1}\cap P, P) - i(\mathcal{T}, B_{r_1}\cap P, P) = -1.$$

Therefore, the operator $\mathcal{T}$ has at least one fixed point in $(B_{R_1}\backslash\overline{B}_{r_1})\cap P$. Thus, (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.3. Suppose that (H0) and (H4)–(H5) hold. Then, (1.1) has at least one positive solution.

Proof. By (H4) there exists a sufficiently small $r_2 > 0$ such that

$$f(t,z) \ge \mu_3 z,\ g(t,z) \ge \nu_3 z,\ I_k(z) \ge \bar{l}_k z,\ z \in [0,r_2], t\in[0,1], k = 1,2,...,m. \tag{3.11}$$

For this r_2, we prove that

$$z - \mathcal{T}z \ne \lambda\bar{z},\ z \in \partial B_{r_2}\cap P,\ \lambda \ge 0, \tag{3.12}$$

where $B_{r_2} = \{z \in P : \|z\| < r_2\}$, and $\bar{z}$ is a fixed element in P_0. If (3.12) is false, then there exist a $z_2 \in \partial B_{r_2}\cap P$, $\lambda_2 \ge 0$ such that

$$z_2 - \mathcal{T}z_2 = \lambda_2\bar{z}.$$

Lemma 2.5 implies that z_2 satisfies (3.2). Moreover, from (3.11) we have

$$z_2(t) \geq (\mathcal{T} z_2)(t) \geq \mu_3 \int_0^1 G(t,s) z_2(s) ds + \nu_3 \frac{t^{\beta-1}}{\beta-1} \int_0^1 z_2(s) d\alpha(s) + \sum_{k=1}^{m} H(t,t_k) \bar{l}_k z_2(t_k). \tag{3.13}$$

Multiplying by $d\psi_{\mu_3,\nu_3}(t)$ on both sides of (3.13) and integrating over [0, 1], from (3.1) we obtain

$$\begin{aligned}
\int_0^1 z_2(t) d\psi_{\mu_3,\nu_3}(t) &\geq \mu_3 \int_0^1 \int_0^1 G(t,s) z_2(s) ds d\psi_{\mu_3,\nu_3}(t) + \nu_3 \int_0^1 \frac{t^{\beta-1}}{\beta-1} \int_0^1 z_2(s) d\alpha(s) d\psi_{\mu_3,\nu_3}(t) \\
&\quad + \sum_{k=1}^{m} \bar{l}_k \int_0^1 H(t,t_k) d\psi_{\mu_3,\nu_3}(t) z_2(t_k) \\
&= \int_0^1 z_2(s) d\left(\mu_3 \int_0^s d\tau \int_0^1 G(t,\tau) d\psi_{\mu_3,\nu_3}(t) + \nu_3 \alpha(s) \int_0^1 \frac{t^{\beta-1}}{\beta-1} d\psi_{\mu_3,\nu_3}(t)\right) \\
&\quad + \sum_{k=1}^{m} \bar{l}_k \int_0^1 H(t,t_k) d\psi_{\mu_3,\nu_3}(t) z_2(t_k) \\
&= \int_0^1 z_2(s) d(\mathcal{L}^*_{\mu_3,\nu_3} \psi_{\mu_3,\nu_3})(s) + \sum_{k=1}^{m} \bar{l}_k \int_0^1 H(t,t_k) d\psi_{\mu_3,\nu_3}(t) z_2(t_k) \\
&= \int_0^1 z_2(s) d(r\left(\mathcal{L}_{\mu_3,\nu_3}\right) \psi_{\mu_3,\nu_3})(s) + \sum_{k=1}^{m} \bar{l}_k \int_0^1 H(t,t_k) d\psi_{\mu_3,\nu_3}(t) z_2(t_k).
\end{aligned}$$

There are two cases to consider.

Cases 1. $r\left(\mathcal{L}_{\mu_3,\nu_3}\right) \geq 1$. From (3.2) we obtain

$$\|z_2\| \left[(r\left(\mathcal{L}_{\mu_3,\nu_3}\right) - 1) \int_0^1 t^{\beta-1} d\psi_{\mu_3,\nu_3}(t) + \sum_{k=1}^{m} \bar{l}_k t_k^{\beta-1} \int_0^1 H(t,t_k) d\psi_{\mu_3,\nu_3}(t) \right] \leq 0,$$

which contradicts $z_2 \in \partial B_{r_2} \cap P,\ r_2 > 0$.

Cases 2. $r\left(\mathcal{L}_{\mu_3,\nu_3}\right) < 1$. By (3.2) we have

$$\|z_2\| \left[(r\left(\mathcal{L}_{\mu_3,\nu_3}\right) - 1) \int_0^1 d\psi_{\mu_3,\nu_3}(t) + \sum_{k=1}^{m} \bar{l}_k t_k^{\beta-1} \int_0^1 H(t,t_k) d\psi_{\mu_3,\nu_3}(t) \right] \leq 0,$$

and it contradicts (H4) unless $\|z_2\| = 0$. We also have a contradiction to $z_2 \in \partial B_{r_2} \cap P,\ r_2 > 0$ if $\|z_2\| = 0$.

Therefore, we obtain that (3.12) holds, and Lemma 2.7 implies that

$$i(\mathcal{T}, B_{r_2} \cap P, P) = 0. \tag{3.14}$$

By (H5) there exist $\overline{c}, \overline{c}_k > 0 (k = 1, 2, ..., m)$ such that

$$f(t,z) \leq \mu_4(z + \overline{c}),\ g(t,z) \leq \nu_4(z + \overline{c}),\ I_k(z) \leq \widehat{l_k} z + \overline{c}_k,\ z \in \mathbb{R}^+, t \in [0,1], k = 1, 2, ..., m. \tag{3.15}$$

Let $S_2 = \{z \in P : z = \lambda \mathcal{T} z, \lambda \in [0,1]\}$. We now prove that S_2 is bounded in P. If $z \in S_2$, then by Lemma 2.5, (3.2) holds, and from (3.15) we have

$$\begin{aligned}
z(t) &\leq (\mathcal{T} z)(t) \\
&\leq \mu_4 \int_0^1 G(t,s)(z(s) + \overline{c}) ds + \nu_4 \frac{t^{\beta-1}}{\beta-1} \int_0^1 (z(s) + \overline{c}) d\alpha(s) + \sum_{k=1}^{m} H(t,t_k) \left(\widehat{l_k} z(t_k) + \overline{c}_k\right).
\end{aligned} \tag{3.16}$$

Multiplying by $d\psi_{\mu_4,\nu_4}(t)$ on both sides of (3.16) and integrating over [0, 1], from (3.1) we obtain

$$\begin{aligned}
\int_0^1 z(t)d\psi_{\mu_4,\nu_4}(t) \le \mu_4 \int_0^1 \int_0^1 G(t,s)\left(z(s)+\overline{c}\right) ds d\psi_{\mu_4,\nu_4}(t) + \nu_4 \int_0^1 \frac{t^{\beta-1}}{\beta-1} \int_0^1 \left(z(s)+\overline{c}\right) d\alpha(s) d\psi_{\mu_4,\nu_4}(t) \\
+ \sum_{k=1}^m \int_0^1 H(t,t_k) d\psi_{\mu_4,\nu_4}(t) \left(\widehat{l_k} z(t_k) + \overline{c}_k\right) \\
= \int_0^1 \left(z(s)+\overline{c}\right) d\left(\mu_4 \int_0^s d\tau \int_0^1 G(t,\tau) d\psi_{\mu_4,\nu_4}(t) + \nu_4 \alpha(s) \int_0^1 \frac{t^{\beta-1}}{\beta-1} d\psi_{\mu_4,\nu_4}(t)\right) \\
+ \sum_{k=1}^m \int_0^1 H(t,t_k) d\psi_{\mu_4,\nu_4}(t) \left(\widehat{l_k} z(t_k) + \overline{c}_k\right) \\
= \int_0^1 \left(z(s)+\overline{c}\right) d(\mathcal{L}^*_{\mu_4,\nu_4}\psi_{\mu_4,\nu_4})(s) + \sum_{k=1}^m \int_0^1 H(t,t_k) d\psi_{\mu_4,\nu_4}(t) \left(\widehat{l_k} z(t_k) + \overline{c}_k\right) \\
= \int_0^1 \left(z(s)+\overline{c}\right) d(r\left(\mathcal{L}_{\mu_4,\nu_4}\right)\psi_{\mu_4,\nu_4})(s) + \sum_{k=1}^m \int_0^1 H(t,t_k) d\psi_{\mu_4,\nu_4}(t) \left(\widehat{l_k} z(t_k) + \overline{c}_k\right).
\end{aligned}$$

Note that (3.2) and $r\left(\mathcal{L}_{\mu_4,\nu_4}\right) < 1$, and we have

$$(1 - r\left(\mathcal{L}_{\mu_4,\nu_4}\right))\|z\| \int_0^1 t^{\beta-1} d\psi_{\mu_4,\nu_4}(t) \le \overline{c} r\left(\mathcal{L}_{\mu_4,\nu_4}\right) \int_0^1 d\psi_{\mu_4,\nu_4})(t) + \sum_{k=1}^m \int_0^1 H(t,t_k) d\psi_{\mu_4,\nu_4}(t) \left(\widehat{l_k}\|z\| + \overline{c}_k\right),$$

and (H5) implies that

$$\|z\| \le \frac{\overline{c} r\left(\mathcal{L}_{\mu_4,\nu_4}\right) \int_0^1 d\psi_{\mu_4,\nu_4})(t) + \sum_{k=1}^m \overline{c}_k \int_0^1 H(t,t_k) d\psi_{\mu_4,\nu_4}(t)}{(1 - r\left(\mathcal{L}_{\mu_4,\nu_4}\right)) \int_0^1 t^{\beta-1} d\psi_{\mu_4,\nu_4}(t) - \sum_{k=1}^m \widehat{l_k} \int_0^1 H(t,t_k) d\psi_{\mu_4,\nu_4}(t)}.$$

This implies that S_2 is a bounded set in P, as required. Therefore, we can choose a large number $R_2 > \max\{\sup S_2, r_2\}$ such that

$$z \ne \lambda \mathcal{T} z,\ z \in \partial B_{R_2} \cap P,\ \lambda \in [0,1],$$

where $B_{R_2} = \{z \in P : \|z\| < R_2\}$. From Lemma 2.8 we have

$$i(\mathcal{T}, B_{R_2} \cap P, P) = 1. \tag{3.17}$$

As a result, from (3.14) and (3.17) we have

$$i(\mathcal{T}, (B_{R_2} \backslash \overline{B}_{r_2}) \cap P, P) = i(\mathcal{T}, B_{R_2} \cap P, P) - i(\mathcal{T}, B_{r_2} \cap P, P) = 1.$$

Therefore, the operator $\mathcal{T}$ has at least one fixed point in $(B_{R_2} \backslash \overline{B}_{r_2}) \cap P$. Thus, (1.1) has at least one positive solution. This completes the proof. □

4. Conclusions

In this paper, we study the existence of positive solutions for the Riemann-Liouville-type impulsive fractional integral boundary value problem (1.1). We first use the Gelfand theorem and the Krein-Rutman theorem to investigate a related positive linear operator, which can include the Riemann-Stieltjes integral condition. Then, the impulsive term is regarded as a perturbation, and we use some conditions concerning the spectral radius of the linear operator to obtain our main results. In this paper we provided a quite different method to study such problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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