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***Research article***

# A note on the degree bounds of the invariant ring

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**Abstract:** Let $G = C_p \times H$ be a finite group, where $C_p$ is a cyclic group of prime order $p$ and $H$ is a $p'$-group. Let $\mathbb{F}$ be an algebraically closed field in characteristic $p$. Let $V$ be a direct sum of $m$ non-trivial indecomposable $G$-modules such that the norm polynomials of the simple $H$-modules are the power of the product of the basis elements of the dual. In previous work, we proved the periodicity property of the polynomial ring $\mathbb{F}[V]$ with actions of $G$. In this paper, by the periodicity property, we showed that $\mathbb{F}[V]^G$ is generated by $m$ norm polynomials together with homogeneous invariants of degree at most $m|G| - \dim(V)$ and transfer invariants, which yields the well-known degree bound $\dim(V) \cdot (|G| - 1)$. More precisely, we found that this bound gets less sharp as the dimensions of simple $H$-modules increase.

**Keywords:** degree bounds; indecomposable module; symmetric algebra; periodicity property
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## 1. Introduction

Let $V$ denote a finite dimensional representation of a finite group $G$ over a field $\mathbb{F}$ in characteristic $p$ such that $p \mid |G|$. Then, $V$ is called a *modular representation*. We choose a basis $\{x_1, \dots, x_n\}$ for the dual space $V^*$. The action of $G$ on $V$ induces an action on $V^*$ and it extends to an action by algebra automorphisms on the symmetric algebra $S(V^*)$, which is equivalent to the polynomial ring

$$\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n].$$

The ring of *invariants*

$$\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] \mid g \cdot f = f \ \forall g \in G\}$$

is an $\mathbb{F}$-subalgebra of $\mathbb{F}[V]$. $\mathbb{F}[V]$ as a graded ring has degree decomposition

$$\mathbb{F}[V] = \bigoplus_{d=0}^{\infty} \mathbb{F}[V]_d$$

and

$$\dim_{\mathbb{F}}\mathbb{F}[V]_d < \infty$$

for each $d$. The group action preserves degree, so $\mathbb{F}[V]_d$ is a finite dimensional $\mathbb{F}G$-module. A classical problem is to determine the structure of $\mathbb{F}[V]^G$, and the construction of generators of the invariant ring mainly relies on its Noether's bound, denoted by $\beta(\mathbb{F}[V]^G)$, which is defined as follows:

$$\beta(\mathbb{F}[V]^G) = \min\{d \mid \mathbb{F}[V]^G \text{ is generated by homogeneous invariants of degree at most } d\}.$$

The bound reduces the task of finding invariant generators to pure linear algebra. If $\mathrm{char}(\mathbb{F}) > |G|$, the famous "Noether bound" says that

$$\beta(\mathbb{F}[V]^G) \leqslant |G| \quad \text{(Noether [1])}.$$

Fleischmann [2] and Fogarty [3] generalized this result to all non-modular characteristic ($|G| \in \mathbb{F}^*$), with a much simplified proof by Benson. However, at first shown by Richman [4, 5], in the modular case ($|G|$ is divisible by $\mathrm{char}(\mathbb{F})$), there is no bound that depends only on $|G|$. Some results have implied that

$$\beta(\mathbb{F}[V]^G) \leqslant \dim(V) \cdot (|G| - 1).$$

It was conjectured by Kemper [6]. It was proved in the case when the ring of invariants is Gorenstein by Campbell et al. [7], then in the Cohen-Macaulay case by Broer [8]. Symonds [9] has established that

$$\beta(\mathbb{F}[V]^G) \leqslant \max\{\dim(V) \cdot (|G| - 1),\ |G|\}$$

for any representation $V$ of any group $G$. This bound cannot be expected to be sharp in most cases. The Noether bound has been computed for every representation of $C_p$ in Fleischmann et al. [10]. It is in fact $2p-3$ for an indecomposable representation $V$ with $\dim(V) > 4$. Also in Neusel and Sezer [11], an upper bound that applies to all indecomposable representations of $C_{p^2}$ is obtained. This bound, as a polynomial in $p$, is of degree two. Sezer [12] provides a bound for the degrees of the generators of the invariant ring of the regular representation of $C_{p^r}$.

In the previous paper Zhang et al. [13], we extended the periodicity property of the symmetric algebra $\mathbb{F}[V]$ to the case

$$G \cong C_p \times H$$

if $V$ is a direct sum of $m$ indecomposable $G$-modules such that the norm polynomials of the simple $H$-modules are the power of the product of the basis elements of the dual. Now, $H$ permutes the power of the basis elements of simple $H$-modules, for example, $H$ is a monomial group. For the cyclic group $C_p$, the periodicity property and the proof of degree bounds mainly relies on the fact that the unique indecomposable projective $\mathbb{F}C_p$-module $V_p$ is isomorphic to the regular representation of $C_p$. Then, every invariant in $V_p^{C_p}$ is in the image of the transfer map (see Hughes and Kemper [14]). Inspired by this, with the assumption above, we find that every invariant in projective $G$-modules is in the image of the transfer. In this paper, since the periodicity leads to degree bounds for the generators of the invariant ring, we show that $\mathbb{F}[V]^G$ is generated by $m$ norm polynomials together with homogeneous invariants of degree at most $m|G| - \dim(V)$ and transfer invariants, which yields the well-known degree bound $\dim(V) \cdot (|G| - 1)$. Moreover, we find that this bound gets less sharp as the dimensions of simple $H$-modules increase, which is presented in the end of the paper.

## 2. Preliminaries

### *2.1. Periodicity property*

Let $G = C_p \times H$ be a finite group, whose form is a direct sum of the cyclic group of order $p$, $C_p$, and a $p'$-group $H$. Let $\mathbb{F}$ be an algebraically closed field in characteristic $p$. The complete set of indecomposable modules of $C_p$ is well-known, which are exactly the Jordan blocks $V_n$ of degree $n$, for $1 \leqslant n \leqslant p$, with 1's in the diagonal. Let $\mathrm{Sim}(H)$ be the complete set of non-isomorphic simple $\mathbb{F}H$-modules. Since $p \nmid |H|$, there is no difference between irreducibility and indecomposibility of $\mathbb{F}H$-modules. Then, by Huppert and Blackburn [15, Chapter VII, Theorem 9.15], the $\mathbb{F}G$-modules

$$V_n \otimes W \quad (1 \leqslant n \leqslant p,\ W \in \mathrm{Sim}(H))$$

form a complete set of non-isomorphic indecomposable $\mathbb{F}G$-modules.

Notice that the $\mathbb{F}H$-module $W$ is simple if and only if $W^*$ is simple, i.e., $\forall\, 0 \neq w \in W^*$, $w$ generates $W^*$ as an $\mathbb{F}H$-module. For a fixed $0 \neq w \in W^*$, let

$$H = \{h_1 = e, h_2, \ldots, h_{|H|}\}$$

and

$$\{w_1 = w, w_2 = h_2 w, \ldots, w_k = h_k w\}$$

be a basis of $W^*$ with $\dim(W^*) = k$. The norm polynomial

$$N^H(w) = \prod_{h \in H} h(w)$$

is an $H$-invariant polynomial.

**Lemma 2.1.** ([13, Lemma 3.1]) *Let $W^*$ be a simple $\mathbb{F}H$-module with a vector space basis $\{w_1, w_2, \ldots, w_k\}$ as above. Then, the norm polynomial $N^H(w_1)$ is of the form $(w_1, w_2, \ldots, w_k)^l f$, where $l \geqslant 1$ and $f$ is a polynomial in $\mathbb{F}[w_1, w_2, \ldots, w_k]$ such that $w_i \nmid f$ for $1 \leqslant i \leqslant k$.*

Choose the triangular basis $\{w_{11}, w_{12}, \ldots, w_{1k}, \ldots, w_{n1}, w_{n2}, \ldots, w_{nk}\}$ for $(V_n \otimes W)^*$, where

$$V_n^* = < v_1, \ldots, v_n >,$$

such that

$$C_p = < \sigma >,\ \ \sigma(v_j) = v_j + v_{j-1},\ \ \sigma(v_1) = v_1,\ \text{ and }\ w_{ji} = v_j \otimes w_i.$$

Then, we obtain

$$\sigma \cdot (w_{ji}) = w_{ji} + \epsilon_j w_{j-1,i} \ \text{ with } \ \epsilon_j = 1 \ \text{ for } \ 2 \leqslant j \leqslant n,\ \ \epsilon_1 = 0.$$

Then,

$$w \in < w_{n1}, w_{n2}, \ldots, w_{nk} >$$

is a distinguished variable in $(V_n \otimes W)^*$, which means that $w$ generates the indecomposable $\mathbb{F}G$-module $(V_n \otimes W)^*$. If

$$k = \frac{|H|}{l}$$

in Lemma 2.1, then

$$N^H(w_{n1}) = (w_{n1}w_{n2}\cdots w_{nk})^l$$

and

$$N_n \triangleq N^{C_p}(N^H(w_{n1})) = N^{C_p}((w_{n1}w_{n2}\cdots w_{nk})^l) \in \mathbb{F}[V_n \otimes W]^G.$$

Note that $\deg_{w_{ni}}(N_n) = lp$ for $1 \leqslant i \leqslant k$.

**Example 2.1.** Consider the monomial group $M$ generated by

$$\pi = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_k \end{pmatrix},$$

where $\pi$ is a $k$-rotation and $\alpha_i$ are $l$th roots of unity in $\mathbb{C}$. $M$ acts on the module $V = \mathbb{C}^k$ and also on the symmetric algebra

$$\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_k]$$

in the standard basis vectors $x_i$ of $V^*$. Then $\{x_1^l, \dots, x_k^l\}$ is an $M$-orbit and $(x_1, \dots, x_k)^l \in \mathbb{C}[V]^M$. For more details about degree bounds and Hilbert series of the invariant ring of monomial groups, it can be referred to Kemper [6] and Stanley [16].

Let $\mathbb{F}[V_n \otimes W]^\sharp$ be the principal ideal of $\mathbb{F}[V_n \otimes W]$ generated by $N_n$. The set $\{N_n\}$ is a Gröbner basis for the ideal it generates. Then, we may divide any given $f \in \mathbb{F}[V_n \otimes W]$ to obtain $f = qN_n + r$ for some $q, r \in \mathbb{F}[V_n \otimes W]$ with $\deg_{w_{ni}}(r) < lp$ for at least one $i$.

We define

$$\mathbb{F}[V_n \otimes W]^\flat := \{r \in \mathbb{F}[V_n \otimes W] \mid \mathrm{deg_{w_{ni}}(r) < lp \text{ for at least one } i}\}.$$

Note that both $\mathbb{F}[V_n \otimes W]^\sharp$ and $\mathbb{F}[V_n \otimes W]^\flat$ are vector spaces and that as vector spaces,

$$\mathbb{F}[V_n \otimes W] = \mathbb{F}[V_n \otimes W]^\sharp \oplus \mathbb{F}[V_n \otimes W]^\flat.$$

Moreover, they are $G$-stable. Therefore, we have the $\mathbb{F}G$-module decomposition

$$\mathbb{F}[V_n \otimes W] = \mathbb{F}[V_n \otimes W]^\sharp \oplus \mathbb{F}[V_n \otimes W]^\flat.$$

**Lemma 2.2.** *With the above notation, we have the $\mathbb{F}G$-module direct sum decomposition*

$$\mathbb{F}[V_n \otimes W] = N_n \cdot \mathbb{F}[V_n \otimes W] \oplus \mathbb{F}[V_n \otimes W]^\flat.$$

*Proof.* It is clear from the definitions of $\mathbb{F}[V_n \otimes W]^\sharp$ and $\mathbb{F}[V_n \otimes W]^\flat$. □

**Lemma 2.3.** ([13, Lemma 3.2]) *The $\mathbb{F}G$-module $\mathbb{F}[V_n \otimes W]_d^\flat$ can be decomposed as a direct sum of indecomposable projective modules if the following conditions are satisfied:*

1) $d + kn \geqslant lkp + 1$*;*

2) $N^H(w_1) = (w_1w_2\cdots w_k)^l$ *for some basis* $\{w_1, w_2, \dots, w_k\}$ *of* $W^*$.

**Theorem 2.1.** ([13, Theorem 3.5]) *Let $G = C_p \times H$ be a finite group described above. Let*

$$V = (V_{n_1} \otimes W_1) \oplus (V_{n_2} \otimes W_2) \oplus \ldots \oplus (V_{n_m} \otimes W_m)$$

*be an $\mathbb{F}G$-module such that the norm polynomial of the simple $H$-module $W_i$ is the power of the product of the basis elements of the dual. $V_{n_i}$ are indecomposable $C_p$-modules. Let $d_1, d_2 \ldots, d_m$ be non-negative integers and write $d_i = q_i(l_i k_i p) + r_i$, where $0 \leqslant r_i \leqslant l_i k_i p - 1$ for $i = 1, 2, \ldots, m$. Then,*

$$\mathbb{F}[V]_{(d_1,d_2,\ldots,d_m)} \cong \mathbb{F}[V]_{(r_1,r_2,\ldots,r_m)} \oplus \Big( \bigoplus_{W \in \mathrm{Sim}(H)} s_W V_p \otimes W \Big)$$

*as $\mathbb{F}G$-modules for some non-negative integers $s_W$.*

*2.2. Dade's algorithm and homogeneous system of parameters*

A set

$$\{f_1, f_2, \ldots, f_n\} \subset \mathbb{F}[V]^G$$

of homogeneous polynomials is called a *homogeneous system of parameters* (in the sequel abbreviated by *hsop*) if the $f_i$ are algebraically independent over $\mathbb{F}$ and $\mathbb{F}[V]^G$ as an $\mathbb{F}[f_1, f_2, \ldots, f_n]$-module is finitely generated. This implies $n = \dim(V)$.

The next lemma provides a criterion about systems of parameters.

**Lemma 2.4.** ([17, Proposition 5.3.7]) *$f_1, f_2, \ldots, f_n \in \mathbb{F}[z_1, z_2, \ldots, z_n]$ are an hsop if and only if for every field extension $\overline{\mathbb{F}} \supset \mathbb{F}$ the variety*

$$\mathcal{V}(f_1, f_2, \ldots, f_n; \overline{\mathbb{F}}) = \{(x_1, x_2, \ldots, x_n) \in \overline{\mathbb{F}} \mid f_i(x_1, x_2, \ldots, x_n) = 0 \ \text{for} \ \ i = 1, 2, \ldots, n\}$$

*consists of the point $(0, 0, \ldots, 0)$ alone.*

With the previous lemma, Dade's algorithm provides an implicit method to obtain an hsop. We refer to Neusel and Smith [18, p. 99-100] for the description of the existence of Dade's bases when $\mathbb{F}$ is infinite.

**Lemma 2.5.** ([18, Proposition 4.3.1]) *Suppose $\rho$: $G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a finite group $G$ over a field $\mathbb{F}$, and set $V = \mathbb{F}^n$. Suppose that there is a basis $z_1, z_2, \ldots, z_n$ for the dual representation $V^*$ that satisfies the condition*

$$z_i \notin \bigcup_{g_1,\ldots,g_{i-1} \in G} \mathrm{Span}_{\mathbb{F}}\{g_1 \cdot z_1, \ldots, g_{i-1} \cdot z_{i-1}\}, \quad i = 2, 3, \ldots, n.$$

*Then the top Chern classes*

$$c_{top}([z_1]), \ldots, c_{top}([z_n]) \in \mathbb{F}[V]^G$$

*of the orbits $[z_1], \ldots, [z_n]$ of the basis elements are a system of parameters.*

*2.3. A degree bound in the non-modular case*

An hsop exists in any invariant ring of a finite group, which is by no means uniquely determined. After an hsop has been chosen, it is important for subsequent computations to have an upper bound for the degrees of homogeneous generators of $\mathbb{F}[V]^G$ as an $\mathbb{F}[f_1, f_2, \ldots, f_n]$-module. In the non-modular case, $\mathbb{F}[V]^G$ is a Cohen-Macaulay algebra. It means that for any chosen hsop $\{f_1, f_2, \ldots, f_n\}$, there exists a set $\{h_1, h_2, \ldots, h_s\} \subset \mathbb{F}[V]^G$ of homogeneous polynomials such that

$$\mathbb{F}[V]^G \cong \bigoplus_{j=1}^{s} \mathbb{F}[f_1, f_2, \ldots, f_n] \cdot h_j,$$

where

$$s = \dim_{\mathbb{F}}(\mathrm{Tot}(\mathbb{F} \otimes_{\mathbb{F}[f_1,f_2,\ldots,f_n]} \mathbb{F}[V]^G)).$$

Using Dade's construction to produce an hsop $f_1, f_2, \ldots, f_n$ of $\mathbb{F}[V]^G$, the following lemmas imply that in the non-modular case, $\mathbb{F}[V]^G$ as $\mathbb{F}[f_1, f_2, \ldots, f_n]$-module is generated by homogeneous polynomials of degree less than or equal to $\dim(V)(|G| - 1)$. This result is very helpful to consider the degree bound for $G = C_p \times H$ in the modular case (see Corollary 3.2).

**Lemma 2.6.** ([16, Proposition 3.8]) *Suppose $\rho\colon G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a finite group $G$ over a field $\mathbb{F}$ whose characteristic is prime to the order of $G$. Let*

$$\mathbb{F}[V]^G \cong \oplus_{j=1}^{s} \mathbb{F}[f_1, f_2, \ldots, f_n] \cdot h_j,$$

*where $\deg(f_i) = d_i$, $\deg(h_j) = e_j$, $0 = e_1 \leqslant e_2 \leqslant \cdots e_s$. Let $\mu$ be the least degree of a $\det^{-1}$ relative invariant of $G$. Then*

$$e_s = \sum_{i=1}^{n} (d_i - 1) - \mu.$$

**Lemma 2.7.** ([17, Proposition 6.8.5]) *With the notations of the preceding lemma, then*

$$e_s \leqslant \sum_{i=1}^{n} (d_i - 1)$$

*with equality if and only if $G \subseteq \mathrm{SL}(n, \mathbb{F})$.*

## 3. Degree bounds for invariants

In this section, with the method of Hughes and Kemper [14], we proceed with the proof of degree bounds for the invariant ring.

Throughout this section, let $G = C_p \times H$ be a direct sum of a cyclic group of prime order and a $p'$-group, $V$ any $\mathbb{F}G$-module with a decomposition

$$V = (V_{n_1} \otimes W_1) \oplus (V_{n_2} \otimes W_2) \oplus \cdots \oplus (V_{n_m} \otimes W_m),$$

where $W_i$'s are simple $H$-modules satisfying that there is $w_{i1} \in W_i^*$ such that

$$N^H(w_{i1}) = (w_{i1} w_{i2} \cdots w_{ik_i})^{l_i}$$

with $k_i = \dim(W_i)$ and

$$\dim(V_{n_i} \otimes W_i) > 1.$$

Then $\{w_{i1}^{l_i}, w_{i2}^{l_i}, \ldots, w_{ik_i}^{l_i}\}$ are $H$-orbits.

Consider the *Chern classes* of

$$O_i = \{w_{i1}^{l_i}, w_{i2}^{l_i}, \ldots, w_{ik_i}^{l_i}\}.$$

Set

$$\varphi_{O_i}(X) = \prod_{t=1}^{k_i}(X + w_{it}^{l_i}) = \sum_{s=0}^{k_i} c_s(O_i) \cdot X^{k_i - s}.$$

Define classes $c_s(O_i) \in \mathbb{F}[W_i]^H$, $1 \leqslant s \leqslant k_i$, the orbit *Chern classes* of the orbit $O_i$.

Consider the twisted derivation

$$\triangle : \mathbb{F}[V] \to \mathbb{F}[V]$$

defined as $\triangle = \sigma - Id$, where $C_p$ is generated by $\sigma$. For a basis $x_1 \ldots, x_{n_i}$ of $V_{n_i}^*$ such that

$$\sigma(x_1) = x_1 \text{ and } \sigma(x_{n_i}) = x_{n_i} + x_{n_i - 1},$$

and a basis $y_1, \ldots, y_{k_i}$ of $W_i^*$, $\{x_s \otimes y_t \,|\, 1 \leqslant s \leqslant n_i, 1 \leqslant t \leqslant k_i\}$ is a basis of $V_{n_i}^* \otimes W_i^*$. On the other hand, since $x_{n_i}$ is a generator of indecomposable $C_p$-module $V_{n_i}^*$ and $y_t$, $1 \leqslant t \leqslant k_i$, are all generators of simple $H$-module $W_i^*$, then

$$z_{it} := x_{n_i} \otimes y_t, \quad 1 \leqslant t \leqslant k_i$$

are all generators of the indecomposable $G$-module $V_{n_i}^* \otimes W_i^*$ such that

$$\begin{aligned} z_{ji1} &:= \triangle^j(z_{i1}) = \triangle^j(x_{n_i}) \otimes y_1 = x_{n_i - j} \otimes y_1, \\ &\vdots \\ z_{jik_i} &:= \triangle^j(z_{ik_i}) = \triangle^j(x_{n_i}) \otimes y_{k_i} = x_{n_i - j} \otimes y_{k_i} \end{aligned}$$

for $0 \leqslant j \leqslant n_i - 1$. Since for $\mathbb{F}G$-modules dual is commutative with direct sum and tensor product, hence

$$\{z_{jit} = \triangle^j(z_{it}) \,|\, 0 \leqslant j \leqslant n_i - 1, 1 \leqslant i \leqslant m, 1 \leqslant t \leqslant k_i\}$$

can be considered as a vector space basis for $V^*$. Moreover,

$$N_i = N^{C_p}((z_{i1} z_{i2} \cdots z_{ik_i})^{l_i})$$

is the norm polynomial of $(V_{n_i} \otimes W_i)^*$. Therefore for the $H$-orbits

$$B_i = \{z_{i1}^{l_i}, z_{i2}^{l_i}, \ldots, z_{ik_i}^{l_i}\},$$

we have

$$N^{C_p}(c_s(B_i)) \in \mathbb{F}[V]^G.$$

Notice that

$$N^{C_p}(c_{k_i}(B_i)) = N_i$$

is the top *Chern class* of the orbit $B_i$.

Consider the transfer

$$\mathrm{Tr}^{\mathrm{G}} : \mathbb{F}[V] \to \mathbb{F}[V]^G$$

defined as

$$f \mapsto \sum_{\sigma \in G} \sigma(f).$$

Note that the transfer is an $\mathbb{F}[V]^G$-module homomorphism. The homomorphism is surjective in the non-modular case, while the image of transfer is a proper ideal of $\mathbb{F}[V]^G$ in the modular case.

**Lemma 3.1.** *Let V be a projective $\mathbb{F}G$-module, then $V^G \subset \mathrm{ImTr}^{\mathrm{G}}$.*

*Proof.* Since $V$ is projective, then it has a direct sum decomposition

$$V = \bigoplus_{W \in \mathrm{Sim}(H)} s_W(V_p \otimes W)$$

as $\mathbb{F}G$-modules for some non-negative integers $s_W$. It is easy to see that

$$(V_p \otimes W)^G = (V_p \otimes W)^{C_p \times H} = V_p^{C_p} \otimes W^H.$$

Since every invariant in $V_p$ is a multiple of the sum over a basis which is permuted by $C_p$ and $W^H = \mathrm{ImTr}^H$, the claim is proved. □

The following result is mainly a consequence of Lemmas 2.2, 2.3 and 3.1.

**Theorem 3.1.** *In the above setting, $\mathbb{F}[V]^G$ is generated as a module over the subalgebra $\mathbb{F}[N_1, N_2, \ldots, N_m]$ by homogeneous invariants of degree less than or equal to $m|G| - \dim(V)$ and $\mathrm{Tr}^G(\mathbb{F}[V])$.*

*Proof.* Let $M \subseteq \mathbb{F}[V]^G$ be the $\mathbb{F}[N_1, N_2, \ldots, N_m]$-module generated by all homogeneous invariants of degree less than or equal to $m|G| - \dim(V)$ and $\mathrm{Tr}^G(\mathbb{F}[V])$. We will prove $\mathbb{F}[V]_d^G \subseteq M$ by induction on $d$. So we can assume $d > m|G| - \dim(V)$. By Lemma 2.2, there is a direct sum decomposition of $\mathbb{F}[V]$ as $\mathbb{F}G$-module

$$\begin{aligned}
\mathbb{F}[V] &\cong \bigotimes_{i=1}^{m} \mathbb{F}[V_{n_i} \otimes W_i] \\
&\cong \bigotimes_{i=1}^{m} \left( N_i \cdot \mathbb{F}[V_{n_i} \otimes W_i] \oplus \mathbb{F}[V_{n_i} \otimes W_i]^\flat \right) \\
&\cong \bigoplus_{I \subseteq \{1,2,\ldots,m\}} \left( \otimes_{i \in I} N_i \cdot \mathbb{F}[V_{n_i} \otimes W_i] \otimes \otimes_{i \in \overline{I}} \mathbb{F}[V_{n_i} \otimes W_i]^\flat \right)
\end{aligned}$$

with $\overline{I} = \{1, 2, \ldots, m\} \setminus I$. Therefore, any $f \in \mathbb{F}[V]_d^G$ can be written as

$$f = \sum_{I \subseteq \{1,2,\ldots,m\}} f_I$$

and

$$f_I \in \bigoplus_{|I| \cdot |G| + \sum_{i \in I} q_i + \sum_{i \in \overline{I}} r_i = d} \left( \otimes_{i \in I} N_i \cdot \mathbb{F}[V_{n_i} \otimes W_i]_{q_i} \otimes \otimes_{i \in \overline{I}} \mathbb{F}[V_{n_i} \otimes W_i]_{r_i}^\flat \right)^G,$$

where $q_i, r_i$ are non-negative integers. For $I \neq \emptyset$, $f_I$ lies in $M$ by the induction hypothesis. For $I = \emptyset$, there is an $i$ such that $r_i > |G| - k_i n_i$ for each summand $\otimes_{i \in \bar{I}} \mathbb{F}[V_{n_i} \otimes W_i]^{\flat}_{r_i}$, since otherwise one would have

$$d \leqslant \sum_{i=1}^{m} (|G| - k_i n_i) = m|G| - \dim(V),$$

which contradicts the hypothesis. By Lemma 2.3, $\mathbb{F}[V_{n_i} \otimes W_i]^{\flat}_{r_i}$ is projective if

$$r_i > l_i k_i p - k_i n_i = |G| - k_i n_i,$$

and so by Alperin [19, Lemma 7.4] the summand $\otimes_{i \in \bar{I}} \mathbb{F}[V_{n_i} \otimes W_i]^{\flat}_{r_i}$ is projective. Then every invariant in it is in the image of transfer by Lemma 3.1. □

**Corollary 3.1.** *Let $G = C_p \times H$ be a finite group, where $H$ is a cyclic group such that $p \nmid |H|$,*

$$V = (V_{n_1} \otimes W_1) \oplus (V_{n_2} \otimes W_2) \oplus \cdots \oplus (V_{n_m} \otimes W_m),$$

*a module over $\mathbb{F}G$ such that $W_i$'s are permutation modules over $\mathbb{F}H$. Then* $\dim(W_i) = |H|$ *and $\mathbb{F}[V]^G$ is generated by $N_1, N_2, \ldots, N_m$ together with homogeneous invariants of degree less than or equal to $\sum_{i=1}^{m} |H|(p - n_i)$ and transfer invariants. Moreover, if*

$$V = (V_p \otimes W_1) \oplus (V_p \otimes W_2) \oplus \cdots \oplus (V_p \otimes W_m),$$

*such that $W_i$'s are permutation modules over $\mathbb{F}H$, then $\mathbb{F}[V]^G$ is generated by $N_1, N_2, \ldots, N_m$ together with transfer invariants.*

**Corollary 3.2.** *Let $G = C_p \times H$ be a finite group,*

$$V = (V_{n_1} \otimes W_1) \oplus (V_{n_2} \otimes W_2) \oplus \cdots \oplus (V_{n_m} \otimes W_m)$$

*be a finite-dimensional vector space in the setting of the beginning of this section. Assume that $\mathbb{F}$ is an infinite field. We have*

$$\beta(\mathbb{F}[V]^G) \leqslant \dim(V) \cdot (|G| - 1).$$

*Proof.* Let $z_{i1}, z_{i2}, \ldots, z_{ik_i}$ be distinguished variables such that

$$N_i = N^{C_p}((z_{i1} z_{i2} \cdots z_{ik_i})^{l_i})$$

are the norm polynomials of $V_{n_i} \otimes W_i$ for $1 \leqslant i \leqslant m$. Let

$$B_i = \{z_{i1}^{l_i}, z_{i2}^{l_i}, \ldots, z_{ik_i}^{l_i}\}, \quad 1 \leqslant i \leqslant m$$

be $H$-orbits. We use Dade's algorithm to extend the set

$$\{N^{C_p}(c_s(B_i)) \mid 1 \leqslant s \leqslant k_i, 1 \leqslant i \leqslant m\} \subset \mathbb{F}[V]^G$$

to an hsop for $\mathbb{F}[V]^G$ in the following way. We extend the sequence

$$\begin{aligned} v_1 &= z_{11}, \\ v_2 &= z_{12}, \\ &\dots \\ v_{k_1} &= z_{1k_1}, \\ &\dots \\ v_{k_1+k_2+\cdots+k_{m-1}+1} &= z_{m1}, \\ v_{k_1+k_2+\cdots+k_{m-1}+2} &= z_{m2}, \\ &\dots \\ v_{k_1+k_2+\cdots+k_m} &= z_{mk_m} \end{aligned}$$

in $V^*$ by vectors $v_{k_1+\cdots+k_m+1}, v_{k_1+\cdots+k_m+2}, \dots, v_{\dim(V)}$ such that

$$v_i \notin \bigcup_{g_1,\dots,g_{i-1}\in G} \mathrm{Span}_{\mathbb{F}}\{g_1 \cdot v_1, \dots, g_{i-1} \cdot v_{i-1}\}$$

for all

$$i \in \{k_1 + \ldots + k_m + 1, k_1 + \ldots + k_m + 2, \dots, \dim(V)\},$$

and set

$$f_j = \prod_{g\in G} g(v_j)$$

for

$$k_1 + \cdots + k_m < j \leqslant \dim(V).$$

Applying Lemmas 2.4 and 2.5, we see that $N^{C_p}(c_s(B_i))$, $1 \leqslant s \leqslant k_i, 1 \leqslant i \leqslant m$, together with $f_{k_1+k_2+\cdots+k_m+1}, \dots, f_{\dim(V)}$ form an hsop of $\mathbb{F}[V]^G$ since their common zero in $\overline{\mathbb{F}}^{\dim(V)}$ is the origin. Let $A$ denote the polynomial algebra generated by the elements of this hsop. Since $\mathbb{F}[V]$ as an $\mathbb{F}[V]^G$-module is finitely generated, then we have that $\mathbb{F}[V]$ is a finitely generated $A$-module. View $\mathbb{F}[V]$ as the invariant ring of the trivial group. By Lemma 2.7 the upper degree bound of module generators satisfies that

$$\beta\left(\mathbb{F}[V]/A\right) \leqslant \sum_{s,i} \deg\left(N^{C_p}(c_s(B_i))\right) + \sum_j \deg(f_j) - \dim(V).$$

Since the transfer preserves the degree of polynomials and it is clear that

$$m|G| = \sum_{i=1}^{m} \deg\left(N^{C_p}(c_{k_i}(B_i))\right) \leqslant \sum_{s,i} \deg\left(N^{C_p}(c_s(B_i))\right) + \sum_j \deg(f_j),$$

we conclude from Theorem 3.1 that the upper degree bound of module generators of $\mathbb{F}[V]^G$ as an

$A$-module satisfies that

$$\begin{aligned}
\beta\left(\mathbb{F}[V]^G/A\right) &\leqslant \sum_{s,i} \deg\left(N^{C_p}(c_s(B_i))\right) + \sum_j \deg(f_j) - \dim(V) \\
&= \sum_{i=1}^{m} pl_i\,(1+2+\cdots+k_i) + |G|\,(\dim(V) - k_1 - k_2 - \cdots - k_m) - \dim(V) \\
&= \sum_{i=1}^{m} \frac{pl_i k_i(k_i+1)}{2} - |G| \sum_{i=1}^{m} k_i + \dim(V)\cdot(|G|-1) \\
&= |G| \sum_{i=1}^{m} \frac{1-k_i}{2} + \dim(V)\cdot(|G|-1) \\
&\leqslant \dim(V)\cdot(|G|-1).
\end{aligned}$$

Note that we have assumed that $|G| \geqslant p$ and $\dim(V_{n_i} \otimes W_i) > 1$. Since

$$\deg\left(N^{C_p}(c_s(B_i))\right) = psl_i \leqslant |G| \text{ and } \deg(f_j) = |G|,$$

we obtain

$$\beta(A) \leqslant |G| \leqslant \dim(V)\cdot(|G|-1),$$

which completes the proof. □

**Remark 3.1.** The proof of the previous corollary implies that Symonds' bound is far too large. Since

$$\beta\left(\mathbb{F}[V]^G/A\right) \leqslant |G| \sum_{i=1}^{m} \frac{1-k_i}{2} + \dim(V)\cdot(|G|-1),$$

it seems that this bound is less sharp as the dimensions of simple $H$-modules increase. Moreover, the construction of $f_j$ for $k_1 + \ldots + k_m < j \leqslant \dim(V)$ is more than necessary to obtain invariants for $\mathbb{F}[V]^G$. It is enough to take the product of the $G$-orbit of $v_j$.

## 4. Conclusions

In this paper, we consider degree bounds of the invariant rings of finite groups $C_p \times H$ with the projectivity property of symmetric algebras, and we show that this bound is sharper than Symonds' bound as the dimensions of simple $H$-modules increase.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest in this paper.

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