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Research article

The uniformly continuous theorem of fractal interpolation surface function and its proof

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Abstract: In order to research uniform continuity of fractal interpolation surface function on a closed rectangular area, the accumulation principle was applied to prove uniform continuity of fractal interpolation surface function on a closed rectangular area. First, fractal interpolation surface function was constructed by affine mapping. Second, the continuous concept of fractal interpolation surface function at a planar point in a three-dimensional cartesian coordinate space system and uniform continuity of fractal interpolation surface function on a closed rectangular area were defined in the paper. Finally, the uniformly continuous theorem of fractal interpolation surface function was proven through accumulation principle in the paper. The conclusion showed that fractal interpolation surface was uniformly continuous function on a closed rectangular area.

Keywords: fractal geometry; affine mapping; fractal interpolation; surface function; accumulation principle; uniform continuity of a function of two variables
Mathematics Subject Classification: 33E99

1. Introduction

As a new branch of research in mathematics and physics, fractal geometry was proposed by American mathematician Mandlebrot in the 1960s and 1970s [1–5]. In nature and society, rather irregular phenomena and things are studied in fractal geometry. Because extremely irregular things and phenomena are ubiquitous in nature and society, fractal geometry is applied in almost every field, such as, chemistry, physics, biology, engineering mechanics, geology, economics, anthropology, sociology, and so on [6–10]. With the development of fractals, many new fractal research methods have also emerged. First, in aspect of fractal theoretical research methods, on the one hand, there is the fractal dimension method, which describes the roughness of extremely irregular curves and surfaces in nature [11–12]. On the other hand, an iterated functional system generated by affine mapping can produce a unique attractor, which has fractal self similar property and other fractal properties. This fractal theory was first proposed by American mathematicians Barnsley and Massopust in the 1980s and 1990s [13–17]. The self similar property of fractal can be applied in describing highly irregular shapes in nature, for example, the irregular shapes of galaxies, clouds, leaves, flowers, mountains, torrents of water, and much else. The advantage of the research method is that as long as a small amount of data information is obtained, the picture shape can be iterated through self similar property. For example, as long as three pairs of interpolation data points' cartesian coordinates are obtained, the curve shape can be iterated by the fractal interpolation curve by three iterations. (refer to: Figure 1). Of course, these fractal graphics are drawn by a computer program. Second, a multi-fractal method describes the state in which the multi-fractal spectrum and generalized fractal dimension change with a probability factor. The multi-fractal method is widely applied in studying thin film growth in material science [18–20]. Finally, the theory of fractal interpolation curves on the two-dimensional plane is extended to that of fractal interpolation surfaces in the three-dimensional space [21–24]. Some properties of fractal interpolation surfaces in three-dimensional space need to be researched. For example, the uniform continuity of a fractal interpolation surface will be researched in the following content of the paper.

The proof of the uniform continuity theorem of fractal interpolation surface can be proven rigorously by the accumulation principle on the two-dimensional plane and the uniform continuity definition of a function of two variables in the three-dimensional space in the paper.

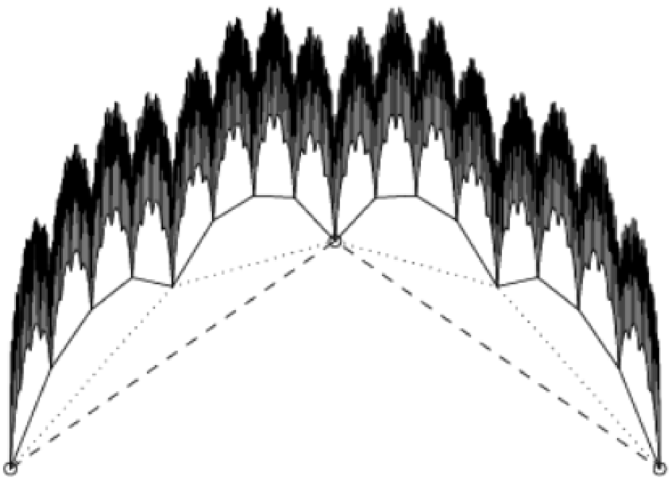

Figure 1. A fractal interpolation curve by three times iteration.

2. Major concepts and lemmas

Definition 1. [25–26] Let E and $A(x, y)$ be a set of planar points and a point on the plane, respectively. $A(x, y)$ is called accumulation point if, for any nonempty neighborhood region $\bigcup^{o}(A)$ of A, there is always a point in E, where $A(x, y)$ belongs to E or $A(x, y)$ does not belong to E.

Definition 2. [25–26] Let f be a function of two variables defined on a plane set D. There is a point $(x_0, y_0) \in D$. The f is called continuous function at the point $(x_0, y_0) \in D$ if, for any given number $\varepsilon > 0$, there is a real number $\delta > 0$ so that

$$|f(x, y) - f(x_0, y_0)| < \varepsilon \tag{1}$$

as long as $d((x, y), (x_0, y_0)) < \delta$ for all $(x, y) \in D$.

Definition 3. [25–26] Let f be a function of two variables defined on a plane set D. The f is called uniformly continuous function on the set D if, for any given number $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$|f(x', y') - f(x'', y'')| < \varepsilon \tag{2}$$

for any point $(x', y'), (x'', y'') \in D$, as long as $d((x', y'), (x'', y'')) < \delta$.

Definition 4 [27–28] Let $I = [a, b]$ and $J = [c, d]$ be two closed intervals. Construct a planar rectangular closed region $D = I \times J = \{(x, y) \mid a \le x \le b, c \le y \le d\}$. Divide D into grids in steps of Δx and Δy. The segmentation points are as follows:

$$\begin{cases} a = x_0 < x_1 < \cdots < x_N = b \\ c = y_0 < y_1 < \cdots < y_M = d \end{cases}. \tag{3}$$

The spacial coordinate data $(x_i, y_i, z_{i,j})$ ($i = 0,1,\cdots N; j = 0,1,\cdots,M$) on a set of grid points are given. The function of two variables $f: D \to R$ is called fractal interpolation function of two variables if it satisfies:

$$f(x_i, y_j) = z_{i,j}, \quad i = 0,1,\cdots N; j = 0,1,\cdots,M. \tag{4}$$

The following discuss is based on spacial region $K = D \times [h_1, h_2]$ $(-\infty < h_1 < h_2 < +\infty)$. The special distance $d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ is defined, for any points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in K$.

Denote $I_n = [x_{n-1}, x_n]$, $J_m = [y_{m-1}, y_m]$, $D_{n,m} = I_n \times J_m$, $(n \in \{1,\cdots N\}; m \in \{1,\cdots,M\})$. Define $\Phi_n: I \to I_n, \Psi_m: J \to J_m$ be contraction mapping and satisfy:

$$\begin{cases} \Phi_n(x_0) = x_{n-1}, \Phi_n(x_N) = x_n \\ \Psi_m(y_0) = y_{m-1}, \Psi_m(y_M) = y_m \end{cases}, \tag{5}$$

and

$$\begin{cases} |\Phi_n(x_1) - \Phi_n(x_2)| < k_1 |x_1 - x_2| \\ |\Psi_m(y_1) - \Psi_m(y_2)| < k_2 |y_1 - y_2| \end{cases}, \tag{6}$$

where $x_1, x_2 \in I$, $y_1, y_2 \in J$, $0 \le k_1 < 1$, $0 \le k_2 < 1$.

Define the two mappings $L_{n,m}: D \to R^2$ and $L_{n,m}(x, y) = (\Phi_n(x), \Psi_m(y))$ that are contraction mappings. The mapping $F_{n,m}: D \to [h_1, h_2]$ is continuous and satisfies:

$$\begin{cases} F_{n,m}(x_0, y_0, z_{0,0}) = z_{n-1,m-1} \\ F_{n,m}(x_N, y_0, z_{N,0}) = z_{n,m-1} \\ F_{n,m}(x_0, y_M, z_{0,M}) = z_{n-1,m} \\ F_{n,m}(x_N, y_M, z_{N,M}) = z_{n,m} \end{cases}. \tag{7}$$

For any points $(x_1, y_1), (x_2, y_2) \in D$, $(z_1, z_2) \in [h_1, h_2]$, $n \in \{1, 2, \cdots N\}; m \in \{1, 2, \cdots, M\}$, and $0 \le k_3 < 1$, the following formula is correct

$$\left| F_{n,m}(x_1, y_1, z_1) - F_{n,m}(x_2, y_2, z_2) \right| \le k_3 \left| z_1 - z_2 \right|. \tag{8}$$

Lemma 1. [25–26] If E is a bounded infinite set of planar points, there is at least one accumulation point in E on the plane R^2.

Lemma 2. [25–26] If the sequence $\{P_n(x_n, y_n)\}$ of planar points is infinite and bounded, there exists a convergent subsequence $\{P_{n_k}(x_{n_k}, y_{n_k})\}$ of $\{P_n(x_n, y_n)\}$.

Lemma 3. [27–28] Let the mapping Φ_n defined by Definition 5 above be a affine mapping, $\Phi_n(x) = a_n x + b_n$, from the Eq (5) above,

$$\begin{cases} a_n x_0 + b_n = x_{n-1} \\ a_n x_N + b_n = x_n \end{cases}. \tag{9}$$

So, the two coefficients a_n and b_n can be solved. The following equation can be obtained

$$\begin{cases} a_n = \dfrac{x_n - x_{n-1}}{x_N - x_0} \\ b_n = \dfrac{x_{n-1} x_N - x_n x_0}{x_N - x_0} \end{cases}. \tag{10}$$

So,

$$\Phi_n(x) = \frac{x_n - x_{n-1}}{x_N - x_0} x + \frac{x_{n-1} x_N - x_n x_0}{x_N - x_0}, \quad n \in \{1, \cdots, N\}. \tag{11}$$

Let the mapping $\Psi_m(y)$ defined by Definition 5 above be a affine mapping, $\Psi_m(y) = c_m y + d_m$ according to the Eq (5) above, the following equations can be obtained

$$\begin{cases} c_m y_0 + d_m = y_{m-1} \\ c_m y_N + d_m = y_m \end{cases}. \tag{12}$$

From the equations above, c_m and d_m can be solved.

$$\begin{cases} c_m = \dfrac{y_m - y_{m-1}}{y_M - y_0} \\ d_m = \dfrac{y_{m-1} y_M - y_m y_0}{y_M - y_0} \end{cases}. \tag{13}$$

So,

$$\Psi_m(y) = \frac{y_m - y_{m-1}}{y_M - y_0} y + \frac{y_{m-1}y_M - y_m y_0}{y_M - y_0}, \quad m \in \{1, \cdots, M\} \ldots \tag{14}$$

Let

$$F_{n,m}(x,y,z) = e_{n,m}x + f_{n,m}y + g_{n,m}xy + s_{n,m}z + k_{n,m} \quad n \in \{1,\cdots,N\}, \quad m \in \{1,\cdots,M\}, \tag{15}$$

according to the Eq (7), the following system of equations can be obtained

$$\begin{cases} e_{n,m}x_0 + f_{n,m}y_0 + g_{n,m}x_0y_0 + s_{n,m}z_{0,0} + k_{n,m} = z_{n-1,m-1} \\ e_{n,m}x_N + f_{n,m}y_0 + g_{n,m}x_Ny_0 + s_{n,m}z_{N,0} + k_{n,m} = z_{n,m-1} \\ e_{n,m}x_0 + f_{n,m}y_M + g_{n,m}x_0y_M + s_{n,m}z_{0,M} + k_{n,m} = z_{n-1,m} \\ e_{n,m}x_N + f_{n,m}y_M + g_{n,m}x_Ny_M + s_{n,m}z_{N,M} + k_{n,m} = z_{n,m} \end{cases}. \tag{16}$$

Let $s_{n,m}$ be free parameter and satisfies $0 \le s_{n,m} < 1$, which is called vertical ratio factor. The every coefficient of the system of equations above can be solved.

$$\begin{cases} g_{n,m} = \dfrac{z_{n-1,m-1} - z_{n-1,m} - z_{n,m-1} + z_{n,m} - s_{n,m}(z_{0,0} - z_{N,0} - z_{0,M} + z_{0,M})}{x_0y_0 - x_Ny_0 - x_0y_M + x_Ny_M} \\ e_{n,m} = \dfrac{z_{n,m-1} - s_{n,m}(z_{0,0} - z_{N,0}) - g_{n,m}(x_0y_0 - x_Ny_0)}{x_0 - x_N} \\ f_{n,m} = \dfrac{z_{n-1,m-1} - z_{n-1,m} - s_{n,m}(z_{0,0} - z_{0,M}) - g_{n,m}(x_0y_0 - x_0y_M)}{x_0y_0 - x_Ny_0 - x_0y_M + x_Ny_M} \\ k_{n,m} = z_{n,m} - e_{n,m}x_N - f_{n,m}y_M - s_{n,m}z_{N,M} - g_{n,m}x_Ny_M \end{cases}, \tag{17}$$

where $n \in \{1,\cdots,N\}$, $m \in \{1,\cdots,M\}$.

Lemma 4 [27–28] For the interpolation function system defined above from Eq (4) to Eq (17), there exists a unique attractor $G = \{(x, y, f(x,y)) | (x,y) \in D\}$ and it is a graph of continuous function f, which satisfies:

$$f(x_i, y_j) = z_{i,j}, \quad i = 0,1,\cdots N;\ j = 0,1,\cdots,M. \tag{18}$$

According to the principle of fractal interpolation on closed rectangular region, the following analytic function of self affine two variables fractal interpolation function can be solved.

$$f(x,y) = e_{n,m}\Phi_n^{-1}(x) + f_{n,m}\Psi_m^{-1}(y) + g_{n,m}\Phi_n^{-1}(x)\Psi_m^{-1}(y) + s_{n,m}f(\Phi_n^{-1}(x), \Psi_m^{-1}(y)) + k_{n,m}, \tag{19}$$

where

$$\Phi_n^{-1}(x) = \frac{x_N - x_0}{x_n - x_{n-1}}(x - x_{n-1}) + x_0, x \in [x_{n-1}, x_n], n \in \{1,\cdots,N\}, \tag{20}$$

$$\Psi_m^{-1}(y) = \frac{y_M - y_0}{y_m - y_{m-1}}(y - y_{m-1}) + y_0, y \in [y_{m-1}, y_m], m \in \{1,\cdots,M\}. \tag{21}$$

The coefficients $e_{n,m}$, $f_{n,m}$, $g_{n,m}$, $k_{n,m}$ of Eq (19) can be solved by Eq (17) and the coefficient $s_{n,m}$ is vertical compress ratio factor, which is artificially given according to roughness of fractal interpolation surface.

3. Uniform continuity of fractal interpolation surface on a closed rectangular area

Theorem. If the fractal interpolation surface function $f(x,y)$ defined by Eq (19) above is a continuous function on a closed rectangular area $D=[a,b]\times[c,d]$. Then $f(x,y)$ is uniformly continuous function on D.

Proof: Here the contradiction proof method can be used. Suppose that $f(x,y)$ is continuous on closed rectangular region D, but it is not uniformly continuous on D. That is to say, $\exists\varepsilon_0>0,\forall\delta>0,\exists P(x,y),Q(x',y')\in D,$ and $d(P,Q)<\delta$, but

$$\left|f(x,y)-f(x',y')\right|\ge\varepsilon_0>0. \tag{22}$$

Because D is a bounded closed domain, from Lemmas 2 and 3, there is a convergent subsequence $\left\{P_{n_k}\left(x_{n_k},y_{n_k}\right)\right\}\subseteq D$, and let

$$\lim_{k\to\infty}P_{n_k}\left(x_{n_k},y_{n_k}\right)=P_0\left(x_0,y_0\right). \tag{23}$$

According to the hypothesis condition of contradiction proof method above, there exists points sequence $\left\{Q_{n_k}\left(x'_{n_k},y'_{n_k}\right)\right\}$ with the same subscript as $\left\{P_{n_k}\left(x_{n_k},y_{n_k}\right)\right\}$ and satisfies:

$$d\left(P_{n_k}\left(x_{n_k},y_{n_k}\right),Q_{n_k}\left(x'_{n_k},y'_{n_k}\right)\right)<\delta. \tag{24}$$

On the one hand,

$$\begin{aligned}&\left|f(x_{n_k},y_{n_k})-f(x'_{n_k},y'_{n_k})\right|\\&=\left|e_{n,m}\Phi_n^{-1}(x_{n_k})+f_{n,m}\Psi_m^{-1}(y_{n_k})+g_{n,m}\Phi_n^{-1}(x_{n_k})\Psi_m^{-1}(y_{n_k})+s_{n,m}f(\Phi_n^{-1}(x_{n_k}),\Psi_m^{-1}(y_{n_k}))+k_{n,m}\right.\\&\left.-(e_{n,m}\Phi_n^{-1}(x'_{n_k})+f_{n,m}\Psi_m^{-1}(y'_{n_k})+g_{n,m}\Phi_n^{-1}(x'_{n_k})\Psi_m^{-1}(y'_{n_k})+s_{n,m}f(\Phi_n^{-1}(x'_{n_k}),\Psi_m^{-1}(y'_{n_k}))+k_{n,m})\right|\\&\le\left|e_{n,m}\right|\left|\Phi_n^{-1}(x_{n_k})-\Phi_n^{-1}(x'_{n_k})\right|+\left|f_{n,m}\right|\left|\Psi_m^{-1}(y_{n_k})-\Psi_m^{-1}(y'_{n_k})\right|+\left|g_{n,m}\right|\left|\Phi_n^{-1}(x_{n_k})\Psi_m^{-1}(y_{n_k})\right.\\&\left.-\Phi_n^{-1}(x'_{n_k})\Psi_m^{-1}(y'_{n_k})\right|+\left|s_{n,m}\right|\left|f(\Phi_n^{-1}(x_{n_k}),\Psi_m^{-1}(y_{n_k}))-f(\Phi_n^{-1}(x'_{n_k}),\Psi_m^{-1}(y'_{n_k}))\right|.\end{aligned} \tag{25}$$

$$\left|\Phi_n^{-1}(x_{n_k})-\Phi_n^{-1}(x'_{n_k})\right|=\left|\frac{x_N-x_0}{x_n-x_{n-1}}\right|\left|x_{n_k}-x'_{n_k}\right|<\left|\frac{x_N-x_0}{x_n-x_{n-1}}\right|\cdot\frac{1}{n_k}\to0\left(n_k\to\infty\right). \tag{26}$$

$$\left|\Psi_m^{-1}(y_{n_k})-\Psi_m^{-1}(y'_{n_k})\right|=\left|\frac{y_M-y_0}{y_m-y_{m-1}}\right|\left|y_{n_k}-y'_{n_k}\right|<\left|\frac{y_M-y_0}{y_m-y_{m-1}}\right|\cdot\frac{1}{n_k}\to0\left(n_k\to\infty\right). \tag{27}$$

$$\left|\Phi_n^{-1}(x_{n_k})\Psi_m^{-1}(y_{n_k})-\Phi_n^{-1}(x'_{n_k})\Psi_m^{-1}(y'_{n_k})\right|$$

$$=\Bigg|\left(x_{n_k}-x_{n-1}\right)\left(y_{n_k}-y_{m-1}\right)\cdot\frac{x_N-x_0}{x_n-x_{n-1}}\cdot\frac{y_M-y_0}{y_m-y_{m-1}}+y_0\left(x_{n_k}-x_{n-1}\right)\cdot\frac{x_N-x_0}{x_n-x_{n-1}}+x_0\left(y_{n_k}-y_{m-1}\right)\frac{y_M-y_0}{y_m-y_{m-1}}$$

$$-\left(x'_{n_k}-x_{n-1}\right)\left(y'_{n_k}-y_{m-1}\right)\frac{x_N-x_0}{x_n-x_{n-1}}\cdot\frac{y_M-y_0}{y_m-y_{m-1}}x_0-y_0\left(x'_{n_k}-x_{n-1}\right)\frac{x_N-x_0}{x_n-x_{n-1}}-x_0\left(y'_{n_k}-y_{m-1}\right)\frac{y_M-y_0}{y_m-y_{m-1}}\Bigg|$$

$$\le\left|\frac{x_N-x_0}{x_n-x_{n-1}}\cdot\frac{y_M-y_0}{y_m-y_{m-1}}\right|\cdot\frac{2}{n_k^2}+|x_0|\left|\frac{y_M-y_0}{y_m-y_{m-1}}\right|\cdot\frac{2}{n_k}+|y_0|\left|\frac{x_N-x_0}{x_n-x_{n-1}}\right|\cdot\frac{2}{n_k}$$

$$=\left(\left|\frac{x_N-x_0}{x_n-x_{n-1}}\cdot\frac{y_M-y_0}{y_m-y_{m-1}}\right|\cdot\frac{1}{n_k}+|x_0|\left|\frac{y_M-y_0}{y_m-y_{m-1}}\right|+|y_0|\left|\frac{x_N-x_0}{x_n-x_{n-1}}\right|\right)\cdot\frac{2}{n_k}\to 0\left(n_k\to\infty\right). \tag{28}$$

$$\left|f(\Phi_n^{-1}(x_{n_k}),\Psi_m^{-1}(y_{n_k}))-f(\Phi_n^{-1}(x'_{n_k}),\Psi_m^{-1}(y'_{n_k}))\right|$$

$$=\Big|e_{n,m}\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k}))+f_{n,m}\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k}))+g_{n,m}\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k}))\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k}))$$

$$+s_{n,m}f(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))+k_{n,m}-(e_{n,m}\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k}))+f_{n,m}\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k}))$$

$$+g_{n,m}\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k}))\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k}))+s_{n,m}f(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))+k_{n,m})\Big|$$

$$\le|e_{n,m}|\left|\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k}))-\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k})\right|+|f_{n,m}|\left|\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k}))-\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k}))\right|$$

$$+|g_{n,m}|\left|\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k}))\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k}))-\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k}))\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k}))\right|$$

$$+|s_{n,m}|\left|f(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))-f(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))\right|$$

$$\le\Bigg(|e_{n,m}|\left(\frac{x_N-x_0}{x_n-x_{n-1}}\right)^2\frac{1}{n_k}+|f_{n,m}|\left(\frac{y_M-y_0}{y_m-y_{m-1}}\right)^2\frac{1}{n_k}+|g_{n,m}|\Bigg[\left(\frac{x_N-x_0}{x_n-x_{n-1}}\right)^2\left(\frac{y_M-y_0}{y_m-y_{m-1}}\right)^2$$

$$\cdot\left(|x_N-x_0|+|y_M-y_0|+|x_{n-1}|+|y_{m-1}|\right)+\left(\frac{x_N-x_0}{x_n-x_{n-1}}\right)^2\left|\frac{y_M-y_0}{y_m-y_{m-1}}\right|\left(|y_0|+|y_{m-1}|\right)$$

$$+\left(\frac{x_N-x_0}{x_n-x_{n-1}}\right)^2|y_0|+\frac{x_N-x_0}{x_n-x_{n-1}}\left(\frac{y_M-y_0}{y_m-y_{m-1}}\right)^2\cdot|x_0-x_{n-1}|+\left(\frac{y_M-y_0}{y_m-y_{m-1}}\right)^2|x_0|\Bigg]\Bigg)\frac{1}{n_k}$$

$$+|s_{n,m}|\left|f(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))-f(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))\right|\to 0\left(n_k\to\infty\right). \tag{29}$$

The following Eq (30) is correct.

$$\left|f(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))-f(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k})),\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))\right|$$

$$\le|e_{n,m}|\left|\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})))-\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k}))\right|+|f_{n,m}|\left|\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))-\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))\right|$$

$$+|g_{n,m}|\left|\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})))\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))-\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k})))\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))\right|$$

$$+\left|s_{n,m}\right|\left|f(\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k}))),\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k}))))-f(\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k}))),\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k}))))\right|$$
$$\to 0\left(n_k\to 0\right), \tag{30}$$

which repeats the steps of Eq (29).

To sum up, Eq (25)

$$\left|f(x_{n_k},y_{n_k})-f(x'_{n_k},y'_{n_k})\right|$$
$$\leq\left|e_{n,m}\right|\left|\frac{x_N-x_0}{x_n-x_{n-1}}\right|\cdot\frac{1}{n_k}+\left|f_{n,m}\right|\left|\frac{y_M-y_0}{y_m-y_{m-1}}\right|\cdot\frac{1}{n_k}+\left|g_{n,m}\right|\left(\left|\frac{x_N-x_0}{x_n-x_{n-1}}\cdot\frac{y_M-y_0}{y_m-y_{m-1}}\right|\cdot\frac{1}{n_k}+\left|x_0\right|\left|\frac{y_M-y_0}{y_m-y_{m-1}}\right|\right.$$
$$\left.+\left|y_0\right|\left|\frac{x_N-x_0}{x_n-x_{n-1}}\right|\right)\cdot\frac{2}{n_k}+\left|s_{n,m}\right|\left[\left|e_{n,m}\right|\left|\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})))-\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k}))\right|\right.$$
$$+\left|f_{n,m}\right|\left|\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))-\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))\right|+\left|g_{n,m}\right|\left|\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k})))\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k})))\right.$$
$$\left.-\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k})))\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k})))\right|+\left|s_{n,m}\right|\left|f(\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x_{n_k}))),\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y_{n_k}))))\right.$$
$$\left.\left.-f(\Phi_n^{-1}(\Phi_n^{-1}(\Phi_n^{-1}(x'_{n_k}))),\Psi_m^{-1}(\Psi_m^{-1}(\Psi_m^{-1}(y'_{n_k}))))\right|\right]\to 0\left(n_k\to 0\right) \tag{31}$$

can be arbitrarily positive small and tend to zero. In other words,

$$\lim_{k\to\infty}\left|f(x_{n_k},y_{n_k})-f(x'_{n_k},y'_{n_k})\right|=0\,. \tag{32}$$

However, according to the not uniform continuity definition and the inequalities of limits, the following inequality is right.

$$\lim_{k\to\infty}\left|f(x_{n_k},y_{n_k})-f(x'_{n_k},y'_{n_k})\right|\geq\varepsilon_0\,. \tag{33}$$

It is very obvious that there is a contradiction between Eqs (32) and (33), which indicates that the previous negative hypothesis is incorrect. Thus, the conclusion above indicates that fractal interpolation surface on a closed rectangular region is uniformly continuous through the proof from Eq (25) to Eq (31) by the definition of uniform continuity. A uniformly continuous fractal interpolation surface on a rectangle can be drawn by a computer program (refer to: Figure 2).

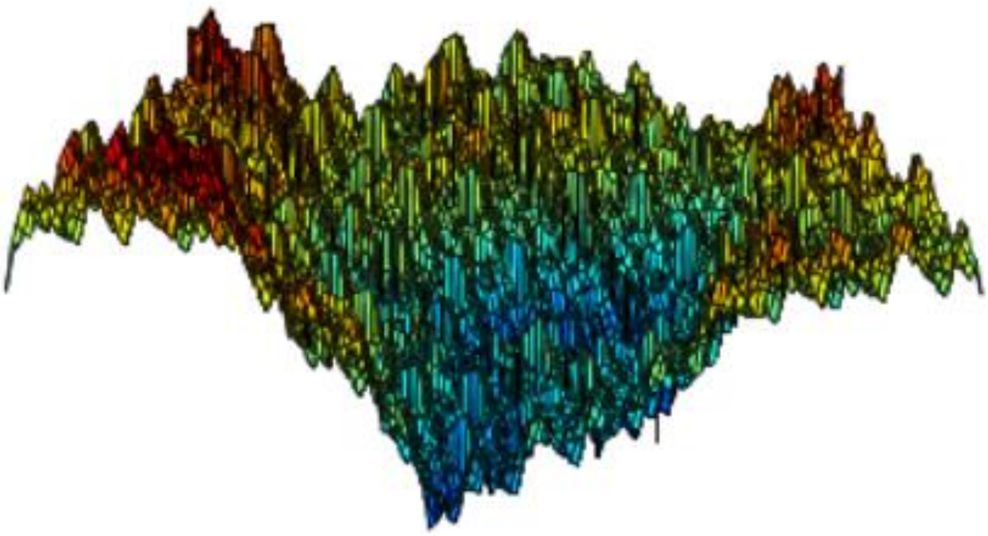

Figure 2. Schematic diagram of a uniformly continuous fractal interpolation surface.

4. Conclusions

First, the definition and relational theorems of fractal interpolation surface function are expounded. Second, the definition and theorem of accumulation point are introduced. Finally, the contradiction proof method and accumulation principle are applied to prove the uniform continuity of fractal interpolation surface function on a closed rectangular area.

In the future, on the one hand, research will be conducted on the variation of uniformly continuous fractal surface. On the other hand, the study of the relationship between the morphology differences of rock fracture surfaces and the fracture mechanics mechanisms of uniformly continuous fractal surfaces will continue.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. B. B. Mandelbrot, How long is the coast of Britain? Statistical self-similarity and fractional dimension, *Science*, **156** (1967), 636–638. https://doi.org/10.1126/science.156.3775.636
2. B. B. Mandelbrot, J. W. V. Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.*, **10** (1968), 422–437. https://doi.org/10.1137/1010093
3. B. B. Mandelbrot, *The fractal geometry of nature*, New York: Macmillan Press, 1983.
4. B. B. Mandelbrot, D. E. Passoja, A. J. Paulley, Fractal character of fracture surfaces of metals, *Nature*, **308** (1984), 721–722. https://doi.org/10.1038/308721a0
5. B. B. Mandelbrot, M Aizenman, Fractals: form, chance, and dimension, *Phys. Today*, **32** (1979), 65–66. https://doi.org/10.1063/1.2995555
6. H. P. Xie, *Fractals in rock mechanics*, London: CRC Press, 1993. https://doi.org/10.1201/9781003077626
7. H. P. Xie, H. Q. Sun, Y. Ju, Study on generation of rock surfaces by using fractal interp*olation*, *Int. J. Solids Struct.*, **38** (2001), 5765–5787. https://doi.org/10.1016/S0020-7683(00)00390-5
8. M. Nasehnejad, G. Nabiyouni, M. G. Shahraki, Thin film growth by 3D multi-particle diffusion limited aggregation model: Anomalous roughening and fractal analysis, *Physica A*, **493** (2018), 135–147. https://doi.org/10.1016/j.physa.2017.09.099
9. S. Bianchi, F. Massimiliano, Fractal stock markets: International evidence of dynamical (in) efficiency, *Chaos*, **27** (2017), 071102. https://doi.org/10.1063/1.4987150
10. S. S. Yang, H. H. Fu, B. M. Yu, Fractal analysis of flow resistance in tree-like branching networks with roughened microchannels, *Fractals*, **25** (2017), 1750008. https://doi.org/10.1142/S0218348X17500086
11. Z. G. Feng, Variation and Minkowski dimension of fractal interpolation surface, *J. Math. Anal. Appl.*, **345** (2008), 322–334. https://doi.org/10.1016/j.jmaa.2008.03.075
12. X. B. Zhang, H. L. Zhu, H. X. Yao, Analysis of a new three-dimensional chaotic system, *Nonlinear Dyn.*, **67** (2012), 335–343. https://doi.org/10.1007/s11071-011-9981-x
13. M. F. Barnsley, Fractal functions and interpolation, *Constr. Approx.*, **2** (1986), 303–329. https://doi.org/10.1007/BF01893434
14. M. F. Barnsley, M. Hegland, P. Massopust, Numerics and fractals, *Bull. Inst. Math. Acad.*, **9** (2014), 389–430.
15. P. R. Massopust, Fractal surfaces, *J. Math. Anal. Appl.*, **151** (1990), 275–290. https://doi.org/10.1016/0022-247X(90)90257-G
16. P. R. Massopust, *Fractal functions, fractal surfaces, and wavelets*, Orlando: Academic Press, 1994. https://doi.org/10.1016/c2009-0-21290-6
17. M. Barnsley, A. J. Hurd, Fractals everywhere, *Am. J. Phys.*, **57** (1989), 1053. https://doi.org/10.1119/1.15823
18. X. Z. Pan, X. D. Shang, M. G. Wang, F. Zuo, The cantor set's multi-fractal spectrum formed by different probability factors in mathematical experiment, *Fractals*, **25** (2017), 1750002. https://doi.org/10.1142/S0218348X17500025
19. S. Iqbal, M. Idrees, A. M. Siddiqui, A. R. Ansari, Some solutions of the linear and nonlinear Klein-Gordon equations using the optimal homotopy asymptotic method, *Appl. Math. Comput.*, **216** (2010), 22898–2909. https://doi.org/10.1016/j.amc.2010.04.001

20. B. Wang, Q. X. Zhu, Stability analysis of semi-Markov switched stochastic systems, *Automatica*, **94** (2018), 72–80. https://doi.org/10.1016/j.automatica.2018.04.016
21. X. Z. Pan, Fractional calculus of fractal interpolation function on [0, b] (b>0), *Abstr. Appl. Anal.*, **2014** (2014), 640628. http://doi.org/10.1155/2014/640628
22. B. J. Hong, D. C. Lu, Modified fractional variational iteration method for solving the generalized time-space fractional schrdinger equation, *Sci. World J.*, **2014** (2014), 964643. http://doi.org/:10.1155/2014/964643
23. X. Z. Pan, R. F. Xu, X. D. Shang, M. G. Wang, The properties of fractional order calculus of fractal interpolation function of broken line segments, *Fractals*, **26** (2018), 1850031. http://doi.org/10.1142/S0218348X18500317
24. Y. Chang, Y. Q. Wang, F. E. Alsaadi, G. D. Zong, Adaptive fuzzy output-feedback tracking control for switched stochastic pure-feedback nonlinear systems, *Int. J. Adapt. Control*, **33** (2019), 1567–1582. http://doi.org/10.1002/acs.3052
25. W. Rubin, *Principles of mathematical analysis*, New York: McGraw-Hill Publishing Company, 1976.
26. Y. Y. Shan, Remarks about two theorems in principles of mathematical analysis, *Journal of Mathematics Research*, **1** (2009), 61–63. http://doi.org/10.5539/jmr.v1n2p61
27. H. P. Xie, H. Q. Sun, The theory of fractal interpolated surface and its applications, *Appl. Math. Mech.*, **19** (1998), 321–331. https://doi.org/10.1007/BF02457536
28. H. P. Xie, H. Q. Sun, The study on bivariate fractal interpolation functions and creation of fractal interpolation surfaces, *Fractals*, **5** (1997), 625–634. http://doi.org/10.1142/S0218348X97000504