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Research article

Some operators in soft primal spaces

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Abstract: The concept of operators in topological spaces occupies a very important place. For this reason, a great deal of work and many results were presented via operators. Herein, we defined a primal local soft closure operator $\Lambda(\cdot)$ using the concept of soft topology and soft primal and reconnoitered its basic characteristics. Then, we found several fundamental results about the behavior of the primal soft closure operator $\lambda(\cdot)$ with the help of $\Lambda(\cdot)$. Among other obtained results, we introduced a new topology induced by the primal soft closure operator. At last, we defined primal soft suitable spaces and gave some equivalent descriptions of it.

Keywords: primal space; soft open; soft local operator; soft primal; soft primal suitable spaces
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1. Introduction

Molodtsov [1] obtained the initial step of soft sets. Many well-known scholars and thinkers have expressed interest in Molodtsov's soft set theory proposal because it compensates for the flaws and shortcomings of existing mathematical model tools and has a significant applicability benefit when dealing with uncertainty. Following the successful introduction of soft sets, the concept was refined and hybridized into soft rough sets, and fuzzy soft sets. Soft set theory has been used to obtain and investigate various structures of mathematics. Soft theory includes soft algebra [1, 2], soft category theory [1, 3], and so on. Shabir and Naz [4] demonstrated two strategies for defining soft topology in 2011.

Soft open sets are known to be the building blocks of soft topology, although other classes of soft sets, such as soft generalized open sets and soft weak structures, can also contribute to the creation of soft topology. Several researchers and philosophers produced a soft version of classical topological concepts and conceptions in the aftermath of Shabir and Naz's work, soft submaximal [5], infra soft structure [6], soft somewhat [7], soft ideal [8], soft grail [9], congruence representation via soft ideals [10], and according to Al-shami et al. in [11], the concept of primal soft topological spaces. Also, he defined a soft operator $(\cdot)^{\diamond}$ using the elements of the soft topology and soft primal in soft primal. After that, in soft contexts, many traditional topological ideas have been introduced, for instance, Baire category soft sets and their symmetric local properties [12] and some classes of soft functions via soft open sets modulo soft sets of the first category [13].

Another fascinating research topic is how to build soft topologies over a universal set. Terepeta [14] revealed two efficient methods for constructing soft topologies from crisp topologies. According to Al-Shami et al. in [15], the soft topology of one of the techniques is identical to the enriched soft topology. One of basic extended soft topologies is produced by Ameen et al. [30] and defined by old soft topology and a soft ideal combined to form the cluster soft topology, which is ideal soft topologies. Similarly, [9] used soft grills to explain the concept of soft topology.

Recently, Acharjee et al. [16] introduced a new structure called primal. They get not just some primal-related fundamental features, but also some links between topological spaces and primal topological spaces. Primals [16] appear to be the dual of the concept of grills, while the duals of filters are ideals. After that, Al-Omari et al. [17, 18] used primal to establish several new operators in primal topological spaces. Moreover, in obtaining the concept of soft primal topology and some results, we show that the set of primal topologies forms a natural class in the lattice of topologies and provide some descriptions for primal soft topology under specific types of soft primal.

The motivations for writing this paper are as follows: The first reason we produce this post is to provide a new form of soft structure that improves soft setting research by establishing distinct frameworks that allow us to design new soft concepts and features. Second, we create a new method for generating soft topology that is inspired by several soft operators.

We arrange the content of the paper as follows: In Section 2, we recall the basic concepts and findings that make this work self-contained. In Section 3, we define the concept of the primal local soft closure operator $\Lambda(\cdot)$ inspired by the concept of soft topology and soft primal. The concept of the primal local soft closure operator $\Lambda(\cdot)$ is modification of a soft operator $(\cdot)^{\diamond}$, which is obtained by Al-Shami et al. in [11], then we study the main properties of this concept. In Section 4, we find a number of fundamental truths about the behavior of the primal soft closure operator $\lambda(\cdot)$, then we introduce a new topology via $\lambda(\cdot)$. Among other obtained results, we define primal soft suitable spaces and give some equivalent descriptions in Section 5.

2. Preliminaries

Definition 2.1. *[1] Let $\Phi : \rho \to 2^{\mathcal{A}}$ be a set-valued function form of parameters set ρ to the power set of a nonempty set $\mathcal{A}$, then the pair (Φ,ρ) is said to be a soft set over $\mathcal{A}$, which is defined as follows: $(\Phi,\rho) = \{(\alpha, \Phi(\alpha)) : \alpha \in \rho$ and $\Phi(\alpha) \in 2^{\mathcal{A}}\}$, and we represented the soft set as Φ_ρ. Throughout this paper, Φ_ρ, Ω_ρ, and Ψ_ρ denote the soft sets over $\mathcal{A}$. We symbolized the family of all soft sets over $\mathcal{A}$ with parameters ρ by $SS(\mathcal{A}_\rho)$.*

Definition 2.2. *[4, 20–23] Let Φ_ρ, Ψ_ρ be two soft sets over $\mathcal{A}$, then:*

(1) The null soft set is $\{(\alpha, \Phi(\alpha)) : \Phi(\alpha) = \emptyset, \forall\, \alpha \in \rho\}$ and symbolized by $\emptyset_\rho$.

(2) The absolute soft set is $\{(\alpha, \Phi(\alpha)) : \Phi(\alpha) = \mathcal{A}, \forall\, \alpha \in \rho\}$ and symbolized by $\mathcal{A}_\rho$.

(3) *If there exists* $\alpha \in \rho$ *and* $a \in \mathcal{A}$ *with* $\Phi(\alpha) = \{a\}$ *and* $\Phi(\beta) = \emptyset \ \ \forall \beta \in \rho - \{\alpha\}$*, then this set is said to be a soft point and denoted by* a_α*. We write* $a_\alpha \in \Phi_\rho$ *if* $a \in \Phi(\alpha)$*.*

(4) *We say that* Φ_ρ *is a soft subset of* Ψ_ρ *if* $\Phi(\alpha) \subseteq \Psi(\alpha)$*,* $\forall \alpha \in \rho$*, and it is written as follows:* $\Phi_\rho \sqsubseteq \Psi_\rho$*.*

(5) *The soft union* $\Phi_\rho \sqcup \Psi_\rho = \Omega_\rho$*, where* $\Omega(\alpha) = \Phi(\alpha) \cup \Psi(\alpha)$*,* $\forall \alpha \in \rho$*.*

(6) *The soft intersection* $\Phi_\rho \sqcap \Psi_\rho = \Omega_\rho$*, where* $\Omega(\alpha) = \Phi(\alpha) \cap \Psi(\alpha)$*,* $\forall \alpha \in \rho$*.*

(7) *The soft difference* $\Phi_\rho \setminus \Psi_\rho = \Omega_\rho$*, where* $\Omega(\alpha) = \Phi(\alpha) \setminus \Psi(\alpha)$*,* $\forall \alpha \in \rho$*.*

(8) *We say that* Φ_ρ *is a soft complement of* Ψ_ρ *if* $\Phi(\alpha) = \mathcal{A} \setminus \Psi(\alpha)$*,* $\forall \alpha \in \rho$*, and written as follows* $\Phi^c_\rho = \Psi_\rho$*.*

Definition 2.3. *[4, 24] A collection of a soft subset* Δ_s *of* $SS(\mathcal{A}_\rho)$ *is called a soft topology on* $\mathcal{A}$ *if the following:*

(1) $\emptyset_\rho$ *and* $\mathcal{A}_\rho$ *are members of* Δ_s*.*

(2) *The finite soft intersection is closed in* Δ_s*.*

(3) *The arbitrary soft union is closed in* Δ_s*.*

The notions $(\mathcal{A}_\rho, \Delta_s)$ *are named a soft topology space over* $\mathcal{A}$ *with parameters* ρ *(soft topology space is briefly,* $STOS$*). A soft set in* $STOS$ *is called soft open and the complement of soft open is soft closed. For* $a_\alpha \in \mathcal{A}\rho$*, the family of all members of* Δ_s *containing* a_α *is denoted by* $\Delta_s(a_\alpha)$*.*

Definition 2.4. *[25] A soft subset* Φ_ρ *of an* $STOS$ $(\mathcal{A}_\rho, \Delta_s)$ *is called a soft neighborhood of a soft point* a_ρ *provided that there exists soft open* $\Psi_\rho \in \Delta_s$ *such that* $a_\rho \in \Psi_\rho \sqsubseteq \Phi_\rho$*.*

Definition 2.5. *[4] Let* $(\mathcal{A}_\rho, \Delta_s)$ *be an* $STOS$*, then*

(1) *The soft closure of a soft set* Φ_ρ *is given by* $Cl(\Phi_\rho) = \sqcap\{\Psi_\rho : \Phi_\rho \sqsubseteq \Psi_\rho,\ \Psi^c_\rho \in \Delta_s\}$*.*

(2) *The soft interior of a soft set* Φ_ρ *is given by* $Int(\Phi_\rho) = \sqcup\{\Psi_\rho : \Psi_\rho \sqsubseteq \Phi_\rho,\ \Psi_\rho \in \Delta_s\}$*.*

Definition 2.6. *[27] A soft subset* Φ_ρ *of* $STOS$ $(\mathcal{A}_\rho, \Delta_s)$ *is called a soft clopen, provided that it is both soft open and soft closed.*

Definition 2.7. *[28] A mapping* $c : SS(\mathcal{A}_\rho) \to SS(\mathcal{A}_\rho)$ *is said to be a soft closure operator on* $\mathcal{A}$ *if it has the following properties for every* $\zeta_\rho, \Psi_\rho \in SS(\mathcal{A}_\rho)$*:*

(1) $c(\phi_\rho) = \phi_\rho$*.*

(2) $\Psi_\rho \sqsubseteq c(\Psi_\rho)$*.*

(3) $c(c(\Psi_\rho)) = c(\Psi_\rho)$*.*

(4) $c(\zeta_\rho \sqcup \Psi_\rho) = c(\zeta_\rho) \sqcup c(\Psi_\rho)$*.*

Definition 2.8. *[26] A collection of* $\mathcal{I}$ *of* $SS(\mathcal{A}_\rho)$ *is said to be a soft ideal over* $\mathcal{A}$ *with parameters* ρ *if the following satisfies*

(1) *If* $\Phi_\rho, \Psi_\rho \in \mathcal{I}$*, then* $\Phi_\rho \sqcup \Psi_\rho \in \mathcal{I}$*.*

(2) If $\Phi_\rho \in \mathcal{I}$ and $\Psi_\rho \sqsubseteq \Phi_\rho$, then $\Psi_\rho \in \mathcal{I}$.

Definition 2.9. *[9] A collection of $\mathcal{F}$ of $SS(\mathcal{A}_\rho)$ is said to be a soft grill over $\mathcal{A}$ with parameters ρ if it obeys the following postulates:*

(1) $\emptyset_\rho \notin \mathcal{F}$.

(2) If $\Phi_\rho \sqcup \Psi_\rho \in \mathcal{F}$, then $\Phi_\rho \in \mathcal{F}$, or $\Psi_\rho \in \mathcal{F}$.

(3) If $\Phi_\rho \in \mathcal{F}$ and $\Phi_\rho \sqsubseteq \Psi_\rho$, then $\Psi_\rho \in \mathcal{F}$.

Definition 2.10. *[11] A subfamily of $\mathcal{P}$ of $SS(\mathcal{A}_\rho)$ is said to be a soft primal over $\mathcal{A}$ with parameters ρ if it satisfies the following postulates:*

(1) $\mathcal{A}_\rho \notin \mathcal{P}$.

(2) If $\Phi_\rho \sqcap \Psi_\rho \in \mathcal{P}$, then $\Phi_\rho \in \mathcal{P}$, or $\Psi_\rho \in \mathcal{P}$.

(3) If $\Phi_\rho \in \mathcal{P}$ and $\Psi_\rho \sqsubseteq \Phi_\rho$, then $\Psi_\rho \in \mathcal{P}$.

Lemma 2.1. *[11] A subfamily of $\mathcal{P}$ of $SS(\mathcal{A}_\rho)$ is a soft primal over $\mathcal{A}$ with parameters ρ if the following holds:*

(1) $\mathcal{A}_\rho \notin \mathcal{P}$.

(2) If $\Phi_\rho \notin \mathcal{P}$ and $\Phi_\rho \sqsubseteq \Psi_\rho$, then $\Psi_\rho \notin \mathcal{P}$.

(3) If $\Phi_\rho \notin \mathcal{P}$ and $\Psi_\rho \notin \mathcal{P}$, then $\Psi_\rho \sqcap \Phi_\rho \notin \mathcal{P}$.

Definition 2.11. *[11] The triple $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ is said to be a primal soft topological space (briefly $PSTOS$), where $(\mathcal{A}_\rho, \Delta_s)$ is a soft topological space and $\mathcal{P}$ is a soft primal on $\mathcal{A}$.*

Definition 2.12. *[11] Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$. The operator $(\cdot)^\diamond : SS(\mathcal{A}_\rho) \to SS(\mathcal{A}_\rho)$ is defined for each soft set Φ_ρ as follows: $\Phi_\rho^\diamond = \{a_\alpha \in \mathcal{A}_\rho : \Phi_\rho^c \sqcup \Psi_\rho^c \in \mathcal{P}$ for all $\Psi_\rho \in \Delta_s(a_\alpha)\}$, and consider the closure operator $Cl^\diamond : SS(\mathcal{A}_\rho) \to SS(\mathcal{A}_\rho)$ as follows: $Cl^\diamond(\Phi_\rho) = \Phi_\rho \sqcup \Phi_\rho^\diamond$, where $\Phi_\rho \in SS(\mathcal{A}_\rho)$.*

Definition 2.13. *[31] Let $(\mathcal{A}_\rho, \Delta_s)$ be an $STOS$ and let $\Phi_\rho \in SS(\mathcal{A}_\rho)$.*

(1) A soft point a_α is in the soft θ-closure of Φ_ρ ($a_\alpha \in Cl_\theta(\Phi_\rho)$) if for every soft open set Ψ_ρ with $a_\alpha \in \Psi_\rho$, we have $Cl(\Psi_\rho) \sqcap \Phi_\rho \neq \emptyset_\rho$.

(2) Φ_ρ is a soft θ-closed in $(\mathcal{A}_\rho, \Delta_s)$ if $Cl_\theta(\Phi_\rho) = \Phi_\rho$.

(3) Φ_ρ is a soft θ-open in $(\mathcal{A}_\rho, \Delta_s)$ if the soft complement of Φ_ρ is soft θ-closed in $(\mathcal{A}_\rho, \Delta_s)$.

(4) A soft point a_α is called a soft θ-interior of Φ_ρ ($a_\alpha \in Int_\theta(\Phi_\rho)$ if there exists a soft open set Ψ_ρ such that $a_\alpha \in \Psi_\rho \sqsubseteq Cl(\Psi_\rho) \sqsubseteq \Phi_\rho$. The soft set of all soft θ-interior points of Φ_ρ is called the soft θ-interior of Φ_ρ and denoted by $Int_\theta(\Phi_\rho)$.

More properties of primal topological space and primal soft topological space can be found in [11, 19].

3. Primal soft closure operators

In this section, we define a primal local soft closure operator using the concept of soft topology and soft primal. We give some characterizations of this concept.

Definition 3.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS. For $\Phi_\rho \in SS(\mathcal{A}_\rho)$, we define the operator $\Lambda(\cdot) : SS(\mathcal{A}_\rho) \to SS(\mathcal{A}_\rho)$ as follows: $\Lambda(\Phi_\rho) = \{a_\alpha \in \mathcal{A}_\rho : \Phi_\rho^c \sqcup [Cl(\Psi_\rho)]^c \in \mathcal{P}$ for all $\Psi_\rho \in \Delta_s(a_\alpha)\}$, and it is called a primal local soft closure operator of Φ_ρ with respect to Δ_s and $\mathcal{P}$.*

Lemma 3.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS, then for any $\Phi_\rho \in SS(\mathcal{A}_\rho)$, we have $\Phi_\rho^\diamond \sqsubseteq \Lambda(\Phi_\rho)$.*

Proof. Let $a_\alpha \in \Phi_\rho^\diamond$, then we have $\Phi_\rho^c \sqcup \Psi_\rho^c \in \mathcal{P}$, for all $\Psi_\rho \in \Delta_s(a_\alpha)$. Since $\Phi_\rho^c \sqcup [Cl(\Psi_\rho)]^c \sqsubseteq \Phi_\rho^c \sqcup \Psi_\rho^c$, we get $\Phi_\rho^c \sqcup [Cl(\Psi_\rho)]^c \in \mathcal{P}$ and, hence, $a_\alpha \in \Lambda(\Phi_\rho)$. So, $\Phi_\rho^\diamond \sqsubseteq \Lambda(\Phi_\rho)$. □

The next example shows that $\Lambda(\Phi_\rho) \not\sqsubseteq \Phi_\rho^\diamond$, in general.

Example 3.1. *Let $\mathcal{A} = \{x, y, z, r\}$ with parameter $\rho = \{\alpha\}$. Consider the following soft sets:*
$\Phi_\rho(1) = (\Phi(1), \rho) = \{(\alpha, \{z\})\}$;
$\Phi_\rho(2) = (\Phi(2), \rho) = \{(\alpha, \{r\})\}$;
$\Phi_\rho(3) = (\Phi(3), \rho) = \{(\alpha, \{y\})\}$;
$\Phi_\rho(4) = (\Phi(4), \rho) = \{(\alpha, \{x, z\})\}$;
$\Phi_\rho(5) = (\Phi(5), \rho) = \{(\alpha, \{x, z, r\})\}$;
$\Phi_\rho(6) = (\Phi(6), \rho) = \{(\alpha, \{z, r\})\}$;
$\Phi_\rho(7) = (\Phi(7), \rho) = \{(\alpha, \{x, y, z\})\}$;
$\Phi_\rho(8) = (\Phi(8), \rho) = \{(\alpha, \{x, y, x, r\})\}$; *and*
$\emptyset_\rho = (\Phi(1), \rho) = \{(\alpha, \emptyset)\}$.
Then, $\Delta_s = \{\emptyset_\rho, \Phi_\rho(2), \Phi_\rho(4), \Phi_\rho(5), \Phi_\rho(8)\}$ is a soft primal topology and $\mathcal{P} = \{\emptyset_\rho, \Phi_\rho(1), \Phi_\rho(2), \Phi_\rho(6)\}$ is a soft primal on $\mathcal{A}$ with parameters ρ. We have $\Lambda(\Phi_\rho(7)) = \Phi_\rho(7)$ and $\Phi_\rho^\diamond(7) = \Phi_\rho(3)$. It is clear that $\Lambda(\Phi_\rho) \not\sqsubseteq \Phi_\rho^\diamond$.

Lemma 3.2. *Let $(\mathcal{A}_\rho, \Delta_s)$ be an STOS, then for any soft subset $\Phi_\rho \in SS(\mathcal{A}_\rho)$, we have*

(1) if Φ_ρ is a soft open set, then $Cl(\Phi_\rho) = Cl_\theta(\Phi_\rho)$,

(2) if Φ_ρ is a soft closed set, then $Int(\Phi_\rho) = Int_\theta(\Phi_\rho)$.

Proof. (1): We know that $Cl(\Phi_\rho) \sqsubseteq Cl_\theta(\Phi_\rho)$ in general. Let $a_\alpha \in Cl_\theta(\Phi_\rho)$, then $\Phi_\rho \sqcap Cl(\Psi_\rho) \neq \emptyset_\rho$ for every soft open set Ψ_ρ containing a_α. Since $\Phi_\rho \sqcap Cl(\Psi_\rho) \neq \emptyset_\rho$, there exists $b_\alpha \in \Phi_\rho \sqcap Cl(\Psi_\rho)$, that is, $b_\alpha \in \Phi_\rho$ and $b_\alpha \in Cl(\Psi_\rho)$. Therefore, $\Psi_\rho \sqcap \Upsilon_\rho \neq \emptyset_\rho$ for every soft open set Υ_ρ containing b_α, and since Φ_ρ is a soft open set containing b_α, $\Psi_\rho \sqcap \Phi_\rho \neq \emptyset_\rho$. Thus, $a_\alpha \in Cl(\Phi_\rho)$ and $Cl(\Phi_\rho) = Cl_\theta(\Phi_\rho)$.
(2): It follows from (1). □

Theorem 3.1. *Let Φ_ρ, Ψ_ρ be soft subsets of a PSTOS $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$, then the following is true:*

(1) $\Lambda(\emptyset_\rho) = \emptyset_\rho$.

(2) If $\Phi_\rho \sqsubseteq \Psi_\rho$, then $\Lambda(\Phi_\rho) \sqsubseteq \Lambda(\Psi_\rho)$.

(3) If $\Phi_\rho^c \notin \mathcal{P}$, then $\Lambda(\Phi_\rho) = \emptyset_\rho$.

(4) $\Lambda(\Phi_\rho) = Cl(\Lambda(\Phi_\rho)) \sqsubseteq Cl_\theta(\Phi_\rho)$ *and* $\Lambda(\Phi_\rho)$ *is a soft closed set.*

(5) If $\Lambda(\Phi_\rho)$ *is a soft open set and* $\Phi_\rho \sqsubseteq \Lambda(\Phi_\rho)$*, then* $\Lambda(\Phi_\rho) = Cl_\theta(\Phi_\rho)$*.*

(6) $\Lambda(\Phi_\rho \sqcup \Psi_\rho) = \Lambda(\Phi_\rho) \sqcup \Lambda(\Psi_\rho)$*.*

(7) $\Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho) \sqcap \Lambda(\Psi_\rho)$*.*

Proof. (1) Since $\emptyset_\rho^c \sqcup [Cl(\Phi_\rho)]^c = \mathcal{A}_\rho$ for any soft open set Φ_ρ, $\mathcal{A}_\rho \notin \mathcal{P}$, hence, $\Lambda(\emptyset_\rho) = \emptyset_\rho$.

(2) Let $a_\alpha \notin \Lambda(\Psi_\rho)$, then there exists $\Omega_\rho \in \Delta_s(a_\alpha) \ni \Psi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \notin \mathcal{P}$, and since $\Psi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \sqsubseteq \Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c$, $\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \notin \mathcal{P}$. Hence, we have $a_\alpha \notin \Lambda(\Phi_\rho)$. Therefore, we get $\Lambda(\Phi_\rho) \sqsubseteq \Lambda(\Psi_\rho)$.

(3) Suppose that there exists $a_\alpha \in \Lambda(\Phi_\rho)$, then for all $\Omega_\rho \in \Delta_s(a_\alpha)$ which means that $\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \in \mathcal{P}$. By assumption, $\Phi_\rho^c \notin \mathcal{P}$; therefore, $\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \notin \mathcal{P}$, for all $\Omega_\rho \in \Delta_s(a_\alpha)$. This is a logical contradiction. Hence, $\Lambda(\Phi_\rho) = \emptyset_\rho$.

(4) First, we know that $\Lambda(\Phi_\rho) \sqsubseteq Cl(\Lambda(\Phi_\rho))$ in general. Let $a_{\alpha_1} \in Cl(\Lambda(\Phi_\rho))$, then $\Lambda(\Phi_\rho) \sqcap \Omega_\rho \neq \emptyset_\rho$ for all soft open sets $\Omega_\rho \in \Delta_s(a_{\alpha_1})$. Therefore, there exists some soft points $a_{\alpha_2} \in \Lambda(\Phi_\rho) \sqcap \Omega_\rho$ and $\Omega_\rho \in \Delta_s(a_{\alpha_2})$. Now, since $a_{\alpha_2} \in \Lambda(\Phi_\rho)$, then $\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \in \mathcal{P}$ and, hence, $a_{\alpha_1} \in \Lambda(\Phi_\rho)$. Therefore, we have $Cl(\Lambda(\Phi_\rho)) \sqsubseteq \Lambda(\Phi_\rho)$ and $Cl(\Lambda(\Phi_\rho)) = \Lambda(\Phi_\rho)$. Now, let $a_{\alpha_1} \in Cl(\Lambda(\Phi_\rho)) = \Lambda(\Phi_\rho)$, then $\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \in \mathcal{P}$, for all soft open set $\Omega_\rho \in \Delta_s(a_{\alpha_1})$. This means that $\Phi_\rho \sqcap Cl(\Omega_\rho) \neq \emptyset_\rho$ for all soft open set $\Omega_\rho \in \Delta_s(a_{\alpha_1})$. Therefore, $a_{\alpha_1} \in Cl_\theta(\Phi_\rho)$. Thus, $\Lambda(\Phi_\rho) = Cl(\Lambda(\Phi_\rho)) \sqsubseteq Cl_\theta(\Phi_\rho)$.

(5) For any soft subset Φ_ρ of $SS(\mathcal{A}_\rho)$. By item (4), we have $\Lambda(\Phi_\rho) = Cl(\Lambda(\Phi_\rho)) \sqsubseteq Cl_\theta(\Phi_\rho)$, and since $\Lambda(\Phi_\rho)$ is soft open and $\Phi_\rho \sqsubseteq \Lambda(\Phi_\rho)$, by Lemma 3.2, $Cl_\theta(\Phi_\rho) \sqsubseteq Cl_\theta(\Lambda(\Phi_\rho)) = Cl(\Lambda(\Phi_\rho)) = \Lambda(\Phi_\rho) \sqsubseteq Cl_\theta(\Phi_\rho)$. Hence, $\Lambda(\Phi_\rho) = Cl_\theta(\Phi_\rho)$.

(6) According to item (2), we have $\Lambda(\Phi_\rho) \sqcup \Lambda(\Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho \sqcup \Psi_\rho)$. Let us demonstrate the reverse of the inclusion if $a_\alpha \notin \Lambda(\Phi_\rho) \sqcup \Lambda(\Psi_\rho)$, then a_α neither belongs to $\Lambda(\Phi_\rho)$ nor to $\Lambda(\Psi_\rho)$. So, there exists two soft open sets $\Omega_\rho, \Upsilon_\rho \in \Delta_s(a_\alpha) \ni \Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c \notin \mathcal{P}$ and $\Psi_\rho^c \sqcup [Cl(\Upsilon_\rho)]^c \notin \mathcal{P}$ by properties of primal soft $\left(\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c\right) \sqcap \left(\Psi_\rho^c \sqcup [Cl(\Upsilon_\rho)]^c\right) \notin \mathcal{P}$. Moreover, since $\mathcal{P}$ is hereditary and

$$\begin{aligned}
&\left(\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c\right) \sqcap \left(\Psi_\rho^c \sqcup [Cl(\Upsilon_\rho)]^c\right)\\
&= \left[\left(\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c\right) \sqcap \Psi_\rho^c\right] \sqcup \left[\left(\Phi_\rho^c \sqcup [Cl(\Omega_\rho)]^c\right) \sqcap [Cl(\Upsilon_\rho)]^c\right]\\
&= \left[\Phi_\rho^c \sqcap \Psi_\rho^c\right] \sqcup \left[[Cl(\Omega_\rho)]^c \sqcap \Psi_\rho^c\right] \sqcup \left[\Phi_\rho^c \sqcap [Cl(\Upsilon_\rho)]^c\right] \sqcup \left[[Cl(\Omega_\rho)]^c \sqcap [Cl(\Upsilon_\rho)]^c\right]\\
&\sqsubseteq \left[\Phi_\rho^c \sqcap \Psi_\rho^c\right] \sqcup [Cl(\Omega_\rho)]^c \sqcup [Cl(\Upsilon_\rho)]^c \sqcup \left([Cl(\Omega_\rho)]^c \sqcap [Cl(\Upsilon_\rho)]^c\right)\\
&\sqsubseteq \left[\Phi_\rho \sqcup \Psi_\rho\right]^c \sqcup [Cl(\Omega_\rho \sqcap \Upsilon_\rho)]^c,
\end{aligned}$$

then $\left[\Phi_\rho \sqcup \Psi_\rho\right]^c \sqcup [Cl(\Omega_\rho \sqcap \Upsilon_\rho)]^c \notin \mathcal{P}$. Since $\Omega_\rho \sqcap \Upsilon_\rho \in \Delta_s(a_\alpha)$, then we get $a_\alpha \notin \Lambda(\Phi_\rho \sqcup \Psi_\rho)$. Therefore, $\Lambda(\Phi_\rho \sqcup \Psi_\rho) = \Lambda(\Phi_\rho) \sqcup \Lambda(\Psi_\rho)$.

(7) Since $\Phi_\rho \sqcap \Psi_\rho \sqsubseteq \Phi_\rho$ and $\Phi_\rho \sqcap \Psi_\rho \sqsubseteq \Psi_\rho$, then by item (2), $\Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho)$ and $\Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Psi_\rho)$. Therefore, $\Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho) \sqcap \Lambda(\Psi_\rho)$. □

The following example discusses some properties of Theorem 3.1.

Example 3.2. *Let* $\mathcal{A} = \{a, b\}$ *and* $\rho = \{\alpha, \epsilon\}$*. Consider the following soft sets:*
$\Phi_\rho(1) = (\Phi(1), \rho) = \{(\alpha, \emptyset), (\epsilon, \{a\})\}$*;*
$\Phi_\rho(2) = (\Phi(2), \rho) = \{(\alpha, \emptyset), (\epsilon, \{b\})\}$*;*

$\Phi_\rho(3) = (\Phi(3), \rho) = \{(\alpha, \{a\}), (\epsilon, \emptyset)\}$;
$\Phi_\rho(4) = (\Phi(4), \rho) = \{(\alpha, \{b\}), (\epsilon, \emptyset)\}$;
$\Phi_\rho(5) = (\Phi(5), \rho) = \{(\alpha, \{b\}), (\epsilon, \{b\})\}$;
$\Phi_\rho(6) = (\Phi(6), \rho) = \{(\alpha, \{a\}), (\epsilon, \{b\})\}$;
$\Phi_\rho(7) = (\Phi(7), \rho) = \{(\alpha, \{a, b\}), (\epsilon, \emptyset)\}$;
$\Phi_\rho(8) = (\Phi(8), \rho) = \{(\alpha, \emptyset), (\epsilon, \{a, b\})\}$;
$\Phi_\rho(9) = (\Phi(9), \rho) = \{(\alpha, \{a, b\}), (\epsilon, \{b\})\}$; *and*
$\Phi_\rho(10) = (\Phi(10), \rho) = \{(\alpha, \{b\}), (\epsilon, \{a\})\}$.
Thus, $\mathcal{P} = \{\emptyset_\rho, \Phi_\rho(i) : i = 1, 2, ..., 9\}$ *is a soft primal on* $\mathcal{A}$ *with parameters* ρ.

(1) *Let* $\Delta_s = \{\emptyset_\rho, \mathcal{A}_\rho, \Phi_\rho(6), \Phi_\rho(10)\}$ *be a soft primal topology on a set* $\mathcal{A}$ *with parameters* ρ, *then the elucidates of the properties* $\Lambda(\Phi_\rho) \sqsubseteq \Phi_\rho$ *and* $\Phi_\rho \sqsubseteq \Lambda(\Phi_\rho)$ *are not true in general. It is easy to check that* $\Phi_\rho(6) \not\sqsubseteq \Lambda(\Phi_\rho(6)) = \emptyset_\rho$. *On the other hand,* $\Lambda(\Phi_\rho(1)) = \Phi_\rho(10) \not\sqsubseteq \Phi_\rho(1)$.

(2) *Let* $\Delta_s = \{\emptyset_\rho, \mathcal{A}_\rho\}$ *be a soft primal topology on a set* $\mathcal{A}$ *with parameters* ρ, *then the properties* $\Lambda(\Phi_\rho \sqcap \Psi_\rho) = \Lambda(\Phi_\rho) \sqcap \Lambda(\Psi_\rho)$ *are not true in general. Obviously,* $\Lambda[\Phi_\rho(1) \sqcap \Phi_\rho(9)] = \Lambda[\emptyset_\rho] = \emptyset_\rho$. *On the other hand,* $\Lambda[\Phi_\rho(1)] \sqcap \Lambda[\Phi_\rho(9)] = \mathcal{A}_\rho \sqcap \mathcal{A}_\rho = \mathcal{A}_\rho$.

Proposition 3.1. *Let* Φ_ρ, Ψ_ρ *be soft subsets of a PSTOS* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ *with* Φ_ρ *as a soft* θ*-open set, then* $\Phi_\rho \sqcap \Lambda(\Psi_\rho) = \Phi_\rho \sqcap \Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho \sqcap \Psi_\rho)$.

Proof. Let $a_\alpha \in \Phi_\rho \sqcap \Lambda(\Psi_\rho)$ and Φ_ρ be a soft θ-open, then $a_\alpha \in \Phi_\rho$ and $a_\alpha \in \Lambda(\Psi_\rho)$. Since Φ_ρ is soft θ-open, then there exists Ω_ρ which a soft open set such that $a_\alpha \in \Omega_\rho \sqsubseteq Cl(\Omega_\rho) \sqsubseteq \Phi_\rho$. Let Υ_ρ be any soft open set, $\ni a_\alpha \in \Upsilon_\rho$, then $\Upsilon_\rho \sqcap \Omega_\rho \in \Delta_s(a_\alpha)$, $[Cl(\Upsilon_\rho \sqcap \Omega_\rho)]^c \sqcup \Psi_\rho^c \in \mathcal{P}$, and $[Cl(\Upsilon_\rho)]^c \sqcup [\Phi_\rho \sqcap \Psi_\rho]^c = [Cl(\Upsilon_\rho)]^c \sqcup \Phi_\rho^c \sqcup \Psi_\rho^c \sqsubseteq [Cl(\Upsilon_\rho)]^c \sqcup [Cl(\Omega_\rho)]^c \sqcup \Psi_\rho^c \sqsubseteq \Psi_\rho^c \sqcup [Cl(\Upsilon_\rho) \sqcap Cl(\Omega_\rho)]^c \sqsubseteq \Psi_\rho^c \sqcup [Cl(\Upsilon_\rho \sqcap \Omega_\rho)]^c \in \mathcal{P}$, and, hence, $[Cl(\Upsilon_\rho)]^c \sqcup [\Phi_\rho \sqcap \Psi_\rho]^c \in \mathcal{P}$. Therefore, $a_\alpha \in \Lambda(\Phi_\rho \sqcap \Psi_\rho)$. As a result, we get $\Phi_\rho \sqcap \Lambda(\Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho \sqcap \Psi_\rho)$. Hence, $\Phi_\rho \sqcap \Lambda(\Psi_\rho) \sqsubseteq \Phi_\rho \sqcap \Lambda(\Phi_\rho \sqcap \Psi_\rho)$. By (2) of Theorem 3.1, $\Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Psi_\rho)$, so $\Phi_\rho \sqcap \Lambda(\Psi_\rho) = \Phi_\rho \sqcap \Lambda(\Phi_\rho \sqcap \Psi_\rho)$. □

Lemma 3.3. *Let* Φ_ρ, Ψ_ρ *be two soft subsets of a PSTOS* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$, *then* $\Lambda(\Phi_\rho) \setminus \Lambda(\Psi_\rho) = \Lambda(\Phi_\rho \setminus \Psi_\rho) \setminus \Lambda(\Psi_\rho)$.

Proof. By item (6) of Theorem 3.1, $\Lambda(\Phi_\rho) = \Lambda([\Phi_\rho \setminus \Psi_\rho] \sqcup [\Phi_\rho \sqcap \Psi_\rho]) = \Lambda(\Phi_\rho \setminus \Psi_\rho) \sqcup \Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho \setminus \Psi_\rho) \sqcup \Lambda(\Phi_\rho)$. Thus, $\Lambda(\Phi_\rho) \setminus \Lambda(\Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho \setminus \Psi_\rho) \setminus \Lambda(\Psi_\rho)$. Also, by item (2) of Theorem 3.1, $\Lambda(\Phi_\rho \setminus \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho)$, and, hence, $\Lambda(\Phi_\rho \setminus \Psi_\rho) \setminus \Lambda(\Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho) \setminus \Lambda(\Psi_\rho)$. So, we have $\Lambda(\Phi_\rho) \setminus \Lambda(\Psi_\rho) = \Lambda(\Phi_\rho \setminus \Psi_\rho) \setminus \Lambda(\Psi_\rho)$. □

Corollary 3.1. *Let* Φ_ρ, Ψ_ρ *be soft subsets of a PSTOS* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ *with* $\Psi_\rho^c \notin \mathcal{P}$, *then* $\Lambda(\Phi_\rho \sqcup \Psi_\rho) = \Lambda(\Phi_\rho) = \Lambda(\Phi_\rho \setminus \Psi_\rho)$.

Proof. Since $\Psi_\rho^c \notin \mathcal{P}$ by item (3) of Theorem 3.1, $\Lambda(\Psi_\rho^c) = \emptyset_\rho$. By Lemma 3.3, $\Lambda(\Phi_\rho) = \Lambda(\Phi_\rho \setminus \Psi_\rho)$, and by item (6) of Theorem 3.1, $\Lambda(\Phi_\rho \sqcup \Psi_\rho) = \Lambda(\Phi_\rho) = \Lambda(\Phi_\rho \setminus \Psi_\rho)$. □

Theorem 3.2. *Let* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ *be a PSTOS, then the following conditions are equivalent:*

(1) $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$;

(2) *If* $\Phi_\rho^c \notin \mathcal{P}$, *then* $Int_\theta(\Phi_\rho) = \emptyset_\rho$;

(3) For every soft clopen subset Ψ_ρ, $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho)$*;*

(4) $\mathcal{A}_\rho = \Lambda(\mathcal{A}_\rho)$*.*

Proof. (1) ⇒ (2): Let $\Phi^c_\rho \notin \mathcal{P}$ and $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$. Suppose that there exists $a_\alpha \in Int_\theta(\Phi_\rho)$, then there exists a soft open set Ω_ρ such that $a_\alpha \in \Omega_\rho \sqsubseteq Cl(\Omega_\rho) \sqsubseteq \Phi_\rho$, so $\Phi^c_\rho \sqsubseteq [Cl(\Omega_\rho)]^c$. Since $\Phi^c_\rho \notin \mathcal{P}$, $[Cl(\Omega_\rho)]^c \notin \mathcal{P}$. This is contrary for $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$. Therefore, $Int_\theta(\Phi_\rho) = \emptyset_\rho$.

(2) ⇒ (3): Let $a_\alpha \in \Psi_\rho$ and assume that $a_\alpha \notin \Lambda(\Psi_\rho)$, then there exists $\Phi_\rho \in \Delta_s(a_\alpha)$ such that, $\Psi^c_\rho \sqcup [Cl(\Phi_\rho)]^c \notin \mathcal{P}$ and hence $[\Psi_\rho \sqcap Cl(\Phi_\rho)]^c \notin \mathcal{P}$. Since isn$\Psi_\rho$ soft clopen, by item (2) and Proposition 3.1, $a_\alpha \in \Psi_\rho \sqcap \Phi_\rho = Int(\Psi_\rho \sqcap \Phi_\rho) \sqsubseteq Int(\Psi_\rho \sqcap Cl(\Phi_\rho)) = Int_\theta(\Psi_\rho \sqcap Cl(\Phi_\rho)) = \emptyset_\rho$. This is a logical contradiction. Hence, $a_\alpha \in \Lambda(\Psi_\rho)$ and $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho)$.

(3) ⇒ (4): Since $\mathcal{A}_\rho$ is a soft clopen set, then $\mathcal{A}_\rho = \Lambda(\mathcal{A}_\rho)$.

(4) ⇒ (1): Since $\mathcal{A}_\rho = \Lambda(\mathcal{A}_\rho) = \{a_\alpha \in \mathcal{A}^c_\rho \sqcup [Cl(\Phi_\rho)]^c = [Cl(\Phi_\rho)]^c \in \mathcal{P}, \forall a_\alpha \in \Phi_\rho \in \Delta_s\}$. Hence, $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$. □

Theorem 3.3. *Let* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ *be a PSTOS and* $\mathcal{A}_\rho = \Lambda(\mathcal{A}_\rho)$ *with* Ψ_ρ *be a soft θ-open set, then* $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho)$*.*

Proof. In case $\Psi_\rho = \emptyset_\rho$, we obviously have $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho) = \emptyset_\rho$. Since $\mathcal{A}_\rho = \Lambda(\mathcal{A}_\rho)$ and by Proposition 3.1, we have that for all Ψ_ρ, which is soft θ-open, $\Psi_\rho = \Lambda(\mathcal{A}_\rho) \sqcap \Psi_\rho \sqsubseteq \Lambda(\mathcal{A}_\rho \sqcap \Psi_\rho) = \Lambda(\Psi_\rho)$. Thus, $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho)$. □

For a soft subset Φ_ρ, Ψ_ρ in $SS(\mathcal{A}_\rho)$ we note that $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$] if $[(\Phi_\rho \setminus \Psi_\rho) \cup (\Psi_\rho \setminus \Phi_\rho)]^c \notin \mathcal{P}$ and observe that = [mod $\mathcal{P}$] is an equivalence relation.

Definition 3.2. *Let* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ *be a PSTOS. A soft subset* Φ_ρ *of* $SS(\mathcal{A}_\rho)$ *is called a soft Baire set with respect to* Δ_s *and* $\mathcal{P}$ *if there exists a soft θ-open set* Ψ_ρ *such that* $\Phi_\rho = \Psi_\rho$ *[mod* $\mathcal{P}$*], and it is denoted by* $\Phi_\rho \in \mathcal{B}_\theta$*.*

Proposition 3.2. *Let* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ *be a PSTOS.*

(1) If $\Phi_\rho \in \mathcal{B}_\theta$ *and* $\Phi^c_\rho \in \mathcal{P}$*, then there exists a non-null soft θ-open set* Ψ_ρ *such that* $\Phi_\rho = \Psi_\rho$ *[mod* $\mathcal{P}$*].*

(2) Let $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$*, then* $\Phi_\rho \in \mathcal{B}_\theta$ *and* $\Phi^c_\rho \in \mathcal{P}$ *if and only if there exists a non-null soft θ-open set* Ψ_ρ *such that* $\Phi_\rho = \Psi_\rho$ *[mod* $\mathcal{P}$*].*

Proof. (1) Assume $\Phi_\rho \in \mathcal{B}_\theta$ and $\Phi^c_\rho \in \mathcal{P}$. Hence, there exists a soft θ-open set Ψ_ρ such that $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$]. If $\Psi_\rho = \emptyset_\rho$, then we have $\Phi_\rho = \emptyset$ [mod $\mathcal{P}$] and $[(\Phi_\rho \setminus \emptyset_\rho) \cup (\emptyset_\rho \setminus \Phi_\rho)]^c \notin \mathcal{P}$. This mean that $\Phi^c_\rho \notin \mathcal{P}$, which is a contradiction.
(2) Assume there exists a non-null soft θ-open set Ψ_ρ such that $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$]; hence, by Definition 3.2, $\Phi_\rho \in \mathcal{B}_\theta$. Thus $\Psi_\rho = (\Psi_\rho \setminus J_\rho) \cup I_\rho$, where $J_\rho = \Phi_\rho \setminus \Psi_\rho$, $I_\rho = \Psi_\rho \setminus \Phi_\rho$, then $(\Psi_\rho \setminus \Phi_\rho)^c$ and $(\Phi_\rho \setminus \Psi_\rho)^c \notin \mathcal{P}$. If $\Phi^c_\rho \notin \mathcal{P}$, then $(\Phi_\rho \setminus J_\rho)^c \notin \mathcal{P}$ and $\Psi^c_\rho \notin \mathcal{P}$. Since Ψ_ρ is a non-null soft θ-open set, there exists a non-null soft open set $\Omega_\rho \ni \Omega_\rho \subseteq Cl(\Omega_\rho) \subseteq \Psi_\rho$. Since $\Psi^c_\rho \notin \mathcal{P}$, then $[Cl(\Omega_\rho)]^c \notin \mathcal{P}$ and $[Cl(\Omega_\rho)]^c$ is a soft open set. This contradicts that $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$. □

4. New topology via primal closure operators

Definition 4.1. *[11] Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS. An operator $\lambda(\cdot) : SS(\mathcal{A}_\rho) \to SS(\mathcal{A}_\rho)$ defined on $\Phi_\rho \in SS(\mathcal{A}_\rho)$ as $\lambda(\Phi_\rho) = \{a_\alpha \in \mathcal{A}_\rho : \exists \Psi_\rho \in \Delta_s(a_\alpha)$ and $[Cl(\Psi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}\}$ for each soft subset Φ_ρ over $\mathcal{A}_\rho$.*

The following theorem includes a number of fundamental truths about the behavior of the operator $\lambda(\cdot)$.

Theorem 4.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be PSTOS, then the below characteristics hold for all $\Phi_\rho, \Psi_\rho \in SS(\mathcal{A}_\rho)$:*

(1) $[\lambda(\Phi_\rho)]^c = \Lambda(\Phi_\rho^c)$.

(2) $\lambda(\Phi_\rho)$ is a soft open set.

(3) If $\Phi_\rho \sqsubseteq \Psi_\rho$, then $\lambda(\Phi_\rho) \sqsubseteq \lambda(\Psi_\rho)$.

(4) $\lambda(\Phi_\rho \sqcap \Psi_\rho) = \lambda(\Phi_\rho) \sqcap \lambda(\Psi_\rho)$.

(5) $\lambda(\Phi_\rho) = \lambda(\lambda(\Phi_\rho))$ if and only if $\Lambda(\Phi_\rho^c) = \Lambda(\Lambda(\Phi_\rho^c))$.

(6) If $\Phi_\rho^c \notin \mathcal{P}$, then $\lambda(\Phi_\rho) = [\Lambda(\mathcal{A}_\rho)]^c$.

(7) If $\Psi_\rho^c \notin \mathcal{P}$, then $\lambda(\Phi_\rho \setminus \Psi_\rho) = \lambda(\Phi_\rho)$.

(8) If $\Psi_\rho^c \notin \mathcal{P}$, then $\lambda(\Phi_\rho \sqcup \Psi_\rho) = \lambda(\Phi_\rho)$.

(9) If $[(\Phi_\rho \setminus \Psi_\rho) \sqcup (\Psi_\rho \setminus \Phi_\rho)]^c \notin \mathcal{P}$, then $\lambda(\Phi_\rho) = \lambda(\Psi_\rho)$.

Proof. (1): Let $a_\alpha \notin [\lambda(\Phi_\rho)]^c$ then $a_\alpha \in \lambda(\Phi_\rho)$. Hence, there exists a soft open set Ψ_ρ containing a_α such that $\Phi_\rho \sqcup [Cl(\Psi_\rho)]^c = [\Phi_\rho^c \sqcap Cl(\Psi_\rho)]^c = [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$. So, $a_\alpha \notin \Lambda(\Phi_\rho^c)$ and, hence, $\Lambda(\Phi_\rho^c) \sqsubseteq [\lambda(\Phi_\rho)]^c$.

Conversely, let $a_\alpha \notin \Lambda(\Phi_\rho^c)$, then there exists a soft open set Ψ_ρ containing a_α such that $[\Phi_\rho^c]^c \sqcup [Cl(\Psi_\rho)]^c = [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$. So, that $a_\alpha \in \lambda(\Phi_\rho)$ and $a_\alpha \notin [\lambda(\Phi_\rho)]^c$. Hence, $[\lambda(\Phi_\rho)]^c = \Lambda(\Phi_\rho^c)$.

(2): This derives from item (4) of Theorem 3.1.

(3): This derives from item (2) of Theorem 3.1.

(4): It is derived from item (3) that $\lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \lambda(\Phi_\rho)$ and $\lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \lambda(\Psi_\rho)$ so that $\lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \lambda(\Phi_\rho) \sqcap \lambda(\Psi_\rho)$. Now, let $a_\alpha \in \lambda(\Phi_\rho) \sqcap \lambda(\Psi_\rho)$, then there exists two soft open sets I_ρ and J_ρ containing a_α such that $[Cl(I_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$ and $[Cl(J_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}$. Let $\Upsilon_\rho = I_\rho \sqcap J_\rho$, which is soft open set containing a_α, $[Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$, and $[Cl(\Upsilon_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}$ by heredity. Thus, $[Cl(\Upsilon_\rho) \setminus (\Phi_\rho \sqcap \Psi_\rho)]^c \sqsubseteq [Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \sqcap [Cl(\Upsilon_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}$ by Lemma 2.1, and, hence, $a_\alpha \in \lambda(\Phi_\rho \sqcap \Psi_\rho)$. Therefore, $\lambda(\Phi_\rho \sqcap \Psi_\rho) = \lambda(\Phi_\rho) \sqcap \lambda(\Psi_\rho)$.

(5): It follows from the facts:

1) $[\lambda(\Phi_\rho)]^c = \Lambda(\Phi_\rho^c)$.

2) $\lambda[\lambda(\Phi_\rho)] = [\Lambda(\Lambda(\Phi_\rho^c))]^c$.

(6): By Corollary 3.1, we obtain that $\Lambda(\Phi^c_\rho) = \Lambda((\mathcal{A}_\rho)$ if $\Phi^c_\rho \notin \mathcal{P}$, then $\lambda(\Phi_\rho) = [\Lambda(\Phi^c_\rho]^c = [\Lambda((\mathcal{A}_\rho)]^c$.

(7): This follows from Corollary 3.1 and $\lambda(\Phi_\rho \setminus \Psi_\rho) = [\Lambda([\Phi_\rho \setminus \Psi_\rho]^c)]^c = [\Lambda(\Phi^c_\rho \sqcup \Psi_\rho)]^c = [\Lambda(\Phi^c_\rho]^c = \lambda(\Phi_\rho)$ by item (1).

(8): This follows by Corollary 3.1 and $\lambda(\Phi_\rho \sqcup \Psi_\rho) = [\Lambda([\Phi_\rho \sqcup \Psi_\rho]^c)]^c = [\Lambda(\Phi^c_\rho \setminus \Psi_\rho)]^c = [\Lambda(\Phi^c_\rho]^c = \lambda(\Phi_\rho)$ by item (1).

(9): Assume that $[(\Phi_\rho \setminus \Psi_\rho) \sqcup (\Psi_\rho \setminus \Phi_\rho)]^c \notin \mathcal{P}$. Let $\Phi_\rho \setminus \Psi_\rho = I_\rho$ and $\Psi_\rho \setminus \Phi_\rho = J_\rho$. We observe that $I^c_\rho, J^c_\rho \notin \mathcal{P}$. Also, we note that $\Psi_\rho = (\Phi_\rho \setminus I_\rho) \sqcup J_\rho$. Thus, $\lambda(\Phi_\rho) = \lambda(\Phi_\rho \setminus I_\rho) = \lambda((\Phi_\rho \setminus I_\rho) \sqcup J_\rho) = \lambda(\Psi_\rho)$ by items (7) and (8). □

Corollary 4.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$ and Φ_ρ be a soft θ-open set, then $\Phi_\rho \sqsubseteq \lambda(\Phi_\rho)$.*

Proof. We know that $[\lambda(\Phi_\rho)]^c = \Lambda(\Phi^c_\rho)$. Now, $\Lambda(\Phi^c_\rho) \sqsubseteq Cl_\theta(\Phi^c_\rho) = \Phi^c_\rho$, since Φ^c_ρ is a soft θ-closed set. Therefore, $\Phi_\rho = (\Phi^c_\rho)^c \sqsubseteq [\Lambda(\Phi^c_\rho)]^c = \lambda(\Phi_\rho)$. □

Proposition 4.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$ $\Phi_\rho \in SS(\mathcal{A}_\rho)$, then the following hold:*

(1) $\lambda(\Phi_\rho) = \sqcup\{\Psi_\rho \in \Delta_s : [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}\}$.

(2) $\lambda(\Phi_\rho) \sqsupseteq \sqcup\{\Psi_\rho \in \Delta_s : [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \sqcup [\Phi_\rho \setminus Cl(\Psi_\rho)]^c \notin \mathcal{P}\}$.

Proof. (1): This comes logically from the definition of λ-operator.

(2): Since $\mathcal{P}$ is heredity, it is clear that $\sqcup\{\Psi_\rho \in \Delta_s : [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \sqcup [\Phi_\rho \setminus Cl(\Psi_\rho)]^c \notin \mathcal{P}\} \sqsubseteq \sqcup\{\Psi_\rho \in \Delta_s : [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \mathcal{P}\} = \lambda(\Phi_\rho)$ for all $\Phi_\rho \in SS(\mathcal{A}_\rho)$. □

We will conclude this part with some technical results relating to the idempotents of the primal soft closure operator and the λ-operator.

Proposition 4.2. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$, and for all $\Phi_\rho \in SS(\mathcal{A}_\rho)$, we have $\Lambda(\Lambda(\Phi_\rho)) \sqsubseteq \Lambda(\Phi_\rho)$ if and only if $\lambda(\Phi^c_\rho) \sqsubseteq \lambda(\lambda(\Phi^c_\rho))$.*

Proof. Let $\Phi_\rho \in SS(\mathcal{A}_\rho)$, then

$$\begin{aligned}
\Lambda(\Lambda(\Phi_\rho)) \sqsubseteq \Lambda(\Phi_\rho) &\text{ if and only if } [\Lambda(\Phi_\rho)]^c \sqsubseteq [\Lambda(\Lambda(\Phi_\rho))]^c \\
&\text{ if and only if } [\Lambda((\Phi^c_\rho)^c)]^c \sqsubseteq [\Lambda([\Lambda(\Phi^c_\rho)^c]^c)^c]^c \\
&\text{ if and only if } \lambda(\Phi^c_\rho) \sqsubseteq [\Lambda(\lambda(\Phi^c_\rho))^c]^c \\
&\text{ if and only if } \lambda(\Phi^c_\rho) \sqsubseteq \lambda[\lambda(\Phi^c_\rho)].
\end{aligned}$$

□

Corollary 4.2. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$ and for all $\Phi_\rho \in SS(\mathcal{A}_\rho)$, then the following criteria are equivalent:*

(1) $\Lambda(\Lambda(\Phi_\rho)) \sqsubseteq \Lambda(\Phi_\rho)$.

(2) $\lambda(\Phi_\rho) \sqsubseteq \lambda(\lambda(\Phi_\rho))$.

Proposition 4.3. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$ with $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$. If $\Phi_\rho \in \mathcal{B}_\theta$ and $\Phi^c_\rho \in \mathcal{P}$, then $\lambda(\Phi_\rho) \sqcap Int_\theta(\Lambda(\Phi_\rho)) \neq \emptyset_\rho$.*

Proof. Let $\Phi_\rho \in \mathcal{B}_\theta$ and $\Phi_\rho^c \in \mathcal{P}$, then by item (1) of Proposition 3.2, there exists a nun-null soft θ-open set Ψ_ρ such that $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$]. By Theorem 3.2 and Proposition 3.1, $\Psi_\rho = \Psi_\rho \sqcap \mathcal{A}_\rho = \Psi_\rho \sqcap \Lambda(\mathcal{A}_\rho) \sqsubseteq \Lambda(\Psi_\rho \sqcap \mathcal{A}_\rho) = \Lambda(\Psi_\rho)$. This means that $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho) = \Lambda((\Phi_\rho \setminus J_\rho) \sqcup I_\rho) = \Lambda(\Phi_\rho)$, where $J_\rho^c = [\Phi_\rho \setminus \Psi_\rho]^c$, $I_\rho^c = [\Psi_\rho \setminus \Phi_\rho]^c \notin \mathcal{P}$ by Corollary 3.1. Since Ψ_ρ is a soft θ-open then, $\Psi_\rho \sqsubseteq Int_\theta(\Lambda(\Phi_\rho))$. Also $\Psi_\rho \sqsubseteq \lambda(\Psi_\rho)$ by Corollary 4.1 and $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$] which implies that $[(\Phi_\rho \setminus \Psi_\rho) \sqcup (\Psi_\rho \setminus \Phi_\rho)]^c \notin \mathcal{P}$, hence, $\Psi_\rho \sqsubseteq \lambda(\Psi_\rho) \sqsubseteq \lambda(\Phi_\rho)$ by item (9) of Theorem 4.1. Consequently, we obtain $\Psi_\rho \sqsubseteq \lambda(\Phi_\rho) \sqcap Int_\theta(\Lambda(\Phi_\rho))$ and $\lambda(\Phi_\rho) \sqcap Int_\theta(\Lambda(\Phi_\rho)) \neq \emptyset_\rho$. □

Theorem 4.2. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS with $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$. If every soft open is a soft θ-open set, then, the following statements are equivalent:*

(1) There is $\Psi_\rho \in \mathcal{B}_\theta$ and $\Psi_\rho^c \in \mathcal{P}$ such that $\Psi_\rho \sqsubseteq \Phi_\rho$;

(2) $\lambda(\Phi_\rho) \sqcap Int_\theta(\Lambda(\Phi_\rho)) \neq \emptyset_\rho$;

(3) $\lambda(\Phi_\rho) \sqcap \Lambda(\Phi_\rho) \neq \emptyset_\rho$;

(4) $\lambda(\Phi_\rho) \neq \emptyset_\rho$;

(5) $\lambda(\Phi_\rho) \sqcap \Phi_\rho \neq \emptyset_\rho$;

(6) There is a non-null soft open Υ_ρ such that $[Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$ and $[Cl(\Upsilon_\rho) \sqcap \Phi_\rho]^c \in \mathcal{P}$.

Proof. (1) ⇒ (2): Let $\Psi_\rho \in \mathcal{B}_\theta$ and $\Psi_\rho^c \in \mathcal{P}$ such that $\Psi_\rho \sqsubseteq \Phi_\rho$, then $Int_\theta(\Lambda(\Psi_\rho)) \sqsubseteq Int_\theta(\Lambda(\Phi_\rho))$ and $\lambda(\Psi_\rho) \sqsubseteq \lambda(\Phi_\rho)$ and, hence, $Int_\theta(\Lambda(\Psi_\rho)) \sqcap \lambda(\Psi_\rho) \sqsubseteq Int_\theta(\Lambda(\Phi_\rho)) \sqcap \lambda(\Phi_\rho)$. By Proposition 4.3, we have $\lambda(\Phi_\rho) \sqcap Int_\theta(\Lambda(\Phi_\rho)) \neq \emptyset_\rho$.

(2) ⇒ (3): The evidence is clear.

(3) ⇒ (4): The evidence is clear.

(4) ⇒ (5): Let $\lambda(\Phi_\rho) \neq \emptyset_\rho$, then there exists a non-null soft open set Ψ_ρ such that $[Cl(\Psi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$. Since $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$, then $[Cl(\Psi_\rho)]^c \in \mathcal{P}$ and $[Cl(\Psi_\rho)]^c = [(Cl(\Psi_\rho) \setminus \Phi_\rho) \sqcup (Cl(\Psi_\rho) \sqcap \Phi_\rho)]^c = [(Cl(\Psi_\rho) \setminus \Phi_\rho)]^c \sqcap [(Cl(\Psi_\rho) \sqcap \Phi_\rho)]^c$, and we get that $[(Cl(\Psi_\rho) \sqcap \Phi_\rho)]^c \in \mathcal{P}$. Now, by Theorem 4.1 and Corollary 4.1, we have $\emptyset_\rho \neq Cl(\Psi_\rho) \sqcap \Phi_\rho \sqsubseteq \lambda(Cl(\Psi_\rho)) \sqcap \Phi_\rho = \lambda[(Cl(\Psi_\rho) \setminus \Phi_\rho) \sqcup (Cl(\Psi_\rho) \sqcap \Phi_\rho)] \sqcap \Phi_\rho = \lambda[Cl(\Psi_\rho) \sqcap \Phi_\rho] \sqcap \Phi_\rho \sqsubseteq \lambda[\Phi_\rho] \sqcap \Phi_\rho$. Hence, $\lambda(\Phi_\rho) \sqcap \Phi_\rho \neq \emptyset_\rho$.

(5) ⇒ (6): Let $\lambda(\Phi_\rho) \sqcap \Phi_\rho \neq \emptyset_\rho$, then $\lambda(\Phi_\rho) \neq \emptyset_\rho$, so there exists a non-null soft open set $\Upsilon_\rho \ni$, $[Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$ and

$[Cl(\Upsilon_\rho)]^c = [(Cl(\Upsilon_\rho) \setminus \Phi_\rho) \sqcup (Cl(\Upsilon_\rho) \sqcap \Phi_\rho)]^c = [Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \sqcap [Cl(\Upsilon_\rho) \sqcap \Phi_\rho]^c$. Since $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$, then $[Cl(\Upsilon_\rho)]^c \in \mathcal{P}$. This mean that $[Cl(\Upsilon_\rho) \sqcap \Phi_\rho]^c \in \mathcal{P}$.

(6) ⇒ (1): Let $\Psi_\rho^c = [Cl(\Upsilon_\rho) \sqcap \Phi_\rho]^c \in \mathcal{P}$ and Υ_ρ is a non-null soft open set, so $[Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$. Thus, $\Psi_\rho \in \mathcal{B}_\theta$ and $\Psi_\rho^c \in \mathcal{P}$. Since $[(\Psi_\rho \setminus Cl(\Upsilon_\rho)) \sqcup (Cl(\Upsilon_\rho) \setminus \Psi_\rho)]^c = [Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$, $[(\Psi_\rho \setminus Cl(\Upsilon_\rho)) \sqcup (Cl(\Upsilon_\rho) \setminus \Psi_\rho)]^c \sqsubseteq [(\Psi_\rho \setminus \Upsilon_\rho) \sqcup (\Upsilon_\rho \setminus \Psi_\rho)]^c$ and $[Cl(\Upsilon_\rho)]^c \in \mathcal{P}$ such that $\Psi_\rho = \Upsilon_\rho$[mod $\mathcal{P}$], then there is $\Psi_\rho \in \mathcal{B}_\theta$ and $\Psi_\rho^c \in \mathcal{P}$ such that $\Psi_\rho \sqsubseteq \Phi_\rho$. □

Now, we introduce a new topology induced by the primal soft closure operator.

Theorem 4.3. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS with $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$ and $\Phi_\rho \in SS(\mathcal{A}_\rho)$, then $\beta = \{\Phi_\rho : \Phi_\rho \sqsubseteq \lambda(\Phi_\rho)\}$ is from a soft topology.*

Proof. Let $\beta = \{\Phi_\rho \in SS(\mathcal{A}_\rho) : \Phi_\rho \sqsubseteq \lambda(\Phi_\rho)\}$. Since $\mathcal{A}_\rho \notin \mathcal{P}$, by item (3) of Theorem 3.1, $\Lambda(\emptyset_\rho) = \emptyset_\rho$ and $\lambda(\mathcal{A}_\rho) = \mathcal{A}_\rho \setminus \Lambda(\mathcal{A}_\rho \setminus \mathcal{A}_\rho) = \mathcal{A}_\rho \setminus \Lambda(\emptyset_\rho) = \mathcal{A}_\rho$. Moreover, $\lambda(\emptyset_\rho) = \mathcal{A}_\rho \setminus \Lambda(\mathcal{A}_\rho \setminus \emptyset_\rho) = \mathcal{A}_\rho \setminus \mathcal{A}_\rho = \emptyset_\rho$ by Theorem 3.2. Therefore, we obtain that $\emptyset_\rho \sqsubseteq \lambda(\emptyset_\rho)$ and $\mathcal{A}_\rho \sqsubseteq \lambda(\mathcal{A}_\rho)$, so $\emptyset_\rho, \mathcal{A}_\rho \in \beta$.

Now, if $\Phi_\rho, \Psi_\rho \in \beta$, then $\Phi_\rho \sqcap \Psi_\rho \sqsubseteq \lambda(\Phi_\rho) \sqcap \lambda(\Psi_\rho) = \lambda(\Phi_\rho \sqcap \Psi_\rho)$, so $\Phi_\rho \sqcap \Psi_\rho \in \beta$.

Let $\{\Phi_\rho(\alpha) : \alpha \in I\} \sqsubseteq \beta$, then $\Phi_\rho(\alpha) \sqsubseteq \lambda(\Phi_\rho(\alpha)) \sqsubseteq \lambda(\sqcup_{\alpha\in I}\Phi_\rho(\alpha))$ for all $\alpha \in I$, and $\sqcup_{\alpha\in I}\Phi_\rho(\alpha) \sqsubseteq \lambda(\sqcup_{\alpha\in I}\Phi_\rho(\alpha))$. Hence, $\beta = \{\Phi_\rho : \Phi_\rho \sqsubseteq \lambda(\Phi_\rho)\}$ is from a soft topology. □

The next example elucidates the properties of Theorem 4.3.

Example 4.1. *Let $\mathcal{A} = \{x, y, x\}$ with parameter $\rho = \{\alpha\}$. Consider the following soft sets:*
$\Phi_\rho(1) = (\Phi(1), \rho) = \{(\alpha, \emptyset)\}$*;*
$\Phi_\rho(2) = (\Phi(2), \rho) = \{(\alpha, \{x\})\}$*;*
$\Phi_\rho(3) = (\Phi(3), \rho) = \{(\alpha, \{y\})\}$*;*
$\Phi_\rho(4) = (\Phi(4), \rho) = \{(\alpha, \{z\})\}$*;*
$\Phi_\rho(5) = (\Phi(5), \rho) = \{(\alpha, \{x, y\})\}$*;*
$\Phi_\rho(6) = (\Phi(6), \rho) = \{(\alpha, \{x, z\})\}$*;*
$\Phi_\rho(7) = (\Phi(7), \rho) = \{(\alpha, \{y, z\})\}$*; and*
$\Phi_\rho(8) = (\Phi(8), \rho) = \{(\alpha, \{x, y, x\})\}$.
Thus, $\Delta_s = \{\emptyset_\rho, \Phi_\rho(2), \Phi_\rho(3), \Phi_\rho(5), \Phi_\rho(8)\}$ is a soft primal topology and $\mathcal{P} = \{\emptyset_\rho, \Phi_\rho(2), \Phi_\rho(3), \Phi_\rho(5)\}$ is a soft primal on $\mathcal{A}$ with parameters ρ. It is clear that $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$ and $\beta = \{\Phi_\rho : \Phi_\rho \sqsubseteq \lambda(\Phi_\rho)\} = \{\emptyset_\rho, \Phi_\rho(4), \Phi_\rho(6), \Phi_\rho(7), \Phi_\rho(8)\}$, as shown by the following table, and it is clear that the soft primal topologies Δ_s and β are independent.

Φ_ρ	$\Lambda(\Phi_\rho^c)$	$\lambda(\Phi_\rho)$
$\Phi_\rho(1)$	$\Phi_\rho(8)$	$\emptyset_\rho$
$\Phi_\rho(2)$	$\Phi_\rho(8)$	$\emptyset_\rho$
$\Phi_\rho(3)$	$\Phi_\rho(8)$	$\emptyset_\rho$
$\Phi_\rho(4)$	$\emptyset_\rho$	$\Phi_\rho(8)$
$\Phi_\rho(5)$	$\Phi_\rho(8)$	$\emptyset_\rho$
$\Phi_\rho(6)$	$\emptyset_\rho$	$\Phi_\rho(8)$
$\Phi_\rho(7)$	$\emptyset_\rho$	$\Phi_\rho(8)$
$\Phi_\rho(8)$	$\emptyset_\rho$	$\Phi_\rho(8)$.

Lemma 4.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$. A soft set $\Phi_\rho \in SS(\mathcal{A}_\rho)$ is soft closed in $(\mathcal{A}_\rho, \beta)$ if and only if $\Lambda(\Phi_\rho) \sqsubseteq \Phi_\rho$.*

Proof. Φ_ρ is soft closed in $(\mathcal{A}_\rho, \beta)$ if and only if Φ_ρ^c is soft open in $(\mathcal{A}_\rho, \beta)$ if and only if $\Phi_\rho^c \sqsubseteq \lambda(\Phi_\rho^c)$ if and only if $\Phi_\rho^c \sqsubseteq [\Lambda([\Phi_\rho^c]^c)]^c$ if and only if $\Phi_\rho^c \sqsubseteq [\Lambda(\Phi_\rho)]^c$ if and only if $\Lambda(\Phi_\rho) \sqsubseteq \Phi_\rho$. □

Theorem 4.4. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$ and $\Lambda(\Lambda(\Phi_\rho)) \sqsubseteq \Lambda(\Phi_\rho)$, then $Cl_\beta(\Phi_\rho) = \Phi_\rho \cup \Lambda(\Phi_\rho)$ for all $\Phi_\rho \in SS(\mathcal{A}_\rho)$.*

Proof. Since $\Lambda(\Phi_\rho \cup \Lambda(\Phi_\rho)) = \Lambda(\Phi_\rho) \cup \Lambda(\Lambda(\Phi_\rho)) = \Lambda(\Phi_\rho) \subseteq \Phi_\rho \cup \Lambda(\Phi_\rho)$, we have that $\Phi_\rho \cup \Lambda(\Phi_\rho)$ is a soft closed set in topology β containing Φ_ρ by Lemma 4.1. Let us prove that $\Phi_\rho \cup \Lambda(\Phi_\rho)$ is a minimal soft closed set in topology β containing Φ_ρ. Let $a_\alpha \in \Lambda(\Phi_\rho) \cup \Phi_\rho$. If $a_\alpha \in \Phi_\rho$, then $a_\alpha \in Cl_\beta(\Phi_\rho)$. If $a_\alpha \in \Lambda(\Phi_\rho)$, then for each soft open set $\Psi_\rho \in \Delta_s(a_\alpha)$, $\Phi_\rho^c \cup [Cl(\Psi_\rho)]^c \in \mathcal{P}$. Since $[Cl_\beta(\Phi_\rho)]^c \subseteq \Phi_\rho^c$ and the property of primal space, we have $[Cl(\Psi_\rho)]^c \cup [Cl_\beta(\Phi_\rho)]^c \in \mathcal{P}$. Therefore, $a_\alpha \in \Lambda[Cl_\beta(\Phi_\rho)]$, and since $Cl_\beta(\Phi_\rho)$ is a soft closed in β, then $\Lambda[Cl_\beta(\Phi_\rho)] \subseteq Cl_\beta(\Phi_\rho)$, and we have $a_\alpha \in Cl_\beta(\Phi_\rho)$. Hence, $Cl_\beta(\Phi_\rho) = \Phi_\rho \cup \Lambda(\Phi_\rho)$ for any $\Phi_\rho \in SS(\mathcal{A}_\rho)$. □

Lemma 4.2. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS for any $\Phi_\rho, \Psi_\rho \in SS(\mathcal{A}_\rho)$, then $Int(Cl(\Phi_\rho \sqcap \Psi_\rho)) = Int(Cl(\Phi_\rho)) \sqcap Int(Cl(\Psi_\rho))$.*

Proof. This is the direct result by Lemma 3.5 of [29]. □

Theorem 4.5. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS with $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$, then the following soft collection set $\varphi = \{\Phi_\rho \in SS(\mathcal{A}_\rho) : \Phi_\rho \sqsubseteq Int(Cl(\lambda(\Phi_\rho)))\}$ from a soft topology.*

Proof. By item (2) of Theorem 4.1 $\lambda(\Phi_\rho)$ is a soft open set for all $\Phi_\rho \in SS(\mathcal{A}_\rho)$ so that $\beta \sqsubseteq \varphi$. Since $\emptyset_\rho, \mathcal{A}_\rho \in \beta$, then $\emptyset_\rho, \mathcal{A}_\rho \in \varphi$.

Let $\Phi_\rho, \Psi_\rho \in \varphi$, then $\Phi_\rho \sqcap \Psi_\rho = Int(Cl(\lambda(\Phi_\rho))) \sqcap Int(Cl(\lambda(\Psi_\rho))) = Int(Cl(\lambda(\Phi_\rho) \sqcap \lambda(\Psi_\rho))) = Int(Cl(\lambda(\Phi_\rho \sqcap \Psi_\rho)))$ by Theorem 4.1 and Lemma 4.2. Therefore, $\Phi_\rho \sqcap \Psi_\rho \in \varphi$.

Let $\Phi_\rho(i) \in \varphi : i \in I$, then $\Phi_\rho(i) \sqsubseteq Int(Cl(\lambda(\Phi_\rho(i)))$ for all $i \in I$. So, $\Phi_\rho(i) \sqsubseteq Int(Cl(\lambda(\Phi_\rho(i))) \sqsubseteq Int(Cl(\lambda(\sqcup_{i\in I}\Phi_\rho(i)))$ and $\sqcup_{i\in I}\Phi_\rho(i) \sqsubseteq Int(Cl(\lambda(\sqcup_{i\in I}\Phi_\rho(i)))$. Hence $\sqcup_{i\in I}\Phi_\rho(i) \in \varphi$ and φ is from a soft topology. □

Remark 4.1. *By Theorem 4.1, we have $\beta \sqsubseteq \varphi$ in general but in Example 4.1, it is clear that the soft primal topologies β and φ are equal.*

The strict inequality between these two topologies has a required condition,

Lemma 4.3. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be PSTOS. If $\beta \sqsubset \varphi$, then there exist a soft set Φ_ρ and a point $a_\alpha \in \Phi_\rho$ such that*

(1) $[Cl(\Psi_\rho) \setminus \Phi_\rho]^c \in \mathcal{P}$, for each $\Psi_\rho \in \Delta_s(a_\alpha)$ and,

(2) there exist $\Omega_\rho \in \Delta_s(a_\alpha)$ and a soft open $\Upsilon_\rho \subseteq \Omega_\rho$ such that $[Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$.

Proof. If $\beta \subsetneq \varphi$, then there exists $\Phi_\rho \in \varphi \setminus \beta$. Since $\Phi_\rho \notin \beta$, there exists $a_\alpha \in \Phi_\rho$ such that

$$\begin{aligned} a_\alpha \notin \lambda(\Phi_\rho) &\text{ iff } a_\alpha \notin [\Lambda(\Phi_\rho^c)]^c \\ &\text{ iff } a_\alpha \in \Lambda(\Phi_\rho^c) \\ &\text{ iff } \forall\ \Psi_\rho \in \Delta_s(a_\alpha),\ [Cl(\Psi_\rho)]^c \cup \Phi_\rho \in \mathcal{P} \\ &\text{ iff } \forall\ \Psi_\rho \in \Delta_s(a_\alpha),\ [Cl(\Psi_\rho) \cap \Phi_\rho^c]^c \in \mathcal{P} \\ &\text{ iff } \forall\ \Psi_\rho \in \Delta_s(a_\alpha),\ [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \in \mathcal{P}. \end{aligned}$$

Since $\Phi_\rho \in \varphi$, for each $y_\rho \in \Phi_\rho$, we have

$$\begin{aligned} & y_\rho \in Int(Cl(\lambda(\Phi_\rho))) \\ & \text{ iff } \exists\ I_\rho \in \Delta_s(y_\rho),\ I_\rho \sqsubseteq Cl(\lambda(\Phi_\rho)) \\ & \text{ iff } \exists\ I_\rho \in \Delta_s(y_\rho),\ \forall\ z_\rho \in I_\rho, \forall\ \Omega_\rho \in \Delta_s(z_\rho),\ \Omega_\rho \sqcap \lambda(\Phi_\rho) \neq \emptyset_\rho \\ & \text{ iff } \exists\ I_\rho \in \Delta_s(y_\rho),\ \forall\ \Omega_\rho \sqsubseteq I_\rho,\ [\Omega_\rho \in \Delta_s \Rightarrow\ \Omega_\rho \sqcap \lambda(\Phi_\rho) \neq \emptyset_\rho] \\ & \text{ iff } \exists\ I_\rho \in \Delta_s(y_\rho),\ \forall\ \Omega_\rho \sqsubseteq I_\rho,\ [\Omega_\rho \in \Delta_s \Rightarrow\ \Omega_\rho \sqcap [\lambda(\Phi_\rho^c)]^c \neq \emptyset_\rho] \end{aligned}$$

$$\text{iff } \exists\ I_\rho \in \Delta_s(y_\rho),\ \ \forall\ \ \Omega_\rho \sqsubseteq I_\rho,\ \ [\Omega_\rho \in \Delta_s \Rightarrow\ \ \Omega_\rho \setminus \lambda(\Phi_\rho^c) \neq \emptyset_\rho]$$
$$\text{iff } \exists\ I_\rho \in \Delta_s(y_\rho), \forall\, \Omega_\rho \sqsubseteq I_\rho, [\Omega_\rho \in \Delta_s \Rightarrow [\ \exists\ \Upsilon_\rho \sqsubseteq \Omega_\rho\ \ (\Upsilon_\rho \in \Delta_s)$$
$$\Rightarrow [Cl(\Upsilon_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P})].$$

□

5. Primal soft suitable spaces

Definition 5.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$. We say that a soft topology Δ_s is suitable with the primal $\mathcal{P}$ say primal soft suitable, if for every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, and $a_\rho \in \Phi_\rho$ there exists $\Psi_\rho \in \Delta_s(a_\rho)$, such that $[Cl(\Psi_\rho)]^c \sqcup \Phi_\rho^c \notin \mathcal{P}$, then $\Phi_\rho^c \notin \mathcal{P}$.*

Remark 5.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$, then we have the following:*

(1) If a soft primal on $\mathcal{A}$ is $SS(\mathcal{A}_\rho) \setminus \mathcal{A}_\rho$ with the family of all soft sets over $\mathcal{A}$ with parameters ρ, then soft topological and primal soft topological spaces are an identical.

(2) If a parameter ρ is singleton, then primal topological and primal soft topological spaces are identical.

(3) If a soft primal on $\mathcal{A}$ is $SS(\mathcal{A}_\rho) \setminus \mathcal{A}_\rho$ and a parameter ρ is singleton, then a topological space and primal soft topological space are equivalent. So, primal soft suitable space and suitable space in classical topology are equivalent if a soft primal on $\mathcal{A}$ is $SS(\mathcal{A}_\rho) \setminus \mathcal{A}_\rho$ and a parameter ρ is singleton.

We now give some equivalent descriptions of this definition.

Theorem 5.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a $PSTOS$. The subsequent properties are equivalent for primal soft suitable:*

(1) Δ_s is primal soft suitable with the primal $\mathcal{P}$.

(2) If a subset $\Phi_\rho \in SS(\mathcal{A}_\rho)$ includes a cover of soft open sets, each with its own soft closure union with Φ_ρ^c is not in $\mathcal{P}$, then $\Phi_\rho^c \notin \mathcal{P}$.

(3) For every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, $\Phi_\rho \cap \Lambda(\Phi_\rho) = \emptyset_\rho$ implies that $\Phi_\rho^c \notin \mathcal{P}$.

(4) For every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, $[\Phi_\rho \setminus \Lambda(\Phi_\rho)]^c \notin \mathcal{P}$.

(5) For every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, if there is non-null subset $\Psi_\rho \sqsubseteq \Phi_\rho$ with $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho)$ then, $\Phi_\rho^c \notin \mathcal{P}$.

Proof. (1) ⇒ (2): It is obvious to prove.
(2) ⇒ (3): Let $a_\alpha \in \Phi_\rho \in SS(\mathcal{A}_\rho)$. Since $\Phi_\rho \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$ then, $a_\alpha \notin \Lambda(\Phi_\rho)$ and there exists $\Psi_\rho(a_\alpha) \in \Delta_s(a_\alpha), \ni [Cl(\Psi_\rho(a_\alpha))]^c \cup \Phi_\rho^c \notin \mathcal{P}$. So, we have $\Phi_\rho \subseteq \sqcup\{\Psi_\rho(a_\alpha) : a_\alpha \in \Phi_\rho\}$ and $\Psi_\rho(a_\alpha) \in \Delta_s(a_\alpha)$, and by item (2), $\Phi_\rho^c \notin \mathcal{P}$.
(3) ⇒ (4): For any $\Phi_\rho \in SS(\mathcal{A}_\rho)$, $\Phi_\rho \setminus \Lambda(\Phi_\rho) \sqsubseteq \Phi_\rho$ and $(\Phi_\rho \setminus \Lambda(\Phi_\rho)) \sqcap \Lambda(\Phi_\rho \setminus \Lambda(\Phi_\rho)) \sqsubseteq (\Phi_\rho \setminus \Lambda(\Phi_\rho)) \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$. By item (3), $(\Phi_\rho \setminus \Lambda(\Phi_\rho))^c \notin \mathcal{P}$.
(4) ⇒ (5): By item (4), for every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, $[\Phi_\rho \setminus \Lambda(\Phi_\rho)]^c \notin \mathcal{P}$. Let $\Phi_\rho \setminus \Lambda(\Phi_\rho) = J_\rho \notin \mathcal{P}$, then $\Phi_\rho = J_\rho \sqcup (\Phi_\rho \cap \Lambda(\Phi_\rho))$ and by items (3) and (6) of Theorem 3.1, $\Lambda(\Phi_\rho) = \Lambda(J_\rho) \sqcup \Lambda(\Phi_\rho \sqcap \Lambda(\Phi_\rho)) =$

$\Lambda(\Phi_\rho \cap \Lambda(\Phi_\rho))$. Therefore, we have $\Psi_\rho = \Phi_\rho \sqcap \Lambda(\Phi_\rho) = \Phi_\rho \sqcap \Lambda(\Phi_\rho \sqcap \Lambda(A)) \sqsubseteq \Lambda(\Phi_\rho \sqcap \Lambda(\Phi_\rho)) = \Lambda(\Psi_\rho)$ and $\Psi_\rho = \Phi_\rho \sqcap \Lambda(\Phi_\rho) \sqsubseteq \Phi_\rho$. By the assumption $\Phi_\rho \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$, $(\Phi_\rho \setminus \Lambda(\Phi_\rho))^c = \Phi_\rho^c \notin \mathcal{P}$.
(5) $\Rightarrow$ (1): Let $\Phi_\rho \in SS(\mathcal{A}_\rho)$ and assume that for every $a_\alpha \in \Phi_\rho$, there exists $\Upsilon \in \Delta_s(a_\alpha)$ such that $[Cl(\Upsilon)]^c \sqcup \Phi_\rho^c \notin \mathcal{P}$. Thus, $\Phi_\rho \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$ (if $a_\alpha \in \Phi_\rho \sqcap \Lambda(\Phi_\rho)$, then for every $\Upsilon \in \Delta_s(a_\alpha)$, we have $[Cl(\Upsilon)]^c \sqcup \Phi_\rho^c \in \mathcal{P}$, which is a contradiction). Suppose that Φ_ρ contains Ψ_ρ such that $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho)$, then $\Psi_\rho = \Psi_\rho \sqcap \Lambda(\Psi_\rho) \sqsubseteq \Phi_\rho \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$. Therefore, Φ_ρ contains a non-null subset Ψ_ρ with $\Psi_\rho \sqsubseteq \Lambda(\Psi_\rho)$. Hence, $\Phi_\rho^c \notin \mathcal{P}$, and Δ_s is primal soft suitable with the primal $\mathcal{P}$. □

The next example elucidates the properties of Theorem 5.1.

Example 5.1. *Take a soft primal topology* $\Delta_s = \{\emptyset_\rho, \Phi_\rho(2), \Phi_\rho(3), \Phi_\rho(5), \Phi_\rho(8)\}$ *and soft primal* $\mathcal{P} = \{\emptyset_\rho, \Phi_\rho(2), \Phi_\rho(3), \Phi_\rho(5)\}$ *on* $\mathcal{A}$ *with parameters* ρ*, which are displayed in Example 4.1. It is clear that* $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$ *and* Δ_s *is primal soft suitable with the primal* $\mathcal{P}$*, as shown by the following table.*

Φ_ρ	$\Lambda(\Phi_\rho)$	$\Phi_\rho \setminus \Lambda(\Phi_\rho)$	$[\Phi_\rho \setminus \Lambda(\Phi_\rho)]^c$	$\in \mathcal{P}$ or $\notin \mathcal{P}$
$\Phi_\rho(1)$	$\emptyset_\rho$	$\emptyset_\rho$	$\Phi_\rho(8)$	$\notin \mathcal{P}$
$\Phi_\rho(2)$	$\emptyset_\rho$	$\Phi_\rho(2)$	$\Phi_\rho(7)$	$\notin \mathcal{P}$
$\Phi_\rho(3)$	$\emptyset_\rho$	$\Phi_\rho(3)$	$\Phi_\rho(6)$	$\notin \mathcal{P}$
$\Phi_\rho(4)$	$\Phi_\rho(8)$	$\emptyset_\rho$	$\Phi_\rho(8)$	$\notin \mathcal{P}$
$\Phi_\rho(5)$	$\emptyset_\rho$	$\Phi_\rho(5)$	$\Phi_\rho(4)$	$\notin \mathcal{P}$
$\Phi_\rho(6)$	$\Phi_\rho(8)$	$\emptyset_\rho$	$\Phi_\rho(8)$	$\notin \mathcal{P}$
$\Phi_\rho(7)$	$\Phi_\rho(8)$	$\emptyset_\rho$	$\Phi_\rho(8)$	$\notin \mathcal{P}$
$\Phi_\rho(8)$	$\Phi_\rho(8)$	$\emptyset_\rho$	$\Phi_\rho(8)$	$\notin \mathcal{P}$

Theorem 5.2. *Let* $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ *be a PSTOS. If* Δ_s *is primal soft suitable with the primal* $\mathcal{P}$, *then for all* $\Phi_\rho \in SS(\mathcal{A}_\rho)$*, the subsequent are equivalent:*

(1) $\Phi_\rho \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$ *implies that* $\Lambda(\Phi_\rho) = \emptyset_\rho$*;*

(2) $\Lambda(\Phi_\rho \setminus \Lambda(\Phi_\rho)) = \emptyset_\rho$*;*

(3) $\Lambda(\Phi_\rho \sqcap \Lambda(\Phi_\rho)) = \Lambda(\Phi_\rho)$*.*

Proof. First, we demonstrate that (1) holds if Δ_s is primal soft suitable with the primal $\mathcal{P}$. Let $\Phi_\rho \in SS(\mathcal{A}_\rho)$ and $\Phi_\rho \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$, then by Theorem 5.1, $\Phi_\rho^c \notin \mathcal{P}$, and by Theorem 3.1 (3), $\Lambda(\Phi_\rho) = \emptyset_\rho$.
(1) $\Rightarrow$ (2): Assume that for every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, $\Phi_\rho \sqcap \Lambda(\Phi_\rho) = \emptyset_\rho$ implies that $\Lambda(\Phi_\rho) = \emptyset_\rho$. Let $\Psi_\rho = \Phi_\rho \setminus \Lambda(\Phi_\rho)$, then

$$\begin{aligned}\Psi_\rho \sqcap \Lambda(\Psi_\rho) &= (\Phi_\rho \setminus \Lambda(\Phi_\rho)) \sqcap \Lambda(\Phi_\rho \setminus \Lambda(\Phi_\rho))\\ &= (\Phi_\rho \sqcap [\Lambda(\Phi_\rho)]^c) \sqcap \Lambda(\Phi_\rho \sqcap [\Lambda(\Phi_\rho)]^c)\\ &\sqsubseteq (\Phi_\rho \sqcap [\Lambda(\Phi_\rho)]^c) \sqcap [\Lambda(\Phi_\rho) \sqcap (\Lambda[\Lambda(\Phi_\rho)]^c)] = \emptyset_\rho.\end{aligned}$$

By item (1), we have $\Lambda(\Psi_\rho) = \emptyset_\rho$. Hence, $\Lambda(\Phi_\rho \setminus \Lambda(\Phi_\rho)) = \emptyset$.
(2) $\Rightarrow$ (3): Assume for every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, $\Lambda(\Phi_\rho \setminus \Lambda(\Phi_\rho)) = \emptyset_\rho$.

$$\begin{aligned}\Phi_\rho &= (\Phi_\rho \setminus \Lambda(\Phi_\rho)) \sqcup (\Phi_\rho \sqcap \Lambda(\Phi_\rho))\\ \Lambda(\Phi_\rho) &= \Lambda[(\Phi_\rho \setminus \Lambda(\Phi_\rho)) \sqcup (\Phi_\rho \sqcap \Lambda(\Phi_\rho))]\end{aligned}$$

$$= \Lambda(\Phi_\rho \setminus \Lambda(\Phi_\rho)) \sqcup \Lambda(\Phi_\rho \sqcap \Lambda(\Phi_\rho))$$
$$= \Lambda(\Phi_\rho \sqcap \Lambda(\Phi_\rho)).$$

(3) ⇒ (1): Assume for every $\Phi_\rho \in SS(\mathcal{A}_\rho)$, $\Lambda(\Phi_\rho) \sqcap \Phi_\rho = \emptyset_\rho$ and $\Lambda(\Lambda(\Phi_\rho) \sqcap \Phi_\rho) = \Lambda(\Phi_\rho)$. This implies that $\emptyset_\rho = \Lambda(\emptyset_\rho) = \Lambda(\Phi_\rho)$. □

Theorem 5.3. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS. If Δ_s is primal soft suitable with the primal $\mathcal{P}$, then, for every soft θ-open Φ_ρ and any subset $\Psi_\rho \in SS(\mathcal{A}_\rho)$, we have $Cl(\Lambda(\Phi_\rho \sqcap \Psi_\rho)) = \Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho \sqcap \Lambda(\Psi_\rho)) \sqsubseteq Cl_\theta(\Phi_\rho \sqcap \Lambda(\Psi_\rho))$.*

Proof. By Theorem 3.1 and Theorem 5.2 (3), we have $\Lambda(\Psi_\rho \sqcap \Phi_\rho) = \Lambda((\Psi_\rho \sqcap \Phi_\rho) \sqcap \Lambda(\Psi_\rho \sqcap \Phi_\rho)) \sqsubseteq \Lambda(\Phi_\rho \sqcap \Lambda(\Psi_\rho))$. Moreover, again by Theorem 3.1, $Cl(\Lambda(\Phi_\rho \cap \Psi_\rho)) = \Lambda(\Phi_\rho \sqcap \Psi_\rho) \sqsubseteq \Lambda(\Phi_\rho \sqcap \Lambda(\Psi_\rho)) \sqsubseteq Cl_\theta(\Phi_\rho \sqcap \Lambda(\Psi_\rho))$. □

Theorem 5.4. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS, then Δ_s is a primal soft suitable with the primal $\mathcal{P}$ if and only if $[\Lambda(\Phi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$ for every $\Phi_\rho \in SS(\mathcal{A}_\rho)$.*

Proof. Necessity. Assume Δ_s is primal soft suitable with the primal $\mathcal{P}$ and let $\Phi_\rho \in SS(\mathcal{A}_\rho)$. Observe that $a_\alpha \in \lambda(\Phi_\rho) \setminus \Phi_\rho$ if and only if $a_\alpha \notin \Phi_\rho$, and $a_\alpha \notin \Lambda(\Phi^c_\rho)$ if and only if $a_\alpha \notin \Phi_\rho$ and there exists $\Psi_\rho \in \Delta_s(a_\alpha) \ni [Cl(\Psi_\rho) \setminus \Phi_\rho]^c = [Cl(\Psi_\rho)]^c \sqcup \Phi_\rho \notin \mathcal{P}$ (since Δ_s is primal soft suitable with the primal $\mathcal{P}$ then $\Phi_\rho \notin \mathcal{P}$) if and only if there exists $\Psi_\rho \in \Delta_s(a_\alpha) \ni a_\alpha \in [Cl(\Psi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$. Now, for each $a_\alpha \in \lambda(\Phi_\rho) \setminus \Phi_\rho$ and $\Psi_\rho \in \Delta_s(a_\alpha)$, $[Cl(\Psi_\rho) \sqcap (\lambda(\Phi_\rho) \setminus \Phi_\rho)]^c = [Cl(\Psi_\rho)]^c \sqcup [(\lambda(\Phi_\rho) \setminus \Phi_\rho)]^c \notin \mathcal{P}$ by heredity and, hence, $[\lambda(\Phi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$ by assumption that Δ_s is primal soft suitable with the primal $\mathcal{P}$.

Sufficiency. Let $\Phi_\rho \in SS(\mathcal{A}_\rho)$ and assume that for each $a_\alpha \in \Phi_\rho$, there exists $\Psi_\rho \in \Delta_s(a_\alpha) \ni Cl(\Psi_\rho)]^c \cup \Phi^c_\rho \notin \mathcal{P}$. Observe that $\lambda(\Phi^c_\rho) \setminus (\Phi^c_\rho) = \Phi_\rho \setminus \Lambda(\Phi_\rho) = \{a_\alpha :$ there exists $\Psi_\rho \in \Delta_s(a_\alpha)$ such that $a_\alpha \in [Cl(\Psi_\rho)]^c \sqcup \Phi^c_\rho \notin \mathcal{P}\}$. Thus, we have $[\Phi_\rho \setminus \Lambda(\Phi_\rho)]^c = [\lambda(\Phi^c_\rho) \setminus (\Phi^c_\rho)]^c \notin \mathcal{P}$ and, hence, Δ_s is primal soft suitable with the primal $\mathcal{P}$. □

Theorem 5.5. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS such that Δ_s is primal soft suitable with the primal $\mathcal{P}$, then $\beta = \{\lambda(\Phi_\rho) \setminus \Psi_\rho : \Phi_\rho, \Psi_\rho \in SS(\mathcal{A}_\rho),\ \Psi^c_\rho \notin \mathcal{P}\}$*

Proof. By Corollary 4.2, we have that $\lambda(\Phi_\rho) \setminus \Psi_\rho \sqsubseteq \lambda(\Phi_\rho) \sqsubseteq \lambda[\lambda(\Phi_\rho)] = \lambda[\lambda(\Phi_\rho) \setminus \Psi_\rho]$ by item (7) Theorem 4.1. So each set of the form $\lambda(\Phi_\rho) \setminus \Psi_\rho$ is in β by Theorem 4.3.

Let $\Phi_\rho \in \beta$. Therefore, $\Phi_\rho \sqsubseteq \lambda(\Phi_\rho)$ but from Δ_s is primal soft suitable with the primal $\mathcal{P}$. By Theorem 5.4, we have $[\lambda(\Phi_\rho) \setminus \Phi_\rho]^c \notin \mathcal{P}$, that is, there exists $\Psi_\rho \in SS(\mathcal{A}_\rho)$ such that $\Psi_\rho = \lambda(\Phi_\rho) \setminus \Phi_\rho$. Hence, $\Phi_\rho = \lambda(\Phi_\rho) \setminus \Psi_\rho$ and $\Psi^c_\rho \notin \mathcal{P}$. So, $\Phi_\rho \in \{\lambda(\Phi_\rho) \setminus \Psi_\rho : \Phi_\rho \in SS(\mathcal{A}_\rho),\ \Psi^c_\rho \notin \mathcal{P}\} = \beta$. □

Proposition 5.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS and Δ_s be primal soft suitable with the primal $\mathcal{P}$ and $\Phi_\rho \in SS(\mathcal{A}_\rho)$. If $\Psi_\rho \sqsubseteq \Lambda(\Phi_\rho) \sqcup \lambda(\Phi_\rho)$ and Ψ_ρ is non-null soft open, then $[\Psi_\rho \setminus \Phi_\rho]^c \notin \mathcal{P}$ and $[Cl(\Psi_\rho)]^c \sqcup \Phi^c_\rho \in \mathcal{P}$.*

Proof. If $\Psi_\rho \sqsubseteq \Lambda(\Phi_\rho) \sqcap \lambda(\Phi_\rho)$, then $[\lambda(\Phi_\rho) \setminus \Phi_\rho]^c \sqsubseteq [\Psi_\rho \setminus \Phi_\rho]^c$ by Theorem 5.4 and hence $[\Psi_\rho \setminus \Phi_\rho]^c \notin \mathcal{P}$ by heredity. Since Ψ_ρ is non-null soft open and $\Psi_\rho \sqsubseteq \Lambda(\Phi_\rho)$, we have $[Cl(\Psi_\rho)]^c \sqcup \Phi^c_\rho \in \mathcal{P}$ by the definition of $\Lambda(\Phi_\rho)$. □

By Theorem 4.1 (9), we have that if $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$], then $\lambda(\Phi_\rho) = \lambda(\Psi_\rho)$.

Lemma 5.1. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS and Δ_s be primal soft suitable with the primal $\mathcal{P}$. If Φ_ρ and Ψ_ρ are soft θ-open and $\lambda(\Phi_\rho) = \lambda(\Psi_\rho)$, then $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$].*

Proof. Since Φ_ρ is a soft θ-open, by Corollary 4.1, we have $\Phi_\rho \sqsubseteq \lambda(\Phi_\rho)$, and, hence, $\Phi_\rho \setminus \Psi_\rho \sqsubseteq \lambda(\Phi_\rho) \setminus \Psi_\rho = \lambda(\Psi_\rho) \setminus \Psi_\rho$ and $[\lambda(\Psi_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}$ by Theorem 5.4. Therefore, $[\Phi_\rho \setminus \Psi_\rho]^c \notin \mathcal{P}$. Similarly, $[\Psi_\rho \setminus \Phi_\rho]^c \notin \mathcal{P}$. Now, $(\Phi_\rho \setminus \Psi_\rho)^c \sqcap (\Psi_\rho \setminus \Phi_\rho)^c = [(\Phi_\rho \setminus \Psi_\rho) \sqcup (\Psi_\rho \setminus \Phi_\rho)]^c \notin \mathcal{P}$ by additivity. Hence, $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$]. □

Theorem 5.6. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS and Δ_s be primal soft suitable with the primal $\mathcal{P}$. If $\Phi_\rho, \Psi_\rho \in \mathcal{B}_\theta$ and $\lambda(\Phi_\rho) = \lambda(\Psi_\rho)$, then $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$].*

Proof. Let $\Phi_\rho, \Psi_\rho \in \mathcal{B}_\theta$, then there is a soft θ-open Υ_ρ and Ω_ρ such that $\Phi_\rho = \Upsilon_\rho$ [mod $\mathcal{P}$] and $\Psi_\rho = \Omega_\rho$ [mod $\mathcal{P}$]. Now, $\lambda(\Phi_\rho) = \lambda(\Upsilon_\rho)$ and $\lambda(\Psi_\rho) = \lambda(\Omega_\rho)$ by item (9) of Theorem 4.1. Since $\lambda(\Phi_\rho) = \lambda(\Psi_\rho)$ implies that $\lambda(\Upsilon_\rho) = \lambda(\Omega_\rho)$, then $\Upsilon_\rho = \Omega_\rho$ [mod $\mathcal{P}$] by Lemma 5.1. Hence, $\Phi_\rho = \Psi_\rho$ [mod $\mathcal{P}$] by transitivity. □

Theorem 5.7. *Let $(\mathcal{A}_\rho, \Delta_s, \mathcal{P})$ be a PSTOS with $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$ and Δ_s be primal soft suitable with the primal $\mathcal{P}$, then for any $\Psi_\rho \in SS(\mathcal{A}_\rho)$, we have $\lambda(\Psi_\rho) \sqsubseteq \Lambda(\Psi_\rho)$.*

Proof. Suppose that $a_\alpha \in \lambda(\Psi_\rho)$ and $a_\alpha \notin \Lambda(\Psi_\rho)$, then there exists a non-null soft open set $\Phi_\rho \ni [Cl(\Phi_\rho) \sqcap \Psi_\rho]^c \notin \mathcal{P}$. Since $a_\alpha \in \lambda(\Psi_\rho)$, by Proposition 4.1, $a_\alpha \in \sqcup\{\Phi_\rho \in \Delta_s : [Cl(\Phi_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}\}$, and there exists a soft open set $\Upsilon_\rho \in \Delta_s(a_\alpha)$, and $[Cl(\Upsilon_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}$. Now, we have $\Phi_\rho \sqcap \Upsilon_\rho \in \Delta_s(a_\alpha)$, $[Cl(\Phi_\rho \sqcap \Upsilon_\rho) \sqcap \Psi_\rho]^c \notin \mathcal{P}$, and $[Cl(\Phi_\rho \sqcap \Upsilon_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}$ by heredity. Hence, by finite additivity, we get $[Cl(\Phi_\rho \sqcap \Upsilon_\rho)]^c = [Cl(\Phi_\rho \sqcap \Upsilon_\rho) \sqcap \Psi_\rho]^c \sqcap [Cl(\Phi_\rho \sqcap \Upsilon_\rho) \setminus \Psi_\rho]^c \notin \mathcal{P}$. Since $[Cl(\Phi_\rho \sqcap \Upsilon_\rho)]^c \in \Delta_s(a_\alpha)$, this is in opposition to $\Delta_s \setminus \{\mathcal{A}_\rho\} \sqsubseteq \mathcal{P}$. So, $a_\alpha \in \Lambda(\Psi_\rho)$. This means that $\lambda(\Psi_\rho) \sqsubseteq \Lambda(\Psi_\rho)$. □

6. Conclusions and future work

Shabir and Naz [4] and Çağman et al. [24] have introduced a soft topology on a universal set, extending the conventional (crisp) topology. This topological generalization has proved to be an intriguing field of research. The literature has several ways of creating soft topologies. Acharjee et al. [16] and Al-Omari et al. [17, 18] have introduced the primal topology, which builds on the conventional (crisp) topology. The study of topological generalization is gaining popularity. Al-shami et al. [11] has contributed to a primal soft topological space that combines a soft topological space with a soft primal. Our investigation focused on some operators for soft primal space and we have introduced a new topology induced by the primal soft closure operator. Several simple procedures on primal spaces were described. This research has focused on soft primal, a companion concept to soft grills, and covers fundamental operations on them. Our findings in this work are early, and further research into the features of the primal soft topology may provide more insights. This study aims to contribute to the trend of merging soft primal structures with rough approximation spaces in both classical and soft contexts.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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