Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 10679–10697.
DOI: 10.3934/math.2024521
Received: 23 January 2024
Revised: 02 March 2024
Accepted: 08 March 2024
Published: 18 March 2024

Research article

On the solutions of the second-order (p, q)-difference equation with an application to the fixed-point theory

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Abstract: In this paper, we examined the existence and uniqueness of solutions to the second-order (p, q)-difference equation with non-local boundary conditions by using the Banach fixed-point theorem. Moreover, we introduced a special case of this equation called the Euler-Cauchy-like (p, q)-difference equation and provide its solution. We also studied the oscillation of solutions for this equation in (p, q)-calculus and proved the (p, q)-Sturm-type separation theorem and (p, q)-Kneser theorem about the oscillation of solutions.

Keywords: (p, q)-derivative; second-order difference equation; time scale; oscillations; fixed-points
Mathematics Subject Classification: 39A13, 39A21, 47H10

1. Introduction

Quantum calculus or q-calculus, known for limitless computation, has been an exciting field throughout its development, where important concepts and results have emerged across various branches of mathematics, including combinatorics, numbers theory, difference equations and sequence spaces; see [1–7]. Together with the studies on q-calculus, post-quantum calculus, which is called (p, q)-calculus, was developed. The concept of (p, q)-calculus was first examined in quantum algebras by Chakrabarti and Jagannathan [8] in 1991. Later, the (p, q)-calculus theory was developed and many studies were carried out. For instance, Sadjang [9] proved the foundational theorem for (p, q)-calculus and some (p, q)-Taylor formulas for polynomials. The behavior of solutions of the (p, q)-sense equations was investigated by Kamsrisuk et al. [10]. Gençtürk [11] also obtained some new existence results for solutions of a boundary value problem in (p, q)-calculus. For further details about (p, q)-calculus, we refer readers to [12–18].

The following two equations were studied in the continuous and discrete cases

$$x''(t) + \rho(t).x(t) = 0, \qquad t \in \mathbb{R} \tag{1.1}$$

and

$$\Delta^2 x(t) + \rho(t).x^{\sigma}(t) = 0, \qquad t \in \mathbb{Z}, \tag{1.2}$$

respectively, where

$$\Delta x(t) = x^{\sigma}(t) - x(t)$$

and

$$x^{\sigma}(t) = x(\sigma(t)) = x(t+1)$$

for all $t \in \mathbb{Z}$. Especially, the research on comparison-type oscillatory and non-oscillatory cases of Eq (1.1) has long been conducted for continuous and discrete cases. The fundamentals of this comparison-type oscillation criteria for Eq (1.1) were first developed by Sturm [19] in 1836, establishing conditions such that $\rho(t) \geq \rho_0 > 0$ for oscillation and $\rho(t) \leq 0$ for non-oscillation. However, the value of Sturm's study was not fully realized until papers by Bôcher [20, 21].

Another well-known comparison-type criterion was proven by Kneser [22]:

$$t^2\rho(t) \geq \frac{1+\varepsilon}{4},$$

which implies oscillation for some $\varepsilon > 0$, while

$$t^2\rho(t) \leq \frac{1}{4}$$

implies non-oscillation. Later, Fite [23] and Hille [24] provided a generalization of Kneser's result and oscillation. For further study, see [25–27]. Much of what has been stated so far is based on two theorems, which we will briefly present below.

Theorem 1. *(i) ([28]) The differential equation*

$$x''(t) + \frac{b}{t.\sigma(t)} x^{\sigma}(t) = 0$$

is oscillatory if and only if $b > \frac{1}{4}$.

(ii) ([29]) The difference equation

$$\Delta^2 x(t) + \frac{b}{t.\sigma(t)} x^{\sigma}(t) = 0$$

is oscillatory if and only if $b > \frac{1}{4}$.

Bohner and Ünal [30] worked on the solution of the second-order difference equations in q-calculus and provided some oscillation criteria. They defined this q-difference equation as

$$D_q^2 x(t) + \rho(t).x^{\sigma}(t) = 0,$$

where

$$t \in \mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$$

with $q > 1$, and they also proved the following theorem.

Theorem 2. *The q-difference equation*

$$D_q^2 x(t) + \frac{b}{t.\sigma(t)} x^{\sigma}(t) = 0$$

is oscillatory if and only if

$$b > \frac{1}{(\sqrt{q}+1)^2}.$$

On the other hand, the Banach fixed-point theorem can be used to demonstrate the existence and uniqueness of solutions of functional equations, integral equations and difference equations ([31–33]). For more information about fixed-point theory, we refer the readers to [34–36]. It may be more beneficial to use non-local boundary conditions instead of classical initial conditions to better describe physical events. For example, Ahmad and Ntouyas [37] investigated the existence and uniqueness of solutions to the q-boundary value problem with non-local and integral boundary conditions. For more information, we refer the readers to [38, 39].

Motivated by the above results, we first study the existence and uniqueness of solutions of the second-order (p, q)-difference equation by using the Banach fixed-point theorem. We also investigate the oscillation of the solutions of the Euler-Cauchy-like (p, q)-difference equation. We have organized this article as follows: In Section 2, we present the basic definitions and theorems that we will need in this article. In Section 3, we examine the existence and uniqueness of solutions of the second-order (p, q)-difference equation with non-local and integral boundary conditions. In Section 4, we obtain the general solution of the Euler-Cauchy-like (p, q)-difference equation. We also prove a theorem about oscillation and give the (p, q)-Kneser theorem. Our results generalize the corresponding results of [28–30] to (p, q)-calculus.

2. Preliminaries

Let us remember some essential concepts that are related to (p, q)-calculus (see [9, 16]).

Let x be any function such that $x\colon D \subseteq \mathbb{R} \to \mathbb{R}$. The (p, q)-derivative of the function x is defined as

$$D_{p,q}x(t) = \frac{x(pt) - x(qt)}{(p-q)t}, \qquad t \neq 0, p \neq q, \tag{2.1}$$

and $(D_{p,q}x)(0) = x'(0)$, provided that x is differentiable at 0. The derivative operator $D_{p,q}$ is also linear. The (p, q)-derivatives of the product and quotient of $x(t)$ and $y(t)$ are given by

$$\begin{aligned} D_{p,q}(x(t).y(t)) &= x(pt).D_{p,q}y(t) + y(qt).D_{p,q}x(t) \\ &= y(pt).D_{p,q}x(t) + x(qt).D_{p,q}y(t) \end{aligned}$$

and

$$\begin{aligned} D_{p,q}\left(\frac{x(t)}{y(t)}\right) &= \frac{y(qt).D_{p,q}x(t) - x(qt).D_{p,q}y(t)}{y(pt).y(qt)} \\ &= \frac{y(pt).D_{p,q}x(t) - x(pt).D_{p,q}y(t)}{y(pt).y(qt)}, \end{aligned} \tag{2.2}$$

respectively.

The (p,q)-derivative is not subject to any general chain rule, unlike the classical derivative. However, the chain rule can be applied as a special case. Let m and n be constants. Consider the function $x(y(t))$ with $y(t) = mt^n$. In this case, the chain rule is obtained as follows:

$$D_{p,q}(x(y(t))) = (D_{p^n,q^n}x).(y(t)).D_{p,q}y(t). \tag{2.3}$$

Example 1. Let $y(t) = qt$. Then, $m = q$ and $n = 1$. According to Eq (2.3), the (p,q)-derivative of the function $x(y(t)) = x(qt)$ becomes as follows:

$$\begin{aligned} D_{p,q}(x(y(t))) &= (D_{p,q}x).(y(t)).D_{p,q}(y(t)) \\ &= \frac{x(pqt) - x(q^2t)}{(p-q)t}. \end{aligned}$$

Now, we recall the definition of (p,q)-integrals [9].

Let x: $[0,T] \to \mathbb{R}$ be a function. Then the (p,q)-integral of x is defined by

$$\int_0^t x(s)\, d_{p,q}s = (p-q).t.\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}.x\left(\frac{q^n}{p^{n+1}}t\right), \quad \text{if} \quad \left|\frac{p}{q}\right| > 1$$

provided that the right-hand side is convergent.

In the following theorem, we give two basic properties of the (p,q)-integral.

Theorem 3. *(I) ([9, Theorem 9]) Suppose that x:* $[0,T] \to \mathbb{R}$, $(T > 0)$ *is a continuous function. In this case, the following formula holds:*

$$\int_0^t D_{p,q}x(s)\, d_{p,q}s = x(t) - x(0).$$

(ii) ([11, Theorem 2.5]) Suppose that a function x: $[0,T] \to \mathbb{R}$, $(T > 0)$. *In this case for* $t \in [0, p^2t]$, *we have*

$$\int_0^t \int_0^s x(r)\, d_{p,q}r\, d_{p,q}s = \frac{1}{p}\int_0^t (t-qs)\, x\left(\frac{s}{p}\right) d_{p,q}s.$$

A time scale is defined as an arbitrary non-empty closed subset of real numbers, denoted by $\mathbb{T}$. Real numbers, integers, natural numbers and non-negative integers, namely $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are the main examples of time scales. Additionally, the forward jump operator σ: $\mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

for $t \in \mathbb{T}$. For more information, we refer the readers to [40, 41]. Throughout this paper, we will consider the set

$$\mathbb{T} = p^{\mathbb{N}_0} = \{p^k : k \in \mathbb{N}_0\}$$

with $p > 1$. In this case, the forward difference operator is $\sigma(t) = pt$. Hence, the concept $D_{p,q}x(t)$ is as follows:

$$D_{p,q}x(t) = \frac{x^\sigma(t) - x(qt)}{pt - qt},$$

where

$$x^{\sigma}(t) = x(\sigma(t)) = x(pt).$$

Note that

$$x^{\sigma}(qt) = x(\sigma(qt)) = x(pqt).$$

Also, the second-order (p, q)-derivative of the function x is as follows:

$$D^2_{p,q}x(t) = \frac{(D_{p,q}x)(pt) - (D_{p,q}x)(qt)}{(p-q)t} = \frac{q.x(p^2t) - (p+q).x(pqt) + p.x(q^2t)}{(p-q)^2pqt^2}. \tag{2.4}$$

3. The existence and uniqueness of second-order (p, q)-difference equations

Let

$$[0, T]_{\mathbb{T}_0} = [0, T] \cap \mathbb{T}_0,$$

where $\mathbb{T}_0 = \mathbb{T} \cup \{0\}$ and $T \in \mathbb{T}$ is a fixed constant.

In this section, we will consider the following equation with non-local and integral boundary conditions

$$\begin{cases} D^2_{p,q}x(t) = \varphi(t, x^{\sigma}(qt)), & t \in [0, \frac{T}{p^2.q^2}]_{\mathbb{T}_0}, \\ x(0) = x_0 + k(x), \quad x(T) = \delta \int_0^T x(s)\, d_{p,q}s, \end{cases} \tag{3.1}$$

where $x_0 \in \mathbb{R}$, $\delta.T \neq p + q$ and $\varphi \in C([0, T]_{\mathbb{T}_0} \times \mathbb{R}; \mathbb{R})$ is (p, q)-differentiable on $[0, \frac{T}{p.q}]_{\mathbb{T}_0}$. Let

$$\mathscr{X} = C([0, T]_{\mathbb{T}_0}; \mathbb{R})$$

denote the Banach space of all continuous real functions with the norm

$$\|x(t)\| = \sup\{|x(t)| : t \in [0, T]_{\mathbb{T}_0}\}.$$

Let k: $\mathscr{X} \to \mathbb{R}$ be any bounded function.

We need the following lemma to prove the existence and uniqueness theorem via the Banach fixed-point theorem.

Lemma 1. *Let $\eta \in \mathscr{X}$ such that $\eta(t) = -\varphi(t, x^{\sigma}(qt))$. In this case, the boundary value problem given by*

$$\begin{cases} D^2_{p,q}x(t) + \eta(t) = 0, & t \in [0, \frac{T}{p^2.q^2}]_{\mathbb{T}_0}, \\ x(0) = x_0 + k(x), \quad x(T) = \delta \int_0^T x(s)\, d_{p,q}s \end{cases} \tag{3.2}$$

is equivalent to the (p, q)-integral equation

$$\begin{aligned} x(t) =& (x_0 + k(x))\frac{[(p+q).(T + t.(\delta.T - 1)) - \delta T^2]}{T(p+q-\delta T)} + \frac{t.(p+q)}{Tp.(p+q-\delta T)}\int_0^T (T - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s \\ & - \frac{1}{p}\int_0^t (t - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s - \frac{\delta.t.(p^2-q^2)}{Tp^3.(p+q-\delta T)}\int_0^T (Ts - qs^2).\eta\left(\frac{s}{p^2}\right) d_{p,q}s. \end{aligned}$$

Proof. Using Theorem 3 (i) and applying the (p, q)-integral to the first equation of Eq (3.2), we get

$$D_{p,q}x(t) = D_{p,q}x(0) - \int_0^t \eta(s)\, d_{p,q}s, \qquad t \in \left[0, \frac{T}{p.q}\right]_{\mathbb{T}_0}. \tag{3.3}$$

Again, by applying the (p, q)-integral to Eq (3.3), we obtain

$$x(t) = x(0) + t.D_{p,q}x(0) - \frac{1}{p}\int_0^t (t - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s, \qquad t \in [0, T]_{\mathbb{T}_0}. \tag{3.4}$$

For convenience, we put the constants $x(0) = c_0$ and $D_{p,q}x(0) = c_1$ in Eq (3.4), and, in this case, we have

$$x(t) = c_0 + c_1.t - \frac{1}{p}\int_0^t (t - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s. \tag{3.5}$$

By applying $t = 0$ in Eq (3.5), we get

$$x(0) = c_0 = x_0 + k(x). \tag{3.6}$$

Substituting Eq (3.6) in Eq (3.5), we obtain

$$x(t) = x_0 + k(x) + c_1.t - \frac{1}{p}\int_0^t (t - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s. \tag{3.7}$$

With the second boundary condition, we have

$$c_1 = \frac{1}{T}(-x_0 - k(x)) + \frac{\delta}{T}\int_0^T x(s)\, d_{p,q}s + \frac{1}{T.p}\int_0^T (T - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s.$$

By incorporating c_1 into Eq (3.7), we obtain

$$\begin{aligned} x(t) =&(x_0 + k(x))\left(\frac{T - t}{T}\right) + \frac{\delta.t}{T}\int_0^T x(s)\, d_{p,q}s + \frac{t}{T.p}\int_0^T (T - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s \\ &- \frac{1}{p}\int_0^t (t - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s. \end{aligned} \tag{3.8}$$

By integrating both sides of Eq (3.8), we get

$$\begin{aligned} \int_0^T x(s)\, d_{p,q}s =&(x_0 + k(x))\frac{T.(p + q - 1)}{p + q - \delta.T} + \frac{T}{p(p + q - \delta.T)}\int_0^T (T - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s \\ &- \frac{p^2 - q^2}{p^3(p + q - \delta.T)}\int_0^T (Ts - qs^2).\eta\left(\frac{s}{p^2}\right) d_{p,q}s. \end{aligned} \tag{3.9}$$

Substituting Eq (3.9) in Eq (3.8), we have

$$\begin{aligned} x(t) =&(x_0 + k(x))\frac{[(p + q).(T + t.(\delta.T - 1)) - \delta T^2]}{T(p + q - \delta T)} + \frac{t.(p + q)}{Tp.(p + q - \delta T)}\int_0^T (T - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s \\ &- \frac{1}{p}\int_0^t (t - qs).\eta\left(\frac{s}{p}\right) d_{p,q}s - \frac{\delta.t.(p^2 - q^2)}{Tp^3.(p + q - \delta T)}\int_0^T (Ts - qs^2).\eta\left(\frac{s}{p^2}\right) d_{p,q}s. \end{aligned}$$

Thus, the desired result is achieved. □

We now introduce an operator $\mathscr{F}$: $\mathscr{X} \to \mathscr{X}$ as follows:

$$\begin{aligned}(\mathscr{F}x)(t) =&(x_0 + k(x))\frac{[(p+q).(T + t.(\delta.T - 1)) - \delta T^2]}{T(p+q-\delta T)}\\ &+ \frac{t.(p+q)}{Tp.(p+q-\delta T)}\int_0^T (T-qs).\varphi(s,x(qs))\,d_{p,q}s - \frac{1}{p}\int_0^t (t-qs).\varphi(s,x(qs))\,d_{p,q}s\\ &- \frac{\delta.t.(p^2-q^2)}{Tp^3.(p+q-\delta T)}\int_0^T (Ts-qs^2).\varphi\left(s,x\left(\frac{qs}{p}\right)\right)\,d_{p,q}s. \end{aligned} \tag{3.10}$$

By Lemma 1, the necessary and sufficient condition to have a solution to Eq (3.1) is that the operator $\mathscr{F}$ has a fixed point. For simplicity, we take a constant Ψ as follows:

$$\Psi = \left\{\frac{T.[(p+q+1).|\delta| + T]}{|p+q-\delta.T|} + \frac{T^2}{p+q} + \frac{T^3.|\delta|.(p-q)}{p.|p+q-\delta.T|.(p^2+pq+q^2)}\right\}. \tag{3.11}$$

The following theorem is based on the Banach fixed-point theorem, which asserts that in a Banach space $\mathscr{X}$, any contraction mapping $\mathscr{F}$: $\mathscr{X} \to \mathscr{X}$, that is, any mapping with $c \in [0, 1)$ such that

$$\|\mathscr{F}x - \mathscr{F}y\| \le c\|x-y\|, \quad \forall x, y \in \mathscr{X}$$

has a unique fixed point.

Theorem 4. *Let* φ: $[0, \frac{T}{p^2.q^2}]_{\mathbb{T}_0} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Suppose that*

(A1) $|\varphi(t,x(t)) - \varphi(t,y(t))| \le L_1.|x(t) - y(t)|$, $\forall t \in [0, \frac{T}{p^2.q^2}]_{\mathbb{T}_0}$ *and* $x(t), y(t) \in \mathbb{R}$.

(A2) k: $\mathscr{X} \to \mathbb{R}$ *exists such that* $|k(x) - k(y)| \le L_2.\|x-y\|$, $\forall x, y \in \mathscr{X}$.

(A3) $L.\Psi < 1$, *where* $L = \max\{L_1, L_2\}$ *and* Ψ *is as in Eq (3.11).*

In this case, the problem given by Eq (3.1) has a unique solution.

Proof. We transform the problem given by Eq (3.1) into a fixed-point problem, i.e., $x = \mathscr{F}x$, where the operator $\mathscr{F}$ is defined by Eq (3.10). Assume that

$$\sup\{|\varphi(t,0)| : t \in [0, \frac{T}{p^2.q^2}]_{\mathbb{T}_0}\} = M_1,$$

$$\sup\{|x_0 + k(x)| : k \in \mathscr{X}\} = M_2$$

and

$$\max\{M_1, M_2\} = M.$$

Also, choose a constant R satisfying

$$R \ge \frac{M.\Psi}{1 - L.\Psi}.$$

First, we will show that $\mathscr{F}(\Omega_R) \subset \Omega_R$, where

$$\Omega_R = \{x \in \mathscr{X} : \|x\| \le R\}.$$

For any $x \in \Omega_R$, we get

$$\begin{aligned}\|\mathscr{F}x\| \leq & M_2. \sup_{t\in[0,T]_{\mathbb{T}_0}} \left| \frac{[(p+q).(T+t(\delta.T-1))-\delta.T^2]}{T.(p+q-\delta.T)} \right| \\ & + \frac{T.(p+q)}{T.p.|p+q-\delta.T|}. \sup_{t\in[0,T]_{\mathbb{T}_0}} \left| \int_0^T (T-qs).(L_1.\|x\|+M_1)\, d_{p,q}s \right| \\ & + \frac{1}{p} \sup_{t\in[0,T]_{\mathbb{T}_0}} \left| \int_0^t (t-qs).(L_1.\|x\|+M_1)\, d_{p,q}s \right| \\ & + \frac{|\delta|.T.(p^2-q^2)}{T.p^3.|p+q-\delta.T|}. \sup_{t\in[0,T]_{\mathbb{T}_0}} \left| \int_0^T (Ts-qs^2).(L_1.\|x\|+M_1)\, d_{p,q}s \right| \\ \leq & (L.R+M) \left\{ \frac{T.[(p+q+1).|\delta|+T]}{|p+q-\delta.T|} + \frac{T^2}{p+q} + \frac{T^3.|\delta|.(p-q)}{p.|p+q-\delta.T|.(p^2+pq+q^2)} \right\} \\ \leq & (L.R+M).\Psi \\ \leq & R.\end{aligned}$$

Hence $\mathscr{F}(\Omega_R) \subset \Omega_R$. Now, we should demonstrate that $\mathscr{F}$ is a contraction. For any $x, y \in \mathscr{X}$ and $\forall t \in [0,T]_{\mathbb{T}_0}$, we get

$$\begin{aligned}\|\mathscr{F}x - \mathscr{F}y\| \leq & \sup_{t\in[0,T]_{\mathbb{T}_0}} \left| (k(x)-k(y)) \frac{[(p+q).(T+t.(\delta.T-1))-\delta.T^2]}{T.(p+q-\delta.T)} \right. \\ & + \frac{t.(p+q)}{T.p.(p+q-\delta.T)}. \int_0^T (T-qs).[\varphi(s,x(qs))-\varphi(s,y(qs))]\, d_{p,q}s \\ & - \frac{1}{p} \int_0^t (t-qs).[\varphi(s,x(qs))-\varphi(s,y(qs))]\, d_{p,q}s \\ & \left. - \frac{\delta.t.(p^2-q^2)}{T.p^3.(p+q-\delta.T)} \int_0^T (Ts-qs^2).\left[\varphi\left(s, x\left(\frac{qs}{p}\right)\right) - \varphi\left(s, y\left(\frac{qs}{p}\right)\right)\right] d_{p,q}s \right| \\ \leq & L_2.\|x-y\| \frac{(p+q+1).|\delta|.T}{|p+q-\delta.T|} \\ & + L_1.\|x-y\| \left\{ \frac{T^2}{|p+q-\delta.T|} + \frac{T^2}{p+q} + \frac{T^3.|\delta|.(p-q)}{p.|p+q-\delta.T|.(p^2+pq+q^2)} \right\} \\ \leq & L.\Psi.\|x-y\|.\end{aligned}$$

Since $L.\Psi < 1$, $\mathscr{F}$ is a contraction. Thus, the proof is completed by using the Banach fixed-point theorem. □

Example 2. Consider the following (p,q)-boundary-value problem:

$$\begin{cases} D^2_{p,q}x(t) = \frac{4t}{27\times10^5}\tan^{-1} x^{\sigma}(qt) + t.e^t, & t \in [0, \frac{T}{p^2.q^2}]_{\mathbb{T}_0} \\ x(0) = 2 + \frac{1}{10^5}x(t), \quad x(T) = \delta \int_0^T x(s)\, d_{p,q}s, \end{cases} \tag{3.12}$$

where $T = 243$, $\delta = 1$, $p = 3$, $q = 2$, $L = L_1 = L_2 = \frac{1}{10^5}$, and

$$\varphi(t, x^{\sigma}(qt)) = \frac{4t}{27\times10^5}\tan^{-1} x(6t) + t.e^t.$$

Since

$$|\varphi(t,x(t)) - \varphi(t,y(t))| \leq \frac{1}{10^5}|\tan^{-1}x(t) - \tan^{-1}y(t)| \leq \frac{1}{10^5}|x(t) - y(t)|,$$

condition (A1) is ensured with $L_1 = \frac{1}{10^5}$. For condition (A2), it is clear that

$$|k(x) - k(y)| \leq \frac{1}{10^5}.\|x - y\| \quad \text{with} \quad L_2 = \frac{1}{10^5}.$$

From Eq (3.11), we deduce that

$$\Psi = \frac{60507}{238} + \frac{59049}{5} + \frac{14348907}{13566} \approx 13121.742.$$

Here, we obtain

$$L.\Psi \approx 0.131 < 1.$$

Thus, condition (A3) is satisfied. From Theorem 4, the problem given by Eq (3.12) has a unique solution.

4. The oscillation of solutions of second-order (p, q)-difference equations

If we take

$$\varphi(t, x^{\sigma}(qt)) = -\rho(t).x^{\sigma}(qt)$$

in the first equation of Eq (3.1), then we get the following equation:

$$D^2_{p,q}x(t) + \rho(t).x^{\sigma}(qt) = 0 \tag{4.1}$$

for $t \in \mathbb{T}$ with $p > q > 1$. In this equation, the concept $\rho(t)$ is as follows:

$$\rho(t) = \frac{b}{q.t.\sigma(t)}.$$

Also, it follows from Eq (2.3) that the (p, q)-derivative of $x(qt)$ yields

$$x^{\sigma}(qt) = x(\sigma(qt)) = x(pqt) = x(q^2t) + (p - q)t.D_{p,q}x(qt) \quad \text{for} \quad t \in \mathbb{T}. \tag{4.2}$$

We will use Eq (4.2) to rewrite Eq (4.1). In this case, we have

$$qt.\sigma(t).D^2_{p,q}x(t) + at.D_{p,q}x(qt) + b.x(q^2t) = 0, \quad \text{where} \quad a = b(p - q), b \in \mathbb{R}, \tag{4.3}$$

with the following condition:

$$\frac{p}{q} - a\left(\frac{p}{q} - 1\right) + bq\left(\frac{p}{q} - 1\right)^2 \neq 0. \tag{4.4}$$

We note that Eq (4.3) is similar to the Euler-Cauchy q-difference equation given in [30]. Hence, we call this equation the Euler-Cauchy-like (p, q)-difference equation. By using Eqs (2.3) and (2.4), we can rewrite Eq (4.3) as follows:

$$x(p^2t) - 2\rho\frac{1}{\gamma}.x(pqt) + (\rho^2 - \ell)\frac{1}{\gamma^2}.x(q^2t) = 0, \tag{4.5}$$

where

$$\rho = \gamma \frac{[(\frac{p}{q}+1) - a(\frac{p}{q}-1)]}{2} \quad \text{and} \quad \ell = \gamma^2 \left[\left(\frac{a-1}{2} \right)^2 - bq \right] \left(\frac{p}{q} - 1 \right)^2. \tag{4.6}$$

Also, the following connections are useful and can be easily controlled:

$$\rho = \gamma \left[1 - \frac{(a-1)(\frac{p}{q}-1)}{2} \right] \quad \text{and} \quad \rho^2 - \ell = \gamma^2 \left[\frac{p}{q} - a\left(\frac{p}{q} - 1 \right) + bq \left(\frac{p}{q} - 1 \right)^2 \right]. \tag{4.7}$$

Now, we can give the following lemma.

Lemma 2. *Let ρ and ℓ be as in Eq (4.6). Let $\gamma = \lambda^{\log_p q}$. Also, let us assume that $\rho^2 - \ell$ is defined by Eq (4.7). If*

$$\lambda^2 - 2\rho\lambda + \rho^2 - \ell = 0, \tag{4.8}$$

then the solution of Eq (4.3) is as below

$$x_\lambda(t) = \lambda^{\log_p qt}, \qquad t \in \mathbb{T}.$$

Proof. Since

$$x(pt) = \lambda^{\log_p q(pt)} = \lambda^{\log_p p + \log_p qt} = \lambda^{1+\log_p qt} = \lambda\lambda^{\log_p qt} = \lambda x(t)$$

and

$$x(qt) = \lambda^{\log_p q(qt)} = \lambda^{\log_p q + \log_p qt} = \lambda^{\log_p q}\lambda^{\log_p qt} = \gamma x(t),$$

we have

$$\begin{aligned} x(p^2 t) &= \lambda^{\log_p q(ppt)} = \lambda^{\log_p p^2 + \log_p qt} = \lambda^2 \lambda^{\log_p qt} = \lambda^2 x(t), \\ x(pqt) &= \lambda x(qt) = \lambda\lambda^{\log_p q} x(t) = \gamma\lambda x(t), \\ x(q^2 t) &= \gamma x(qt) = \gamma^2 x(t). \end{aligned}$$

Here, we get

$$x(p^2 t) - 2\rho\frac{1}{\gamma}.x(pqt) + (\rho^2 - \ell)\frac{1}{\gamma^2}.x(q^2 t) = (\lambda^2 - 2\rho\lambda + (\rho^2 - \ell)).x(t) = 0,$$

where $x = x_\lambda$. Thus, the desired result is achieved. □

Note that, since $\lambda \neq 0$, we can rewrite x_λ as follows:

$$x_\lambda(t) = \lambda^{\log_p qt} = [(\operatorname{sgn}\lambda)|\lambda|]^{\log_p qt} = (\operatorname{sgn}\lambda)^{\log_p qt}|\lambda|^{\log_p qt} = (\operatorname{sgn}\lambda)^{\log_p qt}(qt)^{\log_p |\lambda|}.$$

We can give the general solution of Eq (4.3) according to the value of ℓ.

Theorem 5. *Let ρ and ℓ be as in Eq (4.6). Let us assume that Eq (4.4) exists. In this case, the general solution of Eq (4.3) is obtained as follows, where $c_1, c_2 \in \mathbb{R}$:*

(i) If $\ell > 0$, substituting for $\lambda_1 = \rho + \sqrt{\ell}$ and $\lambda_2 = \rho - \sqrt{\ell}$, then

$$x(t) = c_1\lambda_1^{\log_p qt} + c_2\lambda_2^{\log_p qt}.$$

(ii) If $\ell = 0$*, substituting for* $\lambda = \rho$*, then*

$$x(t) = (c_1 \ln t + c_2)\lambda^{\log_p qt}.$$

(iii) If $\ell < 0$*, substituting for* $\lambda = \rho + i\sqrt{-\ell}$*, then*

$$x(t) = |\lambda|^{\log_p qt}(c_1 \cos(\theta. \log_p qt) + c_2 \sin(\theta. \log_p qt)),$$

where

$$\cos(\theta - \theta. \log_p q) = \frac{Re\lambda}{|\lambda|}.$$

Proof. Since λ_1 and λ_2 are the solutions of Eq (4.8) when $\ell > 0$, it can be ascertained from Lemma 2 that x_{λ_1} and x_{λ_2} are solutions of Eq (4.3). Second, if $\ell = 0$, due to Lemma 2, the solution of Eq (4.3) is x_λ. Now, we define $x(t) = x_\lambda(t) \ln t$. In this case,

$$\begin{aligned}
x(pt) &= \lambda.[\lambda^{\log_p qt} \ln p + \lambda^{\log_p qt} \ln t] = \lambda.[x(t) + x_\lambda(t) \ln p],\\
x(qt) &= \lambda^{\log_p qqt}[\ln t + \ln q] = \gamma.x_\lambda(t)[\ln t + \ln q] = \gamma.[x(t) + x_\lambda(t) \ln q],\\
x(pqt) &= x(p(qt)) = \lambda.[x(qt) + x_\lambda(qt) \ln p] = \lambda.\gamma[x(t) + x_\lambda(t) \ln pq],
\end{aligned}$$

and

$$\begin{aligned}
&x(p^2t) - 2\rho\frac{1}{\gamma}.x(pqt) + (\rho^2 - \ell)\frac{1}{\gamma^2}.x(q^2t) = x(p^2t) - 2\rho\frac{1}{\gamma}.x(pqt) + \rho^2\frac{1}{\gamma^2}.x(q^2t)\\
&= \lambda^2[x(t) + 2x_\lambda(t). \ln p] - 2\rho.\frac{1}{\gamma}.\lambda\gamma[x(t) + x_\lambda(t). \ln pq] + \rho^2.\frac{1}{\gamma^2}.\gamma^2[(x(t) + 2x_\lambda(t). \ln q)]\\
&= (\lambda^2 - 2\rho\lambda + \rho^2).x(t) + [2\lambda^2 - 2\rho\lambda].x_\lambda(t). \ln p - 2\rho\lambda.x_\lambda(t). \ln q + 2\rho^2.x_\lambda(t). \ln q\\
&= (\lambda^2 - 2\rho\lambda + \rho^2).x(t) + 2(\lambda - \rho).x_\lambda(t).[\lambda \ln p - \rho \ln q]\\
&= 0
\end{aligned}$$

yield that x also leads to Eq (4.3).

Last, suppose that $\ell < 0$. We note that

$$\frac{Re\lambda}{|\lambda|} \in (-1, 1),$$

so that there exists $\theta \in (0, \pi)$ with

$$\cos(\theta - \theta. \log_p q) = \frac{Re\lambda}{|\lambda|}.$$

We set

$$u(t) = \cos(\theta. \log_p qt), \quad v(t) = \sin(\theta. \log_p qt), \quad x(t) = x_{|\lambda|}(t).u(t), \quad y(t) = x_{|\lambda|}(t).v(t).$$

In addition, for convenience, it is important to obtain the following expressions and calculations. Now,

$$\begin{aligned}
u(t) =&u(pt). \cos\theta + v(pt). \sin\theta, \qquad v(t) = v(pt). \cos\theta - u(pt). \sin\theta,\\
u(p^2t) =&u(pqt)[\cos\theta. \cos(\theta. \log_p q) + \sin\theta. \sin(\theta. \log_p q)]
\end{aligned}$$

$$
\begin{aligned}
&+ v(pqt)[\cos\theta.\sin(\theta.\log_p q) - \sin\theta.\cos(\theta.\log_p q)],\\
u(q^2t) =&u(pqt)[\cos\theta.\cos(\theta.\log_p q) + \sin\theta.\sin(\theta.\log_p q)]\\
&- v(pqt)[\cos\theta.\sin(\theta.\log_p q) - \sin\theta.\cos(\theta.\log_p q)],
\end{aligned}
$$

and

$$
\begin{aligned}
v(p^2t) =&v(pqt)[\sin\theta.\sin(\theta.\log_p q) + \cos\theta.\cos(\theta.\log_p q)]\\
&+ u(pqt)[\sin\theta.\cos(\theta.\log_p q) - \cos\theta.\sin(\theta.\log_p q)],\\
v(q^2t) =&v(pqt)[\sin\theta.\sin(\theta.\log_p q) + \cos\theta.\cos(\theta.\log_p q)]\\
&- u(pqt)[\sin\theta.\cos(\theta.\log_p q) - \cos\theta.\sin(\theta.\log_p q)],
\end{aligned}
$$

so that

$$
\begin{aligned}
&x(p^2t) - 2\rho\frac{1}{\gamma}.x(pqt) + (\rho^2 - \ell).\frac{1}{\gamma^2}.x(q^2t) = x(p^2t) - 2\rho.\frac{1}{\gamma}.x(pqt) + |\lambda|^2\frac{1}{\gamma^2}.x(q^2t)\\
&= |\lambda|^2.x_{|\lambda|}(t).u(p^2t) - 2\rho.|\lambda|.x_{|\lambda|}(t) + |\lambda|^2.x_{|\lambda|}(t).u(q^2t)\\
&= 2|\lambda.|x_{|\lambda|}(t).u(pqt)\left[|\lambda|\left(\cos\theta.\cos(\theta.\log_p q) + \sin\theta.\sin(\theta.\log_p q)\right) - \rho\right]\\
&= 2|\lambda|.x_{|\lambda|}(t).u(pqt)\left[|\lambda|\cos(\theta - \theta.\log_p q) - \rho\right]\\
&= 0,
\end{aligned}
$$

and, similarly,

$$
y(p^2t) - 2\rho.\frac{1}{\gamma}.y(pqt) + (\rho^2 - \ell).\frac{1}{\gamma^2}.y(q^2t) = 2|\lambda|.x_{|\lambda|}(t).v(pqt)\left[|\lambda|\cos(\theta - \theta.\log_p q) - \rho\right] = 0.
$$

Therefore, $x(t)$ and $y(t)$ lead to Eq (4.3). Now, we must demonstrate the linear independence of solutions x and y to complete the proof. Here, the Wronskian (see [40, Definition 3.5]) for the (p, q)-calculus can be easily defined as follows:

$$
W(x, y) = x(D_{p,q}y) - y(D_{p,q}x).
$$

In this case, for both solutions, we get $(p - q).t.W(x, y)$ as follows:

$$
[2\sqrt{\ell} - (\rho + \sqrt{\ell})^{\log_p q} + (\rho - \sqrt{\ell})^{\log_p q}](\rho^2 - \ell)^{\log_p qt} \quad \text{for} \quad \ell > 0,
$$

$$
\rho^{2\log_p qt}[\rho\ln p - \ln q\rho^{\log_p q}] \quad \text{for} \quad \ell = 0,
$$

and

$$
(\rho^2 - \ell)^{2\log_p qt}\,[(\rho^2 - \ell)\sin\theta - (\rho^2 - \ell)^{\log_p q}\sin(\theta.\log_p q)] \quad \text{for} \quad \ell < 0,
$$

respectively. Considering all situations, none of these Wronskians are zero. Thus, each of the three cases mentioned above forms a foundational set of solutions for their situations. Finally, the solution $s(t)$ of the initial value problem given by

$$
\begin{cases}
D^2_{p,q}s(t) + \rho(t).s^{\sigma}(qt) = 0,\\
s(t_0) = s_0, D_{p,q}s(t_0) = \tilde{s}_0, \qquad t_0 \in \mathbb{T}
\end{cases}
$$

can be easily expressed as follows:

$$s(t) = \frac{s_0.D_{p,q}y(t_0) - y(t_0).\tilde{s}_0}{W(x,y)(t_0)}x(t) + \frac{x(t_0).\tilde{s}_0 - s_0.D_{p,q}x(t_0)}{W(x,y)(t_0)}y(t).$$

This ends the theorem. □

Remark 1. *In this theorem, we observe that* $\frac{Re\lambda}{|\lambda|}$ *is different from Theorem 4 in [30]. We also generalize the general solutions in Theorem 4 to* (p,q)*-calculus. That is, Theorem 5 is reduced to the q-version for* $p \to 1$.

We give some basic definitions and concepts about oscillation.

Definition 1. *(i) We recall that the solution* $x(t)$ *of Eq (4.1) has a generalized zero at t if* $x(t) = 0$. *Now, we say that* $x(t)$ *has a generalized zero in the interval* $(qt, \sigma(t))$ *if* $x(qt).x^{\sigma}(t)) < 0$.

(ii) We also say that Eq (4.1) is non-conjugate on the interval $[a,b]$ *if there is no non-trivial solution of Eq (4.1) with two (or more) generalized zeros in* $[a,b]$.

(iii) We say that Eq (4.1) is non-oscillatory on $[\zeta,\infty)$ *if there exists* $a \in [\zeta,\infty)$ *such that this equation is non-conjugate on* $[a,b]$ *for every* $a < b$. *In other cases, we will mean that Eq (4.1) is oscillatory on* $[\zeta,\infty)$.

(iv) We can also define that an $x(t)$ *solution of Eq (4.1) is non-oscillatory if* $x(qt).x^{\sigma}(t) > 0$ *on* $[T,\infty)$ *for some* $T > 0$. *On the contrary, we will mean that the solution* $x(t)$ *is oscillatory on* $[T,\infty)$.

Remark 2. *In this definition, we state that (ii) and (iii) are as in [30]. However, (i) and (iv) are generalizations of the situation in q-calculus. We observe that (i) and (iv) are reduced to the q-case in [30] for* $p \to 1$.

Now, we can give the (p,q)-calculus version of the Sturm-type separation theorem.

Applying linear independence of the solutions, it is easy to see that one solution of Eq (4.1) is (non)oscillatory if and only if every solution of (4.1) is (non)oscillatory. To prove this, let us assume that $x(t)$ is a non-oscillatory solution of Eq (4.1). In this case, it is $x(qt).x^{\sigma}(t) > 0$ on $[T,\infty)$ for some $T > 0$ from the definition of oscillatory. Let $y(t)$ be any solution of Eq (4.1). Also, let $x(t)$ and $y(t)$ be linearly independent. Thus, $D_{p,q}\left(\frac{y(t)}{x(t)}\right) \neq 0$. Then, $y(t)/x(t)$ is exactly monotone and hence has a single signum. Thus,

$$(y(qt).y^{\sigma}(t))/(x(qt).x^{\sigma}(t)) = (y(qt)/x(qt)).(y^{\sigma}(t)/x^{\sigma}(t))$$

is ultimately positive and $y(qt).y^{\sigma}(t) > 0$. This means that $y(t)$ is also non-oscillatory.

Next, we can give the theorem about oscillation.

Theorem 6. *The* (p,q)*-difference equation*

$$D^2_{p,q}x(t) + \frac{b}{q.t.\sigma(t)}x^{\sigma}(qt) = 0$$

is oscillatory if and only if

$$b > \frac{1}{(\sqrt{p} + \sqrt{q})^2}.$$

Proof. To prove the theorem, we first start by rewriting the equation as follows:

$$D^2_{p,q}x(t) + \frac{b}{pqt^2}x(pqt) = 0. \tag{4.9}$$

If we use Eq (4.2), the equation is obtained as an Euler-Cauchy-like (p, q)-difference equation such that

$$pqt^2.D^2_{p,q}x(t) + b(p - q)t.D_{p,q}x(qt) + b.x(q^2t) = 0. \tag{4.10}$$

Note that Eq (4.10) is the form of Eq (4.3) with $a = (p - q)b$ and $b \in \mathbb{R}$. From Eq (4.6), we get

$$\begin{aligned}
\rho &= \gamma\frac{[(\frac{p}{q} + 1) - a(\frac{p}{q} - 1)]}{2} = \gamma\frac{[\frac{p}{q} + 1 - bq(\frac{p}{q} - 1)^2]}{2} \\
&= \gamma\left[\sqrt{\frac{p}{q}} - \frac{(\frac{p}{q} - 1)^2}{2}q\left(b - \frac{1}{(\sqrt{p} + \sqrt{q})^2}\right)\right] \\
&= \gamma\left[-\sqrt{\frac{p}{q}} - \frac{(\frac{p}{q} - 1)^2}{2}q\left(b - \frac{1}{(\sqrt{p} - \sqrt{q})^2}\right)\right].
\end{aligned} \tag{4.11}$$

Also, we calculate the crucial quantity ℓ as follows:

$$\begin{aligned}
\ell &= \gamma^2\left(\frac{p}{q} - 1\right)^2\left[\left(\frac{a-1}{2}\right)^2 - bq\right] \\
&= \gamma^2\frac{(\frac{p}{q} - 1)^2}{4}[b^2(p - q)^2 - 2b(p - q) + 1 - 4bq] \\
&= \gamma^2\frac{(\frac{p}{q} - 1)^2}{4}[b^2(p - q)^2 - 2b(p + q) + 1] \\
&= \gamma^2q^2\frac{(\frac{p}{q} - 1)^4}{4}\left[b^2 - b\frac{2(p+q)}{(p-q)^2} + \frac{1}{(p-q)^2}\right] \\
&= \gamma^2q^2\frac{(\frac{p}{q} - 1)^4}{4}\left[b - \frac{1}{(\sqrt{p} + \sqrt{q})^2}\right]\left[b - \frac{1}{(\sqrt{p} - \sqrt{q})^2}\right].
\end{aligned}$$

Now we can prove the oscillation of Eq (4.9) by using Theorem 5 and the quantity ℓ. If $\ell = 0$, then b becomes either

$$b = \frac{1}{(\sqrt{p} + \sqrt{q})^2} \quad \text{or} \quad b = \frac{1}{(\sqrt{p} - \sqrt{q})^2}.$$

If

$$b = 1/(\sqrt{p} + \sqrt{q})^2,$$

then

$$\rho = \gamma\sqrt{\frac{p}{q}}$$

by Eq (4.11). From Theorem 5 (ii), the two solutions

$$\left(\gamma\sqrt{\frac{p}{q}}\right)^{\log_p qt} = \sqrt{qt}$$

and

$$\left(\gamma\sqrt{\frac{p}{q}}\right)^{\log_p qt}.\ln t = \sqrt{qt}.\ln t$$

are non-oscillatory; hence, Eq (4.10) is non-oscillatory. If

$$b = 1/(\sqrt{p} - \sqrt{q})^2,$$

then

$$\rho = -\gamma\sqrt{\frac{p}{q}}$$

by Eq (4.11). From Theorem 5 (ii), the two solutions

$$\left(-\gamma\sqrt{\frac{p}{q}}\right)^{\log_p qt} = (-1)^{\log_p qt}\sqrt{qt}$$

and

$$\left(-\gamma\sqrt{\frac{p}{q}}\right)^{\log_p qt}.\ln t = (-1)^{\log_p qt}\sqrt{qt}.\ln t$$

are oscillatory; hence, Eq (4.10) is oscillatory. If $\ell > 0$, then it becomes either

$$b < \frac{1}{(\sqrt{p} + \sqrt{q})^2} \quad \text{or} \quad b > \frac{1}{(\sqrt{p} - \sqrt{q})^2}.$$

If

$$b < 1/(\sqrt{p} + \sqrt{q})^2,$$

then

$$\rho > \gamma\sqrt{p/q}$$

by Eq (4.11), and, considering part (i) of Theorem 5, the solution

$$(\rho + \sqrt{\ell})^{\log_p qt} = (qt)^{\log_p(\rho+\sqrt{\ell})}$$

is non-oscillatory; therefore, Eq (4.10) is non-oscillatory. If

$$b > 1/(\sqrt{p} - \sqrt{q})^2,$$

then

$$\rho < -\gamma\sqrt{p/q}$$

by Eq (4.11), and, from Theorem 5 (i), the solution

$$(\rho - \sqrt{\ell})^{\log_p qt} = (-1)^{\log_p qt}(qt)^{\log_p(\sqrt{\ell}-\rho)}$$

is oscillatory; for this reason, Eq (4.10) is oscillatory. Last, if $\ell < 0$, then it becomes

$$\frac{1}{(\sqrt{p} + \sqrt{q})^2} < b < \frac{1}{(\sqrt{p} - \sqrt{q})^2}.$$

From Theorem 5 (iii), we get that the two solutions

$$(\sqrt{qt})^{\log_p|\lambda|}\cos(\theta.\log_p qt) \quad \text{and} \quad (\sqrt{qt})^{\log_p|\lambda|}\sin(\theta.\log_p qt)$$

are oscillatory; hence, Eq (4.10) is oscillatory, where $\rho \in (-\gamma\sqrt{p/q}, \gamma\sqrt{p/q})$,

$$|\lambda| = \gamma\sqrt{\frac{p}{q}}, \qquad \cos(\theta.(1-\log_p q)) = \frac{\rho}{\gamma\sqrt{p/q}} = \frac{[(\frac{p}{q}+1) - bq(\frac{p}{q}-1)^2]}{2\sqrt{p/q}}.$$

Considering this all together, Eq (4.10) and, thus, Eq (4.9), is oscillatory if and only if

$$b > \frac{1}{(\sqrt{p}+\sqrt{q})^2}.$$

Hence, the proof is completed. □

Remark 3. *Theorem 6 is reduced to Theorem 2 for $p \to 1$. In addition, we observe that Theorem 6 is reduced to the continuous case in Theorem 1 when we take the limit as $q \to 1 = p$ and note that the constant $\frac{1}{(\sqrt{p}+\sqrt{q})^2}$ becomes $\frac{1}{4}$.*

Now, we can give the proof of the (p, q)-version of Kneser's theorem.

Theorem 7. *((p, q)-Kneser theorem).*

(1) If

$$\lim_{t\to\infty}\sup\{q.t.\sigma(t).\rho(t)\} < \frac{1}{(\sqrt{p}+\sqrt{q})^2},$$

then Eq (4.1) is non-oscillatory on $p^{\mathbb{N}_0}$.

(2) If

$$\lim_{t\to\infty}\inf\{q.t.\sigma(t).\rho(t)\} > \frac{1}{(\sqrt{p}+\sqrt{q})^2},$$

then Eq (4.1) is oscillatory on $p^{\mathbb{N}_0}$.

Proof. To prove the first option, it is enough to show that Eq (4.9) for

$$b < 1/(\sqrt{p}+\sqrt{q})^2$$

is non-oscillatory. To prove the second part of the theorem, it is enough to show that

$$b > 1/(\sqrt{p}+\sqrt{q})^2,$$

then, Eq (4.1) is oscillatory. These cases can be deduced again from the proof of Theorem 6. □

Remark 4. *Theorem 7 is reduced to Theorem 6 in [30] for $p \to 1$. We also note that the constant, which is $\frac{1}{(1+\sqrt{q})^2}$ in q-calculus, is $\frac{1}{(\sqrt{p}+\sqrt{q})^2}$ in (p, q)-calculus.*

Remark 5. *If we take*

$$\varphi(t, x^\sigma(qt)) = \frac{-b}{q.t.\sigma(t)}x^\sigma(qt)$$

in Eq (3.1), we can establish the existence and uniqueness of solutions for Eq (4.9).

5. Conclusions

First, we studied the second-order (p, q)-difference equation with integral and local boundary conditions and investigated the existence and uniqueness of solutions with the help of the Banach fixed-point theorem. Moreover, we have obtained the general solution of the Euler-Cauchy-like (p, q)-difference equation, which is a special case of Eq (3.1). Also, we have proven the Sturm-type separation theorem to examine the oscillation of the (p, q)-difference equation and given the (p, q)-Kneser theorem. We can see that the constant in Theorem 1, which is 1/4 in the continuous and discrete cases, is to be $1/(\sqrt{p} + \sqrt{q})^2$. Consequently, when $p \to 1 = q$, Theorem 6 is reduced to the continuous case in Theorem 1.

Second, it may be considered as an open problem to study the oscillation of the equation

$$D^2_{p,q}x(t) = \varphi(t, x^\sigma(qt))$$

in Eq (3.1) by taking $\varphi(t, x^\sigma(qt))$, unlike in Eq (4.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence tools (AI) in the creation of this article.

Acknowledgments

The authors are highly grateful to the editors and referees for their valuable comments and suggestions to improve the paper.

Conflict of interest

The authors declare that they have no conflicts of interest.

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