Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 10638–10658.
DOI: 10.3934/math.2024519
Received: 17 January 2024
Revised: 04 March 2024
Accepted: 08 March 2024
Published: 18 March 2024

Research article

Positive almost periodic solution for competitive and cooperative Nicholson's blowflies system

Wentao Wang*

School of Mathematics, Physics and Statistics, Shanghai University of Engineering Science, Shanghai 201620, China

* **Correspondence:** Email: 21170001@sues.edu.cn.

Abstract: In this paper, we investigate a class of competitive and cooperative Nicholson's blowfly equations. By applying the Lyapunov functional and analysis technique, the conditions for the existence and exponential convergence of a positive almost periodic solution are derived. Moreover, an example and numerical simulation for justifying the theoretical analysis are also provided.

Keywords: positive almost periodic solution; exponential convergence; Nicholson's blowfly model; M-matrix; exponential dichotomy
Mathematics Subject Classification: 34A34, 34C27, 34K14

1. Introduction

Based on the experimental data which were observed and summarized by Nicholson [1], Gurney et al. [2] presented a classic biological dynamical system model

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}. \tag{1.1}$$

Here, $N(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. The research on the Nicholson's blowfly model and its modifications has realized a remarkable progress in the past fifty years and an abundance of results on the existence of positive solutions, persistence, oscillation, stability, periodic solutions, almost periodic solutions, pseudo almost periodic solutions, etc. (see [3–21]) have been obtained. Furthermore, Berezansky et al. [22] systematically collected and compared the results in the above-mentioned studies and put forward several open problems that have been partially answered in recent works [23–30], such as Nicholson's blowfly model with impulsive perturbation, harvesting term and nonlinear density-dependent mortality term. On the other hand, Liu [31] considered the following cooperative Nicholson's blowfly equation with a patch structure:

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \beta x_i(t-r_i)e^{-x_i(t-r_i)} - d_i x_i(t), \quad i = 1, 2, \cdots, n. \tag{1.2}$$

Also, Berezansky et al. [32] studied the following cooperative Nicholson-type delay differential system

$$\begin{cases} x_1'(t) &= -a_1x_1(t) + b_1x_2(t) + c_1x_1(t-\tau)e^{-x_1(t-\tau)}, \\ x_2'(t) &= -a_2x_2(t) + b_2x_1(t) + c_2x_2(t-\tau)e^{-x_2(t-\tau)}. \end{cases} \tag{1.3}$$

In this paper, we will discuss the following competitive and cooperative Nicholson's blowfly system

$$\begin{cases} x_1'(t) &= -\delta_1(t)x_1(t) + a_1(t)x_2(t) + \sum\limits_{j=1}^{n} c_{1j}(t)x_1(t-\tau_{1j}(t))e^{-b_{1j}(t)x_1(t-\tau_{1j}(t))} \\ &\quad -k_1(t)x_1(t)x_2(t), \\ x_2'(t) &= -\delta_2(t)x_2(t) + a_2(t)x_1(t) + \sum\limits_{j=1}^{n} c_{2j}(t)x_2(t-\tau_{2j}(t))e^{-b_{2j}(t)x_2(t-\tau_{2j}(t))} \\ &\quad -k_2(t)x_1(t)x_2(t) \end{cases} \tag{1.4}$$

where $\delta_i, a_i, b_{ij}, c_{ij}, \tau_{ij}, k_i : R^1 \longrightarrow [0, +\infty)$ are almost periodic functions $i = 1, 2, j = 1, 2, \cdots, n$. $x_1(t), x_2(t)$ denote the sizes of the different populations at time t, c_{ij} denotes the maximum per capita daily egg production of x_i, $\frac{1}{b_{ij}}$ represents the size at which the population x_i reproduces at its maximum rate, δ_i is the per capita daily adult death rate of x_i , and τ_{ij} denotes the generation time of x_i. $a_1(t)$ represents the rate at which x_2 contributes to x_1 and $a_2(t)$ represents the rate at which x_1 contributes to x_2 at time t. $k_i(t)$ denotes the death rate of x_i due to the competition between x_1 and x_2 at time t. The cooperative terms $a_1(t)x_2(t)$ and $a_2(t)x_1(t)$ and the competitive terms $k_1(t)x_1(t)x_2(t)$ and $k_1(t)x_1(t)x_2(t)$ reflect the degree at which they cooperate and compete with each other, respectively.

Recently, there have been wide-ranging results obtained on competitive and cooperative systems added the literatures [33–38] due to its extensive applicability. However, to the best of our knowledge, few results are presented in literatures about the existence of positive almost periodic solutions for competitive and cooperative Nicholson's blowfly system. In the real world, since the competition is inevitable and there exists an almost periodically changing environment, it is worth studying the positive almost periodic solution for competitive and cooperative Nicholson's blowfly system. Based on the above idea, we shall consider the existence and exponential convergence of positive almost periodic solutions of system (1.4) which possesses obvious dynamics significance.

For convenience, we introduce some notations. Throughout this paper, given a bounded continuous function g defined on R^1, let g^+ and g^- be defined as follows:

$$g^- = \inf_{t\in R^1} g(t), \quad g^+ = \sup_{t\in R^1} g(t).$$

It will be assumed that

$$\begin{aligned} &\delta_i^- > 0, b_{ij}^- > 0, r_i = \max_{1\le j\le n}\{\tau_{ij}^+\} > 0 \\ &i = 1, 2, j = 1, 2, \cdots, n. \end{aligned} \tag{1.5}$$

Let $R^n(R^n_+)$ be the set of all (nonnegative) real vectors; we will use $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. we let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \cdots, |x_n|)^T$ and define $\|x\| = \max_{1\le i\le n} |x_i|$. For a matrix $A = (a_{ij})_{n\times n}$, A^T, A^{-1}, $|A|$ and $\rho(A)$ denote the transpose, inverse, the absolute-value matrix and the spectral radius of A respectively. A matrix or vector $A \ge 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A and B, $A \ge B$ (resp. $A > B$) means that $A - B \ge 0$ (resp. $A - B > 0$). Denote $C = \prod_{i=1}^{2} C([-r_i, 0], R^1)$ and $C_+ = \prod_{i=1}^{2} C([-r_i, 0], R^1_+)$ as Banach spaces equipped with the supremum norm defined by

$$\|\varphi\| = \sup_{-r_i\le t\le 0} \max_{1\le i\le 2} |\varphi_i(t)| \quad \text{for all } \varphi(t) = (\varphi_1(t), \varphi_2(t))^T \in C(\text{or} \in C_+).$$

If $x_i(t)$ is defined on $[t_0 - r_i, \nu)$ with $t_0, \nu \in R^1$ and $i = 1, 2$, then we define $x_t \in C$ as $x_t = (x^1_t, x^2_t)^T$ where $x^i_t(\theta) = x_i(t + \theta)$ for all $\theta \in [-r_i, 0]$ and $i = 1, 2$.

The initial conditions associated with system (1.4) are of the following form:

$$x_{t_0} = \varphi, \ \varphi = (\varphi_1, \varphi_2)^T \in C_+. \tag{1.6}$$

We write $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ for a solution of the initial value problems (1.4) and (1.6). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of the existence of $x_t(t_0, \varphi)$.

The remaining part of this paper is organized as follows. In Section 2, we shall give some definitions and preliminary results. In Section 3, we shall derive sufficient conditions for checking the existence, uniqueness and exponential convergence of the positive almost periodic solution of (1.4). In Section 4, we shall give an example and numerical simulation to illustrate the results obtained in the previous section.

2. Preliminary results

In this section, some lemmas and definitions will be presented, which are of importance in proving our main results in Section 3.

Definition 2.1. [39,40] Let $u(t) : R^1 \longrightarrow R^n$ be continuous in t. $u(t)$ is said to be almost periodic on R^1, if for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t + \delta) - u(t)| < \varepsilon \text{ for all } t \in R^1\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, such that for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for all $t \in R^1$.

Definition 2.2. [39,40] Let $x \in R^n$ and $Q(t)$ be an $n \times n$ continuous matrix defined on R^1. The linear system

$$x'(t) = Q(t)x(t) \tag{2.1}$$

is said to admit an exponential dichotomy on R^1 if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (2.1) satisfying

$$\|X(t)PX^{-1}(s)\| \le ke^{-\alpha(t-s)} \quad \text{for all } t \ge s,$$
$$\|X(t)(I - P)X^{-1}(s)\| \le ke^{-\alpha(s-t)} \quad \text{for all } t \le s.$$

Definition 2.3. A real $n \times n$ matrix $K = (k_{ij})$ is said to be an M-matrix if $k_{ij} \le 0, i, j = 1, \ldots, n, i \ne j$ and $K^{-1} \ge 0$.

Set

$$B = \{\varphi|\varphi = (\varphi_1(t), \varphi_2(t))^T \text{ is an almost periodic vector function on } R^1\}.$$

For any $\varphi \in B$, we define an induced module $||\varphi||_B = \sup_{t\in R^1} \max_{1\le i\le 2} |\varphi_i(t)|$; then, B is a Banach space.

Lemma 2.1. [39,40] If the linear system (2.1) admits an exponential dichotomy, then the almost periodic system

$$x'(t) = Q(t)x + g(t) \tag{2.2}$$

has a unique almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)ds. \tag{2.3}$$

Lemma 2.2. [39,40] Let $c_i(t)$ be an almost periodic function on R^1 and

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} c_i(s)ds > 0, \quad i = 1, 2, \cdots, n.$$

Then the linear system

$$x'(t) = diag(-c_1(t), -c_2(t), \cdots, -c_n(t))x(t)$$

admits an exponential dichotomy on R^1.

Lemma 2.3. [41,42] Let $A \ge 0$ be an $n\times n$ matrix and $\rho(A) < 1$; then, $(I_n - A)^{-1} \ge 0$, where I_n denotes the identity matrix of size n.

Lemma 2.4. Suppose that there exist positive constants E_{i1} and E_{i2} such that

$$E_{i1} > E_{i2}, \sum_{j=1}^{n} \frac{c_{1j}^+}{\delta_1^- b_{1j}^- e} + \frac{a_1^+ E_{21}}{\delta_1^-} < E_{11}, \sum_{j=1}^{n} \frac{c_{2j}^+}{\delta_2^- b_{2j}^- e} + \frac{a_2^+ E_{11}}{\delta_2^-} < E_{21}, \tag{2.4}$$

$$\frac{a_1^-}{\delta_1^+}E_{22} + \sum_{j=1}^{n} \frac{c_{1j}^-}{\delta_1^+}E_{11}e^{-b_{1j}^+ E_{11}} - \frac{k_1^+}{\delta_1^+}E_{11}E_{21} > E_{12} \ge \frac{1}{\min_{1\le j\le m} b_{1j}^-}, \tag{2.5}$$

$$\frac{a_2^-}{\delta_2^+}E_{12} + \sum_{j=1}^{n} \frac{c_{2j}^-}{\delta_2^+}E_{21}e^{-b_{2j}^+ E_{21}} - \frac{k_2^+}{\delta_2^+}E_{11}E_{21} > E_{22} \ge \frac{1}{\min_{1\le j\le m} b_{2j}^-}, \tag{2.6}$$

where $i = 1, 2$. Let

$$C^0 := \{\varphi|\varphi \in C, E_{i2} < \varphi_i(t) < E_{i1}, \text{ for all } t \in [-r_i, 0], \ i = 1, 2\}.$$

Moreover, assume that $x(t; t_0, \varphi)$ is the solution of (1.4) with $\varphi \in C^0$. Then,

$$E_{i2} < x_i(t; t_0, \varphi) < E_{i1}, \text{ for all } t \in [t_0, \eta(\varphi)), i = 1, 2, \tag{2.7}$$

and $\eta(\varphi) = +\infty$.

Proof. We rewrite the system (1.4) as

$$x'(t) = f(t, x_t),$$

where $x(t) = (x_1(t), x_2(t))^T$, $f(t,\varphi) = (f_1(t,\varphi), f_2(t,\varphi))^T$, $f_1(t,\varphi) = -\delta_1(t)\varphi_1(0) + a_1(t)\varphi_2(0) + \sum_{j=1}^{n} c_{1j}(t)\varphi_1(-\tau_{1j}(t))e^{-b_{1j}(t)\varphi_1(-\tau_{1j}(t))} - k_1(t)\varphi_1(0)\varphi_2(0)$, $f_2(t,\varphi) = -\delta_2(t)\varphi_2(0) + a_2(t)\varphi_1(0) + \sum_{j=1}^{n} c_{2j}(t)\varphi_2(-\tau_{2j}(t))e^{-b_{2j}(t)\varphi_2(-\tau_{2j}(t))} - k_2(t)\varphi_1(0)\varphi_2(0)$, $\varphi(t) = (\varphi_1(t), \varphi_2(t))^T \in C^0$. It is obvious that $f : R^1 \times C^0 \to R^2$ is continuous and $C^0 \subset C$ is open. Let $\phi, \psi \in C^0$; then, considering that $\sup_{u\geq 0} |\frac{1-u}{e^u}| = 1$ and the inequality

$$\begin{aligned} |xe^{-x} - ye^{-y}| &= |\frac{1-(x+\theta(y-x))}{e^{x+\theta(y-x)}}||x-y| \\ &\leq |x-y| \text{ where } x, y \in [0,+\infty), 0 < \theta < 1, \end{aligned} \tag{2.8}$$

we obtain

$$\begin{aligned} |f_1(t,\phi) - f_1(t,\psi)| &\leq \delta_1(t)|\phi_1(0) - \psi_1(0)| + a_1(t)|\phi_2(0) - \psi_2(0)| + \sum_{j=1}^{n} c_{1j}(t) \times \\ &\quad |\phi_1(-\tau_{1j}(t))e^{-b_{1j}(t)\phi_1(-\tau_{1j}(t))} - \psi_1(-\tau_{1j}(t))e^{-b_{1j}(t)\psi_1(-\tau_{1j}(t))}| \\ &\quad +k_1(t)|\phi_1(0)\phi_2(0) - \psi_1(0)\psi_2(0)| \\ &\leq (\delta_1^+ + a_1^+)\|\phi - \psi\| + \sum_{j=1}^{n} \frac{c_{1j}(t)}{b_{1j}(t)} \times |b_{1j}(t)\phi_1(-\tau_{1j}(t))e^{-b_{1j}(t)\phi_1(-\tau_{1j}(t))} \\ &\quad -b_{1j}(t)\psi_1(-\tau_{1j}(t))e^{-b_{1j}(t)\psi_1(-\tau_{1j}(t))}| \\ &\quad +k_1^+(|\phi_1(0)||\phi_2(0) - \psi_2(0)| + |\psi_2(0)||\phi_1(0) - \psi_1(0)|) \\ &\leq (\delta_1^+ + a_1^+)\|\phi - \psi\| + \sum_{j=1}^{n} \frac{c_{1j}^+}{b_{1j}(t)} \times \\ &\quad |b_{1j}(t)\phi_1(-\tau_{1j}(t)) - b_{1j}(t)\psi_1(-\tau_{1j}(t))| \\ &\quad +k_1^+(E_{11}\|\phi - \psi\| + E_{21}\|\phi - \psi\|) \\ &\leq (\delta_1^+ + a_1^+ + \sum_{j=1}^{n} c_{1j}^+ + k_1^+ E_{11} + k_1^+ E_{21})\|\phi - \psi\|. \end{aligned} \tag{2.9}$$

In the same way, we also get

$$|f_2(t,\phi) - f_2(t,\psi)| \leq (\delta_2^+ + a_2^+ + \sum_{j=1}^{n} c_{2j}^+ + k_2^+ E_{11} + k_2^+ E_{21}))\|\phi - \psi\|. \tag{2.10}$$

Then (2.9) and (2.10) imply that f satisfies the Lipschitz condition in its second argument on each compact subset of $R^1 \times C^0$. Moreover, since $\varphi \in C_+$, it is easy to get that $x_t(t_0,\varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$ by using Theorem 5.2.1 from [43, p. 81]. Set $x(t) = x(t; t_0, \varphi)$ for all $t \in [t_0, \eta(\varphi))$.

We claim that

$$0 \leq x_i(t) < E_{i1} \text{ for all } t \in [t_0, \eta(\varphi)), i = 1, 2. \tag{2.11}$$

By contradiction, assume that (2.11) does not hold. Then, there exists $t_1 \in (t_0, \eta(\varphi))$ such that one of the following two cases must occur:

$$(1)\ x_1(t_1) = E_{11},\ 0 \leq x_i(t) < E_{i1} \text{ for all } t \in [t_0 - r_i, t_1),\ i = 1, 2; \tag{2.12}$$

$$(2)\;\; x_2(t_1) = E_{21},\;\; 0 \le x_i(t) < E_{i1} \;\text{ for all }\; t \in [t_0 - r_i, t_1),\; i = 1, 2. \tag{2.13}$$

In the sequel, we consider two cases.

Case (i). Suppose that (2.12) holds. Considering the derivative of $x_1(t)$, together with (2.4) and the fact that $\sup\limits_{u\ge 0} ue^{-u} = \frac{1}{e}$, we have

$$\begin{aligned}
0 &\le x_1'(t_1) \\
&= -\delta_1(t_1)x_1(t_1) + a_1(t_1)x_2(t_1) + \sum_{j=1}^{n} c_{1j}(t_1)x_1(t_1 - \tau_{1j}(t_1))e^{-b_{1j}(t_1)x_1(t_1-\tau_{1j}(t_1))} \\
&\quad -k_1(t_1)x_1(t_1)x_2(t_1) \\
&\le -\delta_1^- x_1(t_1) + a_1^+ x_2(t_1) + \sum_{j=1}^{n} \frac{c_{1j}^+}{b_{1j}^-}\frac{1}{e} \\
&\le -\delta_1^- E_{11} + a_1^+ E_{21} + \sum_{j=1}^{n} \frac{c_{1j}^+}{b_{1j}^-}\frac{1}{e} \\
&= \delta_1^-(-E_{11} + \sum_{j=1}^{n} \frac{c_{1j}^+}{\delta_1^- b_{1j}^- e} + \frac{a_1^+ E_{21}}{\delta_1^-}) \\
&< 0,
\end{aligned}$$

which is a contradiction.

Case (ii). Suppose that (2.13) holds. Considering the derivative of $x_2(t)$, together with (2.4) and the fact that $\sup\limits_{u\ge 0} ue^{-u} = \frac{1}{e}$, we have

$$\begin{aligned}
0 &\le x_2'(t_1) \\
&= -\delta_2(t_1)x_2(t_1) + a_2(t_1)x_1(t_1) + \sum_{j=1}^{n} c_{2j}(t_1)x_2(t_1 - \tau_{1j}(t_1))e^{-b_{2j}(t_1)x_2(t_1-\tau_{1j}(t_1))} \\
&\quad -k_2(t_1)x_1(t_1)x_2(t_1) \\
&\le -\delta_2^- x_2(t_1) + a_2^+ x_1(t_1) + \sum_{j=1}^{n} \frac{c_{2j}^+}{b_{2j}^-}\frac{1}{e} \\
&\le -\delta_2^- E_{21} + a_2^+ E_{11} + \sum_{j=1}^{n} \frac{c_{2j}^+}{b_{2j}^-}\frac{1}{e} \\
&= \delta_2^-(-E_{21} + \sum_{j=1}^{n} \frac{c_{2j}^+}{\delta_2^- b_{2j}^- e} + \frac{a_2^+ E_{11}}{\delta_2^-}) \\
&< 0,
\end{aligned}$$

which is a contradiction. Together with Cases (i) and (ii), (2.11) holds for $t \in [t_0, \eta(\varphi))$.

We next show that

$$x_i(t) > E_{i2},\;\; \text{for all }\; t \in (t_0, \eta(\varphi)), i = 1, 2. \tag{2.14}$$

Suppose, for the sake of contradiction, that (2.14) does not hold. Then, there exists $t_2 \in (t_0, \eta(\varphi))$ such that one of the following two cases must occur:

$$(1)\;\; x_1(t_2) = E_{12},\;\; E_{i2} < x_i(t) < E_{i1} \;\text{ for all }\; t \in [t_0 - r_i, t_2),\; i = 1, 2; \tag{2.15}$$

$$(2)\ \ x_2(t_2) = E_{22},\ \ E_{i2} < x_i(t) < E_{i1}\ \text{ for all }\ t \in [t_0 - r_i, t_2),\ i = 1, 2. \tag{2.16}$$

If (2.15) holds, from (2.5), (2.6), (2.11) and (2.15), we get

$$E_{i2} < x_i(t) < E_{i1},\ \ b_{ij}^+ x_i(t) \ge b_{ij}^+ E_{i2} \ge b_{ij}^+ \frac{1}{\min\limits_{1\le j\le n} b_{ij}^-} \ge 1, \tag{2.17}$$

for all $t \in [t_0 - r_i, t_2), i = 1, 2,\ j = 1, 2 \cdots, n$. Calculating the derivative of $x_1(t)$, together with (2.5) and the fact that $\min\limits_{1\le u\le\kappa} ue^{-u} = \kappa e^{-\kappa}$, (1.4), (2.15) and (2.17) imply that

$$\begin{aligned}
0 &\ge x_1'(t_2)\\
&= -\delta_1(t_2)x_1(t_2) + a_1(t_2)x_2(t_2) + \sum_{j=1}^{n} c_{1j}(t_2)x_1(t_2 - \tau_{1j}(t_2))e^{-b_{1j}(t_2)x_1(t_2-\tau_{1j}(t_2))}\\
&\quad -k_1(t_2)x_1(t_2)x_2(t_2)\\
&\ge -\delta_1^+ x_1(t_2) + a_1^- x_2(t_2) + \sum_{j=1}^{n} \frac{c_{1j}(t_2)}{b_{1j}^+} b_{1j}^+ x_1(t_2 - \tau_{1j}(t_2))e^{-b_{1j}^+ x_1(t_2-\tau_{1j}(t_2))}\\
&\quad -k_1^+ x_1(t_2)x_2(t_2)\\
&> -\delta_1^+ E_{12} + a_1^- E_{22} + \sum_{j=1}^{n} c_{1j}^- E_{11} e^{-b_{1j}^+ E_{11}} - k_1^+ E_{11}E_{21}\\
&= \delta_1^+\Big(-E_{12} + \frac{a_1^-}{\delta_1^+}E_{22} + \sum_{j=1}^{n} \frac{c_{1j}^-}{\delta_1^+} E_{11} e^{-b_{1j}^+ E_{11}} - \frac{k_1^+}{\delta_1^+} E_{11}E_{21}\Big)\\
&> 0,
\end{aligned}$$

which is absurd and implies that (2.14) holds. If (2.16) holds, we can prove that (2.14) also holds in a similar way.

It follows from (2.11) and (2.14) that (2.7) is true. From Theorem 2.3.1 in [44], we easily obtain $\eta(\varphi) = +\infty$. This completes the proof.

3. Main results

Theorem 3.1. Let (2.4)–(2.6) hold. Moreover, suppose that

$$\rho(A^{-1}B) < 1. \tag{3.1}$$

where

$$A = \begin{pmatrix} \delta_1^- & 0 \\ 0 & \delta_2^- \end{pmatrix},\ \ B = \begin{pmatrix} \sum\limits_{j=1}^{n} \frac{c_{1j}^+}{e^2} + k_1^+ E_{21} & a_1^+ + k_1^+ E_{11} \\ a_2^+ + k_2^+ E_{21} & \sum\limits_{j=1}^{n} \frac{c_{2j}^+}{e^2} + k_2^+ E_{11} \end{pmatrix}.$$

Then, there exists a unique positive almost periodic solution of system (1.4) in the region $B^* = \{\varphi | \varphi \in B, E_{i2} \le \varphi_i(t) \le E_{i1},\ \text{for all}\ t \in R^1, i = 1, 2, \cdots, n\}$.

Proof. For any $\phi \in B$, we consider the following auxiliary system

$$\begin{cases} x_1'(t) &= -\delta_1(t)x_1(t) + a_1(t)\phi_2(t) + \sum\limits_{j=1}^{n} c_{1j}(t)\phi_1(t-\tau_{1j}(t))e^{-b_{1j}(t)\phi_1(t-\tau_{1j}(t))} \\ &\quad -k_1(t)\phi_1(t)\phi_2(t) \\ x_2'(t) &= -\delta_2(t)x_2(t) + a_2(t)\phi_1(t) + \sum\limits_{j=1}^{n} c_{2j}(t)\phi_2(t-\tau_{2j}(t))e^{-b_{2j}(t)\phi_2(t-\tau_{2j}(t))} \\ &\quad -k_2(t)\phi_1(t)\phi_2(t). \end{cases} \tag{3.2}$$

Since $M[\delta_i] > 0$ $(i = 1, 2)$, it follows from Lemma 2.2 that the linear system

$$x_i'(t) = -\delta_i(t)x_i(t), \qquad i = 1, 2 \tag{3.3}$$

admits an exponential dichotomy on R^1. Thus, by Lemma 2.1, we obtain that the system (3.2) has exactly one almost periodic solution $x^\phi(t) = (x_1^\phi(t), x_2^\phi(t))^T$:

$$\begin{cases} x_1^\phi(t) &= \int_{-\infty}^{t} e^{-\int_s^t \delta_1(u)du}[a_1(s)\phi_2(s) + \sum\limits_{j=1}^{n} c_{1j}(s)\phi_1(s-\tau_{1j}(s))e^{-b_{1j}(s)\phi_1(s-\tau_{1j}(s))} \\ &\quad -k_1(s)\phi_1(s)\phi_2(s)]ds \\ x_2^\phi(t) &= \int_{-\infty}^{t} e^{-\int_s^t \delta_2(u)du}[a_2(s)\phi_1(s) + \sum\limits_{j=1}^{n} c_{2j}(s)\phi_2(s-\tau_{2j}(s))e^{-b_{2j}(s)\phi_2(s-\tau_{2j}(s))} \\ &\quad -k_2(s)\phi_1(s)\phi_2(s)]ds. \end{cases} \tag{3.4}$$

Define a mapping $T : B \longrightarrow B$ by setting

$$T(\phi(t)) = x^\phi(t), \ \forall\, \phi \in B.$$

Since $B^* = \{\varphi | \varphi \in B, E_{i2} \le \varphi_i(t) \le E_{i1}, \ \text{for all} \ t \in R^1, i = 1, 2\}$, it is obvious that B^* is a closed subset of B. For $i = 1, 2$ and any $\phi \in B^*$, from (2.4), (3.4) and the fact that $\sup\limits_{u \ge 0} ue^{-u} = \frac{1}{e}$, we have

$$\begin{aligned} x_1^\phi(t) &\le \int_{-\infty}^{t} e^{-\int_s^t \delta_1(u)du}[a_1^+E_{21} + \sum_{j=1}^{n} c_{1j}(s)\frac{1}{b_{1j}(s)e}]ds \\ &\le \frac{1}{\delta_1^-}[a_1^+E_{21} + \sum_{j=1}^{n} \frac{c_{1j}^+}{b_{1j}^-e}] \\ &= \sum_{j=1}^{n} \frac{c_{1j}^+}{\delta_1^- b_{1j}^- e} + \frac{a_1^+E_{21}}{\delta_1^-} < E_{11} \qquad \text{for all} \ \ t \in R^1, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} x_2^\phi(t) &\le \int_{-\infty}^{t} e^{-\int_s^t \delta_2(u)du}[a_2^+E_{11} + \sum_{j=1}^{n} c_{2j}(s)\frac{1}{b_{2j}(s)e}]ds \\ &\le \frac{1}{\delta_2^-}[a_2^+E_{11} + \sum_{j=1}^{n} \frac{c_{2j}^+}{b_{2j}^-e}] \end{aligned}$$

$$= \sum_{j=1}^{n} \frac{c_{2j}^{+}}{\delta_2^{-} b_{2j}^{-} e} + \frac{a_2^{+} E_{11}}{\delta_2^{-}} < E_{21} \quad \text{for all } t \in R^1. \tag{3.6}$$

In view of the fact that $\min_{1 \le u \le \kappa} u e^{-u} = \kappa e^{-\kappa}$, from (2.5)–(2.7) and (3.4), we obtain

$$\begin{aligned}
x_1^{\phi}(t) &\ge \int_{-\infty}^{t} e^{-\int_s^t \delta_1(u)du} [a_1^{-} E_{22} + \sum_{j=1}^{n} c_{1j}(s) \frac{1}{b_{1j}^{+}} b_{1j}^{+} \phi_1(s - \tau_{1j}(s)) e^{-b_{1j}^{+} \phi_1(s - \tau_{1j}(s))} \\
&\quad - k_1^{+} \phi_1(s) \phi_2(s)] ds \\
&\ge \frac{1}{\delta_1^{+}} [a_1^{-} E_{22} + \sum_{j=1}^{n} c_{1j}^{-} E_{11} e^{-b_{1j}^{+} E_{11}} - k_1^{+} E_{11} E_{21}] \\
&= \frac{a_1^{-}}{\delta_1^{+}} E_{22} + \sum_{j=1}^{n} \frac{c_{1j}^{-}}{\delta_1^{+}} E_{11} e^{-b_{1j}^{+} E_{11}} - \frac{k_1^{+}}{\delta_1^{+}} E_{11} E_{21} > E_{12} \ \text{ for all } t \in R^1,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
x_2^{\phi}(t) &\ge \int_{-\infty}^{t} e^{-\int_s^t \delta_2(u)du} [a_2^{-} E_{12} + \sum_{j=1}^{n} c_{2j}(s) \frac{1}{b_{2j}^{+}} b_{2j}^{+} \phi_2(s - \tau_{2j}(s)) e^{-b_{2j}^{+} \phi_1(s - \tau_{2j}(s))} \\
&\quad - k_2^{+} \phi_1(s) \phi_2(s)] ds \\
&\ge \frac{1}{\delta_2^{+}} [a_2^{-} E_{12} + \sum_{j=1}^{n} c_{2j}^{-} E_{21} e^{-b_{2j}^{+} E_{21}} - k_2^{+} E_{11} E_{21}] \\
&= \frac{a_2^{-}}{\delta_2^{+}} E_{12} + \sum_{j=1}^{n} \frac{c_{2j}^{-}}{\delta_2^{+}} E_{21} e^{-b_{2j}^{+} E_{21}} - \frac{k_2^{+}}{\delta_2^{+}} E_{11} E_{21} > E_{22} \ \text{ for all } t \in R^1.
\end{aligned} \tag{3.8}$$

Therefore, (3.5)–(3.8) show that the mapping T is a self-mapping from B^* to B^*.

Let $\varphi, \psi \in B^*$; for $i = 1, 2$, we get

$$\begin{aligned}
&\sup_{t \in R^1} |(T(\varphi(t)) - T(\psi(t)))_1| \\
= &\sup_{t \in R^1} | \int_{-\infty}^{t} e^{-\int_s^t \delta_1(u)du} [a_1(s)(\varphi_2(s) - \psi_2(s)) \\
&+ \sum_{j=1}^{n} c_{1j}(s)(\varphi_1(s - \tau_{1j}(s)) e^{-b_{1j}(s)\varphi_1(s - \tau_{1j}(s))} - \psi_1(s - \tau_{1j}(s)) e^{-b_{1j}(s)\psi_1(s - \tau_{1j}(s))}) \\
&- k_1(s)(\varphi_1(s)\varphi_2(s) - \psi_1(s)\psi_2(s))] ds| \\
= &\sup_{t \in R^1} | \int_{-\infty}^{t} e^{-\int_s^t \delta_1(u)du} [a_1(s)(\varphi_2(s) - \psi_2(s)) + \sum_{j=1}^{n} \frac{c_{1j}(s)}{b_{1j}(s)} \times \\
&(b_{1j}(s)\varphi_1(s - \tau_{1j}(s)) e^{-b_{1j}(s)\varphi_1(s - \tau_{1j}(s))} - b_{1j}(s)\psi_1(s - \tau_{1j}(s)) e^{-b_{1j}(s)\psi_1(s - \tau_{1j}(s))}) \\
&- k_1(s)(\varphi_1(s)\varphi_2(s) - \varphi_1(s)\psi_2(s) + \varphi_1(s)\psi_2(s) - \psi_1(s)\psi_2(s))] ds|,
\end{aligned} \tag{3.9}$$

and

$$\sup_{t \in R^1} |(T(\varphi(t)) - T(\psi(t)))_2|$$

$$\begin{aligned}
&= \sup_{t\in R^1}|\int_{-\infty}^{t} e^{-\int_s^t \delta_2(u)du}[a_2(s)(\varphi_1(s)-\psi_1(s)) \\
&\quad +\sum_{j=1}^{n} c_{2j}(s)(\varphi_2(s-\tau_{2j}(s))e^{-b_{2j}(s)\varphi_2(s-\tau_{2j}(s))}-\psi_2(s-\tau_{2j}(s))e^{-b_{2j}(s)\psi_2(s-\tau_{2j}(s))}) \\
&\quad -k_2(s)(\varphi_1(s)\varphi_2(s)-\psi_1(s)\psi_2(s))]ds| \\
&= \sup_{t\in R^1}|\int_{-\infty}^{t} e^{-\int_s^t \delta_2(u)du}[a_2(s)(\varphi_1(s)-\psi_1(s))+\sum_{j=1}^{n}\frac{c_{2j}(s)}{b_{2j}(s)}\times \\
&\quad (b_{2j}(s)\varphi_2(s-\tau_{2j}(s))e^{-b_{2j}(s)\varphi_2(s-\tau_{2j}(s))}-b_{2j}(s)\psi_2(s-\tau_{2j}(s))e^{-b_{2j}(s)\psi_2(s-\tau_{2j}(s))}) \\
&\quad -k_2(s)(\varphi_1(s)\varphi_2(s)-\varphi_1(s)\psi_2(s)+\varphi_1(s)\psi_2(s)-\psi_1(s)\psi_2(s))]ds|. \qquad (3.10)
\end{aligned}$$

Since

$$b_{ij}(s)\varphi_i(s-\tau_{ij}(s)) \geq b_{ij}^{-}E_{i2} \geq b_{ij}^{-}\frac{1}{\min\limits_{1\leq j\leq n} b_{ij}^{-}} \geq 1, \quad \text{for all } \ s\in R^1, i=1,2, j=1,2,\cdots,n,$$

and

$$b_{ij}(s)\psi_i(s-\tau_{ij}(s)) \geq b_{ij}^{-}E_{i2} \geq b_{ij}^{-}\frac{1}{\min\limits_{1\leq j\leq n} b_{ij}^{-}} \geq 1, \quad \text{for all } \ s\in R^1, i=1,2, j=1,2,\cdots,n.$$

According to (1.4), (2.5), (3.5), (3.7) and (3.9), together with $\sup\limits_{u\geq 1}|\frac{1-u}{e^u}| = \frac{1}{e^2}$ and the inequality

$$\begin{aligned}
|xe^{-x}-ye^{-y}| &= |\frac{1-(x+\theta(y-x))}{e^{x+\theta(y-x)}}||x-y| \\
&\leq \frac{1}{e^2}|x-y| \ \text{ where } \ x,y\in[1,+\infty), 0<\theta<1, \qquad (3.11)
\end{aligned}$$

we have

$$\begin{aligned}
&\sup_{t\in R^1}|(T(\varphi(t))-T(\psi(t)))_1| \\
&\leq \frac{a_1^+}{\delta_1^-}\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)| \\
&\quad +\sup_{t\in R^1}\int_{-\infty}^{t} e^{-\int_s^t \delta_1(u)du}\sum_{j=1}^{n} c_{1j}^{+}\frac{1}{e^2}|\varphi_1(s-\tau_{1j}(s))-\psi_1(s-\tau_{1j}(s))|ds \\
&\quad +\sup_{t\in R^1}\int_{-\infty}^{t} e^{-\int_s^t \delta_1(u)du}k_1(s)(|\varphi_1(s)||\varphi_2(s)-\psi_2(s)|+|\psi_2(s)||\varphi_1(s)-\psi_1(s)|)ds \\
&\leq \frac{a_1^+}{\delta_1^-}\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|+\sum_{j=1}^{n}\frac{c_{1j}^+}{\delta_1^- e^2}\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)| \\
&\quad +\frac{k_1^+}{\delta_1^-}E_{11}\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|+\frac{k_1^+}{\delta_1^-}E_{21}\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|,
\end{aligned}$$

$$= \Big(\sum_{j=1}^{n}\frac{c_{1j}^{+}}{\delta_1^{-}e^{2}}+\frac{k_1^{+}}{\delta_1^{-}}E_{21}\Big)\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|+\Big(\frac{a_1^{+}}{\delta_1^{-}}+\frac{k_1^{+}}{\delta_1^{-}}E_{11}\Big)\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|. \quad (3.12)$$

Similarly, we also get

$$\begin{aligned}
&\sup_{t\in R^1}|(T(\varphi(t))-T(\psi(t)))_2|\\
\le\ & \Big(\sum_{j=1}^{n}\frac{c_{2j}^{+}}{\delta_2^{-}e^{2}}+\frac{k_2^{+}}{\delta_2^{-}}E_{11}\Big)\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|+\Big(\frac{a_2^{+}}{\delta_2^{-}}+\frac{k_2^{+}}{\delta_2^{-}}E_{21}\Big)\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|. \quad (3.13)
\end{aligned}$$

Hence

$$\begin{aligned}
&(\sup_{t\in R^1}|(T(\varphi(t))-T(\psi(t)))_1|,\sup_{t\in R^1}|(T(\varphi(t))-T(\psi(t)))_2|)^T\\
\le\ & \Big(\Big(\sum_{j=1}^{n}\frac{c_{1j}^{+}}{\delta_1^{-}e^{2}}+\frac{k_1^{+}}{\delta_1^{-}}E_{21}\Big)\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|+\Big(\frac{a_1^{+}}{\delta_1^{-}}+\frac{k_1^{+}}{\delta_1^{-}}E_{11}\Big)\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|,\\
& \Big(\sum_{j=1}^{n}\frac{c_{2j}^{+}}{\delta_2^{-}e^{2}}+\frac{k_2^{+}}{\delta_2^{-}}E_{11}\Big)\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|+\Big(\frac{a_2^{+}}{\delta_2^{-}}+\frac{k_2^{+}}{\delta_2^{-}}E_{21}\Big)\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|\Big)^T\\
=\ & F(\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|,\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|)^T\\
=\ & F(\sup_{t\in R^1}|(\varphi(t)-\psi(t))_1|,\sup_{t\in R^1}|(\varphi(t)-\psi(t))_2|)^T, \quad (3.14)
\end{aligned}$$

where $F = A^{-1}B$. Let μ be a positive integer. Then, from (3.14) we get

$$\begin{aligned}
&(\sup_{t\in R^1}|(T^{\mu}(\varphi(t))-T^{\mu}(\psi(t)))_1|,\sup_{t\in R^1}|(T^{\mu}(\varphi(t))-T^{\mu}(\psi(t)))_2|)^T\\
=\ & (\sup_{t\in R^1}|(T(T^{\mu-1}(\varphi(t)))-T(T^{\mu-1}(\psi(t))))_1|,\\
& \sup_{t\in R^1}|(T(T^{\mu-1}(\varphi(t)))-T(T^{\mu-1}(\psi(t))))_2|)^T\\
\le\ & F(\sup_{t\in R^1}|(T^{\mu-1}(\varphi(t))-T^{\mu-1}(\psi(t)))_1|,\sup_{t\in R^1}|(T^{\mu-1}(\varphi(t))-T^{\mu-1}(\psi(t)))_2|)^T\\
& \vdots\\
\le\ & F^{\mu}(\sup_{t\in R^1}|(\varphi(t)-\psi(t))_1|,\sup_{t\in R^1}|(\varphi(t)-\psi(t))_2|)^T\\
=\ & F^{\mu}(\sup_{t\in R^1}|\varphi_1(t)-\psi_1(t)|,\sup_{t\in R^1}|\varphi_2(t)-\psi_2(t)|)^T. \quad (3.15)
\end{aligned}$$

Since $\rho(F) < 1$, we obtain

$$\lim_{\mu\to+\infty}F^{\mu}=0,$$

which implies that there exist a positive integer N and a positive constant $r < 1$ such that

$$F^N = (A^{-1}B)^N = (g_{ij})_{2\times 2} \ \text{ and } \ \sum_{j=1}^{2} g_{ij} \le r, i = 1, 2. \quad (3.16)$$

In view of (3.15) and (3.16), we have

$$\begin{aligned}|(T^N(\varphi(t)) - T^N(\psi(t)))_i| &\le \sup_{t\in R^1} |(T^N(\varphi(t)) - T^N(\psi(t)))_i| \\ &\le \sum_{j=1}^{2} g_{ij} \sup_{t\in R^1} |\varphi_j(t) - \psi_j(t)| \\ &\le \sup_{t\in R^1} \max_{1\le j\le 2} |\varphi_j(t) - \psi_j(t)| \sum_{j=1}^{2} g_{ij} \\ &\le r\|\varphi(t) - \psi(t)\|_B,\end{aligned}$$

for all $t \in R, i = 1, 2$. It follows that

$$\|T^N(\varphi(t)) - T^N(\psi(t))\|_B = \sup_{t\in R^1} \max_{1\le i\le 2} |(T^N(\varphi(t)) - T^N(\psi(t)))_i| \le r\|\varphi(t) - \psi(t)\|_B. \tag{3.17}$$

This implies that the mapping $T^N : B^* \to B^*$ is a contraction mapping.

By the fixed point theorem for Banach space, T possesses a unique fixed point $\varphi^* \in B^*$ such that $T\varphi^* = \varphi^*$. By (3.2), φ^* satisfies (1.4). So φ^* is an almost periodic solution of (1.4) in B^*. The proof of Theorem 3.1 is now completed.

Theorem 3.2. Let $x^*(t)$ be the positive almost periodic solution of system (1.4) in the region B^*. Suppose that (2.4)–(2.6) and (3.1) hold. Then, the solution $x(t; t_0, \varphi)$ of (1.4) with $\varphi \in C^0$ converges exponentially to $x^*(t)$ as $t \to +\infty$.

Proof. Since $\rho(A^{-1}B) < 1$, it follows from Theorem 3.1 that system (1.4) has a unique almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ in the region B^*. Set $x(t) = x(t; t_0, \varphi)$, $x^*(t) = x^*(t; t_0, \varphi^*)$ and $y_i(t) = x_i(t) - x_i^*(t)$, where $\varphi, \varphi^* \in C^0$, $t \in [t_0 - r_i, +\infty)$, $i = 1, 2$. Then, we have the following:

$$\begin{cases} y_1'(t) = -\delta_1(t)y_1(t) + a_1(t)y_2(t) + \sum_{j=1}^{n} c_{1j}(t)(x_1(t-\tau_{1j}(t))e^{-b_{1j}(t)x_1(t-\tau_{1j}(t))} \\ \qquad -x_1^*(t-\tau_{1j}(t))e^{-b_{1j}(t)x_1^*(t-\tau_{1j}(t))}) - k_1(t)(x_1(t)x_2(t) - x_1^*(t)x_2^*(t)), \\ y_2'(t) = -\delta_2(t)y_2(t) + a_2(t)y_1(t) + \sum_{j=1}^{n} c_{2j}(t)(x_2(t-\tau_{2j}(t))e^{-b_{2j}(t)x_2(t-\tau_{2j}(t))} \\ \qquad -x_2^*(t-\tau_{2j}(t))e^{-b_{2j}(t)x_2^*(t-\tau_{2j}(t))}) - k_2(t)(x_1(t)x_2(t) - x_1^*(t)x_2^*(t)). \end{cases} \tag{3.18}$$

Again from $\rho(A^{-1}B) < 1$, it follows from Lemma 2.3 that $I_2 - A^{-1}B$ is an M-matrix; we obtain that there exists a constant $\bar{\mu} > 0$ and a vector $\xi = (\xi_1, \xi_2)^T > (0, 0)^T$ such that

$$(I_2 - A^{-1}B)\xi > (\bar{\mu}, \bar{\mu})^T.$$

Therefore,

$$\begin{cases} \xi_1 - (\sum_{j=1}^{n} \frac{c_{1j}^+}{\delta_1^- e^2} + \frac{k_1^+}{\delta_1^-}E_{21})\xi_1 - (\frac{a_1^+}{\delta_1^-} + \frac{k_1^+}{\delta_1^-}E_{11})\xi_2 > \overline{\mu}, \\ \xi_2 - (\sum_{j=1}^{n} \frac{c_{2j}^+}{\delta_2^- e^2} + \frac{k_2^+}{\delta_2^-}E_{11})\xi_2 - (\frac{a_2^+}{\delta_2^-} + \frac{k_2^+}{\delta_2^-}E_{21})\xi_1 > \overline{\mu}, \end{cases}$$

which implies that

$$\begin{cases} (-\delta_1^- + \sum_{j=1}^{n} \frac{c_{1j}^+}{e^2} + k_1^+E_{21})\xi_1 + (a_1^+ + k_1^+E_{11})\xi_2 < -\delta_1^-\overline{\mu}, \\ (-\delta_2^- + \sum_{j=1}^{n} \frac{c_{2j}^+}{e^2} + k_2^+E_{11})\xi_2 + (a_2^+ + k_2^+E_{21})\xi_1 < -\delta_2^-\overline{\mu}. \end{cases} \tag{3.19}$$

We can choose a positive constant $\eta < 1$ such that

$$\begin{cases} \eta\xi_1 + (-\delta_1^- + \sum_{j=1}^{n} \frac{c_{1j}^+}{e^2} e^{\eta r_1} + k_1^+ E_{21})\xi_1 + (a_1^+ + k_1^+ E_{11})\xi_2 < 0, \\ \eta\xi_2 + (-\delta_2^- + \sum_{j=1}^{n} \frac{c_{2j}^+}{e^2} e^{\eta r_2} + k_2^+ E_{11})\xi_2 + (a_2^+ + k_2^+ E_{21})\xi_1 < 0. \end{cases} \tag{3.20}$$

In the sequel, we consider the following Lyapunov function:

$$V_i(t) = |y_i(t)| e^{\eta(t-t_0)}, \quad i = 1, 2. \tag{3.21}$$

In view of (2.5)–(2.7), for $i \in \{1, 2\}$ and $j \in \{1, 2, \cdots, n\}$, we obtain

$$b_{ij}(t)x_i(t - \tau_{ij}(t)) \geq b_{ij}^- E_{i2} \geq b_{ij}^- \frac{1}{\min\limits_{1\leq j\leq m} b_{ij}^-} \geq 1, \text{ for all } t \in [t_0 - r_i, +\infty),$$

and

$$b_{ij}(t)x_i^*(t - \tau_{ij}(t)) \geq b_{ij}^- E_{i2} \geq b_{ij}^- \frac{1}{\min\limits_{1\leq j\leq m} b_{ij}^-} \geq 1, \quad \text{for all } t \in R^1,$$

which, together with (3.11) and (3.18), imply that

$$\begin{aligned} D^- V_1(t) \leq\ & -\delta_1(t)|y_1(t)|e^{\eta(t-t_0)} + a_1(t)|y_2(t)|e^{\eta(t-t_0)} + \sum_{j=1}^{n} c_{1j}(t)e^{\eta(t-t_0)} \times \\ & |x_1(t - \tau_{1j}(t))e^{-b_{1j}(t)x_1(t-\tau_{1j}(t))} - x_1^*(t - \tau_{1j}(t))e^{-b_{1j}(t)x_1^*(t-\tau_{1j}(t))}| \\ & + k_1(t)e^{\eta(t-t_0)}|x_1(t)x_2(t) - x_1^*(t)x_2^*(t)| + \eta|y_1(t)|e^{\eta(t-t_0)} \\ \leq\ & (\eta - \delta_1(t))V_1(t) + a_1^+ V_2(t) + \sum_{j=1}^{n} \frac{c_{1j}(t)}{b_{1j}(t)} e^{\eta(t-t_0)} \times \\ & |b_{1j}(t)x_1(t - \tau_{1j}(t))e^{-b_{1j}(t)x_1(t-\tau_{1j}(t))} \\ & - b_{1j}(t)x_1^*(t - \tau_{1j}(t))e^{-b_{1j}(t)x_1^*(t-\tau_{1j}(t))}| \\ & + k_1(t)|x_1(t)||x_2(t) - x_2^*(t)|e^{\eta(t-t_0)} + k_1(t)|x_2^*(t)||x_1(t) - x_1^*(t)|e^{\eta(t-t_0)} \\ \leq\ & (\eta - \delta_1^-)V_1(t) + a_1^+ V_2(t) + \sum_{j=1}^{n} \frac{c_{1j}(t)}{e^2} |y_1(t - \tau_{1j}(t))| e^{\eta(t-t_0)} \\ & + k_1^+ E_{11} V_2(t) + k_1^+ E_{21} V_1(t) \\ \leq\ & (\eta - \delta_1^- + k_1^+ E_{21})V_1(t) + \sum_{j=1}^{n} \frac{c_{1j}^+}{e^2} e^{\eta r_1} V_1(t - \tau_{1j}(t)) \\ & + (a_1^+ + k_1^+ E_{11})V_2(t), \end{aligned} \tag{3.22}$$

and

$$
\begin{aligned}
D^{-}V_2(t) \;\le\; & -\delta_2(t)|y_2(t)|e^{\eta(t-t_0)} + a_2(t)|y_1(t)|e^{\eta(t-t_0)} + \sum_{j=1}^{n} c_{2j}(t)e^{\eta(t-t_0)}\times \\
& |x_2(t-\tau_{2j}(t))e^{-b_{2j}(t)x_2(t-\tau_{2j}(t))} - x_2^*(t-\tau_{2j}(t))e^{-b_{2j}(t)x_2^*(t-\tau_{2j}(t))}| \\
& +k_2(t)e^{\eta(t-t_0)}|x_1(t)x_2(t) - x_1^*(t)x_2^*(t)| + \eta|y_2(t)|e^{\eta(t-t_0)} \\
\le\; & (\eta - \delta_2(t))V_2(t) + a_2^+V_1(t) + \sum_{j=1}^{n} \frac{c_{2j}(t)}{b_{2j}(t)}e^{\eta(t-t_0)}\times \\
& |b_{2j}(t)x_2(t-\tau_{2j}(t))e^{-b_{2j}(t)x_2(t-\tau_{2j}(t))} \\
& -b_{2j}(t)x_2^*(t-\tau_{2j}(t))e^{-b_{2j}(t)x_2^*(t-\tau_{2j}(t))}| \\
& +k_2(t)|x_1(t)||x_2(t) - x_2^*(t)|e^{\eta(t-t_0)} + k_2(t)|x_2^*(t)||x_1(t) - x_1^*(t)|e^{\eta(t-t_0)} \\
\le\; & (\eta - \delta_2^-)V_2(t) + a_2^+V_1(t) + \sum_{j=1}^{n} \frac{c_{2j}(t)}{e^2}|y_2(t-\tau_{2j}(t))|e^{\eta(t-t_0)} \\
& +k_2^+E_{11}V_2(t) + k_2^+E_{21}V_1(t) \\
\le\; & (\eta - \delta_2^- + k_2^+E_{11})V_2(t) + \sum_{j=1}^{n} \frac{c_{2j}^+}{e^2}e^{\eta r_2}V_2(t-\tau_{2j}(t)) \\
& +(a_2^+ + k_2^+E_{21})V_1(t). \qquad (3.23)
\end{aligned}
$$

Let $\varsigma > 1$ denote an arbitrary real number such that

$$
\varsigma\xi_i > \|\varphi - \varphi^*\| = \sup_{-r_i\le s\le 0}\max_{1\le i\le 2}|\varphi_i(s) - \varphi_i^*(s)| > 0,\ i = 1,2.
$$

It follows from (3.21) that

$$
V_i(t) = |y_i(t)|e^{\eta(t-t_0)} < \varsigma\xi_i,\ \ \text{for all } t \in [t_0 - r_i, t_0],\ i = 1,2.
$$

We claim that

$$
V_i(t) = |y_i(t)|e^{\eta(t-t_0)} < \varsigma\xi_i,\ \ \text{for all } t > t_0,\ i = 1,2. \qquad (3.24)
$$

In contrast, there must exist $i \in \{1,2\}$ and $r^* > t_0$ such that

$$
V_i(r^*) = \varsigma\xi_i,\ \text{and } V_j(t) < \varsigma\xi_j,\ \ \text{for all } t \in [t_0 - r_j, r^*),\ j = 1,2. \qquad (3.25)
$$

Thus,

$$
V_1(r^*) - \varsigma\xi_1 = 0,\ \text{and } V_j(t) - \varsigma\xi_j < 0,\ \ \text{for all } t \in [t_0 - r_j, r^*),\ j = 1,2, \qquad (3.26)
$$

or

$$
V_2(r^*) - \varsigma\xi_2 = 0,\ \text{and } V_j(t) - \varsigma\xi_j < 0,\ \ \text{for all } t \in [t_0 - r_j, r^*),\ j = 1,2. \qquad (3.27)
$$

Together with (3.20), (3.22), (3.23), (3.26) and (3.27), we obtain

$$
\begin{aligned}
0 \;\le\; & D^{-}(V_1(r^*) - \varsigma\xi_1) \\
=\; & D^{-}(V_1(r^*)) \\
\le\; & (\eta - \delta_1^- + k_1^+E_{21})V_1(r^*) + \sum_{j=1}^{n} \frac{c_{1j}^+}{e^2}e^{\eta r_1}V_1(r^* - \tau_{1j}(r^*)) \\
& +(a_1^+ + k_1^+E_{11})V_2(r^*) \\
\le\; & \varsigma[\eta\xi_1 + (-\delta_1^- + \sum_{j=1}^{n} \frac{c_{1j}^+}{e^2}e^{\eta r_1} + k_1^+E_{21})\xi_1 + (a_1^+ + k_1^+E_{11})\xi_2] \\
<\; & 0, \qquad (3.28)
\end{aligned}
$$

or

$$
\begin{aligned}
0 &\leq D^-(V_2(r^*) - \varsigma\xi_2) \\
&= D^-(V_2(r^*)) \\
&\leq (\eta - \delta_2^- + k_2^+ E_{11}) V_2(r^*) + \sum_{j=1}^{n} \frac{c_{2j}^+}{e^2} e^{\eta r_2} V_2(r^* - \tau_{2j}(r^*)) \\
&\quad + (a_2^+ + k_2^+ E_{21}) V_1(r^*) \\
&\leq \varsigma[\eta\xi_2 + (-\delta_2^- + \sum_{j=1}^{n} \frac{c_{2j}^+}{e^2} e^{\eta r_2} + k_2^+ E_{11})\xi_2 + (a_2^+ + k_2^+ E_{21})\xi_1] \\
&< 0, \qquad (3.29)
\end{aligned}
$$

which are both contradictory. Hence, (3.24) holds. Let $M > 1$ such that

$$\varsigma\xi_i \leq M\|\varphi - \varphi^*\|, \ i = 1, 2. \qquad (3.30)$$

In view of (3.24) and (3.30), we get

$$|x_i(t) - x_i^*(t)| = |y_i(t)| \leq \varsigma\xi_i e^{-\eta(t-t_0)} \leq M\|\varphi - \varphi^*\| e^{-\eta(t-t_0)}, \ \text{ for all } \ t > t_0 \ \ i = 1, 2.$$

This completes the proof.

Corollary 3.1. Let (2.4)–(2.6) hold. Suppose that $I_2 - A^{-1}B$ is an M-matrix. Then system (1.4) has exactly one almost periodic solution $x^*(t)$. Moreover, the solution $x(t; t_0, \varphi)$ of (1.4) with $\varphi \in C^0$ converges exponentially to $x^*(t)$ as $t \to +\infty$.

Proof. Since $I_2 - A^{-1}B$ is an M-matrix, it follows that there exists a vector $d = (d_1, d_2)^T > (0, 0)^T$ such that

$$(I_2 - A^{-1}B)d > 0, \qquad (3.31)$$

hence

$$
\begin{cases}
-\delta_1^- d_1 + (\sum_{j=1}^{n} \frac{c_{1j}^+}{e^2} + k_1^+ E_{21}) d_1 + (a_1^+ + k_1^+ E_{11}) d_2 < 0 \\
-\delta_2^- d_2 + (\sum_{j=1}^{n} \frac{c_{2j}^+}{e^2} + k_2^+ E_{11}) d_2 + (a_2^+ + k_2^+ E_{21}) d_1 < 0.
\end{cases} \qquad (3.32)
$$

For any matrix norm $\|\cdot\|$ and nonsingular matrix D, $\|A\|_D = \|D^{-1}AD\|$ also defines a matrix norm. Let $D = diag(d_1, d_2)$. Then (3.32) implies that the row norm of matrix $D^{-1}A^{-1}BD$ is less than 1. Hence $\rho(A^{-1}B) < 1$. Corollary 3.1 follows immediately from Theorems 3.1 and 3.2.

4. Example and numerical simulation

In this section, we give an example and present a numerical simulation to demonstrate the results obtained in previous sections.

Example 4.1. Consider the following competitive and cooperative Nicholson's blowfly system:

$$\begin{cases} x_1'(t) &= -(18+\cos^2\sqrt{5}t)x_1(t)+(1+0.7\sin^2 t)e^{e-2}x_2(t) \\ &\quad +e^{e-1}(9.5+0.005|\sin\sqrt{2}t|)x_1(t-e^{|\sin t|+|\sin\sqrt{2}t|})e^{-x_1(t-e^{|\sin t|+|\sin\sqrt{2}t|})}, \\ &\quad +e^{e-1}(9.5+0.005|\sin\sqrt{5}t|)x_1(t-e^{|\cos\sqrt{3}t|+|\cos t|})e^{-x_1(t-e^{|\cos\sqrt{3}t|+|\cos t|})} \\ &\quad -0.1e^{-2}\cos^2 t x_1(t)x_2(t) \\ x_2'(t) &= -(18+\sin^2\sqrt{5}t)x_2(t)+(1+0.7\cos^2 t)e^{e-2}x_1(t) \\ &\quad +e^{e-1}(9.5+0.005|\cos\sqrt{2}t|)x_2(t-e^{|\cos t|+|\cos\sqrt{7}t|})e^{-x_2(t-e^{|\cos t|+|\cos\sqrt{7}t|})}, \\ &\quad +e^{e-1}(9.5+0.005|\sin\sqrt{6}t|)x_2(t-e^{|\cos\sqrt{7}t|+|\cos\sqrt{3}t|})e^{-x_2(t-e^{|\cos\sqrt{7}t|+|\cos\sqrt{3}t|})} \\ &\quad -0.1e^{-2}\sin^2 t x_1(t)x_2(t). \end{cases} \tag{4.1}$$

Obviously, $\delta_i^- = 18, \delta_i^+ = 19, a_i^- = e^{e-2}, a_i^+ = 1.7e^{e-2}, b_{ij}^- = b_{ij}^+ = 1,\ c_{ij}^- = 9.5e^{e-1},\ c_{ij}^+ = 9.505e^{e-1}, k_i^+ = 0.1e^{-2}$ and $r_i = e^2$ $(i, j = 1, 2)$. Let $E_{i1} = e$ and $E_{i2} = 1$ for $i = 1, 2$; we obtain

$$A = \begin{pmatrix} 18 & 0 \\ 0 & 18 \end{pmatrix}, B = \begin{pmatrix} 19.01e^{e-3}+0.1e^{-1} & 1.7e^{e-2}+0.1e^{-1} \\ 1.7e^{e-2}+0.1e^{-1} & 19.01e^{e-3}+0.1e^{-1} \end{pmatrix}$$

$$\sum_{j=1}^{2}\frac{c_{1j}^+}{\delta_1^- b_{1j}^- e}+\frac{a_1^+E_{21}}{\delta_1^-} = \frac{19.01e^{e-2}+1.7e^{e-1}}{18} < e \tag{4.2}$$

$$\sum_{j=1}^{n}\frac{c_{2j}^+}{\delta_2^- b_{2j}^- e}+\frac{a_2^+E_{11}}{\delta_2^-} = \frac{19.01e^{e-2}+1.7e^{e-1}}{18} < e \tag{4.3}$$

$$\frac{a_1^-}{\delta_1^+}E_{22}+\sum_{j=1}^{n}\frac{c_{1j}^-}{\delta_1^+}E_{11}e^{-b_{1j}^+E_{11}}-\frac{k_1^+}{\delta_1^+}E_{11}E_{21} = \frac{19+e^{e-2}-0.1}{19} > 1 \tag{4.4}$$

$$\frac{a_2^-}{\delta_2^+}E_{12}+\sum_{j=1}^{n}\frac{c_{2j}^-}{\delta_2^+}E_{21}e^{-b_{2j}^+E_{21}}-\frac{k_2^+}{\delta_2^+}E_{11}E_{21} = \frac{19+e^{e-2}-0.1}{19} > 1 \tag{4.5}$$

$$\rho(A^{-1}B) \approx 0.9946 < 1. \tag{4.6}$$

Then (4.2)–(4.6) imply that the competitive and cooperative Nicholson's blowfly system (4.1) satisfies (2.4)–(2.6) and (3.1). Hence, from Theorems 3.1 and 3.2, system (4.1) has a positive almost periodic solution

$$x^*(t) \in B^* = \{\varphi|\varphi \in B, 1 \le \varphi_i(t) \le e, \text{ for all } t \in R, i = 1, 2\}.$$

Moreover, if $\varphi \in C^0 = \{\varphi|\varphi \in C, 1 < \varphi_i(t) < e, \text{ for all } t \in [-e^2, 0], i = 1, 2\}$, then $x(t; t_0, \varphi)$ converges exponentially to $x^*(t)$ as $t \to +\infty$. The fact is verified by the numerical simulation illustrated in Figure 1.

Remark 4.1. To the best of our knowledge, few authors have considered the problems related to positive almost periodic solutions of competitive and cooperative Nicholson's blowfly systems. Therefore, the main results in [31, 32] and the references therein can not be appled to prove that all solutions of (4.1) with initial the value $\varphi \in C^0$ converge exponentially to the positive almost periodic solution. This implies that the results in this paper are new and this complements previously obtained results.

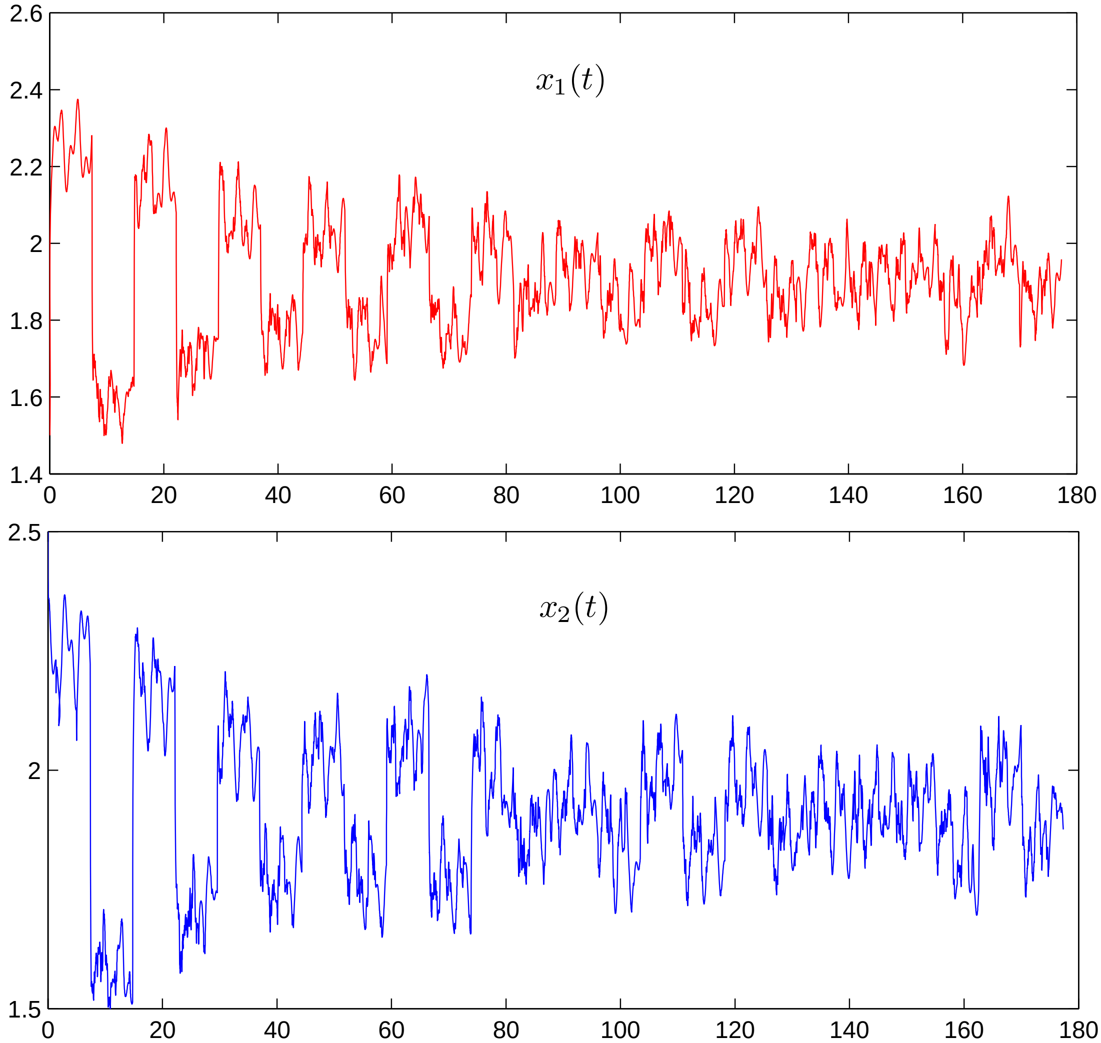

Figure 1. Numerical solution $x(t) = (x_1(t), x_2(t))^T$ of system (4.1) for the initial value $\varphi(t) \equiv (1.5, 2.5)^T$.

5. Conclusions

This article investigated a class of competitive and cooperative Nicholson's blowfly system. Unlike what has been done for some known cooperative Nicholson's blowfly systems [31,32], we have introduced the competitive terms to describe two distinct blowfly populations that compete with each other. By constructing invariant sets and applying the fixed point theorem, we derived some sufficient conditions to ensure that the addressed system has a unique exponential stable positive almost periodic solution. Inspired by the latest Nicholson's blowfly models [3–10], our future works will be devoted to competitive and cooperative Nicholson's blowfly systems involving distinct delays, distributed delays and mixed delays.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by Natural Scientific Research Fund of Zhejiang Provincial of China (grant nos. LY18A010019, LY16A010018).

Conflict of interest

The author declares no conflicts of interest.

References

1. A. J. Nicholson, An outline of the dynamics of animal populations, *Aust. J. Zool.,* **2** (1954), 9–65. https://doi.org/10.1071/ZO9540009
2. W. S. C. Gurney, S. P. Blythe, R. M. Nisbet, Nicholsons blowflies revisited, *Nature,* **287** (1980), 17–21. https://doi.org/10.1038/287017a0
3. T. Faria, Periodic solutions for a non-monotone family of delayed differential equations with applications to Nicholson systems, *J. Differ. Equations,* **263** (2017), 509–533. https://doi.org/10.1016/j.jde.2017.02.042
4. C. Huang, J. Wu, Global exponential stability analysis for tick population model with delayed sub-linear Gamma-Ricker nonlinearities, *J. Differ. Equations,* **355** (2023), 369–385. https://doi.org/10.1016/j.jde.2022.12.025
5. C. Huang, X. Ding, Dynamics of the diffusive Nicholson's blowflies equation with two distinct distributed delays, *Appl. Math. Lett.,* **145** (2023), 108741. https://doi.org/10.1016/j.aml.2023.108741
6. X. Long, S. Gong, New results on stability of Nicholson's blowflies equation with multiple pairs of time-varying delays, *Appl. Math. Lett.,* **100** (2020), 10627. https://doi.org/10.1016/j.aml.2019.106027
7. F. Chérif, Pseudo almost periodic solution of Nicholson's blowflies model with mixed delays, *Appl. Math. Model.,* **39** (2015), 10627. http://dx.doi.org/10.1016/j.apm.2015.03.043
8. X. Zhao, C. Huang, B. Liu, J. Cao, Stability analysis of delay patch-constructed Nicholson's blowflies system, *Math. Comput. Simulat.,* 2024, 10627. (in press). https://doi.org/10.1016/j.matcom.2023.09.012
9. C. Huang, B. Liu, Traveling wave fronts for a diffusive Nicholson's Blowflies equation accompanying mature delay and feedback delay, *Appl. Math. Lett.,* **134** (2022), 108321. https://doi.org/10.1016/j.aml.2022.108321

10. J. Sugie, Y. Yan, M. Qu, Effect of decimation on positive periodic solutions of discrete generalized Nicholson's blowflies models with multiple time-varying delays, *Commun. Nonlinear Sci. Numer. Simulat.,* **97** (2021), 105731. https://doi.org/10.1016/j.cnsns.2021.105731

11. W. Chen, B. Liu, Positive almost periodic solution for a class of Nicholson's blowflies model with multiple time-varying delays, *J. Comput. Appl. Math.,* **235** (2011), 2090–2097. https://doi.org/10.1016/j.cam.2010.10.007

12. W. Xiong, New results on positive pseudo-almost periodic solutions for a delayed Nicholson's blowflies model, *Nonlinear Dynam.,* **85** (2016), 653–571. https://doi.org/10.1007/s11071-016-2706-4

13. C. Xu, M. Liao, P. Li, Q. Xiao, S. Yuan, A new method to investigate almost periodic solutions for an Nicholson's blowflies model with time-varying delays and a linear harvesting term, *Math. Biosci. Eng.,* **16** (2019), 3830–3840. https://doi.org/10.3934/mbe.2019189

14. C. Huang, H. Zhang, L. Huang, Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term, *Commun. Pure Appl. Anal.,* **18** (2019), 3337–3349. https://doi.org/10.3934/cpaa.2019150

15. W. Chen, W. Wang, Almost periodic solutions for a delayed Nicholson's blowflies system with nonlinear density-dependent mortality terms and patch structure, *Adv. Differ. Equ.,* **205** (2014). https://doi.org/10.1186/1687-1847-2014-205

16. C. Huang, B. Liu, H. Yang, J. Cao, Positive almost periodicity on SICNNs incorporating mixed delays and D operator, *Nonlinear Anal.-Model.,* **27** (2022), 719–739. https://doi.org/10.15388/namc.2022.27.27417

17. C. Huang, X. Ding, Existence of traveling wave fronts for a diffusive Mackey-Glass model with two delays, *Nonlinear Anal.-Real,* **76** (2024), 104024. https://doi.org/10.1016/j.nonrwa.2023.104024

18. Y. Chen, Periodic solutions of delayed periodic Nicholson's blowflies models, *Can. Appl. Math. Q.,* **11** (2003), 23–28.

19. E. Braverman, D. Kinzebulatov, Nicholson's blowflies equation with a distributed delay, *Can. Appl. Math. Q.,* **14** (2006), 107–128.

20. J. Li, C. Du, Existence of positive periodic solutions for a generalized Nicholson blowflies model, *J. Comput. Appl. Math.,* **221** (2008), 226–233. https://doi.org/10.1016/j.cam.2007.10.049

21. W. T. Li, Y. H. Fan, Existence and global attractivity of positive periodic solutions for the impulsive delay Nicholson's blowflies model, *J. Comput. Appl. Math.,* **201** (2007), 55–68. https://doi.org/10.1016/j.cam.2006.02.001

22. L. Berezansky, E. Braverman, L. Idels, Nicholson's blowflies differential equations revisited: Main results and open problems, *Appl. Math. Model.,* **34** (2010), 1405–1417. https://doi.org/10.1016/j.apm.2009.08.027

23. W. Zhao, C. Zhu, H. Zhu, On positive periodic solution for the delay Nicholson's blowflies model with a harvesting term, *Appl. Math. Model.,* **36** (2012), 3336–3340. https://doi.org/10.1016/j.apm.2011.10.011

24. H. Zhou, Z. Zhou, Z. Qiao, Mean-square almost periodic solution for impulsive stochastic Nicholson's blowflies model with delays, *Appl. Math. Comput.,* **219** (2013), 5943–5948. https://doi.org/10.1016/j.amc.2012.12.005

25. P. Amster, A. Déboli, Existence of positive T-periodic solutions of a generalized Nicholson's blowflies model with a nonlinear harvesting term, *Appl. Math. Lett.,* **25** (2012), 1203–1207. https://doi.org/10.1016/j.aml.2012.02.040

26. F. Long, Positive almost periodic solution for a class of Nicholson's blowflies model with a linear harvesting term, *Nonlinear Anal.-Real,* **13** (2012), 686–693. https://doi.org/10.1016/j.nonrwa.2011.08.009

27. J. O. Alzabut, Almost periodic solutions for an impulsive delay Nicholson's blowflies model, *J. Comput. Appl. Math.,* **234** (2010), 233–239. https://doi.org/10.1016/j.cam.2009.12.019

28. W. Wang, Positive periodic solutions of delayed Nicholson's blowflies models with a nonlinear density-dependent mortality term, *Appl. Math. Model.,* **36** (2012), 4708–4713. https://doi.org/10.1016/j.apm.2011.12.001

29. W. Chen, L. Wang, Positive periodic solutions of Nicholson-type delay systems with nonlinear density-dependent mortality terms, *Abstr. Appl. Anal.,* **2012** (2012), 843178. https://doi.org/10.1155/2012/843178

30. W. Wang, Exponential extinction of Nicholson's blowflies system with nonlinear density-dependent mortality terms, *Abstr. Appl. Anal.,* **2012** (2012), 302065. https://doi.org/10.1155/2012/302065

31. B. Liu, Global stability of a class of delay differential systems, *J. Comput. Appl. Math.,* **233** (2009), 217–223. https://doi.org/10.1016/j.cam.2009.07.024

32. L. Berezansky, L. Idels, L. Troib, Global dynamics of Nicholson-type delay systems with applications, *Nonlinear Anal.-Real,* **12** (2011), 436–445. https://doi.org/10.1016/j.nonrwa.2010.06.028

33. M. Sotomayor, Connecting the cooperative and competitive structures of the multiple-partners assignment game, *J. Econom. Theory,* **134** (2007), 155–174. https://doi.org/10.1016/j.jet.2006.02.005

34. K. Yuan, J. Cao, Periodic oscillatory solution in delayed competitive-cooperative neural networks: A decomposition approach, *Chaos Soliton. Fract.,* **27** (2006), 223–231. https://doi.org/10.1016/j.chaos.2005.04.016

35. G. Lv, M. Wang, Traveling wave front in diffusive and competitive Lotka-Volterra system with delays, *Nonlinear Anal.-Real,* **11** (2010), 1323–1329. https://doi.org/10.1016/j.nonrwa.2009.02.020

36. T. Chu, Z. Zhang, Z. Wang, A decomposition approach to analysis of competitive cooperative neural networks with delay, *Phys. Lett. A,* **312** (2003), 339–347. https://doi.org/10.1016/S0375-9601(03)00692-3

37. P. Liu, S. N. Elaydi, Discrete competitive and cooperative models of Lotka-Volterra type, *J. Comput. Anal. Appl.,* **3** (2001), 53–73. https://doi.org/10.1023/A:1011539901001

38. L. Huang, T. Yi, X. Zou, On dynamic of generalized competitive and cooperative systems, *Tohoku Math. J.,* **58** (2006), 89–100. https://doi.org/10.2748/tmj/1145390207
39. A. M. Fink, *Almost periodic differential equations, lecture notes in mathematics*, Berlin: Springer, **377** (1974).
40. C. Y. He, *Almost periodic differential equation*, Beijing: Higher Education Publishing House, 1992, (in Chinese).
41. A. Berman, R. J. Plemmons, *Nonnegative matrices in the mathematical science*, New York: Academic Press, 1979.
42. J. P. Lasalle, *The stability of dynamical system*, Philadelphia: SIAM, 1976.
43. H. L. Smith, *The Monotone dynamical systems, in: Math. Surveys Monogr*, Providence: American Mathematical Society, 1995.
44. J. K. Hale, S. M. V. Lunel, *Introduction to functional differential equations*, New York: Springer-Verlag, 1993.

AIMS Press