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Research article

A mathematical fractional model of waves on Shallow water surfaces: The Korteweg-de Vries equation

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Abstract: The homotopy perturbation transform method was examined in the present research to address the nonlinear time-fractional Korteweg-de Vries equations using a nonsingular kernel fractional derivative that Caputo-Fabrizio recently developed. We devoted our research to the nonlinear time-fractional Korteweg-de Vries equation and certain associated phenomena because of some physical applications of this equation. The results are significant and necessary for illuminating a range of physical processes. This paper considered an innovative method and fractional operator in this context to obtain satisfactory approximations to the provided issues. To solve nonlinear time-fractional Korteweg-de Vries equations, we first considered the Yang transform of the Caputo-Fabrizio fractional derivative. In order to confirm the applicability and efficacy of the provided method, we took into consideration two cases of the nonlinear time-fractional Korteweg-de Vries equation. He's polynomials were useful in order to manage nonlinear terms. In this method, the outcome was calculated as a convergent series, and it was demonstrated that the homotopy perturbation transform method solutions converge to the exact solutions. The main benefit of the suggested method was that it offered solutions with a high degree of precision while requiring minimal computation. Graphs were also used to illustrate the series solution for a certain non-integer orders. Finally, a comparison of both examples outcomes were examined using diagrams and numerical data. These graphs showed how the approximated solution's graph and the precise solution's graph eventually converged as the non-integer order gets closer to integer order. When $\varsigma = 1$, several numerical comparisons were conducted with the exact solutions. The numerical simulation was offered to illustrate the efficiency and reliability of the proposed approach. In addition, the behavior of the provided solutions was explained using a number of fractional orders. The theoretical analysis matched with the findings obtained using the current technique, and the suggested technique can be extended to tackle many higher-order nonlinear dynamics problems.

Keywords: Caputo-Fabrizio operator; Yang transform; homotopy perturbation transform method
Mathematics Subject Classification: 26A33, 34A34, 35A22

1. Introduction

A 300-year-old branch of mathematics recognized as fractional calculus (FC) was created in 1823 by Euler, Abel, and Liouville after being defined as "the generalization of the ordinary derivative to non-integer values" by Rieman and Liouville in the nineteenth century. It was found that many applications, especially multidisciplinary ones, can be precisely modeled using fractional derivatives [1–4]. Numerous people in the areas of engineering and the natural sciences are interested in learning fractional calculus because of its significant applications, such as those found in electrodynamics [1, 5], nanotechnology [6], biotechnology [7], signal and image processing [8], chaos theory [9], viscoelasticity [10], random walk [11], and other numerous areas [3, 12–15]. Remember as well that there are only two main definitions of the fractional derivative: The first is the derivative of the convolution of a given function and a power law kernel (Riemann and Liouville), and the second is the convolution of the local derivative of a given function with a power law function (Caputo). On the other hand, these categories come with built-in limitations. The Caputo derivative has addressed this limitation, but it is still insufficient to fully describe the phenomena of index law. The Riemann-Liouville derivative is not enough for understanding the significance of the initial requirements. Indeed, every fractional derivative has benefits and drawbacks. The singularity of the kernel is one flaw in the Caputo and Riemann-Liouville derivative. This flaw is fixed by a nonlocal derivative developed by Caputo and Fabrizio. Caputo and Fabrizio have proposed an alternative idea of differentiation utilizing the exponential decay as the kernel instead of the power law because the power law cannot be utilized to address all physical situations. It is observed that in nature, many problems also follow the exponential decay law, which does not in fact possess a singularity; thus, this derivative is considerably suitable in modeling such real-life problems. This novel differentiation has also captured the consideration of several researchers, but was disqualified also for fractional derivative arrangement due to the nonlocality of kernel.

The theory of fractional differential equations effectively and methodically represents the fact of nature [16]. Alternative models to nonlinear differential equations are fractional differential equations. In order to develop the mathematical modeling of numerous physical events, a variety of them play significant roles and serve as tools not only in mathematics, but also in control systems, dynamical systems, engineering, and physics. Additionally, they are used in social scientific fields like economics, dietary supplements, and climate change [17]. Physical science makes use of the mathematical physics governed by nonlinear partial deferential dynamical equations. Numerous phenomena in optics, fluid mechanics, plasma physics, and hydrodynamics [18–20] depend on the analytical solutions to these dynamical equations. It inspires many academics to develop innovative techniques for solving fractional partial differential equations (FPDEs) precisely and approximatively. Over the past few years, a large number of scholars have developed efficient direct procedures for the analytical solution of fractional nonlinear differential equations. Several effective methods have been used for solving FPDEs including the Sine-Gordon expansion method [21], Yang transform decomposition method [22], q-homotopy analysis Sumudu transform method [23], reduced differential transform method [24], Existence and stability analysis [25], fractional variational iteration method [26], Elzaki transform decomposition method [27], variational iteration method [28], F-expansion method [29], homotopy perturbation Sumudu transform method [30], and many more [31–35].

The Korteweg-de Vries (KdV) equations are discovered in the investigation of nonlinear dispersive waves [36]. For the purpose of simulating shallow water waves in a canal, Korteweg and de Vries invented them in 1895 [37]. Hydrodynamics, water waves, quantum field theory, and plasma physics are just a few of the many applied sciences and engineering fields where the suggested KdV equations are essential. They discuss how two long waves with various dispersion relations interact. The study of the KdV-type equations has received considerable attention. In order to solve the KdV problem in the 1980s, the linearized technique was merged with the Adomian decomposition method, the finite element method, and the finite difference schemes [38–40]. Wang [41] derived lump solutions of the (2+1)-dimensional KdV equation using a method based on quadratic functions, while Wazwaz [42] derived solitons and periodic solutions of the KdV, modified KdV, and generalized KdV equations using various trustworthy methods. Recently, a number of systematic methods have been used in cite41,42 to examine the fractional versions of the KdV-type equations. Wang and Kara [45] in 2019 were the first to present the classical type of the new two dimensional modified KdV (2D-mKdV) equation. They did this by employing the extended Lax pair. Through the Lie symmetry technique, Wang and Kara in [45] extracted a collection of solitary wave solutions of the novel 2D-mKdV problem (the classical type).

He created the homotopy perturbation technique (HPM) in 1999, which combines the homotopy approach and the traditional perturbation methodology [46]. The work [47, 48] shows that both linear and nonlinear problems can be solved satisfactorily using this method. HPM has less restrictions than traditional perturbation techniques because it avoids the need for a small parameter in the equation. This study's main objective is to handle fractional nonlinear PDEs employing Yang's recently developed integral transform [49], also referred to as the "Yang Transform", which has been demonstrated with HPM using the fractional derivative of Caputo-Fabrizio (CF). The CF fractional derivative has added a new dimension to the analysis of fractional differential equations. One of the most interesting aspects of the new derivative is its nonsingular kernel. Two well-known nonlinear PDEs can be solved using the given technique. The recommended approach generates reliable outcomes that provide accurate solutions to the required problems. We acquired a power series solution in the setting of a rapidly convergent series, and only a small number of iterations are required to achieve outstanding results. After only a few iterations, a solution that can be quickly established using this method can be reached, eliminating the requirement for techniques like discretization or linearization of the nonlinear issue. This study can be used as a basic reference by researchers to explore this approach and implement it in many applications to obtain precise and approximate outcomes in a few simple steps.

The organization of the paper is described below: The core idea behind CF definitions is illustrated in Section 2. Section 3 introduced the fractional CF derivative and the Yang-Laplace duality property. In Section 4, we show convergence analysis as well as the general applicability of the proposed technique. In Section 5, there are several test problems that demonstrate the viability of the suggested approach. In Section 6, the conclusion is presented.

2. Preleminaries

Here, we address basic ideas associated to our study. Also, we give the exponential decay as a kernel as, $P(\vartheta, \varrho) = \exp[-\wp(\vartheta - \varrho/1 - \wp)]$.

Definition 2.1. If $\mathbb{F}(\vartheta) \in \mathbf{H}^1[0, T], T > 0$, then the CF fractional derivative is stated as:

$$^{CF}D_t^{\wp}[\mathbb{F}(\vartheta)] = \frac{Q(\wp)}{1-\wp}\int_0^{\vartheta} \mathbb{F}'(\varrho)P(\vartheta, \varrho)d\varrho, \tag{2.1}$$

with $Q(\wp)$ illustrating the normalization function with $Q(0) = Q(1) = 1$. In addition, if $\mathbb{F}(\vartheta) \notin \mathbf{H}^1[0, T]$, then:

$$^{CF}D_t^{\wp}[\mathbb{F}(\vartheta)] = \frac{Q(\wp)}{1-\wp}\int_0^{\vartheta} [\mathbb{F}(\vartheta) - \mathbb{F}(\varrho)]P(\vartheta, \varrho)d\varrho. \tag{2.2}$$

Definition 2.2. The CF integral with non-integer order is given as:

$$^{CF}I_t^{\wp}[\mathbb{F}(\vartheta)] = \frac{1-\wp}{Q(\wp)}\mathbb{F}(\vartheta) + \frac{\wp}{Q(\wp)}\int_0^{\vartheta} \mathbb{F}(\varrho)d\varrho, \quad \vartheta \geq 0, \wp \in (0, 1]. \tag{2.3}$$

Definition 2.3. For $Q(\wp) = 1$, the CF derivative in terms of Laplace transform (LT) is as:

$$L\left[^{CF}D_t^{\wp}[\mathbb{F}(\vartheta)]\right] = \frac{\varpi L[\mathbb{F}(\vartheta) - \mathbb{F}(0)]}{\varpi + \wp(1-\varpi)}. \tag{2.4}$$

Definition 2.4. The Yang transform (YT) of $\mathbb{F}(\vartheta)$ is stated as:

$$Y[\mathbb{F}(\vartheta)] = \zeta(\varpi) = \int_0^{\infty} \mathbb{F}(\vartheta)e^{-\frac{t}{\varpi}}d\vartheta. \ \ \vartheta > 0, \tag{2.5}$$

where ϖ is the transform variable.

Remark 2.1. Some properties of YT are given below.

$$\begin{aligned} Y[1] &= \varpi, \\ Y[\vartheta] &= \varpi^2, \\ Y[t^q] &= \Gamma(q+1)\varpi^{q+1}. \end{aligned} \tag{2.6}$$

3. Core idea

We initially create the YT formula for the CF fractional derivative using the Yang-Laplace duality principle. At the end of this part, we offer a few cases with comprehensive results to demonstrate the precision and efficacy of the suggested technique.

Lemma 3.1. *(Laplace-Yang duality) Assume the LT of* $\mathbb{F}(\vartheta)$ *is* $F(\varpi)$*, then* $\zeta(v) = F(\frac{1}{\varpi})$.

Proof. By putting $\frac{\vartheta}{u} = \imath$ in Eq (2.5), we determine YT another form as.

$$Y\left[\mathbb{F}(\vartheta)\right] = \zeta(\varpi) = \varpi \int_0^\infty \mathbb{F}(\varpi\imath)e^{\imath}d\imath. \;\; \imath > 0. \tag{3.1}$$

As $L\left[\mathbb{F}(\vartheta)\right] = F(\varpi)$, we have

$$F(\varpi) = L\left[\mathbb{F}(\vartheta)\right] = \int_0^\infty \mathbb{F}(\vartheta)e^{-\varpi\vartheta}d\vartheta. \tag{3.2}$$

Put $\vartheta = \imath/\varpi$ in (3.2), and we have

$$F(\varpi) = \frac{1}{\varpi}\int_0^\infty \mathbb{F}\left(\frac{\imath}{\varpi}\right)e^{\imath}d\imath. \tag{3.3}$$

From Eq (3.1), we get

$$F(\varpi) = \zeta\left(\frac{1}{\varpi}\right). \tag{3.4}$$

From Eqs (2.5) and (3.2), we have

$$F\left(\frac{1}{\varpi}\right) = \zeta\left(\varpi\right). \tag{3.5}$$

So, (3.4) and (3.5) illustrate the duality relation among the YT and Laplace.

Lemma 3.2. *The YT of the fractional CF derivative is stated as:*

$$Y\left[\mathbb{F}(\vartheta)\right] = \frac{Y[\mathbb{F}(\vartheta) - \varpi\mathbb{F}(0)]}{1 + \wp(\varpi - 1)}. \tag{3.6}$$

Proof. The LT of the CF derivative is illustrated as

$$L\left[\mathbb{F}(\vartheta)\right] = \frac{L[\varpi\mathbb{F}(\vartheta) - \mathbb{F}(0)]}{\varpi + \wp(1 - \varpi)}. \tag{3.7}$$

Also, the Laplace and Yang properties are related, as shown by the equation $\zeta(\varpi) = F\frac{1}{\varpi}$. To get the desired outcome, the variable ϖ in Eq (3.7) is changed to $\frac{1}{\varpi}$, and we get.

$$\begin{aligned} Y\left[\mathbb{F}(\vartheta)\right] &= \frac{\frac{1}{\varpi}Y[\mathbb{F}(\vartheta) - \mathbb{F}(0)]}{\frac{1}{\varpi} + \wp(1 - \frac{1}{\varpi})}, \\ Y\left[\mathbb{F}(\vartheta)\right] &= \frac{Y[\mathbb{F}(\vartheta) - \varpi\mathbb{F}(0)]}{1 + \wp(\varpi - 1)}, \end{aligned} \tag{3.8}$$

which is proved.

4. General procedure of the proposed technique

Assume the general arbitrary order differential equation, in order to determine the fundamental solution procedure.

4.1. General procedure of fractional differential equations in terms of CF derivative

Assume the following nonlinear general PDE as:

$$\begin{cases} {}^{CF}D_t^{\wp}\mathcal{J}(\imath,\vartheta) + M(\mathcal{J}(\imath,\vartheta)) + N(\mathcal{J}(\imath,\vartheta)) = g(\imath,\vartheta), \\ \mathcal{J}(\imath,0) = h(\imath), \end{cases} \tag{4.1}$$

with $M(\mathcal{J}(\imath,\vartheta))$ and $N(\mathcal{J}(\imath,\vartheta))$ illustrating the linear and nonlinear terms, and $g(\imath,\vartheta)$ as the source term.

By implementing YT to Eq (4.2), we obtain

$$\frac{Y[\mathcal{J}(\imath,\vartheta) - \varpi\mathcal{J}(\imath,0)]}{1+\wp(\varpi-1)} = -Y[M(\mathcal{J}(\imath,\vartheta)) + N(\mathcal{J}(\imath,\vartheta))] + Y[g(\imath,\vartheta)],$$

$$Y[\mathcal{J}(\imath,\vartheta)] = \varpi h(\imath) - (1+\wp(\varpi-1))[Y[M(\mathcal{J}(\imath,\vartheta)) + N(\mathcal{J}(\imath,\vartheta))]] + Y[g(\imath,\vartheta)]. \tag{4.2}$$

Now, in terms of the inverse Yang transform, we have

$$\mathcal{J}(\imath,\vartheta) = \mathcal{G}(\imath,\vartheta) - Y^{-1}[(1+\wp(\varpi-1))[Y[M(\mathcal{J}(\imath,\vartheta)) + N(\mathcal{J}(\imath,\vartheta))]] + Y[g(\imath,\vartheta)]], \tag{4.3}$$

with $G(\imath,\vartheta)$ illustrating the combined form of initial guess and source term. By using HPM:

$$\mathcal{J}(\imath,\vartheta) = \sum_{q=0}^{\infty}\rho^q\mathcal{J}_q(\imath,\vartheta). \tag{4.4}$$

The nonlinear term $N(\mathcal{J}(\imath,\vartheta))$ is decomposed as

$$N(\mathcal{J}(\imath,\vartheta)) = \sum_{q=0}^{\infty}\rho^q H_q(\mathcal{J}), \tag{4.5}$$

with $H_q(\mathcal{J})$ illustrating the He's polynomial, which is calculated as:

$$H_q(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3,\cdots,\mathcal{J}_q) = \frac{1}{\Gamma(q+1)}\frac{\partial^q}{\partial\rho^q}\left[N\left(\sum_{i=0}^{\infty}\rho^i\mathcal{J}_i\right)\right]_{\rho=0}, \quad q=1,2,3,\cdots. \tag{4.6}$$

By using Eqs (4.4) and (4.5) in Eq (4.3), we obtain

$$\sum_{q=0}^{\infty}\rho^q\mathcal{J}_q(\imath,\vartheta) = \mathcal{G}(\imath,\vartheta) - \rho\left(Y^{-1}\left[(1+\wp(\varpi-1))Y\left[M\sum_{q=0}^{\infty}\rho^q\mathcal{J}_q(\imath,\vartheta) + N\sum_{q=0}^{\infty}\rho^q H_q(\mathcal{J})\right]\right]\right). \tag{4.7}$$

By comparison of the ρ coefficients in (4.7), we have:

$$\begin{aligned} \rho^0 &: \mathcal{J}_0(\imath,\vartheta) = \mathcal{G}(\imath,\vartheta), \\ \rho^1 &: \mathcal{J}_1(\imath,\vartheta) = Y^{-1}\left[(1+\wp(\varpi-1))Y\left[M(\mathcal{J}_0(\imath,\vartheta)) + H_0(\mathcal{J})\right]\right], \\ \rho^2 &: \mathcal{J}_2(\imath,\vartheta) = Y^{-1}\left[(1+\wp(\varpi-1))Y\left[M(\mathcal{J}_1(\imath,\vartheta)) + H_1(\mathcal{J})\right]\right], \\ \rho^3 &: \mathcal{J}_3(\imath,\vartheta) = Y^{-1}\left[(1+\wp(\varpi-1))Y\left[M(\mathcal{J}_2(\imath,\vartheta)) + H_2(\mathcal{J})\right]\right], \\ &\vdots \\ \rho^q &: \mathcal{J}_q(\imath,\vartheta) = Y^{-1}\left[(1+\wp(\varpi-1))Y\left[M(\mathcal{J}_q(\imath,\vartheta)) + H_q(\mathcal{J})\right]\right]. \end{aligned} \tag{4.8}$$

Thus, we obtain the solution as:

$$\mathcal{J}(\imath,\vartheta) = \mathcal{J}_0(\imath,\vartheta) + \mathcal{J}_1(\imath,\vartheta) + \cdots. \tag{4.9}$$

4.2. Convergence analysis

The given theorems, which depend on the principle of the technique described in [50], deal with the convergence analysis of the original problem (4.1).

Theorem 4.1. *Assume the accurate solution of (4.1) is $\mathcal{J}(\iota,\vartheta)$ and assume $\mathcal{J}(\iota,\vartheta)$, $\mathcal{J}_n(\iota,\vartheta) \in H$, and $\theta \in (0,1)$, with H illustrating the Hilbert space. The result obtains that $\sum_{q=0}^{\infty}\mathcal{J}_q(\iota,\vartheta)$ will converge with $\mathcal{J}(\iota,\vartheta)$ if $\mathcal{J}_q(\iota,\vartheta) \le \mathcal{J}_{q-1}(\iota,\vartheta)$ $\forall q > A$, i.e., for all $\omega > 0 \exists A > 0$, with $\|\mathcal{J}_{q+n}(\iota,\vartheta)\| \le \beta, \forall q, n \in N$.*

Proof. Assume a sequence of $\sum_{q=0}^{\infty}\mathcal{J}_q(\iota,\vartheta)$.

$$\begin{aligned}
\mathcal{K}_0(\iota,\vartheta) =&\mathcal{J}_0(\iota,\vartheta),\\
\mathcal{K}_1(\iota,\vartheta) =&\mathcal{J}_0(\iota,\vartheta) + \mathcal{J}_1(\iota,\vartheta),\\
\mathcal{K}_2(\iota,\vartheta) =&\mathcal{J}_0(\iota,\vartheta) + \mathcal{J}_1(\iota,\vartheta) + \mathcal{J}_2(\iota,\vartheta),\\
\mathcal{K}_3(\iota,\vartheta) =&\mathcal{J}_0(\iota,\vartheta) + \mathcal{J}_1(\iota,\vartheta) + \mathcal{J}_2(\iota,\vartheta) + \mathcal{J}_3(\iota,\vartheta),\\
&\vdots\\
\mathcal{K}_q(\iota,\vartheta) =&\mathcal{J}_0(\iota,\vartheta) + \mathcal{J}_1(\iota,\vartheta) + \mathcal{J}_2(\iota,\vartheta) + \cdots + \mathcal{J}_q(\iota,\vartheta).
\end{aligned} \tag{4.10}$$

To attain the required result, we must verify that $\mathcal{K}_q(\iota,\vartheta)$ forms a "Cauchy sequence". Moreover, let's take

$$\begin{aligned}
\|\mathcal{K}_{q+1}(\iota,\vartheta) - \mathcal{K}_q(\iota,\vartheta)\| = \|\mathcal{J}_{q+1}(\iota,\vartheta)\| \le \theta\|\mathcal{J}_q(\iota,\vartheta)\| \le \theta^2\|\mathcal{J}_{q-1}(\iota,\vartheta)\| \le \theta^3\|\mathcal{J}_{q-2}(\iota,\vartheta)\|\cdots\\
\le \theta_{q+1}\|\mathcal{J}_0(\iota,\vartheta)\|.
\end{aligned} \tag{4.11}$$

For $q, n \in N$, we obtain

$$\begin{aligned}
\|\mathcal{K}_q(\iota,\vartheta) - \mathcal{K}_n(\iota,\vartheta)\| =&\|\mathcal{J}_{q+n}(\iota,\vartheta)\| = \|\mathcal{K}_q(\iota,\vartheta) - \mathcal{K}_{q-1}(\iota,\vartheta) + (\mathcal{K}_{q-1}(\iota,\vartheta) - \mathcal{K}_{q-2}(\iota,\vartheta))\\
&+ (\mathcal{K}_{q-2}(\iota,\vartheta) - \mathcal{K}_{q-3}(\iota,\vartheta)) + \cdots + (\mathcal{K}_{n+1}(\iota,\vartheta) - \mathcal{K}_n(\iota,\vartheta))\|\\
\le&\|\mathcal{K}_q(\iota,\vartheta) - \mathcal{K}_{q-1}(\iota,\vartheta)\| + \|(\mathcal{K}_{q-1}(\iota,\vartheta) - \mathcal{K}_{q-2}(\iota,\vartheta))\|\\
&+ \|(\mathcal{K}_{q-2}(\iota,\vartheta) - \mathcal{K}_{q-3}(\iota,\vartheta))\| + \cdots + \|(\mathcal{K}_{n+1}(\iota,\vartheta) - \mathcal{K}_n(\iota,\vartheta))\|\\
\le&\theta^q\|\mathcal{J}_0(\iota,\vartheta)\| + \theta^{q-1}\|\mathcal{J}_0(\iota,\vartheta)\| + \cdots + \theta^{q+1}\|\mathcal{J}_0(\iota,\vartheta)\|\\
=&\|\mathcal{J}_0(\iota,\vartheta)\|(\theta^q + \theta^{q-1} + \theta^{q+1})\\
=&\|\mathcal{J}_0(\iota,\vartheta)\|\frac{1-\theta^{q-n}}{1-\theta^{q+1}}\theta^{n+1}.
\end{aligned} \tag{4.12}$$

As $0 < \theta < 1$, and $\mathcal{J}_0(\iota,\vartheta)$ are bounded, assume $\beta = 1 - \theta/(1-\theta_{q-n})\theta^{n+1}\|\mathcal{J}_0(\iota,\vartheta)\|$, and we get

$$\|\mathcal{J}_{q+n}(\iota,\vartheta)\| \le \beta, \forall q, n \in N. \tag{4.13}$$

Thus, $\{\mathcal{J}_q(\iota,\vartheta)\}_{q=0}^{\infty}$ forms a "Cauchy sequence" in H. It shows that $\{\mathcal{J}_q(\iota,\vartheta)\}_{q=0}^{\infty}$ is a convergent sequence having the limit $\lim_{q\to\infty}\mathcal{J}_q(\iota,\vartheta) = \mathcal{J}(\iota,\vartheta)$ for $\exists\mathcal{J}(\iota,\vartheta) \in \mathcal{H}$.

Theorem 4.2. *Assume that $\sum_{h=0}^{k}\mathcal{J}_h(\iota,\vartheta)$ is finite and that $\mathcal{J}(\iota,\vartheta)$ illustrates the series solution. Taking $\theta > 0$ such that $\|\mathcal{J}_{h+1}(\iota,\vartheta)\| \le \|\mathcal{J}_h(\iota,\vartheta)\|$, the maximum absolute error is as:*

$$\|\mathcal{J}(\iota,\vartheta) - \sum_{h=0}^{k}\mathcal{J}_h(\iota,\vartheta)\| < \frac{\theta^{k+1}}{1-\theta}\|\mathcal{J}_0(\iota,\vartheta)\|. \tag{4.14}$$

Proof. Suppose $\sum_{h=0}^{k}\mathcal{J}_h(\imath,\vartheta)$ is finite, which illustrates that $\sum_{h=0}^{k}\mathcal{J}_h(\imath,\vartheta)<\infty$.
Assume

$$\begin{aligned}
\|\mathcal{J}(\imath,\vartheta)-\sum_{h=0}^{k}\mathcal{J}_h(\imath,\vartheta)\| =&\|\sum_{h=k+1}^{\infty}\mathcal{J}_h(\imath,\vartheta)\|\\
\leq&\sum_{h=k+1}^{\infty}\|\mathcal{J}_h(\imath,\vartheta)\|\\
\leq&\sum_{h=k+1}^{\infty}\theta^h\|\mathcal{J}_0(\imath,\vartheta)\|\\
\leq&\theta^{k+1}(1+\theta+\theta^2+\cdots)\|\mathcal{J}_0(\imath,\vartheta)\|\\
\leq&\frac{\theta^{k+1}}{1-\theta}\|\mathcal{J}_0(\imath,\vartheta)\|,
\end{aligned} \tag{4.15}$$

which proves the theorem.

5. Numerical applications

The HPTM is used to solve fractional nonlinear KdV equations in this work.

Problem 5.1. Consider the fractional KdV equation as

$${}^{CF}D_{\vartheta}^{\wp}z(\imath,\vartheta)=-zz_{\imath}-z_{\imath\imath\imath}+\imath^2+2\imath^3\vartheta^2,\quad \wp\in(0,1], \tag{5.1}$$

with initial guess

$$z(\imath,0)=0.$$

Solution 5.1. By employing YT to Eq (5.1), we obtain

$$Y[z(\imath,\vartheta)]=(1+\wp\varpi-\wp)\left[\varpi\imath^2+\varpi^2 2\imath^3\vartheta^2\right]-(1+\wp\varpi-\wp)Y\left[zz_{\imath}+z_{\imath\imath\imath}\right]. \tag{5.2}$$

Now in terms of inverse YT, we get

$$z(\imath,\vartheta)=\imath^3\Big(\frac{2\wp\vartheta^3}{3}-2\wp\vartheta^2+2\vartheta^2\Big)+\imath^2(1+\wp\vartheta-\wp)-Y^{-1}\left[(1+\wp\varpi-\wp)Y\left[zz_{\imath}+z_{\imath\imath\imath}\right]\right]. \tag{5.3}$$

By using the homotopy perturbation technique, we have

$$\begin{aligned}
&\sum_{q=0}^{\infty}\rho^q z_q(\imath,\vartheta)=\imath^3\Big(\frac{2\wp\vartheta^3}{3}-2\wp\vartheta^2+2\vartheta^2\Big)+\imath^2(1+\wp\vartheta-\wp)-Y^{-1}\Bigg[(1+\wp v-\wp)Y\Bigg[\sum_{q=0}^{\infty}\rho^q H_q(z)+\\
&\left(\sum_{q=0}^{\infty}\rho^q z_q(\imath,\vartheta)\right)_{\imath\imath\imath}\Bigg]\Bigg],
\end{aligned} \tag{5.4}$$

where $H_q(z)$ represents the nonlinear terms and is determined as

$$\begin{aligned}
H_0(z) =&z_0 z_{0_{\imath}},\\
H_1(z) =&z_0 z_{1_{\imath}}+z_1 z_{0_{\imath}},\\
&\vdots
\end{aligned} \tag{5.5}$$

By comparison of the ρ coefficients in (5.5), we have:

$$
\begin{aligned}
\rho^0 : z_0(\imath,\vartheta) =& \imath^3(\frac{2\wp\vartheta^3}{3} - 2\wp\vartheta^2 + 2\vartheta^2) + \imath^2(1 + \wp\vartheta - \wp),\\
\rho^1 : z_1(\imath,\vartheta) =& - Y^{-1}\left[(1 + \wp v - \wp)\left[H_0(z) + \frac{\partial^3}{\partial \imath^3} z_0\right]\right],\\
=& -\frac{4\wp^3\vartheta^7\imath^5}{21} - 6\vartheta(\wp^3\imath^3 - 2\wp^2\imath^3 + \wp\imath^3) + 2(\wp^3\imath^3 - 3\wp^2\imath^3 + 3\wp\imath^3 - \imath^3)\\
&+ 2\vartheta^2(-6\wp^3 + 12\wp + 5\wp^3\imath^4 - 15\wp^2\imath^4 + 15\wp\imath^4 - 5\imath^4 + 2\wp^3\imath^3 - 2\wp^2\imath^3 - 6)\\
&+ \frac{8\vartheta^6}{3}(\wp^3\imath^5 - \wp^2\imath^5) - \frac{2\vartheta^3}{3}(-12\wp^2 + 12\wp + 25\wp^3\imath^4 - 50\wp^2\imath^4 + 25\wp\imath^4 + \wp^3\imath^3)\\
&+ \frac{\vartheta^4}{3}(-3\wp^2 + 36\wp^3\imath^5 - 108\wp^2\imath^5 + 108\wp\imath^5 - 36\imath^5 + 20\wp^3\imath^4 - 20\wp^2\imath^4)\\
&\frac{2\vartheta^5}{15}(78\wp^3\imath^5 - 156\wp^2\imath^5 + 78\wp\imath^5 + 5\wp^3\imath^4).
\end{aligned}
\tag{5.6}
$$

Finally, the series form solution is taken as:

$$
\begin{aligned}
&z(\imath,\vartheta) = z_0(\imath,\vartheta) + z_1(\imath,\vartheta) + \cdots\\
&z(\imath,\vartheta) = \imath^3(\frac{2\wp\vartheta^3}{3} - 2\wp\vartheta^2 + 2\vartheta^2) + \imath^2(1 + \wp\vartheta - \wp) - \frac{4\wp^3\vartheta^7\imath^5}{21} - 6\vartheta(\wp^3\imath^3 - 2\wp^2\imath^3 + \wp\imath^3) + 2\\
&(\wp^3\imath^3 - 3\wp^2\imath^3 + 3\wp\imath^3 - \imath^3) + 2\vartheta^2(-6\wp^3 + 12\wp + 5\wp^3\imath^4 - 15\wp^2\imath^4 + 15\wp\imath^4 - 5\imath^4 + 2\wp^3\imath^3 - 2\\
&\wp^2\imath^3 - 6) + \frac{8\vartheta^6}{3}(\wp^3\imath^5 - \wp^2\imath^5) - \frac{2\vartheta^3}{3}(-12\wp^2 + 12\wp + 25\wp^3\imath^4 - 50\wp^2\imath^4 + 25\wp\imath^4 + \wp^3\imath^3)\\
&+ \frac{\vartheta^4}{3}(-3\wp^2 + 36\wp^3\imath^5 - 108\wp^2\imath^5 + 108\wp\imath^5 - 36\imath^5 + 20\wp^3\imath^4 - 20\wp^2\imath^4)\frac{2\vartheta^5}{15}(78\wp^3\imath^5 - 156\wp^2\imath^5 +\\
&78\wp\imath^5 + 5\wp^3\imath^4) + \cdots .
\end{aligned}
\tag{5.7}
$$

If we take $\wp = 1$, the solution approaches to:

$$z(\imath,\vartheta) = \imath^2\vartheta. \tag{5.8}$$

Problem 5.2. Assume the nonlinear fractional mKdV equation as

$${}^{CF}D^{\wp}_{\vartheta} z(\imath,\xi,\vartheta) = 6z^2 z_\imath - 6z^2 z_\xi + z_{\imath\imath\imath} - z_{\xi\xi\xi} - 3z_{\imath\imath\xi} + z_{\imath\xi\xi}, \quad \wp \in (0, 1], \tag{5.9}$$

with initial guess

$$z(\imath,\xi, 0) = -4\frac{e^{\imath-\xi}}{1 + e^{2(\imath-\xi)}}.$$

Solution 5.2. By employing YT to Eq (5.9), we obtain

$$Y[z(\imath,\xi,\vartheta)] = \varpi z(\imath,\xi, 0) + (1 + \wp\varpi - \wp)Y\left[6z^2 z_\imath - 6z^2 z_\xi + z_{\imath\imath\imath} - z_{\xi\xi\xi} - 3z_{\imath\imath\xi} + z_{\imath\xi\xi}\right]. \tag{5.10}$$

Now, in terms of inverse YT, we get

$$z(\imath,\xi,\vartheta) = -4\frac{e^{\imath-\xi}}{1 + e^{2(\imath-\xi)}} + Y^{-1}\left[(1 + \wp\varpi - \wp)Y\left[6z^2 z_\imath - 6z^2 z_\xi + z_{\imath\imath\imath} - z_{\xi\xi\xi} - 3z_{\imath\imath\xi} + z_{\imath\xi\xi}\right]\right]. \tag{5.11}$$

By using the HPT technique, we have

$$\sum_{q=0}^{\infty}\rho^q z_q(\imath,\xi,\vartheta) = -4\frac{e^{\imath-\xi}}{1+e^{2(\imath-\xi)}} + Y^{-1}\Bigg[(1+\wp v-\wp)Y\Bigg[6\sum_{q=0}^{\infty}\rho^q H_q^1(z) - 6\sum_{q=0}^{\infty}\rho^q H_q^2(z) + \left(\sum_{q=0}^{\infty}\rho^q z_q(\imath,\xi,\vartheta)\right)_{\imath\imath\imath} - \left(\sum_{q=0}^{\infty}\rho^q z_q(\imath,\xi,\vartheta)\right)_{\xi\xi\xi} - 3\left(\sum_{q=0}^{\infty}\rho^q z_q(\imath,\xi,\vartheta)\right)_{\imath\imath\xi} + 3\left(\sum_{q=0}^{\infty}\rho^q z_q(\imath,\xi,\vartheta)\right)_{\imath\xi\xi}\Bigg]\Bigg], \tag{5.12}$$

where $H_q(z)$ represents the nonlinear terms and is determined as

$$\begin{aligned}
H_0^1(z) &= z_0^2(z_0)_\imath,\\
H_1^1(z) &= z_0^2(z_1)_\imath + 2z_0z_1(z_0)_\imath,\\
&\vdots\\
H_0^2(z) &= z_0^2(z_0)_\xi,\\
H_1^2(z) &= z_0^2(z_1)_\xi + 2z_0z_1(z_0)_\xi,\\
&\vdots
\end{aligned} \tag{5.13}$$

By comparison of the ρ coefficients in (5.13), we have:

$$\begin{aligned}
\rho^0 : z_0(\imath,\xi,\vartheta) &= -4\frac{e^{\imath-\xi}}{1+e^{2(\imath-\xi)}},\\
\rho^1 : z_1(\imath,\xi,\vartheta) &= Y^{-1}\left[(1+\wp v-\wp)\left[6H_0^1(z) - 6H_0^2(z) + (z_0)_{\imath\imath\imath} - (z_0)_{\xi\xi\xi} - 3(z_0)_{\imath\imath\xi} + (z_0)_{\imath\xi\xi}\right]\right],\\
&= 32\frac{e^{\imath-\xi}(\wp e^{2(\imath-\xi)}\vartheta - \wp e^{2(\imath-\xi)} - \wp\vartheta + \wp - 1)}{(1+e^{2(\imath-\xi)})^2}.
\end{aligned} \tag{5.14}$$

Finally the series form solution is taken as:

$$\begin{aligned}
z(\imath,\xi,\vartheta) &= z_0(\imath,\xi,\vartheta) + z_1(\imath,\xi,\vartheta) + \cdots\\
z(\imath,\xi,\vartheta) &= -4\frac{e^{\imath-\xi}}{1+e^{2(\imath-\xi)}} + 32\frac{e^{\imath-\xi}(\wp e^{2(\imath-\xi)}\vartheta - \wp e^{2(\imath-\xi)} - \wp\vartheta + \wp - 1)}{(1+e^{2(\imath-\xi)})^2} + \cdots.
\end{aligned} \tag{5.15}$$

If we take $\wp = 1$, the solution approaches to:

$$z(\imath,\xi,\vartheta) = -4\frac{e^{\imath-\xi+8\vartheta}}{1+e^{2(\imath-\xi+8\vartheta)}}. \tag{5.16}$$

6. Results and discussion

The numerical results for the KdV equation are shown with the homotopy perturbation transform method (HPTM) via the CF derivative in this section. With the help of Maple, the above problems are displayed in tabular and graphical form. The numerical results demonstrate the technique's applicability, and the precision of the approach is assessed in light of the precise results. The behavior of the approximate solution and exact solution for example 1 at $\wp = 1$ is depicted in Figure 1. The behavior of the proposed method solution in terms of CF at $\wp = 0.90, 0.95$ is depicted in Figure 2.

The nature of absolute error, as well as the variation in fractional order $\wp$ for the solution in example 1 employing the suggested method, is displayed in Figure 3. The suggested approach solution of the first example at $\xi = 0.5, \vartheta = 0.01$ is compared with exact and various fractional orders in Table 1. The behavior of the approximate solution and exact solution for example 2 at $\wp = 1$ is depicted in Figure 4. The behavior of the proposed method solution in terms of CF at $\wp = 0.90, 0.95$ is depicted in Figure 5. The nature of absolute error, as well as the variation in fractional order $\wp$ for the solution in example 2 employing the suggested method, is displayed in Figure 6. The suggested approach solution of the second example at $\xi = 0.5, \vartheta = 0.01$ is compared with exact and various fractional orders in Table 2. The absolute error comparison of the suggested approach with HATM is shown in Table 3. The graph illustrates how different fractional orders impact the behavior of the solution and shows how variations in $\wp$ impact the solution with respect to the given parameters. Also, the numerical results are briefly specified in the tables, which enables a direct comparison of the performance of the solution at various fractional orders. It needs to be noted that throughout the calculations, we used different approximations, and that using accurate results for the problem gave us a better estimate. By increasing the order of the approximation, which adds more terms to the solution, we could have been able to get approximation solutions that were more accurate.

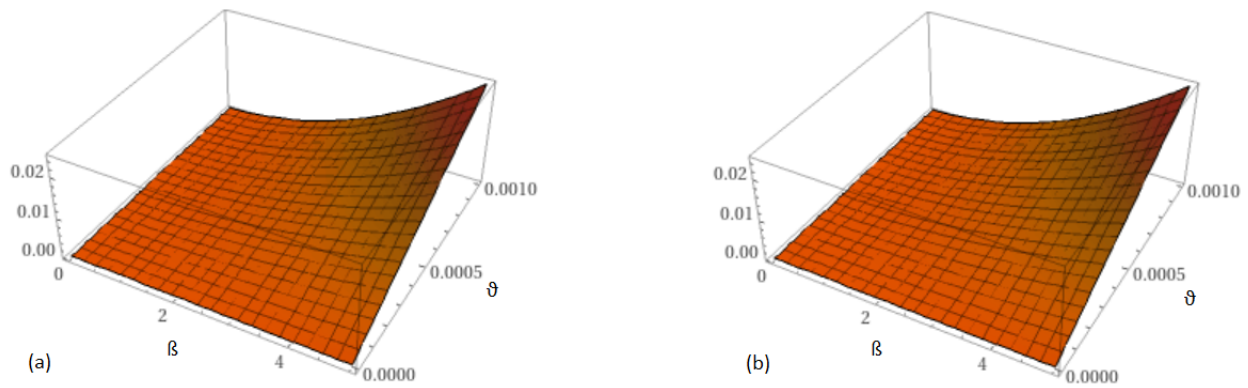

Figure 1. Behavior of (a) exact solution and (b) our technique solution for Problem 5.1.

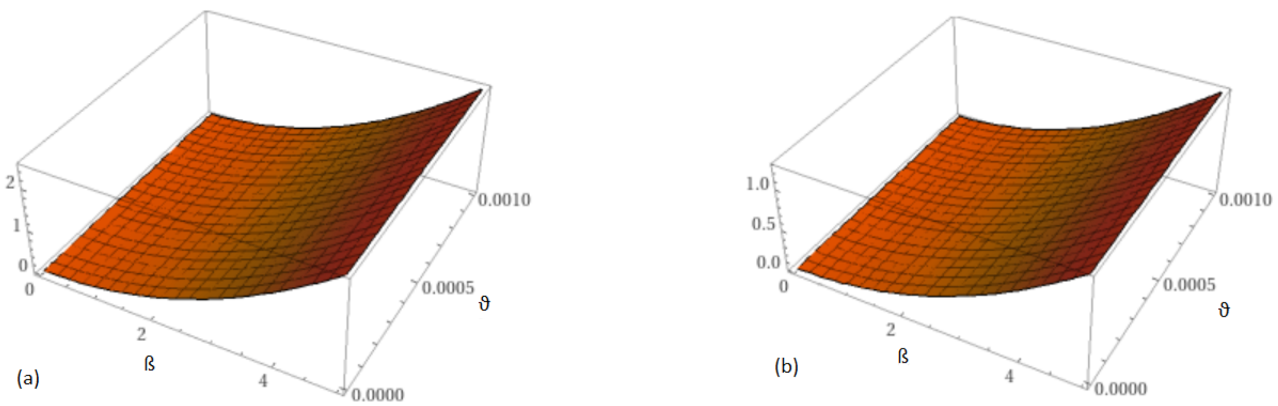

Figure 2. Surface of (a) analytical solution at $\wp = 0.90$ (b) analytical solution at $\wp = 0.95$.

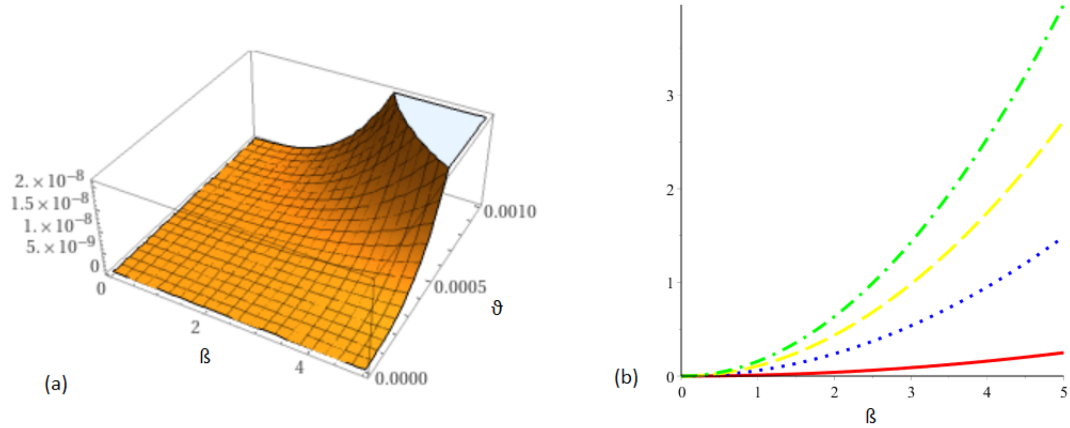

Figure 3. The (a) 3D plot of absolute error and (b) 2D solution graph of our technique solution at different values of $\wp$.

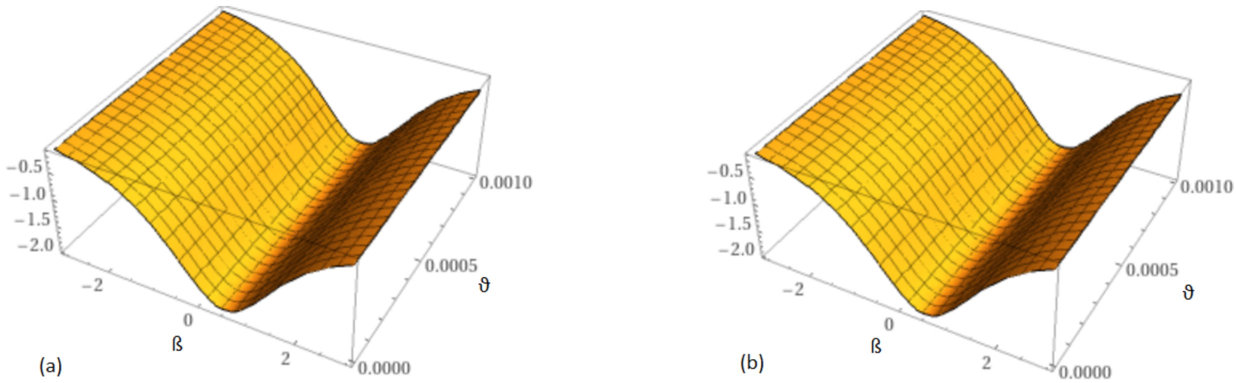

Figure 4. The HPTM solution $z(1, \xi, \vartheta)$ (a) second order approximate solution (b) exact solution for Problem 5.2.

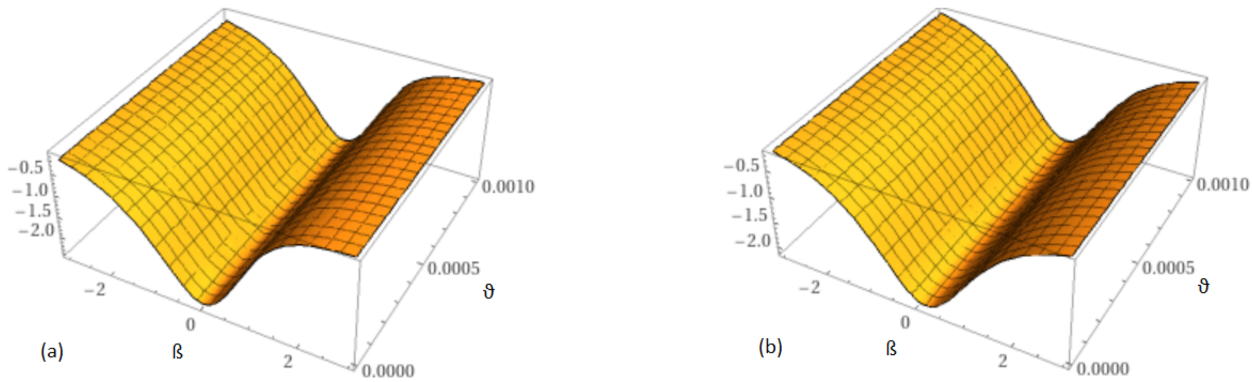

Figure 5. Behavior of the suggested approach solution at (a) $\wp = 0.90$ and (b) $\wp = 0.95$.

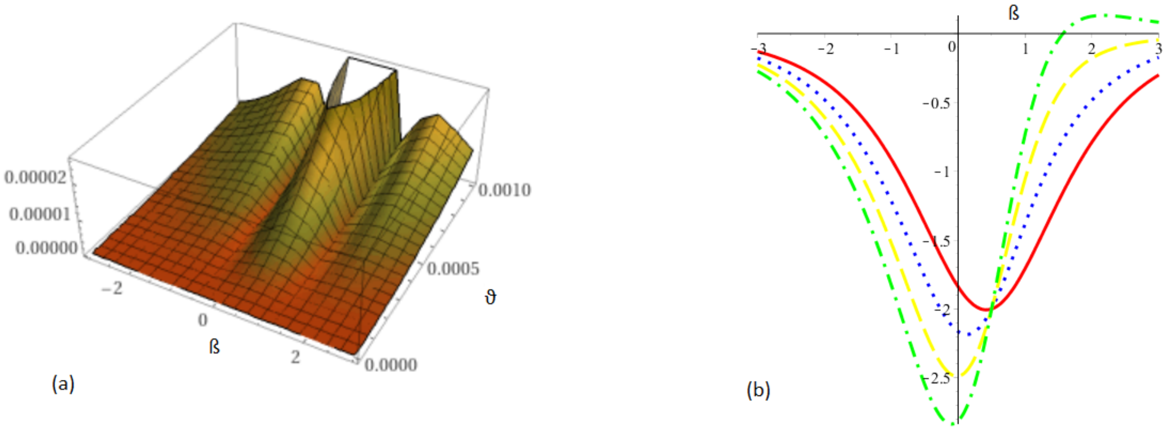

Figure 6. The (a) absolute error between exact solution and second order approximate solution and (b) analytical solution behavior at numerous values of $\wp$.

Table 1. Accurate solution as well as second-order approximate solution at numerous values of $\wp$.

λ	$\wp = 0.97$	$\wp = 0.98$	$\wp = 0.99$	$\wp = 1(appro)$	$\wp = 1(exact)$
0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.00039700	0.00029800	0.00019900	0.00010000	0.00010000
0.2	0.00158805	0.00119203	0.00079602	0.00040000	0.00040000
0.3	0.00357317	0.00268212	0.00179107	0.00090001	0.00090000
0.4	0.00635242	0.00476829	0.00318417	0.00160004	0.00160000
0.5	0.00992583	0.00745058	0.00497533	0.00250008	0.00250000
0.6	0.01429343	0.01072900	0.00716457	0.00360014	0.00360000
0.7	0.01945527	0.01460359	0.00975191	0.00490022	0.00490000
0.8	0.02541140	0.01907438	0.01273736	0.00640034	0.00640000
0.9	0.03216184	0.02414139	0.01612093	0.00810048	0.00810000
1.0	0.03970664	0.02980465	0.01990266	0.01000066	0.01000000

Table 2. Numerical comparison between the second term approximate solution at different fractional orders and exact solution.

λ	$\wp = 0.97$	$\wp = 0.98$	$\wp = 0.99$	$\wp = 1(appro)$	$\wp = 1(exact)$
0.0	-1.97670891	-1.91120420	-1.84569950	-1.78019479	-1.78016202
0.1	-2.02416819	-1.96799152	-1.91181486	-1.85563819	-1.85559583
0.2	-2.05134623	-2.00680238	-1.96225852	-1.91771466	-1.91766362
0.3	-2.05653542	-2.02560757	-1.99467972	-1.96375187	-1.96369386
0.4	-2.03918307	-2.02333146	-2.00747985	-1.99162824	-1.99156574
0.5	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-1.99993600
0.6	-1.94089992	-1.95675153	-1.97260314	-1.98845475	-1.98839241
0.7	-1.86477656	-1.89570441	-1.92663226	-1.95756011	-1.95750242
0.8	-1.77516541	-1.81970926	-1.86425312	-1.90879698	-1.90874636
0.9	-1.67586161	-1.73203828	-1.78821494	-1.84439161	-1.84434975
1.0	-1.57056662	-1.63607133	-1.70157603	-1.76708074	-1.76704848

Table 3. Comparison between HATM [51] and our method in terms of absolute error.

ϑ	HATM at $\imath = 0.5$	HPTM at $\imath = 0.5$	HATM at $\xi = 0.5$	HPTM at $\xi = 0.5$
0.0	9.921881×10^{-07}	$3.2500000000\times10^{-07}$	2.607245×10^{-07}	$3.2500000000\times10^{-07}$
0.5	1.702236×10^{-05}	$6.4000000000\times10^{-07}$	1.702236×10^{-05}	$6.4000000000\times10^{-07}$
0.1	2.607245×10^{-07}	$3.2500000000\times10^{-07}$	9.921881×10^{-07}	$3.2500000000\times10^{-07}$
1.5	6.913822×10^{-06}	$6.6000000000\times10^{-08}$	7.076190×10^{-06}	$6.7000000000\times10^{-08}$
2.0	2.534162×10^{-06}	$1.7410000000\times10^{-07}$	2.783063×10^{-06}	$1.7350000000\times10^{-07}$
2.5	2.305783×10^{-07}	$1.46500000000\times10^{-07}$	3.045477×10^{-07}	$1.4590000000\times10^{-07}$
3.0	2.741436×10^{-07}	$9.9000000000\times10^{-08}$	2.651408×10^{-07}	$9.9100000000\times10^{-08}$
3.5	2.706178×10^{-07}	$6.2300000000\times10^{-08}$	2.751423×10^{-07}	$6.2400000000\times10^{-08}$
4.0	1.884674×10^{-07}	$3.8400000000\times10^{-08}$	1.936185×10^{-07}	$3.8300000000\times10^{-08}$
4.5	1.198279×10^{-07}	$2.3390000000\times10^{-08}$	1.235044×10^{-07}	$2.3370000000\times10^{-08}$
5.0	7.391757×10^{-08}	$1.4210000000\times10^{-08}$	7.627198×10^{-08}	$1.4190000000\times10^{-08}$

7. Conclusions

In order to provide an analytical solution to the nonlinear time-fractional KdV problems, we employ the HPTM along with the CF fractional derivative. We have considered two cases of the KdV equations together with different initial conditions. The potential application for physicists and engineers working in diverse fields of the natural sciences has inspired the development of the approach as a significant mathematical instrument. In light of the fractional operator of the CF type, this work demonstrates that the HPTM approach is an effective tool for solving nonlinear FPDEs. With only a few steps, the HPTM produces a quickly converging series solution. The suggested method has been used to compute the solutions for both examples. The effectiveness of HPTM has also been shown via various figures and tables. The efficacy as well as reliability of the proposed approach are illustrated by the investigation of HPTM approximations for various fractional order KdV equations with exact solutions. We consequently come to the conclusion that the presented approach is significant nonsophisticated effective tools that generate good quality approximations for nonlinear partial differential equations using straightforward computations and achieve convergence with a minimal number of terms. The simple operation of the suggested solutions led to the conclusion that they are appropriate for handling every physical issue arising up in engineering and the sciences. Thus, the expansion will be significantly valued to add other operators and approaches in the future, especially in light of the advantages of the current operator. The results can be of great use to many authors in evaluating and understanding their experimental and observational data, particularly those working in nonlinear sciences like nonlinear optics and plasma physics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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