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Research article

A new approach in handling one-dimensional time-fractional Schrödinger equations

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Abstract: Our aim of this paper was to present the accurate analytical approximate series solutions to the time-fractional Schrödinger equations via the Caputo fractional operator using the Laplace residual power series technique. Furthermore, three important and interesting applications were given, tested, and compared with four well-known methods (Adomian decomposition, homotopy perturbation, homotopy analysis, and variational iteration methods) to show that the proposed technique was simple, accurate, efficient, and applicable. When there was a pattern between the terms of the series, we could obtain the exact solutions; otherwise, we provided the approximate series solutions. Finally, graphical results were presented and analyzed. Mathematica software was used to calculate numerical and symbolic quantities.

Keywords: fractional operators; fractional Schrödinger equation; multiple fractional power series; Laplace residual power series method
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1. Introduction

Fractional operators are concerned with the derivatives and integrals of non-integer orders, and there were many ancient and modern definitions of the fractional derivative and integral. Some of them have received attention from researchers because they have been used to represent many natural phenomena in all sciences, such as the definition of Grünwald-Letnikov, Riemann-Liouville, and Caputo as well [1–5]. Since there are no exact analytical solutions for most nonlinear fractional differential equations, many analytic and numerical methods have been proposed to solve them. For example, the Laplace transform methodology [6,7], Adomian decomposition method [8–10], operational matrix method [11,12], partial differential transformation method [13], homotopy analysis method [14–16], homotopy perturbation method [10,17,18], variational iteration method [18–20], residual power series (RPS) technique [21–25], analytical spectral method [26], Adam's-Bashforth numerical technique [27], Newton polynomials of interpolation [28], Haar wavelets method [29], analytical generalized exponential rational function method [30–32], Laplace transform with decomposition method [33], and meshless numerical technique [34].

Eriqat et al. [35] introduced the Laplace residual power series (LRPS) method to adapt the Laplace transform to solve the nonlinear fractional differential equations based on the RPS method. The LRPS method is a simple and effective analytical technique that is used to construct the series solutions of fractional differential equations by applying the Laplace transform and solving the obtained result using the concepts of the Laurent expansion and the limit at infinity. The inverse Laplace transform returns the solution to the original space and obtains the series solution to the target equation. El-Ajou [36] obtained the solitary solutions for the nonlinear time-fractional dispersive PDEs, El-Ajou and Al-Zhour [37] created a vector series solution for a class of hyperbolic system of Caputo-time-fractional PDEs with variable coefficients, Burqan et al. [38] constructed the series solution to the time-fractional Navier-Stokes equations, Oqielat et al. [39] solved the fuzzy quadratic Riccati differential equations, and Saadeh et al. [40] introduced the reliable solutions to the fractional Lane-Emden equations via the LRPS approach. Moreover, several works on solving different types of fractional differential equations using the LRPS method can be found in the literature [41–44].

The Schrödinger equation [10,16,38,45] is a fundamental equation in quantum mechanics that describes how quantum systems, such as electrons or atoms, evolve in time. It is named after the Austrian physicist Erwin Schrödinger, who first proposed it in 1926. This equation is a partial differential equation (PDE) that relates the wave function (or state function) of a quantum system to its position, momentum, and energy, and it describes the wave-like nature of quantum particles and the probability of finding a particle in a specific location at a specific time. The wave function describes the probability amplitude of a particle, and it is a key concept in quantum mechanics. The Schrödinger equation is a key tool in understanding the behavior of quantum systems and is used to calculate various properties such as energy levels, transition probabilities, and wave functions. It plays a central role in many areas of physics, including quantum mechanics and quantum chemistry. Quantum mechanics with the Schrödinger equation is a subset of quantum theory whose primary flaw is that it is a non-relativistic theory. This theory explains the significant phenomena such as the quantization of energy state in hydrogen atoms. The indeterminacy principle can be derived from Schrödinger's quantum mechanics, which is of philosophical relevance [45,46]. Finally, for more details about the reasons and physical meaning of fractional calculus, kindly refer to the literature [47].

The fractional Schrödinger equation (FSE) [48–51] is a generalization of the conventional Schrödinger equation, which is derived from fractional quantum mechanics, according to the expansion using Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths [48]. Instead of the first-order time derivative in the regular Schrödinger equation, the FSE incorporates an α-fractional-order time derivative. The fundamental theory behind the solution concept for time-FSE is considerable and intriguing, and it is a topic of current scientific, theoretical, and engineering studies in the broad sense [49–51].

One of the most general forms of the one-dimensional FSE is the following form [52]:

$$iD_t^{\alpha}\psi(x,t)+\delta\psi_{xx}(x,t)+\gamma|\psi(x,t)|^2\psi(x,t)+\phi(x)\psi(x,t)=0,\ x\in\mathbb{R},\ t\geq t_0,\ 0<\alpha\leq 1, \quad (1.1)$$

subject to the initial condition:

$$\psi(x,t_0)=\zeta(x), \quad (1.2)$$

where $i^2=-1$, D_t^{α} is the Caputo time-fractional derivative of order α, the constants $\delta,\gamma\in\mathbb{R}$, and $|.|$ is the modulus. The multivariable function ψ is the macroscopic wave function of the condensate, the single variable $\phi(x)$ is the external trapping potential analytic function, and $\zeta(x)$ is an analytic displacement function.

As is known to the research community, there is no single way to solve differential equations. Therefore, researchers' efforts focus on searching for new ways to solve various types of differential and non-differential equations. Researchers strive for these methods to provide speed, less effort, and accuracy, in addition to their ability to solve a wider range of equations. Problem (1.1) with (1.2) has been solved in several ways such as Adomian decomposition [10], homotopy perturbation [10], homotopy analysis [16], variational iteration [20], and RPS [21] methods. Each method has its advantages and disadvantages. They all arrived at the same solution, but each method had different advantages.

We aim to adapt and test the LRPS method to generate exact and accurate approximate analytical solutions for the time-FSE of one dimension in the context of the initial value problem (IVP) (1.1) with (1.2). The purpose is to demonstrate that the LRPS method is a simple, efficient, and applicable way to solve the IVP (1.1) with (1.2), without requiring differentiation, linearization, or discretization like other methods. One of the advantages of this method is that it can adapt the Laplace transform to solve non-linear equations. The solution is expressed as an infinite series that rapidly converges to the exact solution. The LRPS method requires only defining the LRPS function and taking the limit at infinity, which can be computed without computer programs. In cases where the Mathematica software is used, the method requires less computational time than other methods. The accuracy of the obtained solutions is tested by analyzing two types of errors: Absolute, and relative errors. To study the behavior of solutions by changing the order of the fractional derivatives in FSE and to estimate the region of convergence for the series solutions, graphical solutions in three dimensions are considered. Additionally, three important and interesting examples are given and discussed to show that the proposed method is accurate, efficient, and applicable.

2. Basic results on fractional and Laplace operators

This section reviews some basic concepts on fractional operators that are very important to get our results in other sections below.

Definition 2.1. [2] The Caputo time-fractional derivative of order $\alpha\in(m-1,m]$ of the multivariable function $\psi(x,t)$ is defined as

$$D_t^{\alpha}\psi(x,t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\xi)^{n-\alpha-1}\partial_{\xi}^{n}\psi(x,\xi)d\xi\,, & m-1<\alpha<m,\\ \partial_t^{m}\psi(x,t)=\frac{\partial^m}{\partial t^m}\psi(x,t), & \alpha=m,\ m\in N.\end{cases} \quad (2.1)$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. [6] If the following improper integral

$$\Psi(x,s)=\int_0^{\infty} e^{-st}\psi(x,t)dt \tag{2.2}$$

exists for all s in some a domain $D\subseteq\mathbb{C}$, then $\Psi(x,s)$ is called the Laplace transform of $\psi(x,t)$, and denoted by $\mathcal{L}[\psi(x,t)](x,s)$. The original function $\psi(x,t)$ can be restored from the Laplace transform $\Psi(x,s)$ with the help of the following inverse Laplace transform:

$$\psi(x,t)=\mathcal{L}^{-1}[\Psi(x,s)]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Psi(x,s)ds,\ c=\mathrm{Re}(s)>c_0, \tag{2.3}$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral.

Here are some of the necessary properties of LT and its inverse that will be used in our work.

Lemma 2.3. [36] Let $\psi(x,t)$ be a piecewise continuous function on $I\times[0,\infty)$ and of exponential order, and $\Psi(x,s)=\mathcal{L}[\psi(x,t)]$. Then, we have

(i) $\lim_{s\to\infty} s\Psi(x,s)=\psi(x,0)$.

(ii) $\mathcal{L}[D_t^{\alpha}\psi(x,t)]=s^{\alpha}\Psi(x,s)-\sum_{k=0}^{m-1}s^{\alpha-k-1}\partial_t^k\psi(x,0)$, $m-1<\alpha<m$.

(iii) $\mathcal{L}[D_t^{n\alpha}\psi(x,t)]=s^{n\alpha}\Psi(x,s)-\sum_{k=0}^{m-1}s^{(n-k)\alpha-1}D_t^{k\alpha}\psi(x,0)$, $0<\alpha<1$.

Definition 2.4. [23] The α-power series about t_0 is defined as follows:

$$\sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha}=f_0(x)+f_1(x)(t-t_0)^{\alpha}+f_2(x)(t-t_0)^{2\alpha}+\cdots,\ \alpha>0,\ t\geq t_0, \tag{2.4}$$

where the coefficients $f_n(x)$ are functions of x.

Theorem 2.5. [23] Suppose that $\psi(x,t)$ has α-power series representation about t_0 as follows:

$$\psi(x,t)=\sum_{n=0}^{\infty}f_n(x)(t-t_0)^{n\alpha},\ 0\leq m-1<\alpha\leq m,\ x\in I,\ t_0\leq t<t_0+R. \tag{2.5}$$

If $D_t^{n\alpha}\psi(x,t)$ are continuous on $I\times(t_0,t_0+R)$, $n=0,1,2,\ldots$, then the coefficients $f_n(x)$ are given as

$$f_n(x)=\frac{(D_t^{n\alpha}\psi)(x,t_0)}{\Gamma(n\alpha+1)},\ n=0,1,2,\ldots, \tag{2.6}$$

where $D_t^{n\alpha}=D_t^{\alpha}.D_t^{\alpha}\ldots D_t^{\alpha}$($n$-times).

Definition 2.6. [36] The α- singular Laurent series about $s=0$ is defined as follows:

$$\sum_{n=0}^{\infty}\frac{f_n(x)}{s^{n\alpha+1}}=\frac{f_0(x)}{s}+\frac{f_1(x)}{s^{\alpha+1}}+\frac{f_2(x)}{s^{2\alpha+1}}+\cdots,\ \alpha>0,\ s>0, \tag{2.7}$$

where the coefficients $f_n(x)$ are functions of x.

Theorem 2.7. [36] Suppose $\psi(x,t)$ has a Laplace transform over $I\times[0,\infty)$, such that $\mathcal{L}[\psi(\mathrm{x},\mathrm{t})]=\Psi(x,s)$. Suppose $\Psi(x,s)$ has the following α- singular Laurent series representation:

$$\Psi(x,s)=\sum_{n=0}^{\infty}\frac{f_n(x)}{s^{n\alpha+1}},\ 0<\alpha\leq 1,\ s>0. \tag{2.8}$$

Then, $f_n(x)=(D_t^{n\alpha}\psi)(x,0)$.

Note that the inverse Laplace transform of the α-singular Laurent series in Theorem 2.6 has the following form:

$$\psi(x,t)=\sum_{n=0}^{\infty}\frac{(D_t^{n\alpha}\psi)(x,0)}{\Gamma(n\alpha+1)}t^{n\alpha},\ 0<\alpha\leq 1,\ t\geq 0, \tag{2.9}$$

which is the α-power series representation of $\psi(x,t)$ about $t=0$ that is given in Theorem 2.5.

Theorem 2.8. [36] Let $\psi(x,t)$ be a piecewise continuous on $I\times[0,\infty)$ and of exponential order $\lambda(x)$ and let $\Psi(x,s)=\mathcal{L}[\psi(x,t)]$can be written as the α- singular Laurent series in the Theorem 2.7. If $\left|s\mathcal{L}\left[D_t^{(n+1)\alpha}\psi(x,t)\right]\right|\leq \mathrm{K}(x)$, on $I\times(\delta,\gamma]$ where $0<\alpha\leq 1$, then the remainder of the α-singular Laurent expansion in Eq (2.8) becomes

$$|\Omega_n(x,s)|\leq\frac{K(x)}{s^{1+(n+1)\alpha}},\ x\in I,\ 0\leq\delta<s\leq\gamma. \tag{2.10}$$

3. Constructing LRPS solutions to the FSE

This section adopts the LRPS method for introducing and constructing a new analytical solution to the FSE. As we mentioned in Section 1, we transpose Eqs (1.1) and (1.2) to the Laplace space, and then we solve the obtained result using the α-Laurent series. To accomplish this idea, we rewrite the complex functions $\psi(x,t)$ and $\zeta(x)$ in terms of the real and imaginary parts as follows:

$$\psi(x,t)=u(x,t)+iv(x,t),\ \zeta(x)=f(x)+ig(x), \tag{3.1}$$

where $u(x,t),v(x,t)$are multivariable real-valued analytic functions defined for each $x\in I\subseteq\mathbb{R}$, $t\geq 0$, and $f(x),g(x)$are real-valued analytic functions defined for each $x\in I\subseteq\mathbb{R}$.

In any case, Eq (1.1) can be rewritten as

$$\left[D_t^{\alpha}v(x,t)-\delta u_{xx}(x,t)-\gamma\left(u^2(x,t)+v^2(x,t)\right)u(x,t)-\phi(x)u(x,t)\right]$$

$$-i\left[D_t^{\alpha}u(x,t)+\delta v_{xx}(x,t)+\gamma\left(u^2(x,t)+v^2(x,t)\right)v(x,t)+\phi(x)v(x,t)\right]=0, \tag{3.2}$$

and the initial conditions in Eq (1.2) are as follows:

$$iu(x,0)+iv(x,0)=f(x)+ig(x), \tag{3.3}$$

Based on that, the time FSE, Eq (1.1), can be transformed into an equivalent system of PDEs as follows using the results in (3.1)–(3.3):

$$\begin{cases}D_t^{\alpha}u(x,t)+\delta v_{xx}(x,t)+\gamma\left(u^2(x,t)+v^2(x,t)\right)v(x,t)+\phi(x)v(x,t)=0,\\ D_t^{\alpha}v(x,t)-\delta u_{xx}(x,t)-\gamma\left(u^2(x,t)+v^2(x,t)\right)u(x,t)-\phi(x)u(x,t)=0,\end{cases} \tag{3.4}$$

subject to the following initial conditions:

$$u(x,0)=f(x),\ v(x,0)=g(x). \tag{3.5}$$

The solution of system (3.4), subject to the initial conditions (3.5), is the solution to Eqs (1.1) and (1.2) completely. Therefore, we will construct the LRPS solution for the system (3.4). The first step is to apply the Laplace transform to the system (3.4) to transfer it to Laplace space and utilize the conditions in (3.5), as follows:

$$\begin{cases} U(x,s) - \frac{f(x)}{s} + \frac{\delta}{s^\alpha} V_{xx}(x,s) + \frac{\phi(x)}{s^\alpha} V(x,s) + \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^2 (\mathcal{L}^{-1}\{V(x,s)\})\} \\ + \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^3\} = 0, \\ V(x,s) - \frac{g(x)}{s} - \frac{\delta}{s^\alpha} U_{xx}(x,s) - \frac{\phi(x)}{s^\alpha} U(x,s) - \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^2 (\mathcal{L}^{-1}\{U(x,s)\})\} \\ - \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^3\} = 0, \end{cases} \tag{3.6}$$

where $U(x,s) = \mathcal{L}\{u(x,t)\}$ and $V(x,s) = \mathcal{L}[v(x,t)]$.

In the next step, we assume the solution of the algebraic system (3.6), $U(x,s)$ and $V(x,s)$, has the following expansions:

$$\begin{cases} U(x,s) = \sum_{i=0}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}}, \ \ 0 < \alpha \le 1, \ \ x \in I, \ \ s > 0, \\ V(x,s) = \sum_{i=0}^{\infty} \frac{g_i(x)}{s^{1+i\alpha}}, \ \ 0 < \alpha \le 1, \ \ x \in I, \ \ s > 0. \end{cases} \tag{3.7}$$

Theorem 2.7 provides the first coefficient of the system (3.7). So, the kth truncated series of the system (3.7) is given by

$$\begin{cases} U_k(x,s) = \frac{f(x)}{s} + \sum_{i=1}^{k} \frac{f_i(x)}{s^{1+i\alpha}}, \ \ 0 < \alpha \le 1, \ \ x \in I, \ \ s > 0, \\ V_k(x,s) = \frac{g(x)}{s} + \sum_{i=1}^{k} \frac{g_i(x)}{s^{1+i\alpha}}, \ \ 0 < \alpha \le 1, \ \ x \in I, \ \ s > 0. \end{cases} \tag{3.8}$$

The third step to constructing an LRPS solution is to define the so-called Laplace residual functions (LRFs) of the algebraic system (3.4) with (3.5) as follows:

$$\begin{cases} LRes^1(x,s) = U(x,s) - \frac{f(x)}{s} + \frac{\delta}{s^\alpha} V_{xx}(x,s) + \frac{\phi(x)}{s^\alpha} V(x,s) \\ \qquad + \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^2 (\mathcal{L}^{-1}\{V(x,s)\})\} + \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^3\}, \\ LRes^2(x,s) = V(x,s) - \frac{g(x)}{s} - \frac{\delta}{s^\alpha} U_{xx}(x,s) - \frac{\phi(x)}{s^\alpha} U(x,s) \\ \qquad - \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^2 (\mathcal{L}^{-1}\{U(x,s)\})\} - \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^3\}, \end{cases} \tag{3.9}$$

then we can define the kth LRFs as follows:

$$\begin{cases} LRes_k^1(x,s) = U_k(x,s) - \frac{f(x)}{s} + \frac{\delta}{s^\alpha} (V_k)_{xx}(x,s) + \frac{\phi(x)}{s^\alpha} V_k(x,s) \\ \qquad + \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{U_k(x,s)\})^2 (\mathcal{L}^{-1}\{V_k(x,s)\})\} + \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{V_k(x,s)\})^3\}, \\ LRes_k^2(x,s) = V_k(x,s) - \frac{g(x)}{s} - \frac{\delta}{s^\alpha} (U_k)_{xx}(x,s) - \frac{\phi(x)}{s^\alpha} U_k(x,s) \\ \qquad - \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{V_k(x,s)\})^2 (\mathcal{L}^{-1}\{U_k(x,s)\})\} - \frac{\gamma}{s^\alpha} \mathcal{L}\{(\mathcal{L}^{-1}\{U_k(x,s)\})^3\}. \end{cases} \tag{3.10}$$

Without a doubt, it is clear that $\text{Lim}_{k\to\infty} LRes_k(s) = LRes(s)$, $LRes(s) = 0$, and thus $s^k LRes(s) = 0$ for $s > 0$ and $k = 0,1,2,3,\ldots$. Therefore, $\text{Lim}_{s\to\infty}\left(s^k LRes(s)\right) = 0$. Moreover, El-Ajou [29] proved that

$$\lim_{s\to\infty} s^{k\alpha+1} LRes_k(s) = 0, \ \ \alpha > 0, \ \ k = 1,2,3, \ldots \tag{3.11}$$

Fact (3.11) is considered the essential tool for finding the unknown coefficients of the expansions in the system (3.7). So, for determining the first unknown coefficient of the system (3.7), substitute the

1st truncated series of the system (3.8) that is given by

$$U_1(x,s)=\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}},\qquad V_1(x,s)=\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}},$$

into the 1st LRF to obtain

$$\begin{cases} LRes_1^1(x,s)=\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}-\frac{f(x)}{s}+\frac{\delta}{s^\alpha}\left(\frac{g''(x)}{s}+\frac{g_1''(x)}{s^{1+\alpha}}\right)+\frac{\phi(x)}{s^\alpha}\left(\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}\right) \\ \qquad +\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left\{\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}\right\}\right)^2\mathcal{L}^{-1}\left\{\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}\right\}\right\} \\ \qquad +\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left\{\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}\right\}\right)^3\right\}, \\ LRes_1^2(x,s)=\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}-\frac{g(x)}{s}-\frac{\delta}{s^\alpha}\left(\frac{f''(x)}{s}+\frac{f_1''(x)}{s^{1+\alpha}}\right)-\frac{\phi(x)}{s^\alpha}\left(\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}\right) \\ \qquad -\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left\{\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}\right\}\right)^2\mathcal{L}^{-1}\left\{\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}\right\}\right\} \\ \qquad -\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left\{\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}\right\}\right)^3\right\}. \end{cases} \tag{3.12}$$

By running the Laplace transforms in the system (3.12) and multiplying both sides by $s^{\alpha+1}$, one can get

$$\begin{cases} s^{\alpha+1}LRes_1^1(x,s)=f_1(x)+\delta g''(x)+\delta\frac{g''(x)}{s^\alpha}+\phi(x)g(x)+\frac{\phi(x)g_1(x)}{s^\alpha}+\gamma f^2(x)g(x) \\ \qquad +\gamma\frac{f^2(x)g_1(x)}{s^\alpha}+2\gamma\frac{f_1(x)f(x)g_0(x)}{s^{\alpha-1}}+2\gamma\frac{f_1(x)f(x)g_1(x)}{\Gamma(1+2\alpha)s^{2\alpha}} \\ \qquad +\cdots+\gamma\frac{g_1^2(x)g(x)}{\Gamma(1+2\alpha)s^{2\alpha}}+\gamma\frac{g_1^3(x)\Gamma(1+3\alpha)}{\Gamma^2(1+2\alpha)\Gamma(1+\alpha)s^{2\alpha}}, \\ s^{\alpha+1}LRes_1^2(x,s)=g_1(x)-\delta f''(x)-\delta\frac{f''(x)}{s^\alpha}-\phi(x)f(x)-\frac{\phi(x)f_1(x)}{s^\alpha}-\gamma g^2(x)f(x) \\ \qquad -\gamma\frac{f_1(x)g^2(x)}{s^\alpha}-2\gamma\frac{g_1(x)g(x)f(x)}{s^{\alpha-1}}-2\gamma\frac{g_1(x)g(x)f_1(x)}{\Gamma(1+2\alpha)s^{2\alpha}} \\ \qquad -\cdots-\gamma\frac{f_1^2(x)f(x)}{\Gamma(1+2\alpha)s^{2\alpha}}-\gamma\frac{f_1^3(x)\Gamma(1+3\alpha)}{\Gamma^2(1+2\alpha)\Gamma(1+\alpha)s^{2\alpha}}. \end{cases} \tag{3.13}$$

Taking the limit of both sides of the system (3.13) as $s\to\infty$ and employing the fact in the system (3.11), we obtain the following algebraic system:

$$\begin{cases} f_1(x)+\delta g''(x)+\phi(x)g(x)+\gamma f^2(x)g(x)+\gamma g^3(x)=0, \\ g_1(x)-\delta f''(x)-\phi(x)f(x)-\gamma g^2(x)f(x)-\gamma f^3(x)=0. \end{cases} \tag{3.14}$$

Solving the resulting algebraic system (3.14) for $f_1(x)$ and $g_1(x)$ gives the form of the first unknown coefficients of the system (3.7) as functions of x as

$$\begin{cases} f_1(x)=-\left(\delta g''(x)+\phi(x)g(x)+\gamma f^2(x)g(x)+\gamma g^3(x)\right), \\ g_1(x)=\delta f''(x)+\phi(x)f(x)+\gamma g^2(x)f(x)+\gamma f^3(x). \end{cases} \tag{3.15}$$

Similarly, to find the value of the second unknown coefficients $U_2(x,s)$ and $V_2(x,s)$, substitute the 2nd truncated series $U_2(x,s)=\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}$ and $V_2(x,s)=\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}$ into the 2nd LRF, $LRes_2^1(x,s)$, and $LRes_2^2(x,s)$ to get

$$\begin{aligned}LRes_2^1(x,s)=&\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{\delta}{s^\alpha}\left(\frac{g''(x)}{s}+\frac{g_1''(x)}{s^{1+\alpha}}+\frac{g_2''(x)}{s^{1+2\alpha}}\right)+\frac{\phi(x)}{s^\alpha}\left(\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}\right)\\&+\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\{\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}\}\right)^3\right\}\\&+\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\{\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}\}\right)^2\mathcal{L}^{-1}\left\{\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}\right\}\right\},\end{aligned}\tag{3.16}$$

$$\begin{aligned}LRes_2^2(x,s)=&\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}-\frac{\delta}{s^\alpha}\left(\frac{f''(x)}{s}+\frac{f_1''(x)}{s^{1+\alpha}}+\frac{f_2''(x)}{s^{1+2\alpha}}\right)-\frac{\phi(x)}{s^\alpha}\left(\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}\right)\\&-\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left\{\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}\right\}\right)^3\right\}\\&-\frac{\gamma}{s^\alpha}\mathcal{L}\left\{\left(\mathcal{L}^{-1}\left\{\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}\right\}\right)^2\mathcal{L}^{-1}\left\{\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}\right\}\right\}.\end{aligned}\tag{3.17}$$

Running the transforms in the system (3.16) and (3.17) and multiplying both sides by $s^{2\alpha+1}$ gives

$$\begin{cases}s^{2\alpha+1}LRes_2^1(x,s)=f_2(x)+\delta g_1''(x)+\phi(x)g_1(x)+\gamma f^2(x)g_1(x)+2\gamma f_1(x)f(x)g(x)\\\qquad+3\gamma g^2(x)g_1(x)+\delta\frac{g_2''(x)}{s^\alpha}+\frac{\phi(x)g_2(x)}{s^\alpha}+\cdots+\gamma\frac{g_2^3(x)\Gamma(1+6\alpha)}{\Gamma^3(1+2\alpha)s^{5\alpha}},\\s^{2\alpha+1}LRes_2^2(x,s)=g_2(x)-\delta f_1''(x)-\phi(x)f_1(x)-\gamma g^2(x)f_1(x)-2\gamma g_1(x)g(x)f(x)\\\qquad-3\gamma f^2(x)f_1(x)-\delta\frac{f_2''(x)}{s^\alpha}-\frac{\phi(x)f_2(x)}{s^\alpha}-\cdots-\gamma\frac{f_2^3(x)\Gamma(1+6\alpha)}{\Gamma^3(1+2\alpha)s^{5\alpha}}.\end{cases}\tag{3.18}$$

Computing the limit as s goes to infinity for both sides of the system (3.18) and using the facts in the system (3.11), we get

$$\begin{cases}f_2(x)=-\left(\delta g_1''(x)+\phi(x)g_1(x)+\gamma f^2(x)g_1(x)+2\gamma f_1(x)f(x)g(x)+3\gamma g^2(x)g_1(x)\right),\\g_2(x)=\delta f_1''(x)+\phi(x)f_1(x)+\gamma g^2(x)f_1(x)+2\gamma g_1(x)g(x)f(x)+3\gamma f^2(x)f_1(x).\end{cases}\tag{3.19}$$

The third unknown coefficients $U_3(x,s)$, and $V_3(x,s)$ can be determined by substituting the 3rd truncated series $U_3(x,s)=\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{f_3(x)}{s^{1+3\alpha}}$ and $V_3(x,s)=\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}+\frac{g_3(x)}{s^{1+3\alpha}}$, into the 3rd LRF, $LRes_3^1(x,s)$ and $LRes_3^2(x,s)$ to get the following functions:

$$\begin{cases}LRes_3^1(x,s)=\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{f_3(x)}{s^{1+3\alpha}}+\frac{\delta}{s^\alpha}\left(\frac{g''(x)}{s}+\frac{g_1''(x)}{s^{1+\alpha}}+\frac{g_2''(x)}{s^{1+2\alpha}}+\frac{g_3''(x)}{s^{1+3\alpha}}\right)\\\qquad+\frac{\phi(x)}{s^\alpha}\left(\frac{g(x)}{s}+\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}+\frac{g_3(x)}{s^{1+3\alpha}}\right)+\frac{\gamma}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{V_3(x,s)\})^3\}\\\qquad+\frac{\gamma}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{U_3(x,s)\})^2\mathcal{L}^{-1}\{V_3(x,s)\}\},\\LRes_3^2(x,s)=\frac{g_1(x)}{s^{1+\alpha}}+\frac{g_2(x)}{s^{1+2\alpha}}+\frac{g_3(x)}{s^{1+3\alpha}}-\frac{\delta}{s^\alpha}\left(\frac{f''(x)}{s}+\frac{f_1''(x)}{s^{1+\alpha}}+\frac{f_2''(x)}{s^{1+2\alpha}}+\frac{f_3''(x)}{s^{1+3\alpha}}\right)\\\qquad-\frac{\phi(x)}{s^\alpha}\left(\frac{f(x)}{s}+\frac{f_1(x)}{s^{1+\alpha}}+\frac{f_2(x)}{s^{1+2\alpha}}+\frac{f_3(x)}{s^{1+3\alpha}}\right)-\frac{\gamma}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{U_3(x,s)\})^3\}\\\qquad-\frac{\gamma}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{V_3(x,s)\})^2\mathcal{L}^{-1}\{U_3(x,s)\}\}.\end{cases}\tag{3.20}$$

Again, by multiplying both sides of the system (3.20) by $s^{3\alpha+1}$ and computing the limit as s goes to infinity by utilizing the facts (3.11), we can get the following:

$$\begin{cases} f_3(x) = -(\delta g_2''(x) + \phi(x)g_2(x) + \gamma f^2(x)g_2(x) \\ \quad +\gamma g(x)\big(2f(x)f_2(x) + 3g(x)g_2(x)\big) \\ \quad +\gamma(2f(x)f_1(x)g_1(x) + 3g(x)(f_1^2(x) + g_1^2(x)))\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\Big), \\ g_3(x) = \delta f_2''(x) + \phi(x)f_2(x) + \gamma g^2(x)f_2(x) \\ \quad +\gamma f(x)\big(2g(x)g_2(x) + 3f(x)f_2(x)\big) \\ \quad +\gamma\left(2g(x)g_1(x)f_1(x) + f(x)\big(g_1^2(x) + 3f_1^2(x)\big)\right)\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}. \end{cases} \tag{3.21}$$

Therefore, by using the obtained coefficients of the series in (3.7), the solution of the system (3.6), can be expressed in an α-Laurent series form as

$$U(x,s) = \sum_{i=0}^{\infty}\frac{f_i(x)}{s^{1+i\alpha}} = \frac{f(x)}{s} - \frac{\big(\delta g''(x) + \phi(x)g(x) + \gamma f^2(x)g(x) + \gamma g^3(x)\big)}{s^{1+\alpha}}$$
$$-\frac{\big(\delta g_1''(x) + \phi(x)g_1(x) + \gamma f^2(x)g_1(x) + 2\gamma f_1(x)f(x)g(x) + 3\gamma g^2(x)g_1(x)\big)}{s^{1+2\alpha}}$$
$$-\Bigg(\delta g_2''(x) + \phi(x)g_2(x) + \gamma f^2(x)g_2(x) + \gamma g(x)\big(2f(x)f_2(x) + 3g(x)g_2(x)\big)$$
$$+\gamma(2f(x)f_1(x)g_1(x) + 3g(x)(f_1^2(x) + g_1^2(x)))\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\Bigg)\frac{1}{s^{1+3\alpha}} + \cdots,$$

$$V(x,s) = \sum_{i=0}^{\infty}\frac{g_i(x)}{s^{1+i\alpha}} = \frac{g(x)}{s} + \frac{\delta f''(x) + \phi(x)f(x) + \gamma g^2(x)f(x) + \gamma f^3(x)}{s^{1+\alpha}}$$
$$+\frac{\big(\delta f_1''(x) + \phi(x)f_1(x) + \gamma g^2(x)f_1(x) + 2\gamma g_1(x)g(x)f(x) + 3\gamma f^2(x)f_1(x)\big)}{s^{1+2\alpha}}$$
$$+\Bigg(\delta f_2''(x) + \phi(x)f_2(x) + \gamma g^2(x)f_2(x) + \gamma f(x)\big(2g(x)g_2(x) + 3f(x)f_2(x)\big)$$
$$+\gamma\left(2g(x)g_1(x)f_1(x) + 3f(x)\big(g_1^2(x) + f_1^2(x)\big)\right)\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\Bigg)\frac{1}{s^{1+3\alpha}} + \cdots.$$

Applying the inverse Laplace transform gives the approximate LRPS solution to system (3.4) with (3.5) in the following series form:

$$u(x,t) = f(x) - \frac{\big(\delta g''(x)+\phi(x)g(x)+\gamma f^2(x)g(x)+\gamma g^3(x)\big)}{\Gamma(1+\alpha)}t^{\alpha} \tag{3.22}$$
$$-\frac{\big(\delta g_1''(x) + \phi(x)g_1(x) + \gamma f^2(x)g_1(x) + 2\gamma f_1(x)f(x)g(x) + 3\gamma g^2(x)g_1(x)\big)}{\Gamma(1+2\alpha)}t^{2\alpha}$$
$$-\Big(\delta g_2''(x) + \phi(x)g_2(x) + \gamma f^2(x)g_2(x) + \gamma g(x)\big(2f(x)f_2(x) + 3g(x)g_2(x)\big) +$$
$$\gamma\left(2f(x)f_1(x)g_1(x) + 3g(x)\big(f_1^2(x) + g_1^2(x)\big)\right)\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - \cdots,$$

$$v(x,t)=\frac{g(x)}{s}+\frac{\delta f''(x)+\phi(x)f(x)+\gamma g^2(x)f(x)+\gamma f^3(x)}{\Gamma(1+\alpha)}t^{\alpha} \tag{3.23}$$
$$+\frac{\left(\delta f_1''(x)+\phi(x)f_1(x)+\gamma g^2(x)f_1(x)+2\gamma g_1(x)g(x)f(x)+3\gamma f^2(x)f_1(x)\right)}{\Gamma(1+2\alpha)}t^{2\alpha}$$
$$+\Big(\delta f_2''(x)+\phi(x)f_2(x)+\gamma g^2(x)f_2(x)+\gamma f(x)\big(2g(x)g_2(x)+3f(x)f_2(x)\big)+$$
$$\gamma\left(2g(x)g_1(x)f_1(x)+3f(x)\big(g_1^2(x)+f_1^2(x)\big)\right)\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots.$$

4. Some applications

In this section, three attractive problems are given and solved to explain that the LRPS algorithm is efficient, simple, and accurate. Mathematica software was used to perform all the symbolic and numerical calculations.

Application 4.1. Consider the following one-dimensional linear time-FSE:

$$iD_t^{\alpha}\psi(x,t)-\psi_{xx}(x,t)=0,\ x\in\mathbb{R},\ t\geq 0,\ 0<\alpha\leq 1. \tag{4.1}$$

subject to the constraint:

$$\psi(x,0)=e^{3ix}. \tag{4.2}$$

The time-FSE in Eq (4.1) can be transformed into an equivalent system of fractional PDEs as follows:

$$\begin{cases} D_t^{\alpha}u(x,t)+v_{xx}(x,t)=0, \\ D_t^{\alpha}v(x,t)-u_{xx}(x,t)=0, \end{cases} \tag{4.3}$$

subject to the following constraints:

$$u(x,0)=\cos(3x),\ v(x,0)=\sin(3x). \tag{4.4}$$

By comparing system (4.3) with (4.4) and system (3.4) with (3.5), we find that $\delta=1$, $\gamma=0$, $\phi(x)=0$, $f(x)=\cos 3x$, and $g(x)=\sin 3x$. Therefore, we can determine the fourth approximate LRPS solution of system (4.3) with (4.4) according to the results in Eqs (3.22) and (3.23). However, to further illustrate the proposed method, we will use the procedures performed in the previous section to arrive at the LRPS solution. The first step for this is to apply the Laplace transform to Eq (4.3) to transfer it to the Laplace space, and using conditions (4.4), one can obtain

$$\begin{cases} U(x,s)-\frac{\cos(3x)}{s}-\frac{1}{s^{\alpha}}V_{xx}(x,s)=0, \\ V(x,s)-\frac{\sin(3x)}{s}+\frac{1}{s^{\alpha}}U_{xx}(x,s)=0. \end{cases} \tag{4.5}$$

Suppose the solution of the algebraic system (4.5), $U(x,s)$ and $V(x,s)$, has α-Laurent series as in Eq (3.7), and using the conditions in (4.4), the kth truncated series of α-Laurent series will have the following expression:

$$\begin{cases} U_k(x,s)=\frac{\cos(3x)}{s}+\sum_{i=1}^{k}\frac{f_i(x)}{s^{1+i\alpha}},\ 0<\alpha\leq 1,\ x\in I,\ s>0, \\ V_k(x,s)=\frac{\sin(3x)}{s}+\sum_{i=1}^{k}\frac{g_i(x)}{s^{1+i\alpha}},\ 0<\alpha\leq 1,\ x\in I,\ s>0. \end{cases} \tag{4.6}$$

So, the kth LRFs of the system (4.5) are defined as

$$\begin{cases} LRes_k^1(x,s) = U_k(x,s) - \frac{\cos(3x)}{s} - \frac{1}{s^\alpha}(V_k)_{xx}(x,s), \\ LRes_k^2(x,s) = V_k(x,s) - \frac{\sin(3x)}{s} + \frac{1}{s^\alpha}(U_k)_{xx}(x,s). \end{cases} \tag{4.7}$$

Now, to determine the first unknown coefficient of Eq (4.6), replace the 1st truncated series given by

$$U_1(x,s) = \frac{\cos(3x)}{s} + \frac{f_1(x)}{s^{1+\alpha}},\ V_1(x,s) = \frac{\sin(3x)}{s} + \frac{g_1(x)}{s^{1+\alpha}}, \tag{4.8}$$

into the 1st LRFs to get

$$\begin{cases} LRes_1^1(x,s) = \frac{\cos(3x)}{s} + \frac{f_1(x)}{s^{1+\alpha}} - \frac{\cos(3x)}{s} - \frac{1}{s^\alpha}\left(\frac{-9\sin(3x)}{s} + \frac{g_1''(x)}{s^{1+\alpha}}\right), \\ LRes_1^2(x,s) = \frac{\sin(3x)}{s} + \frac{g_1(x)}{s^{1+\alpha}} - \frac{\sin(3x)}{s} + \frac{1}{s^\alpha}\left(\frac{-9\cos(3x)}{s} + \frac{f_1''(x)}{s^{1+\alpha}}\right). \end{cases} \tag{4.9}$$

Multiply both sides of Eq (4.9) by $s^{\alpha+1}$ and taking the limit of both sides as $s \to \infty$, and considering the fact in Eq (3.11), then we have

$$f_1(x) = -9\sin(3x),\quad g_1(x) = 9\cos(3x). \tag{4.10}$$

Similarly, to find the value of the second unknown coefficients of the expansions of $U(x,s)$ and $V(x,s)$, substitute the 2nd truncated series, $U_2(x,s) = \frac{\cos(3x)}{s} - \frac{9\sin(3x)}{s^{1+\alpha}} + \frac{f_2(x)}{s^{1+2\alpha}}$ and $V_2(x,s) = \frac{\sin(3x)}{s} + \frac{9\cos(3x)}{s^{1+\alpha}} + \frac{g_2(x)}{s^{1+2\alpha}}$, into the 2nd LRFs, $LRes_2^1(x,s)$ and $LRes_2^2(x,s)$, to get the following:

$$\begin{cases} LRes_2^1(x,s) = \frac{-9\sin(3x)}{s^{1+\alpha}} + \frac{f_2(x)}{s^{1+2\alpha}} - \frac{1}{s^\alpha}\left(\frac{-9\sin(3x)}{s} - \frac{81\cos(3x)}{s^{1+\alpha}} + \frac{g_2''(x)}{s^{1+2\alpha}}\right), \\ LRes_2^2(x,s) = \frac{9\cos(3x)}{s^{1+\alpha}} + \frac{g_2(x)}{s^{1+2\alpha}} + \frac{\delta}{s^\alpha}\left(\frac{-9\cos(3x)}{s} + \frac{81\sin(3x)}{s^{1+\alpha}} + \frac{f_2''(x)}{s^{1+2\alpha}}\right). \end{cases} \tag{4.11}$$

Employing fact (3.11) after multiplying both sides of Eq (4.12) by $s^{1+2\alpha}$ and taking the limit as s goes to infinity gives the values of the following coefficients:

$$f_2(x) = -81\cos(3x),\quad g_2(x) = -81\sin(3x). \tag{4.12}$$

By applying the same procedure for $k = 3$ and $k = 4$ and taking into account the forms of $f_0(x), f_1(x), f_2(x)$ and $g_0(x), g_1(x), g_2(x)$, we can determine the values of the coefficients following:

$$\begin{aligned} f_3(x) &= 729\sin(3x), \qquad g_3(x) = -729\cos(3x), \\ f_4(x) &= 6561\cos(3x), \qquad g_4(x) = 6561\sin(3x). \end{aligned} \tag{4.13}$$

So, the 4th approximate solution of system (4.5) can be expressed as

$$\begin{cases} U_4(x,s) = \frac{\cos(3x)}{s} - \frac{9\sin(3x)}{s^{1+\alpha}} - \frac{81\cos(3x)}{s^{1+2\alpha}} + \frac{729\sin(3x)}{s^{1+3\alpha}} + \frac{6561\cos(3x)}{s^{1+4\alpha}}, \\ V_4(x,s) = \frac{\sin(3x)}{s} + \frac{9\cos(3x)}{s^{1+\alpha}} - \frac{81\sin(3x)}{s^{1+2\alpha}} - \frac{729\cos(3x)}{s^{1+3\alpha}} + \frac{6561\sin(3x)}{s^{1+4\alpha}}. \end{cases} \tag{4.14}$$

Applying the inverse Laplace transform on Eq (4.14) gives the 4th approximate LRPS solution to the system (4.3) with (4.4) in the following series form:

$$\begin{cases} u_4(x,t) = \cos(3x) - \frac{9\sin(3x)}{\Gamma(1+\alpha)}t^{\alpha} - \frac{(9)^2\cos(3x)}{\Gamma(1+2\alpha)}t^{2\alpha} + \frac{(9)^3\sin(3x)}{\Gamma(1+3\alpha)}t^{3\alpha} + \frac{(9)^4\cos(3x)}{\Gamma(1+4\alpha)}t^{4\alpha}, \\ v_4(x,t) = \sin(3x) + \frac{9\cos(3x)}{\Gamma(1+\alpha)}t^{\alpha} - \frac{(9)^2\sin(3x)}{\Gamma(1+2\alpha)}t^{2\alpha} - \frac{(9)^3\cos(3x)}{\Gamma(1+2\alpha)}t^{3\alpha} + \frac{(9)^4\sin(3x)}{\Gamma(1+4\alpha)}t^{4\alpha}. \end{cases} \tag{4.15}$$

Consequently, the components of LRPS solutions are obtained as much as we like. Checking the pattern of the coefficients, we can write the exact solution of $u(x,t)$ and $v(x,t)$ in series form as follows:

$$u(x,t) = \cos 3x \sum_{k=0}^{\infty}\left(\frac{(-1)^k(9)^{2k}t^{2k\alpha}}{\Gamma(1+2k\alpha)}\right) + \sin 3x \sum_{k=0}^{\infty}\left(\frac{(-1)^{k+1}(9)^{(2k+1)}t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}\right), \tag{4.16}$$

$$v(x,t) = \sin 3x \sum_{k=0}^{\infty}\left(\frac{(-1)^k(9)^{2k}t^{2k\alpha}}{\Gamma(1+2k\alpha)}\right) + \cos 3x \sum_{k=0}^{\infty}\left(\frac{(-1)^{k}(9)^{(2k+1)}t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}\right). \tag{4.17}$$

One can verify that the expansion of the exact solution can be separated and collected using Euler's formula for complex numbers to discover that the approximate solution of system (4.1) with (4.2) has a general pattern that exactly coincides with the series form of the solution as follows:

$$\psi(x,t) = e^{3ix}\sum_{n=0}^{\infty}(-9i)^n \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{4.18}$$

Also, we mention here that Eq (4.18) can be expressed in a term of Mittag-Leffler function as follows:

$$\psi(x,t) = e^{3ix}E_{\alpha}(-9it^{\alpha}). \tag{4.19}$$

It should be noted that the results obtained in Eqs (4.16) and (4.17) agree with the outcomes obtained by the RPS method [52]. Moreover, when $\alpha = 1$, we get the following solution to the system (4.1) with (4.2):

$$\psi(x,t) = e^{3i(x+3t)}, \tag{4.20}$$

which coincides with the solutions obtained by the Adomian decomposition technique [10], the homotopy perturbation method [10], the homotopy analysis method [16], and the variational iteration method [20].

To review some numerical results of the solution obtained in Eq (4.18), and to test the accuracy of the approximate solution of the method used, we use two types of error, namely, absolute error and relative error, which are defined respectively, as follows:

$$Abs.Err\,(x,t) = |\varphi(x,t) - \varphi_k(x,t)|, \tag{4.21}$$

where $\varphi(x,t)$ is the exact solution and $\varphi_k(x,t)$ is an approximate solution of order k. Sometimes the exact solution may not be available, so we can replace it with an approximation of a higher order than k such as $\varphi_{2k}(x,t)$.

Tables 1 and 2 display some numerical results for the solution of Application 4.1 given in Eqs (4.16) and (4.17) at $\alpha = 1$, and the corresponding absolute and relative errors. The results indicate the accuracy of the 10th approximate LRPS solutions obtained. The tables show that the convergent region is $\mathbb{R} \times [0,0.4]$ when $\alpha = 1$.

Table 1. Numerical comparisons between the exact value of $u(x,t)$ and the 10th-approximation of $u(x,t)$ at $\alpha = 1$.

x	t	$u_{10}(x,t)$	$u(x,t)$	**Re. Err.**	**Abs. Err.**
0		0.62161	0.62161	9.4433×10^{-10}	5.8701×10^{-10}
2.5	0.1	−0.519289	−0.519289	14.51896×10^{-9}	75.39531×10^{-10}
5		−0.981618	−0.981618	4.72682×10^{-9}	4.63993×10^{-9}
0		−0.227204	−0.227202	10.44299×10^{-6}	23.72669×10^{-7}
2.5	0.2	−0.992241	−0.992225	1.573828×10^{-5}	15.615924×10^{-6}
5		−0.460687	−0.460679	1.834986×10^{-5}	8.45339×10^{-6}
0		−0.904373	−0.904072	3.33126×10^{-4}	30.11701×10^{-5}
2.5	0.3	−0.715617	−0.714266	1.892477×10^{-3}	13.5173×10^{-4}
5		0.408257	0.408893	1.555287×10^{-3}	6.35945×10^{-4}
0		−0.905983	−0.896758	1.028624×10^{-2}	9.22427×10^{-3}
2.5	0.4	0.0725272	0.104236	0.30420×10^{-2}	3.31708×10^{-2}
5		0.956264	0.969022	1.3166×10^{-2}	1.2759×10^{-2}

Table 2. Numerical comparisons between the exact value of $v(x,t)$ and the 10th-approximation of $v(x,t)$ at $\alpha = 1$.

x	t	$v_{10}(x,t)$	$v(x,t)$	**Re. Err.**	**Abs. Err.**
0		0.783327	0.783327	9.984278×10^{-9}	7.820954×10^{-9}
2.5	0.1	0.854599	0.854599	2.527978×10^{-9}	2.160407×10^{-9}
5		−0.190859	−0.190859	3.313032×10^{-8}	6.323206×10^{-9}
0		0.97386	0.973848	1.619482×10^{-5}	1.577129×10^{-5}
2.5	0.2	0.124550	0.124454	2.604426×10^{-5}	3.241323×10^{-6}
5		−0.887517	−0.887567	1.523736×10^{-5}	1.352417×10^{-5}
0		0.428710	0.427380	3.111474×10^{-3}	1.329781×10^{-3}
2.5	0.3	−0.699696	−0.699875	2.549767×10^{-4}	1.784517×10^{-4}
5		−0.913789	−0.912582	1.321597×10^{-3}	1.206066×10^{-3}
0		−0.412125	−0.442520	6.868819×10^{-2}	3.039593×10^{-2}
2.5	0.4	−0.992669	−0.994553	1.894255×10^{-3}	1.883936×10^{-3}
5		−0.276064	−0.246974	1.177852×10^{-1}	2.908985×10^{-2}

Figure 1 shows the surface graph of the 10th approximate LRPS solutions, $u_{10}(x,t)$ and $v_{10}(x,t)$ of the system (4.1) with (4.2) at different values of α that are given in Eqs (4.16) and (4.17) in addition the exact solution when $\alpha = 1$. The graphs illustrate the behavior of the solutions at $\alpha = 1$, $\alpha = 0.9$, and $\alpha = 0.8$. Figure 1 illustrates the agreement between the 10th approximate solution and the exact solution for $\alpha = 1$, while all graphs depict the consistency of the behavior of the solutions at different values of α.

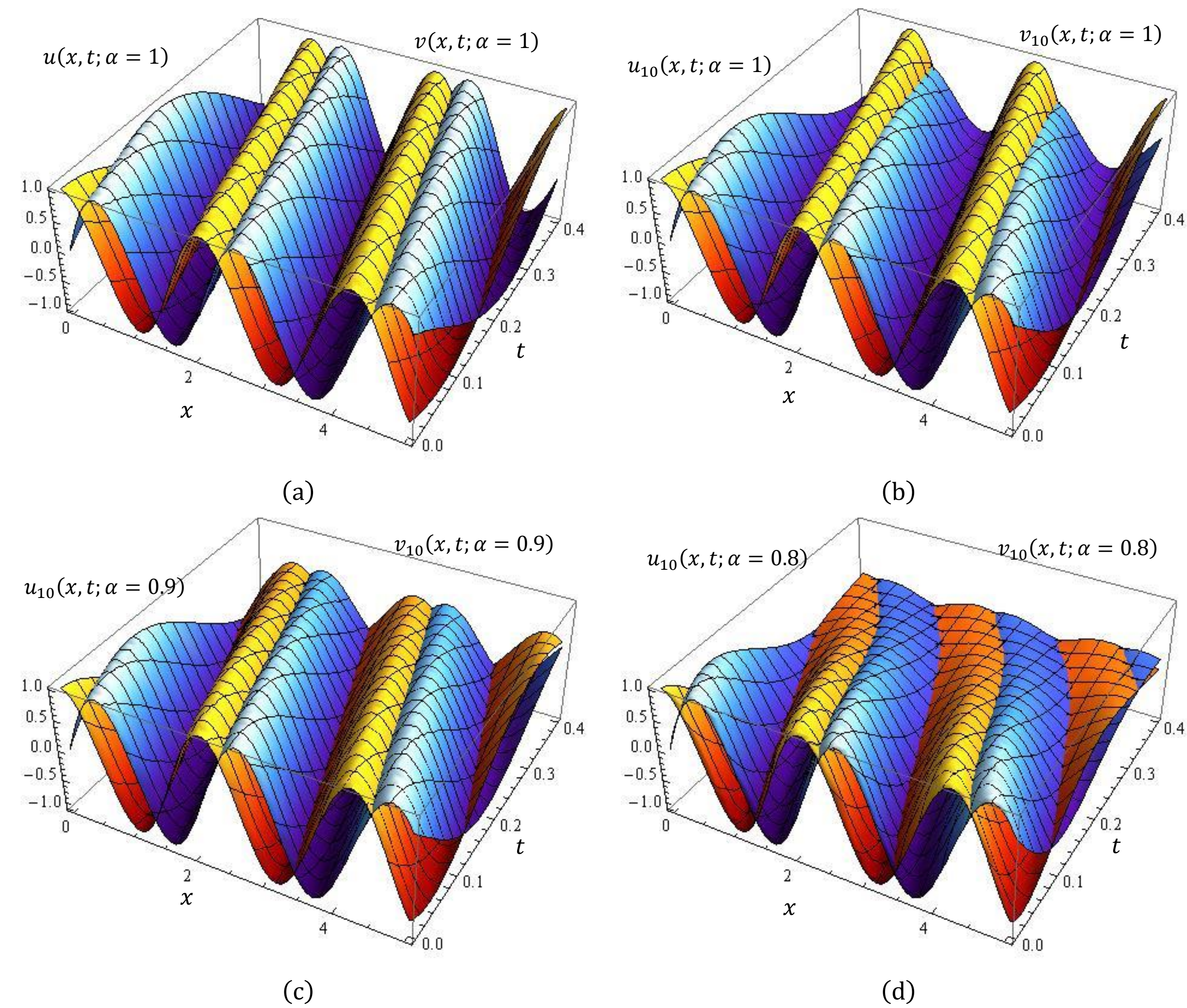

Figure 1. The surface graphs of the exact solution of $\boldsymbol{u(x,t)}$ and $\boldsymbol{v(x,t)}$ at $\boldsymbol{\alpha = 1}$, and the 10th approximations, $\boldsymbol{u_{10}(x,t)}$ and $\boldsymbol{v_{10}(x,t)}$, at different values of $\boldsymbol{\alpha}$.

Application 4.2. Consider the following one-dimensional nonlinear time-FSE:

$$iD_t^\alpha \psi(x,t) + \psi_{xx}(x,t) + 2|\psi(x,t)|^2\psi(x,t) = 0,\ x \in \mathbb{R},\ t \geq 0,\ 0 < \alpha \leq 1, \tag{4.22}$$

subject to the constraint:

$$\psi(x,0) = e^{ix}. \tag{4.23}$$

The time-FSE in Eq (4.22) can be transformed into an equivalent system of PDEs as follows:

$$\begin{cases} D_t^\alpha u(x,t) + v_{xx}(x,t) + 2\big(u^2(x,t) + v^2(x,t)\big)v(x,t) = 0, \\ D_t^\alpha v(x,t) - u_{xx}(x,t) - 2\big(u^2(x,t) + v^2(x,t)\big)u(x,t) = 0, \end{cases} \tag{4.24}$$

subject to the following constraints:

$$u(x,0) = \cos(x)\,,\ v(x,0) = \sin(x), \tag{4.25}$$

where $\psi(x,t) = u(x,t) + iv(x,t)$.

According to the LRPS approach, the Laplace transform of the system (4.24) with (4.25) into a Laplace space will lead to the following system:

$$\begin{cases} U(x,s) - \frac{\cos(x)}{s} + \frac{1}{s^\alpha} V_{xx}(x,s) + \frac{2}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^2(\mathcal{L}^{-1}\{V(x,s)\})\} \\ +\frac{2}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^3\} = 0, \\ V(x,s) - \frac{\sin(x)}{s} - \frac{1}{s^\alpha} U_{xx}(x,s) - \frac{2}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^2(\mathcal{L}^{-1}\{U(x,s)\})\} \\ -\frac{2}{s^\alpha}\mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^3\} = 0. \end{cases} \tag{4.26}$$

The series solution to the system (4.26) in the new space assumed to be of the form:

$$\begin{cases} U(x,s) = \frac{\cos(x)}{s} + \sum_{i=1}^{\infty} \frac{f_i(x)}{s^{1+i\alpha}},\ 0 < \alpha \le 1,\ x \in I,\ s > 0, \\ V(x,s) = \frac{\sin(x)}{s} + \sum_{i=1}^{\infty} \frac{g_i(x)}{s^{1+i\alpha}},\ 0 < \alpha \le 1,\ x \in I,\ s > 0. \end{cases} \tag{4.27}$$

By comparing the standard form of FSE (1.1) with Eq (4.22), we find that $\delta = 1,\ \gamma = 2$, and $\phi\ (x) = 0$. So, according to the results in (3.15), (3.19), and (3.21), the first few coefficients of the series (4.27) have the following form:

$$f_1(x) = -\sin(x),\ \ g_1(x) = \cos(x),$$

$$f_2(x) = -\cos(x),\quad g_2(x) = -\sin(x), \tag{4.28}$$

$$f_3(x) = \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\sin(x),\ \ g_3(x) = -\left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right)\cos(x).$$

Calculating an additional coefficient of the expansions in (4.27) utilizing the LRPS approach used in Section 3, we get

$$\begin{cases} f_4(x) = \left(5 - \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\cos(x)\,, \\ g_4(x) = \left(5 - \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\sin(x)\,. \end{cases} \tag{4.29}$$

So, the 4th approximate solution of system (4.27) can be expressed as follows:

$$U_4(x,s) = \frac{\cos(x)}{s} - \frac{\sin(x)}{s^{1+\alpha}} - \frac{\cos(x)}{s^{1+2\alpha}} + \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\frac{\sin(x)}{s^{1+3\alpha}}$$

$$+\left(5 - \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\frac{\cos(x)}{s^{1+3\alpha}},$$

$$V_4(x,s) = \frac{\sin(x)}{s} + \frac{\cos(x)}{s^{1+\alpha}} - \frac{\sin(x)}{s^{1+2\alpha}} - \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\frac{\cos(x)}{s^{1+3\alpha}}$$
$$+\left(5 - \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\frac{\sin(x)}{s^{1+3\alpha}}.$$

Therefore, based on the results in Eqs (3.22) and (3.23), the 4th approximate LRPS solution to the system (4.24) with (4.25) is given by

$$u_4(x,t) = \cos(x) - \frac{\sin(x)\,t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\cos(x)\,t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\frac{\sin(x)\,t^{3\alpha}}{\Gamma(1+3\alpha)}$$
$$+\left(5 - \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\frac{\cos(x)\,t^{4\alpha}}{\Gamma(1+4\alpha)}, \tag{4.30}$$

$$v_4(x,t) = \sin(x) + \frac{\cos(x)\,t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\sin(x)\,t^{2\alpha}}{\Gamma(1+2\alpha)} - \left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right)\frac{\cos(x)\,t^{3\alpha}}{\Gamma(1+2\alpha)}$$
$$+\left(5 - \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\frac{\sin(x)\,t^{4\alpha}}{\Gamma(1+4\alpha)}. \tag{4.31}$$

With a little bit of focus and scrutiny, one can get the general pattern of the series solution in Eqs (4.30) and (4.31). Using Euler's formula, we can write the solution to FSE in Eq (4.22) in the following infinite series form:

$$\psi(x,t) = e^{ix}\Bigg(1 + i\frac{t^{\alpha}}{\Gamma(1+\alpha)} + i^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + i^3\left(5 - 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$
$$+ i^4\left(5 - \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + \frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3}\right)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \tag{4.32}$$
$$+\cdots\Bigg).$$

It should be noted that the results obtained in Eq (4.32) are in complete agreement with the results obtained by the RPS method [44]. Moreover, when $\alpha = 1$, the closed form of the exact solution to the IVP (4.22) with (4.23) is $\psi(x,t) = e^{i(x+t)}$, which coincides with the solutions obtained by the Adomian decomposition technique [10], and the variational iteration method [20]. Figure 2 shows the surface graphs of the 4th approximate LRPS solutions, $u_4(x,t)$ and $v_4(x,t)$, of the IVP (4.22) with (4.23) at different values of α that are given in Eqs (4.30) and (4.31) in addition the exact solution when $\alpha = 1$. The graphs illustrate the behavior of the solutions at $\alpha = 1$, $\alpha = 0.9$, and $\alpha = 0.8$.

Figure 2 illustrates the agreement between the 4th approximate solution and the exact solution at $\alpha = 1$, while all graphs depict the consistency of the behavior of the solutions at different values of α.

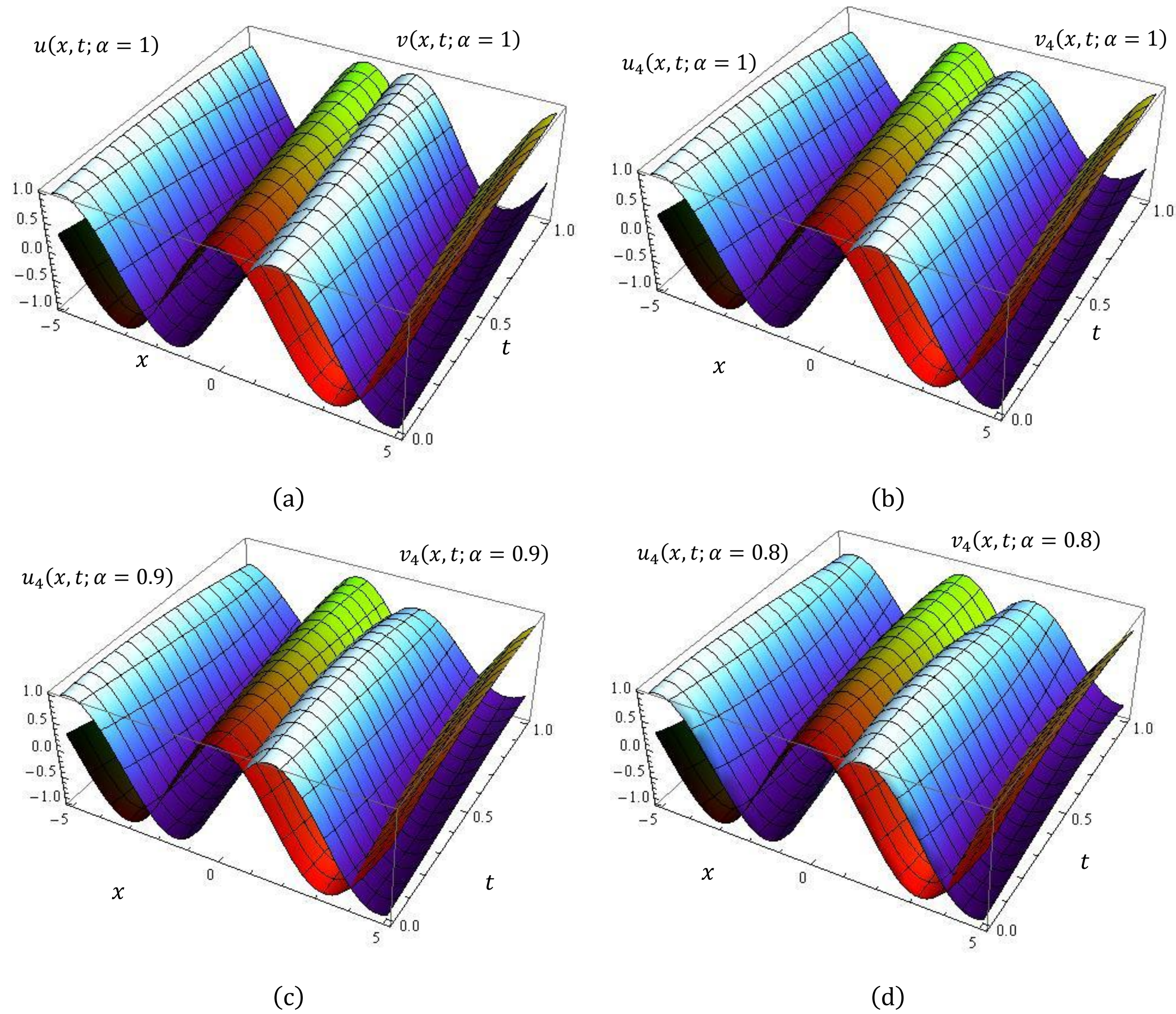

Figure 2. The surface graphs of the exact solution of $\boldsymbol{u(x,t)}$ and $\boldsymbol{v(x,t)}$ at $\boldsymbol{\alpha=1}$, and the 4th approximations, $\boldsymbol{u_4(x,t)}$ and $\boldsymbol{v_4(x,t)}$, at different values of $\boldsymbol{\alpha}$.

In the following example, we will deal with another form of the FSE that differs from the expression of Eq (1.1) by replacing the exponent of the modulus with the number 4 instead of 2; thus we will need to reconstruct the solution in detail to determine the coefficients of the series solution.

Application 4.3. Consider the following one-dimensional nonlinear time-FSE:

$$iD_t^{\alpha}\psi(x,t)+\psi_{xx}(x,t)+2|\psi(x,t)|^4\psi(x,t)=0,\ x\in\mathbb{R},\ t\geq 0,\ 0<\alpha\leq 1, \tag{4.33}$$

subject to the constraint:

$$\psi(x,0)=(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}}. \tag{4.34}$$

The time-FSE (4.33) can be reformulated into an equivalent system of PDEs as follows:

$$\begin{cases} D_t^{\alpha}u(x,t)+v_{xx}(x,t)+2\big(u^4(x,t)+2u^2(x,t)v^2(x,t)+v^4(x,t)\big)v(x,t)=0,\\ D_t^{\alpha}v(x,t)-u_{xx}(x,t)-2\big(u^4(x,t)+2u^2(x,t)v^2(x,t)+v^4(x,t)\big)u(x,t)=0, \end{cases} \tag{4.35}$$

subject to the following constraints:

$$\begin{cases} f(x)=u(x,0)=(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}},\\ g(x)=v(x,0)=0, \end{cases} \tag{4.36}$$

where $\psi(x,t)=u(x,t)+iv(x,t)$.

Following the same steps as the previous two examples, the corresponding system of the system (4.35) with (4.36) in Laplace space will be as follows:

$$\begin{cases} U(x,s)-\frac{(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}}}{s}+\frac{1}{s^{\alpha}}V_{xx}(x,s)\\ +\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^4\mathcal{L}^{-1}\{V(x,s)\}+2(\mathcal{L}^{-1}\{U(x,s)\})^2(\mathcal{L}^{-1}\{V(x,s)\})^3\}\\ +\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^5\}=0,\\ V(x,s)-\frac{1}{s^{\alpha}}U_{xx}(x,s)\\ -\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{V(x,s)\})^4\mathcal{L}^{-1}\{U(x,s)\}+2(\mathcal{L}^{-1}\{V(x,s)\})^2(\mathcal{L}^{-1}\{U(x,s)\})^3\}\\ -\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{U(x,s)\})^5\}=0, \end{cases} \tag{4.37}$$

and so, the kth LRFs of Eq (4.37) will be as follows:

$$\begin{cases} LRes_k^1(x,s)=U_k(x,s)-\frac{(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}}}{s}+\frac{1}{s^{\alpha}}(V_k)_{xx}(x,s)\\ \qquad +\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{U_k(x,s)\})^4\mathcal{L}^{-1}\{V_k(x,s)\}\\ \qquad +2(\mathcal{L}^{-1}\{U_k(x,s)\})^2(\mathcal{L}^{-1}\{V_k(x,s)\})^3\}+\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{V_k(x,s)\})^5\},\\ LRes_k^2(x,s)=V_k(x,s)-\frac{1}{s^{\alpha}}(U_k)_{xx}(x,s)-\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{V_k(x,s)\})^4\mathcal{L}^{-1}\{U_k(x,s)\}\\ \qquad +2(\mathcal{L}^{-1}\{V_k(x,s)\})^2(\mathcal{L}^{-1}\{U_k(x,s)\})^3\}-\frac{2}{s^{\alpha}}\mathcal{L}\{(\mathcal{L}^{-1}\{U_k(x,s)\})^5\}, \end{cases} \tag{4.38}$$

where U_k and V_k are the kth truncated series of the expansion (3.7) that are assumed, as the approach of the LRPS method, to be a solution to the system (4.37).

Solving the system

$$\begin{cases} \lim_{s\to\infty} s^{k\alpha+1}LRes_k^1(x,s)=0,\ k=0,1,2,\dots,\\ \lim_{s\to\infty} s^{k\alpha+1}LRes_k^2(x,s)=0,\ k=0,1,2,\dots, \end{cases} \tag{4.39}$$

recursively, yields the following first few coefficients of the series solution (3.7):

$$g_0(x)=0,\qquad f_0(x)=(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}},$$

$$f_1(x)=0,\qquad g_1(x)=4(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}},$$

$$g_2(x)=0,\qquad f_2(x)=-(4)^2(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}},$$

$$f_3(x) = 0, \;\; g_3(x) = -(4)^3 (6\,\mathrm{sech}^2(4x))^{\frac{1}{4}} \left(25 + \cosh(8x) - \frac{12\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} \right) \frac{\mathrm{sech}^2(4x)}{2},$$

$$g_4(x) = 0,$$

$$f_4(x) = (4)^4 (6\,\mathrm{sech}^2(4x))^{\frac{1}{4}} \left(601 + 12 \frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3} \left(\frac{2\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)} - 1 \right) \right.$$

$$\left. + \cosh(8x) - 768\,\mathrm{sech}^2(4x) + \frac{12\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} (32\,\mathrm{sech}^2(4x) - 25) \right) \frac{\mathrm{sech}^2(4x)}{2}.$$

So, the 4th approximate solution of the system (4.37) will be as

$$\begin{aligned} U_4(x,s) = {} & \frac{(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}}}{s} - \frac{(4)^2 \left(6\,\mathrm{sech}^2(4x)\right)^{1/4}}{s^{1+2\alpha}} \\ & + \frac{(4)^4 (6\,\mathrm{sech}^2(4x))^{\frac{1}{4}}}{s^{1+4\alpha}} \left(601 + 12 \frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3} \left(\frac{2\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)} - 1 \right) \right. \\ & + \cosh(8x) - 768\,\mathrm{sech}^2(4x) + \frac{12\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} (32\,\mathrm{sech}^2(4x) \\ & \left. - 25) \right) \frac{\mathrm{sech}^2(4x)}{2}, \end{aligned} \tag{4.40}$$

$$\begin{aligned} V_4(x,s) = {} & \frac{4(6\,\mathrm{sech}^2(4x))^{\frac{1}{4}}}{s^{1+\alpha}} \\ & - \frac{(4)^3 (6\,\mathrm{sech}^2(4x))^{\frac{1}{4}}}{s^{1+3\alpha}} \left(25 + \cosh(8x) - \frac{12\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} \right) \frac{\mathrm{sech}^2(4x)}{2}. \end{aligned} \tag{4.41}$$

Applying the inverse Laplace transform on Eqs (4.40) and (4.41), and combining the two resultant formulas as the real and imaginary parts of a complex function yields the 4th approximate LRPS of the FSE in the following form:

$$\begin{aligned} \psi(x,t) = {} & (6\,\mathrm{sech}^2(4x))^{\frac{1}{4}} \left(1 + 4i \frac{t^{\alpha}}{\Gamma(1+\alpha)} + (4i)^2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right. \\ & + (4i)^3 \left(25 + \cosh(8x) - \frac{12\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} \right) \frac{\mathrm{sech}^2(4x)}{2} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \Bigg) \\ & + (4i)^4 (6\,\mathrm{sech}^2(4x))^{\frac{1}{4}} \left(601 + 12 \frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^3} \left(\frac{2\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)} - 1 \right) + \cosh(8x) \right. \\ & \left. - 768\,\mathrm{sech}^2(4x) + \frac{12\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} (32\,\mathrm{sech}^2(4x) - 25) \right) \frac{\mathrm{sech}^2(4x)}{2} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}. \end{aligned} \tag{4.42}$$

It should be noted that the results obtained in Eq (4.42) are in complete agreement with the results obtained by the RPS method [52]. Further, when $\alpha = 1$, we obtain the following solution for the IVP (4.33) and (4.34):

$$\psi(x,t) = (6\,\mathrm{sech}^2(4x))^{\frac{1}{4}} e^{4it}, \tag{4.43}$$

which agrees with the results obtained by the variational iteration method [20].

5. Conclusions

In this paper, we have presented a new method for solving fractional Schrödinger equations, called the LRPS method. This method is powerful and efficient and can be used to find analytical solutions in the form of a fast-converging series. The coefficients of the series are usually easily computed, and the method can be applied to both linear and nonlinear equations. It is shown that the LRPS method is very effective and can produce approximate solutions that are indistinguishable from the exact solutions. The major advantages of the LRPS method are:

- It can be used to solve nonlinear equations, while other methods are limited to linear equations.
- It simplifies the processing of fractional differential equations by converting them to algebraic equations.
- The iterators can be computed easily using the concept of limit at infinity.
- It does not require modeling assumptions such as linearization, perturbation, or discretization.

It is shown that the LRPS method is a valuable tool for solving fractional Schrödinger equations. We have demonstrated its effectiveness in this paper, and we believe that it can be applied to a wide range of problems [53–55]. In future work, we plan to apply the LRPS method to other fractional differential equations [56–60], such as the space-time fractional Schrödinger equations and systems of fractional Schrödinger equations. We also plan to modify the method to construct solutions to fractional differential equations with other concepts of fractional derivatives [61,62].

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

References

1. G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, *Comput. Math. Appl.*, **51** (2006), 1367–1376. https://doi.org/10.1016/j.camwa.2006.02.001

2. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85.
3. M. C. Caputo, D. F. M. Torres, Duality for the left and right fractional derivatives, *Signal Process.*, **107** (2015), 265–271. https://doi.org/10.1016/j.sigpro.2014.09.026
4. Y. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2014** (2014), 1–12. https://doi.org/10.1186/1687-1847-2014-10
5. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
6. S. Kazem, Exact solution of some linear fractional differential equations by Laplace transform, *Int. J. Nonlinear Sci.*, **16** (2013), 3–11.
7. A. Burqan, M. Shqair, A. El-Ajou, Z. Al-Zhour, Analytical solutions to the coupled fractional neutron diffusion equations with delayed neutrons system using Laplace transform method, *AIMS Math.*, **8** (2023), 19297–19312. https://doi.org/10.3934/math.2023984
8. Y. Z. Hu, Y. Luo, Z. Y. Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method, *J. Comput. Appl. Math.*, **215** (2008), 220–229. https://doi.org/10.1016/j.cam.2007.04.005
9. A. Bibi, A. Kamran, U. Hayat, S. T. Mohyud-Din, New iterative method for time-fractional Schrödinger equations, *World J. Model. Simul.*, **9** (2013), 89–95.
10. A. Sadighi, D. D. Ganji, Analytic treatment of linear and nonlinear Schrödinger equations: a study with homotopy-perturbation and Adomian decomposition methods, *Phys. Lett. A*, **372** (2008), 465–469. https://doi.org/10.1016/j.physleta.2007.07.065
11. I. Podlubny, A. Chechkin, T. Skovranek, Y. Q. Chen, B. M. Vinagre, Matrix approach to discrete fractional calculus II: Partial fractional differential equations, *J. Comput. Phys.*, **228** (2009), 3137–3153. https://doi.org/10.1016/j.jcp.2009.01.014
12. A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Comput. Math. Appl.*, **59** (2010), 1326–1336. https://doi.org/10.1016/j.camwa.2009.07.006
13. X. J. Yang, J. A. T. Machado, H. M. Srivastava, A new numerical technique for solving the local fractional diffusion equation: two-dimensional extended differential transform approach, *Appl. Math. Comput.*, **274** (2016), 143–151. https://doi.org/10.1016/j.amc.2015.10.072
14. F. Saba, S. Jabeen, S. T. Mohyud-Din, Homotopy analysis transform method for time-fractional Schrödinger equations, *Int. J. Modern Math. Sci.*, **7** (2013), 26–40.
15. N. A. Khan, M. Jamil, A. Ara, Approximate solutions to time-fractional Schrödinger equation via homotopy analysis method, *Int. Scholarly Res. Notices*, **2012** (2012), 1–11. https://doi.org/10.5402/2012/197068

16. A. K. Alomari, M. S. M. Noorani, R. Nazar, Explicit series solutions of some linear and nonlinear Schrödinger equations via the homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simul.*, **14** (2009), 1196–1207. https://doi.org/10.1016/j.cnsns.2008.01.008
17. A. Yildirim, An algorithm for solving the fractional nonlinear Schrödinger equation by means of the homotopy perturbation method, *Int. J. Nonlinear Sci. Numer. Simul.*, **10** (2009), 445–450. https://doi.org/10.1515/IJNSNS.2009.10.4.445
18. S. Momani, Z. Odibat, Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations, *Comput. Math. Appl.*, **54** (2007), 910–919. https://doi.org/10.1016/j.camwa.2006.12.037
19. B. J. Hong, D. C. Lu, Modified fractional variational iteration method for solving the generalized time-space fractional Schrödinger equation, *Sci. World J.*, **2014** (2014), 1–6. https://doi.org/10.1155/2014/964643
20. A. M. Wazwaz, A study on linear and nonlinear Schrodinger equations by the variational iteration method, *Chaos Solitons Fract.*, **37** (2008), 1136–1142. https://doi.org/10.1016/j.chaos.2006.10.009
21. M. N. Oqielat, A. El-Ajou, Z. Al-Zhour, R. Alkhasawneh, H. Alrabaiah, Series solutions for nonlinear time-fractional Schrödinger equations: comparisons between conformable and Caputo derivatives, *Alex. Eng. J.*, **59** (2020), 2101–2114. https://doi.org/10.1016/j.aej.2020.01.023
22. A. El-Ajou, M. N. Oqielat, Z. Al-Zhour, S. Momani, A class of linear non-homogenous higher order matrix fractional differential equations: analytical solutions and new technique, *Fract. Calc. Appl. Anal.*, **23** (2020), 356–377. https://doi.org/10.1515/fca-2020-0017
23. A. El-Ajou, Taylor's expansion for fractional matrix functions: theory and applications, *J. Math. Comput. Sci.*, **21** (2020), 1–17. http://dx.doi.org/10.22436/jmcs.021.01.01
24. A. El-Ajou, M. Al-Smadi, M. N. Oqielat, S. Momani, S. Hadid, Smooth expansion to solve high-order linear conformable fractional PDEs via residual power series method: applications to physical and engineering equations, *Ain Shams Eng. J.*, **11** (2020), 1243–1254. https://doi.org/10.1016/j.asej.2020.03.016
25. A. Qazza, A. Burqan, R. Saadeh, Application of ARA-residual power series method in solving systems of fractional differential equations, *Math. Probl. Eng.*, **2022** (2022), 1–17. https://doi.org/10.1155/2022/6939045
26. K. Shah, H. Naz, M. Sarwar, T. Abdeljawad, On spectral numerical method for variable-order partial differential equations, *AIMS Math.*, **7** (2022), 10422–10438. http://dx.doi.org/10.3934/math.2022581

27. K. Shah, M. Arfan, A. Ullah, Q. Al-Mdallal, K. J. Ansari, T. Abdeljawad, Computational study on the dynamics of fractional order differential equations with applications, *Chaos Solitons Fract.*, **157** (2022), 111955. https://doi.org/10.1016/j.chaos.2022.111955
28. K. Shah, T. Abdeljawad, A. Ali, Mathematical analysis of the Cauchy type dynamical system under piecewise equations with Caputo fractional derivative, *Chaos Solitons Fract.*, **161** (2022), 112356. https://doi.org/10.1016/j.chaos.2022.112356
29. S. Ali, A. Khan, K. Shah, M. A. Alqudah, T. Abdeljawad, Siraj-ul-Islam, On computational analysis of highly nonlinear model addressing real world applications, *Results Phys.*, **36** (2022), 105431. https://doi.org/10.1016/j.rinp.2022.105431
30. B. Ghanbari, D. Baleanu, New optical solutions of the fractional Gerdjikov-Ivanov equation with conformable derivative, *Front. Phys.*, **8** (2020), 167. https://doi.org/10.3389/fphy.2020.00167
31. B. Ghanbari, D. Baleanu, M. Al Qurashi, New exact solutions of the generalized Benjamin-Bona-Mahony equation, *Symmetry*, **11** (2019), 1–12. https://doi.org/10.3390/sym11010020
32. B. Ghanbari, C. K. Kuo, New exact wave solutions of the variable-coefficient (1 +1)-dimensional Benjamin-Bona-Mahony and (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equations via the generalized exponential rational function method, *Eur. Phys. J. Plus*, **134** (2019), 334. https://doi.org/10.1140/epjp/i2019-12632-0
33. K. Shah, A. R. Seadawy, M. Arfan, Evaluation of one dimensional fuzzy fractional partial differential equations, *Alex. Eng. J.*, **59** (2020), 3347–3353. https://doi.org/10.1016/j.aej.2020.05.003
34. A. Mohebbi, M. Abbaszadeh, M. Dehghan, The use of a meshless technique based on collocation and radial basis functions for solving the time fractional nonlinear Schrödinger equation arising in quantum mechanics, *Eng. Anal. Bound. Elem.*, **37** (2013), 475–485. https://doi.org/10.1016/j.enganabound.2012.12.002
35. T. Eriqat, A. El-Ajou, M. Oqielat, Z. Al-Zhour, S. Momani, A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations, *Chaos Solitons Fract.*, **138** (2020), 109957. https://doi.org/10.1016/j.chaos.2020.109957
36. A. El-Ajou, Adapting the Laplace transform to create solitary solutions for the nonlinear time-fractional dispersive PDEs via a new approach, *Eur. Phys. J. Plus*, **136** (2021), 229. https://doi.org/10.1140/epjp/s13360-020-01061-9
37. A. El-Ajou, Z. Al-Zhour, A vector series solution for a class of hyperbolic system of Caputo time-fractional partial differential equations with variable coefficients, *Front. Phys.*, **9** (2021), 525250. https://doi.org/10.3389/fphy.2021.525250
38. A. Burqan, A. El-Ajou, R. Saadeh, M. Al-Smadi, A new efficient technique using Laplace transforms and smooth expansions to construct a series solution to the time-fractional Navier-Stokes equations, *Alex. Eng. J.*, **61** (2022), 1069–1077. https://doi.org/10.1016/j.aej.2021.07.020

39. M. N. Oqielat, A. El-Ajou, Z. Al-Zhour, T. Eriqat, M. Al-Smadi, A new approach to solving fuzzy quadratic Riccati differential equations, *Int. J. Fuzzy Logic Intell. Syst.*, **22** (2022), 23–47. https://doi.org/10.5391/IJFIS.2022.22.1.23
40. R. Saadeh, A. Burqan, A. El-Ajou, Reliable solutions to fractional Lane-Emden equations via Laplace transform and residual error function, *Alex. Eng. J.*, **61** (2022), 10551–10562. https://doi.org/10.1016/j.aej.2022.04.004
41. A. Qazza, R. Saadeh, On the analytical solution of fractional SIR epidemic model, *Appl. Comput. Intell. Soft Comput.*, **2023** (2023), 1–16. https://doi.org/10.1155/2023/6973734
42. A. El-Ajou, H. Al-ghananeem, R. Saadeh, A. Qazza, M. N. Oqielat, A modern analytic method to solve singular and non-singular linear and non-linear differential equations, *Front. Phys.*, **11** (2023), 271. https://doi.org/10.3389/fphy.2023.1167797
43. E. Salah, A. Qazza, R. Saadeh, A. El-Ajou, A hybrid analytical technique for solving multi-dimensional time-fractional Navier-Stokes system, *AIMS Math.*, **8** (2023), 1713–1736. https://doi.org/10.3934/math.2023088
44. E. Salah, R. Saadeh, A. Qazza, R. Hatamleh, Direct power series approach for solving nonlinear initial value problems, *Axioms*, **12** (2023), 1–21. https://doi.org/10.3390/axioms12020111
45. D. Baleanu, M. Inc, A. I. Aliyu, A. Yusuf, Dark optical solitons and conservation laws to the resonance nonlinear Schrödinger's equation with Kerr law nonlinearity, *Optik*, **147** (2017), 248–255. https://doi.org/10.1016/j.ijleo.2017.08.080
46. A. Qazza, M. Abdoon, R. Saadeh, M. Berir, A new scheme for solving a fractional differential equation and a chaotic system, *Eur. J. Pure Appl. Math.*, **16** (2023), 1128–1139. https://doi.org/10.29020/nybg.ejpam.v16i2.4769
47. R. Saadeh, M. A. Abdoon, A. Qazza, M. Berir, A numerical solution of generalized Caputo fractional initial value problems, *Fractal Fract.*, **7** (2023), 1–12. https://doi.org/10.3390/fractalfract7040332
48. N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A*, **268** (2000), 298–305. https://doi.org/10.1016/S0375-9601(00)00201-2
49. N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E*, **66** (2002), 056108. https://doi.org/10.1103/PhysRevE.66.056108
50. M. Naber, Time fractional Schrödinger equation, *J. Math. Phys.*, **45** (2004), 3339–3352. https://doi.org/10.1063/1.1769611
51. X. Y. Jiang, Time-space fractional Schrödinger like equation with a nonlocal term, *Eur. Phys. J. Spec. Top.*, **193** (2011), 61–70. https://doi.org/10.1140/epjst/e2011-01381-7
52. O. Abu Arqub, Application of residual power series method for the solution of time-fractional Schrödinger equations in one-dimensional space, *Fund. Inform.*, **166** (2019), 87–110.

53. G. W. Wang, Q. Zhou, A. S. Alshormani, A. Biswas, Explicit optical dromions with Kerr law having fractional temporal evolution, *Fractals*, **31** (2023), 2350056. https://doi.org/10.1142/S0218348X23500561
54. G. W. Wang, A new (3+1)-dimensional Schrödinger equation: derivation, soliton solutions and conservation laws, *Nonlinear Dyn.*, **104** (2021), 1595–1602. https://doi.org/10.1007/s11071-021-06359-6
55. R. Santana-Carrillo, J. M. V. Peto, G. H. Sun, S. H. Dong, Quantum information entropy for a hyperbolic double well potential in the fractional Schrödinger equation, *Entropy*, **25** (2023), 1–10. https://doi.org/10.3390/e25070988
56. A. Qazza, R. Hatamleh, The existence of a solution for semi-linear abstract differential equations with infinite *B*-chains of the characteristic sheaf, *Int. J. Appl. Math.*, **31** (2018), 611–620. https://doi.org/10.12732/ijam.v31i5.7
57. J. W. Wang, X. X. Jiang, X. H. Yang, H. X. Zhang, A nonlinear compact method based on double reduction order scheme for the nonlocal fourth-order PDEs with Burgers' type nonlinearity, *J. Appl. Math. Comput.*, **70** (2024), 489–511. https://doi.org/10.1007/s12190-023-01975-4
58. W. Xiao, X. H. Yang, Z. Y. Zhou, Pointwise-in-time α-robust error estimate of the ADI difference scheme for three-dimensional fractional subdiffusion equations with variable coefficients, *Commun. Anal. Mech.*, **16** (2024), 53–70. https://doi.org/10.3934/cam.2024003
59. A. M. Qazza, R. M. Hatamleh, N. A. Alodat, About the solution stability of Volterra integral equation with Random kernel, *Far East J. Math. Sci. (FJMS)*, **100** (2016), 671–680. https://doi.org/10.17654/ms100050671
60. L. D. Zhao, Y. H. Chen, Comments on "A novel approach to approximate fractional derivative with uncertain conditions", *Chaos Solitons Fract.*, **154** (2022), 111651. https://doi.org/10.1016/j.chaos.2021.111651
61. L. D. Zhao, A note on "Cluster synchronization of fractional-order directed networks via intermittent pinning control", *Phys. A*, **561** (2021), 125150. https://doi.org/10.1016/j.physa.2020.125150
62. J. B. Hu, G. P. Lu, S. B. Zhang, L. D. Zhao, Lyapunov stability theorem about fractional system without and with delay, *Commun. Nonlinear Sci. Numer. Simul.*, **20** (2015), 905–913. https://doi.org/10.1016/j.cnsns.2014.05.013

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