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Research article

$\mathcal{N} = 2$ double graded supersymmetric quantum mechanics via dimensional reduction

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Abstract: We presented a novel $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-graded supersymmetric quantum mechanics ($\mathbb{Z}_2^2$-SQM) which has different features from those introduced so far. It is a two-dimensional (two-particle) system and was the first example of the quantum mechanical realization of an eight-dimensional irreducible representation (irrep) of the $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetry algebra. The $\mathbb{Z}_2^2$-SQM was obtained by quantizing the one-dimensional classical system derived by dimensional reduction from the two-dimensional $\mathbb{Z}_2^2$-supersymmetric Lagrangian of $\mathcal{N} = 1$, which we constructed in our previous work. The ground states of the $\mathbb{Z}_2^2$-SQM were also investigated.

Keywords: $\mathbb{Z}_2^2$-supersymmetry; constrained system; Dirac-Bergmann method; quantization; quantum mechanics
Mathematics Subject Classification: 81Q60, 81Q80, 81S08

1. Introduction

In our previous work [1], a $\mathbb{Z}_2^2$-graded supersymmetric Lagrangian in two-dimensional spacetime was constructed by the $\mathbb{Z}_2^2$-extension of the superfield formalism. The $\mathbb{Z}_2^2$-supersymmetry is a higher graded extension of the supersymmetry, based on the $\mathbb{Z}_2^2$-graded superalgebras introduced by Bruce [2]. The Lagrangian given in [1], which is $\mathcal{L}$ in (2.5) of the present paper, has very general interaction terms, and appropriate choices of them give $\mathbb{Z}_2^2$-graded supersymmetric extensions of the two-dimensional integrable systems, for example, the sine(h)-Gordon equation and Liouville equation. The integrability of these $\mathbb{Z}_2^2$-extended equations is an open problem; however, one may expect the existence of a novel class of integrable systems characterized by the $\mathbb{Z}_2^2$-supersymmetry. Indeed, a $\mathbb{Z}_2^2$-graded extension of the sine-Gordon equation, which is different from the one in [1], was introduced and its integrability is shown by Bruce [3]; this is the only integrable classical system having the $\mathbb{Z}_2^2$-supersymmetry known so far. Therefore, in order to open up a new field of integrable systems, the study of the classical systems obtained from $\mathcal{L}$ in (2.5) is important.

It is also important to quantize the Lagrangian $\mathcal{L}$, which will give quantum integrable systems. In the present paper, however, instead of quantizing $\mathcal{L}$, we study the simpler but highly nontrivial case, that is, $\mathbb{Z}_2^2$-supersymmetric quantum mechanics ($\mathbb{Z}_2^2$-SQM) obtained from $\mathcal{L}$ via dimensional reduction. The $\mathbb{Z}_2^2$-SQM was first introduced by Bruce and Duplij [4], which is $\mathcal{N} = 1$ in our terminology (see Section 2 for the definition of $\mathcal{N}$). The operators of this $\mathbb{Z}_2^2$-SQM close in ordinary the one-dimensional (1D) super-Poincaré algebra, as well as its $\mathbb{Z}_2^2$-counterpart. However, this does not mean the $\mathbb{Z}_2^2$-SQM is trivial, since the $\mathbb{Z}_2^2$-SQM shows detectable difference from the ordinary SQM in multiparticle sectors [5, 6]. The Bruce-Duplij $\mathbb{Z}_2^2$-SQM is also extended to $\mathcal{N} > 1$ [7], $\mathbb{Z}_2^n$-grading [8], and conformal symmetries [9].

Our Lagrangian $\mathcal{L}$ in (2.5) is $\mathcal{N} = 1$ and defined in two-dimensional spacetime. Reduction of it to one-dimension gives $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetric classical mechanics. We quantize 1D system using a $\mathbb{Z}_2^2$-graded extension of the Dirac-Bergmann method of constrained systems [10,11]. This gives us a $\mathbb{Z}_2^2$-SQM, which has different features from the $\mathbb{Z}_2^2$-SQMs mentioned above (see Section 4). In particular, the $\mathbb{Z}_2^2$-SQM obtained is a two-dimensional or two-particle (with the same mass) quantum mechanics and the left and right movers are separated in the light cone coordinates. Furthermore, it is realized by 8×8 matrix differential operators, which correspond to the eight-dimensional irrep of $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetry algebra [12], and it is the first example of the quantum mechanical realization of the irrep.

It is well known that SQMs are closely related to solvable potentials through the factorization of Hamiltonian (see, e.g., [13–15]). It is also known that $\mathbb{Z}_2^2$-graded algebraic structure appears in simple solvable systems in quantum mechanics [16–18]. We therefore expect the $\mathbb{Z}_2^2$-SQMs to have a deep connection with solvable quantum mechanical systems.

Before proceeding further, we mention some works discussing $\mathbb{Z}_2^2$-graded algebraic structure in physics. Vasiliev pointed out that the symmetry group of SUGRA in de Sitter spacetime is enhanced to $\mathbb{Z}_2^2$-graded superalgebra [19]. The quasi-spin formalism is generalized to higher graded algebra in [20], and the superconformal symmetry in two-dimension is also generalized to $\mathbb{Z}_2^2$-graded setting [21]. Equivalence between algebraic structures generated by parastatistics triple relations of Green and Greenberg–Messiah and certain orthosymplectic $\mathbb{Z}_2^2$-graded superalgebras is pointed out in [22]. This observation of $\mathbb{Z}_2^2$-graded superalgebras in parastatistics leads to further development of parastatistics representations of $\mathbb{Z}_2^2$-graded superalgebras [23–25]. We also comment that the paraparticles are simulated recently by using a trapped ion [26].

There are some proposals of $\mathbb{Z}_2^n$-graded extensions of the spacetime supersymmetry [27–30], which are related to higher graded SQMs. Regarding the higher graded supersymmetry, we mention the bosonization [31], sigma model [32], and n-bit extension of parastatistics [33]. A precise analysis of the $\mathbb{Z}_2^2$-graded superfield formulation of $\mathbb{Z}_2^2$-supersymmetry has recently been done [12, 34, 35]. The $\mathbb{Z}_2^2$-graded superfield formulation is the simplest example of higher supergeometry, which was started in [36] (see [37, 38] for a concise review of the higher supergeometry). Integration over the $\mathbb{Z}_2^2$-superspace is a necessary ingredient of the superfield formulation. There are some different ideas of integration, and one of them recently proposed by two of the present authors is suitable for the superfield formulation [39].

This paper is organized as follows: In the next section, we recall the definition of the $\mathbb{Z}_2^2$-graded Lie superalgebras and collect the results from [1], which we need in the present work. In Section 3, we investigate the classical aspects of the 1D system obtained by dimensional reduction. The 1D Lagrangian is derived from the 2D one, and equations of motion and conserved Noether charges are computed explicitly. The Lagrangian is singular, but all the constraints are second class. We thus develop a $\mathbb{Z}_2^2$-extension of the Dirac-Bargmann method suitable to the present model to quantize the system. We also observe the increase of the $\mathbb{Z}_2^2$-supersymmetry from $\mathcal{N} = 1$ to $\mathcal{N} = 2$. Section 4 is devoted to the study of the quantized system. The quantum operators are realized in terms of the eight-dimensional real irrep of the Clifford algebra $Cl(4, 2)$. The use of light cone coordinates provides separation of variables. This allows us to easily study the ground states of the Hamiltonian. We close the paper with a short summary and some remarks in Section 5.

2. Preliminaries

Let us first recall the definition of $\mathbb{Z}_2^2$-graded Lie superalgebras [40, 41] (see also [42, 43]). A $\mathbb{Z}_2^2$-graded vector space (over $\mathbb{R}$ or $\mathbb{C}$) is the direct sum of homogeneous vector subspaces labeled by an element of $\mathbb{Z}_2^2$:

$$\mathfrak{g} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}.$$

An element of $\mathfrak{g}_{\vec{a}}$ is said to have the $\mathbb{Z}_2^2$-*degree* $\vec{a} \in \mathbb{Z}_2^2$. We define the $\mathbb{Z}_2^2$-Lie bracket by

$$[\![X, Y]\!] = XY - (-1)^{\vec{a}\cdot\vec{b}} YX, \quad X \in \mathfrak{g}_{\vec{a}}, \; Y \in \mathfrak{g}_{\vec{b}}, \tag{2.1}$$

where $\vec{a} \cdot \vec{b}$ is the standard scalar product of two-dimensional vectors, namely, the $\mathbb{Z}_2^2$-Lie bracket is the commutator (anti-commutator) for $\vec{a} \cdot \vec{b}$ is even (odd). A $\mathbb{Z}_2^2$-graded vector space is said to be a $\mathbb{Z}_2^2$-graded Lie superalgebra if $[\![X, Y]\!] \in \mathfrak{g}_{\vec{a}+\vec{b}}$, and the Jacobi identity is satisfied:

$$[\![X, [\![Y, Z]\!]]\!] = [\![[\![X, Y]\!], Z]\!] + (-1)^{\vec{a}\cdot\vec{b}} [\![Y, [\![X, Z]\!]]\!].$$

If $[\![X, Y]\!] = 0$, we say that X and Y are $\mathbb{Z}_2^2$-*commutative*. We also define the even and odd subspaces of $\mathfrak{g}$ by $\mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)}$ and $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$, respectively.

The $\mathbb{Z}_2^2$-graded Lie superalgebra considered in [1], which is denoted simply by $\mathfrak{g}$, is five-dimensional and the $\mathbb{Z}_2^2$-degree assignment is as follows:

$$H \in \mathfrak{g}_{(0,0)}, \quad Z, \; L_{11} \in \mathfrak{g}_{(1,1)}, \quad Q_{10} \in \mathfrak{g}_{(1,0)}, \quad Q_{01} \in \mathfrak{g}_{(0,1)}. \tag{2.2}$$

Their nonvanishing $\mathbb{Z}_2^2$-Lie brackets, in terms of commutator or anti-commutator, are given by

$$\begin{aligned} &\{Q_{10}, Q_{10}\} = \{Q_{01}, Q_{01}\} = 2H, && [Q_{10}, Q_{01}] = iZ, \\ &[L_{11}, H] = \frac{i}{2} Z, && [L_{11}, Z] = 2iH, \\ &\{L_{11}, Q_{10}\} = -\frac{1}{2} Q_{01}, && \{L_{11}, Q_{01}\} = \frac{1}{2} Q_{10}. \end{aligned} \tag{2.3}$$

The subalgebra $\langle H, Q_{10}, Q_{01}, Z\rangle$ is the $\mathbb{Z}_2^2$-supersymmetry algebra introduced in [2]. We refer to this algebra as $\mathcal{N} = 1$ since each odd subspace has only one element.

We consider the eight real fields with $\mathbb{Z}_2^2$-grading defined in two-dimensional spacetime

$$\begin{aligned} &\varphi_{00}(t,x), \quad A_{00}(t,x), \quad A_{11}(t,x), \quad \varphi_{11}(t,x), \\ &\psi_{10}(t,x), \quad \lambda_{10}(t,x), \quad \psi_{01}(t,x), \quad \lambda_{01}(t,x), \end{aligned} \tag{2.4}$$

where the suffices indicate their $\mathbb{Z}_2^2$-degree and the fields are $\mathbb{Z}_2^2$-commutative. It is shown in [1] that the following action is invariant under the transformations generated by $\mathfrak{g}$:

$$\begin{aligned} S &= \int dt\, dx\, \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{int}, \\ \mathcal{L}_{kin} &= \frac{1}{2}(\dot{\varphi}_{00}^2 - {\varphi'_{00}}^2 + \dot{\varphi}_{11}^2 - {\varphi'_{11}}^2) + 2A_{00}^2 + 2A_{11}^2 \\ &\quad + i(\psi_{10}\dot{\psi}_{10} + \psi_{01}\dot{\psi}_{01} + \lambda_{10}\dot{\lambda}_{10} + \lambda_{01}\dot{\lambda}_{01}) - i(\psi_{10}\lambda'_{10} - \psi'_{10}\lambda_{10} - \psi_{01}\lambda'_{01} + \psi'_{01}\lambda_{01}), \\ \mathcal{L}_{int} &= -2\alpha(A_{11}V_{00} + A_{00}V_{11}) + 2\alpha((\psi_{10}\psi_{01} + \lambda_{10}\lambda_{01})\partial_{00}V_{00} + i(\psi_{10}\lambda_{10} + \psi_{01}\lambda_{01})\partial_{00}V_{11}), \end{aligned} \tag{2.5}$$

where α is a degree $(1,1)$ coupling constant and V_{00}, V_{11} are functions of $\varphi_{00}, \varphi_{11}$ satisfying

$$\partial_{00}V_{00}(\varphi_{00},\varphi_{11}) = \partial_{11}V_{11}(\varphi_{00},\varphi_{11}), \quad \partial_{11}V_{00}(\varphi_{00},\varphi_{11}) = \partial_{00}V_{11}(\varphi_{00},\varphi_{11}) \tag{2.6}$$

with

$$\partial_{00} := \frac{\partial}{\partial\varphi_{00}}, \qquad \partial_{11} := \frac{\partial}{\partial\varphi_{11}}. \tag{2.7}$$

H and Z are the generator of the translations of t and x, respectively. Q_{10} and Q_{01} are supercharges mixing up even (bosonic) and odd (fermionic) fields and changing the degree by $(1,0)$ and $(0,1)$, respectively. L_{11} is the degree $(1,1)$ Lorentz transformation, which gives rise to mixture among bosonic (fermionic) fields with different degrees. The transformation generated by Z and L_{11} disappears after the reduction to one-dimensional spacetime due to the nonexistence of space translationa and Lorentz transformation. Explicit form of the transformations are given in the equations (3.28)–(3.32) of [1]. The matrix presentation of the generators is found in (3.34)–(3.37) of [1].

As is seen from the Lagrangian (2.5), A_{00}, A_{11} are auxiliary, i.e., their equations of motion are given by the algebraic equation

$$A_{00} = \frac{\alpha}{2}V_{11}, \qquad A_{11} = \frac{\alpha}{2}V_{00}. \tag{2.8}$$

Using these relations, we remove the auxiliary fields.

3. 1D: classical mechanics

3.1. Lagrangian and equations of motion

We make the dimensional reduction $(t,x) \to (t)$, then, we have the world-line $\mathbb{Z}_2^2$-supersymmetric Lagrangian from (2.5):

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\dot{\varphi}_{00}^2 + \dot{\varphi}_{11}^2) + i(\psi_{10}\dot{\psi}_{10} + \psi_{01}\dot{\psi}_{01} + \lambda_{10}\dot{\lambda}_{10} + \lambda_{01}\dot{\lambda}_{01}) \\ &\quad + 2A_{00}^2 + 2A_{11}^2 - 2\alpha(A_{11}V_{00} + A_{00}V_{11}) \\ &\quad + 2\alpha((\psi_{10}\psi_{01} + \lambda_{10}\lambda_{01})\partial_{00}V_{00} + i(\psi_{10}\lambda_{10} + \psi_{01}\lambda_{01})\partial_{00}V_{11}). \end{aligned} \tag{3.1}$$

This Lagrangian is invariant under the following transformations generated by $\mathfrak{g}$ which is the one-dimensional reduction of (3.28)–(3.32) of [1]:

(i) Transformations by H and Z

$$\delta_{00} f(t,x) = -\frac{\epsilon_{00}}{2}\partial_t f(t,x), \quad \delta_{11} f(t,x) = 0, \qquad \text{for any component fields.} \tag{3.2}$$

(ii) Transformations by Q_{10}

$$\begin{aligned}
\delta_{10}\varphi_{00} &= -i\epsilon_{10}\psi_{10}, & \delta_{10}\varphi_{11} &= \epsilon_{10}\lambda_{01},\\
\delta_{10}\psi_{10} &= \frac{1}{2}\epsilon_{10}\dot{\varphi}_{00}, & \delta_{10}\lambda_{01} &= -\frac{i}{2}\epsilon_{10}\dot{\varphi}_{11},\\
\delta_{10}\psi_{01} &= i\epsilon_{10}A_{11}, & \delta_{10}\lambda_{10} &= \epsilon_{10}A_{00},\\
\delta_{10}A_{11} &= -\frac{1}{2}\epsilon_{10}\dot{\psi}_{01}, & \delta_{10}A_{00} &= -\frac{i}{2}\epsilon_{10}\dot{\lambda}_{10}.
\end{aligned} \tag{3.3}$$

(iii) Transformations by Q_{01}

$$\begin{aligned}
\delta_{01}\varphi_{00} &= -i\epsilon_{01}\psi_{01}, & \delta_{01}\varphi_{11} &= \epsilon_{01}\lambda_{10},\\
\delta_{01}\psi_{10} &= i\epsilon_{01}A_{11}, & \delta_{01}\lambda_{01} &= \epsilon_{01}A_{00},\\
\delta_{01}\psi_{01} &= \frac{1}{2}\epsilon_{01}\dot{\varphi}_{00}, & \delta_{01}\lambda_{10} &= -\frac{i}{2}\epsilon_{01}\dot{\varphi}_{11},\\
\delta_{01}A_{11} &= -\frac{1}{2}\epsilon_{01}\dot{\psi}_{10}, & \delta_{01}A_{00} &= -\frac{i}{2}\epsilon_{01}\dot{\lambda}_{01}.
\end{aligned} \tag{3.4}$$

Using the equations of motion (2.8), we get rid of V_{00}, V_{11} (instead of A's) so that the coupling constant is absorbed into A's and does not appear in $\mathcal{L}$. We change the notations $W_{00} = A_{00}, W_{11} = A_{11}$ as they will be the potentials of our model, then our Lagrangian reads

$$\begin{aligned}
\mathcal{L} = {}& \frac{1}{2}(\dot{\varphi}_{00}^2 + \dot{\varphi}_{11}^2) + i(\psi_{10}\dot{\psi}_{10} + \lambda_{10}\dot{\lambda}_{10} + \psi_{01}\dot{\psi}_{01} + \lambda_{01}\dot{\lambda}_{01})\\
& - 2W_{00}^2 - 2W_{11}^2 + 4(\psi_{10}\psi_{01} + \lambda_{10}\lambda_{01})\partial_{00}W_{11} + 4i(\psi_{10}\lambda_{10} + \psi_{01}\lambda_{01})\partial_{00}W_{00},
\end{aligned} \tag{3.5}$$

and the constraints (2.6) are given by

$$\partial_{00}W_{00} = \partial_{11}W_{11}, \qquad \partial_{00}W_{11} = \partial_{11}W_{00}. \tag{3.6}$$

We present here an example of the potentials satisfying the constraints:

$$W_{00} = e^{\varphi_{00}}\cosh\varphi_{11}, \qquad W_{11} = e^{\varphi_{00}}\sinh\varphi_{11}. \tag{3.7}$$

The conserved charges corresponding to the transformations (3.2)–(3.4) are obtained from the formulas (3.47)–(3.50) of [1]:

$$\begin{aligned}
H &= \frac{1}{2}(\dot{\varphi}_{00}^2 + \dot{\varphi}_{11}^2) + 2W_{00}^2 + 2W_{11}^2 - 4(\psi_{10}\psi_{01} + \lambda_{10}\lambda_{01})\partial_{00}W_{11} - 4i(\psi_{10}\lambda_{10} + \psi_{01}\lambda_{01})\partial_{00}W_{00},\\
Z &= 0,
\end{aligned}$$

$$Q_{10} = \sqrt{2}(\dot{\varphi}_{00}\psi_{10} - i\dot{\varphi}_{11}\lambda_{01} + 2W_{00}\lambda_{10} + 2iW_{11}\psi_{01}),$$
$$Q_{01} = \sqrt{2}(\dot{\varphi}_{00}\psi_{01} - i\dot{\varphi}_{11}\lambda_{10} + 2W_{00}\lambda_{01} + 2iW_{11}\psi_{10}). \tag{3.8}$$

The charge Z vanishes as the operator Z does not generate any transformation, cf. (3.2).

We now introduce the complex femionic variables:

$$\xi := \psi_{10} + i\lambda_{10}, \qquad \eta := \psi_{01} + i\lambda_{01}. \tag{3.9}$$

The Lagrangian (3.5) becomes (up to total time derivative)

$$\mathcal{L} = \frac{1}{2}(\dot{\varphi}_{00}^2 + \dot{\varphi}_{11}^2) + i(\bar{\xi}\dot{\xi} + \bar{\eta}\dot{\eta}) - 2W_{00}^2 - 2W_{11}^2$$
$$+ 2(\bar{\eta}\xi + \bar{\xi}\eta)\partial_{00}W_{11} + 2(\bar{\xi}\xi + \bar{\eta}\eta)\partial_{00}W_{00}. \tag{3.10}$$

The equations of motion derived from the Lagrangian are given by

$$\ddot{\varphi}_{00} + 4W_{00}\partial_{00}W_{00} + 4W_{11}\partial_{00}W_{11} - 2(\bar{\eta}\xi + \bar{\xi}\eta)\partial_{00}^2W_{11} - 2(\bar{\xi}\xi + \bar{\eta}\eta)\partial_{00}^2W_{00} = 0,$$
$$\ddot{\varphi}_{11} + 4W_{00}\partial_{11}W_{00} + 4W_{11}\partial_{11}W_{11} - 2(\bar{\eta}\xi + \bar{\xi}\eta)\partial_{00}^2W_{00} - 2(\bar{\xi}\xi + \bar{\eta}\eta)\partial_{00}^2W_{11} = 0,$$
$$i\dot{\psi}_{10} + 2\psi_{01}\partial_{00}W_{11} + 2i\lambda_{10}\partial_{00}W_{00} = 0,$$
$$i\dot{\lambda}_{10} + 2\lambda_{01}\partial_{00}W_{11} - 2i\psi_{10}\partial_{00}W_{00} = 0,$$
$$i\dot{\psi}_{01} + 2\psi_{10}\partial_{00}W_{11} + 2i\lambda_{01}\partial_{00}W_{00} = 0,$$
$$i\dot{\lambda}_{01} + 2\lambda_{10}\partial_{00}W_{11} - 2i\psi_{01}\partial_{00}W_{00} = 0. \tag{3.11}$$

In terms of the complex fermions, the conserved Noether charges Q_{10} and Q_{01} split into two parts which are conjugate to each other; see (3.18).

3.2. *Hamiltonian formalism*

When we switch from Lagrangian theory to Hamiltonian theory, we have to be careful about the order of $\mathbb{Z}_2^2$-commutative variables and their derivatives, since the derivatives are also $\mathbb{Z}_2^2$-commutative among themselves and have nontrivial relations with the $\mathbb{Z}_2^2$-graded variables [11]. We describe our conventions below.

First, we define the conjugate momentum by

$$p_q := \mathcal{L}\overleftarrow{\partial}_q, \quad q \in \{\varphi_{00}, \varphi_{11}, \xi, \bar{\xi}, \eta, \bar{\eta}\}. \tag{3.12}$$

Explicitly,

$$p_{00} = \dot{\varphi}_{00}, \quad p_{11} = \dot{\varphi}_{11}, \quad p_\xi = i\bar{\xi}, \quad p_\eta = i\bar{\eta}, \quad p_{\bar{\xi}} = p_{\bar{\eta}} = 0. \tag{3.13}$$

We see that, as the standard supersymmetry, our model is a constrained system. Here, we employ the Dirac-Bergman method for constrained systems. The constraints are given by

$$\phi_\xi = p_\xi - i\bar{\xi}, \qquad \phi_{\bar{\xi}} = p_{\bar{\xi}}, \qquad \phi_\eta = p_\eta - i\bar{\eta}, \qquad \phi_{\bar{\eta}} = p_{\bar{\eta}}. \tag{3.14}$$

The Hamiltonian and the total Hamiltonian involving the constraints are defined by

$$\mathcal{H} = \sum_q p_q\dot{q} - \mathcal{L} = \frac{1}{2}(p_{00}^2 + p_{11}^2) + 2W_{00}^2 + 2W_{11}^2 - 2(\bar{\eta}\xi + \bar{\xi}\eta)\partial_{00}W_{11} - \Big([\bar{\xi},\xi] + [\bar{\eta},\eta]\Big)\partial_{00}W_{00} \quad (3.15)$$

and

$$\mathcal{H}_T := \mathcal{H} + \alpha_\xi\phi_\xi + \alpha_{\bar{\xi}}\phi_{\bar{\xi}} + \alpha_\eta\phi_\eta + \alpha_{\bar{\eta}}\phi_{\bar{\eta}}, \quad (3.16)$$

where the Lagrange multiplier α_q has the same degree as q. The Hamilton's equations of motion equivalent to the Euler-Lagrange Eq (3.11) are given by

$$\dot{q} = \overrightarrow{\partial}_{p_q}\mathcal{H}, \qquad \dot{p}_q = -\mathcal{H}\overleftarrow{\partial}_q. \quad (3.17)$$

The Hamiltonian (3.15) is, of course, identical to the conserved Noether charge H in (3.8). The supercharges in complex notations split into two parts:

$$Q_{10} = \mathcal{Q}_{10} + \bar{\mathcal{Q}}_{10}, \qquad Q_{01} = \mathcal{Q}_{01} + \bar{\mathcal{Q}}_{01} \quad (3.18)$$

with

$$\begin{aligned}
\mathcal{Q}_{10} &= \frac{1}{\sqrt{2}}((p_{00} - 2iW_{00})\xi - (p_{11} - 2iW_{11})\eta),\\
\bar{\mathcal{Q}}_{10} &= \frac{1}{\sqrt{2}}((p_{00} + 2iW_{00})\bar{\xi} + (p_{11} + 2iW_{11})\bar{\eta}),\\
\mathcal{Q}_{01} &= \frac{1}{\sqrt{2}}((p_{00} - 2iW_{00})\eta - (p_{11} - 2iW_{11})\xi),\\
\bar{\mathcal{Q}}_{01} &= \frac{1}{\sqrt{2}}((p_{00} + 2iW_{00})\bar{\eta} + (p_{11} + 2iW_{11})\bar{\xi}).
\end{aligned} \quad (3.19)$$

Now we introduce the $\mathbb{Z}_2^2$-version of the Poisson bracket

$$\{A,B\}_{\mathrm{PB}} := A\hat{\Gamma}B - (-1)^{\vec{a}\cdot\vec{b}}B\hat{\Gamma}A, \qquad \hat{\Gamma} := \sum_q \overleftarrow{\partial}_q\overrightarrow{\partial}_{p_q}, \qquad \vec{a} := \deg A. \quad (3.20)$$

It is straightforward to verify that the Poisson bracket satisfies the following relations:

$$\begin{aligned}
\{A,B\}_{\mathrm{PB}} &= -(-1)^{\vec{a}\cdot\vec{b}}\{B,A\}_{\mathrm{PB}},\\
\{A,BC\}_{\mathrm{PB}} &= \{A,B\}_{\mathrm{PB}}C + (-1)^{\vec{a}\cdot\vec{b}}B\{A,C\}_{\mathrm{PB}},\\
\{A,\{B,C\}_{\mathrm{PB}}\}_{\mathrm{PB}} &= \{\{A,B\}_{\mathrm{PB}},C\}_{\mathrm{PB}} + (-1)^{\vec{a}\cdot\vec{b}}\{B,\{A,C\}_{\mathrm{PB}}\}_{\mathrm{PB}}.
\end{aligned} \quad (3.21)$$

The constraints (3.14) are the second class as there exist nonvanishing Poisson brackets:

$$\{\phi_\xi,\phi_{\bar{\xi}}\}_{\mathrm{PB}} = \{\phi_\eta,\phi_{\bar{\eta}}\}_{\mathrm{PB}} = -i. \quad (3.22)$$

The time evolution of the constraints determined by the equation $\dot{\phi}_q = \{\phi_q,\mathcal{H}_T\}_{\mathrm{PB}}$ is summarized as

$$(\dot{\phi}_\xi,\dot{\phi}_{\bar{\xi}},\dot{\phi}_\eta,\dot{\phi}_{\bar{\eta}}) = (\{\phi_\xi,\mathcal{H}\}_{\mathrm{PB}},\{\phi_{\bar{\xi}},\mathcal{H}\}_{\mathrm{PB}},\{\phi_\eta,\mathcal{H}\}_{\mathrm{PB}},\{\phi_{\bar{\eta}},\mathcal{H}\}_{\mathrm{PB}}) + (\alpha_\xi,\alpha_{\bar{\xi}},\alpha_\eta,\alpha_{\bar{\eta}})\Delta = 0, \quad (3.23)$$

where

$$\Delta := \left(\begin{array}{cc|cc} -\{\phi_\xi, \phi_\xi\}_{\rm PB} & -\{\phi_{\bar\xi}, \phi_\xi\}_{\rm PB} & \{\phi_\eta, \phi_\xi\}_{\rm PB} & \{\phi_{\bar\eta}, \phi_\xi\}_{\rm PB} \\ -\{\phi_\xi, \phi_{\bar\xi}\}_{\rm PB} & -\{\phi_{\bar\xi}, \phi_{\bar\xi}\}_{\rm PB} & \{\phi_\eta, \phi_{\bar\xi}\}_{\rm PB} & \{\phi_{\bar\eta}, \phi_{\bar\xi}\}_{\rm PB} \\ \hline \{\phi_\xi, \phi_\eta\}_{\rm PB} & \{\phi_{\bar\xi}, \phi_\eta\}_{\rm PB} & -\{\phi_\eta, \phi_\eta\}_{\rm PB} & -\{\phi_{\bar\eta}, \phi_\eta\}_{\rm PB} \\ \{\phi_\xi, \phi_{\bar\eta}\}_{\rm PB} & \{\phi_{\bar\xi}, \phi_{\bar\eta}\}_{\rm PB} & -\{\phi_\eta, \phi_{\bar\eta}\}_{\rm PB} & -\{\phi_{\bar\eta}, \phi_{\bar\eta}\}_{\rm PB} \end{array}\right) = i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{3.24}$$

This relation determines the Lagrange multiplier

$$\begin{aligned}(\alpha_\xi, \alpha_{\bar\xi}, \alpha_\eta, \alpha_{\bar\eta}) &= -(\{\phi_\xi, \mathcal{H}\}_{\rm PB}, \{\phi_{\bar\xi}, \mathcal{H}\}_{\rm PB}, \{\phi_\eta, \mathcal{H}\}_{\rm PB}, \{\phi_{\bar\eta}, \mathcal{H}\}_{\rm PB})\Delta^{-1} \\ &= i(\{\phi_{\bar\xi}, \mathcal{H}\}_{\rm PB}, \{\phi_\xi, \mathcal{H}\}_{\rm PB}, \{\phi_{\bar\eta}, \mathcal{H}\}_{\rm PB}, \{\phi_\eta, \mathcal{H}\}_{\rm PB}).\end{aligned} \tag{3.25}$$

More explicitly, we have the expressions:

$$\begin{aligned}\alpha_\xi &= -2i\eta\, \partial_{00} W_{11} - 2i\xi\, \partial_{00} W_{00}, \\ \alpha_{\bar\xi} &= -2i\bar\eta\, \partial_{00} W_{11} + 2i\bar\xi\, \partial_{00} W_{00} = \overline{\alpha}_\xi, \\ \alpha_\eta &= -2i\xi\, \partial_{00} W_{11} - 2i\eta\, \partial_{00} W_{00}, \\ \alpha_{\bar\eta} &= -2i\bar\xi\, \partial_{00} W_{11} + 2i\bar\eta\, \partial_{00} W_{00} = \overline{\alpha}_\eta.\end{aligned} \tag{3.26}$$

With this data, one may define a $\mathbb{Z}_2^2$-version of the Dirac bracket by

$$\begin{aligned}\{A, B\}_{\rm DB} &:= \{A, B\}_{\rm PB} + \sum_{q,q'} \{A, \phi_q\}_{\rm PB}\, \Delta^{-1}_{qq'}\, \{\phi_{q'}, B\}_{\rm PB} \\ &= \{A, B\}_{\rm PB} - i\{A, \phi_\xi\}_{\rm PB}\{\phi_{\bar\xi}, B\}_{\rm PB} - i\{A, \phi_{\bar\xi}\}_{\rm PB}\{\phi_\xi, B\}_{\rm PB} \\ &\quad - i\{A, \phi_\eta\}_{\rm PB}\{\phi_{\bar\eta}, B\}_{\rm PB} - i\{A, \phi_{\bar\eta}\}_{\rm PB}\{\phi_\eta, B\}_{\rm PB}.\end{aligned} \tag{3.27}$$

It is not difficult to verify that the Dirac bracket satisfies the same relations in (3.21) as the $\mathbb{Z}_2^2$-Poisson bracket.

One may easily find that the nonvanishing Dirac brackets for the canonical variables are the following

$$\{\varphi_{00}, p_{00}\}_{\rm DB} = \{\varphi_{11}, p_{11}\}_{\rm DB} = \{\xi, p_\xi\}_{\rm DB} = \{\eta, p_\eta\}_{\rm DB} = 1. \tag{3.28}$$

Using (3.13), the Dirac brackets for the fermionic variables are converted into the form:

$$\{\xi, \bar\xi\}_{\rm DB} = \{\eta, \bar\eta\}_{\rm DB} = -i. \tag{3.29}$$

We introduce the quantity of $\mathbb{Z}_2^2$-degree $(1,1)$:

$$\mathcal{Z} = -p_{00}p_{11} - 4W_{00}W_{11} + 2\partial_{00}W_{00}(\bar\xi\eta + \bar\eta\xi) + \partial_{00}W_{11}([\bar\xi, \xi] + [\bar\eta, \eta]). \tag{3.30}$$

One may verify that $\mathcal{H}, Q_a, \bar{Q}_a, \mathcal{Z}$ close in the $\mathcal{N} = 2$ extended $\mathbb{Z}_2^2$-supersymmetry algebra whose nonvanishing Dirac brackets are given by

$$\begin{aligned}\{Q_{10}, \bar{Q}_{10}\}_{\rm DB} &= \{Q_{01}, \bar{Q}_{01}\}_{\rm DB} = -i\mathcal{H}, \\ \{\bar{Q}_{10}, Q_{01}\}_{\rm DB} &= -\{Q_{10}, \bar{Q}_{01}\}_{\rm DB} = i\mathcal{Z}.\end{aligned} \tag{3.31}$$

The combined $\mathcal{N} = 1$ supercharges (3.18) satisfy the $\mathcal{N} = 1$ $\mathbb{Z}_2^2$-supersymmetry algebra with vanishing Z:

$$\{Q_{10}, Q_{10}\}_{\rm DB} = \{Q_{01}, Q_{01}\}_{\rm DB} = -2i\mathcal{H}, \qquad \{Q_{10}, Q_{01}\}_{\rm DB} = 0. \tag{3.32}$$

4. $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetric quantum mechanics

We quantize the system discussed in Section 3.2, which means that the Dirac bracket is replaced with the $\mathbb{Z}_2^2$-Lie bracket ($\hbar = 1$):

$$\{A, B\}_{\rm DB} \;\to\; \frac{1}{i}[\![A, B]\!] \tag{4.1}$$

This gives the following nonvanishing (anti) commutators

$$[\varphi_{00}, p_{00}] = [\varphi_{11}, p_{11}] = i, \qquad \{\xi, \xi^\dagger\} = \{\eta, \eta^\dagger\} = 1, \tag{4.2}$$

and all the following vanishes:

$$\begin{array}{lllll} \{\xi, \xi\}, & \{\xi^\dagger, \xi^\dagger\}, & \{\eta, \eta\}, & \{\eta^\dagger, \eta^\dagger\}, & \\ {[\xi, \eta]}, & [\xi, \eta^\dagger], & [\xi^\dagger, \eta], & [\xi^\dagger, \eta^\dagger], & \\ \{c_{11}, \xi\}, & \{c_{11}, \eta\}, & \{c_{11}, \xi^\dagger\}, & \{c_{11}, \eta^\dagger\}, & c_{11} = \varphi_{11}, p_{11}, \end{array} \tag{4.3}$$

where and in what follows, we use "dagger" instead of "bar" for the hermitian conjugation of the quantum operators.

By using the real representation of the Clifford algebra $Cl\,(4, 2)$ [44–46], the relations (4.2) and (4.3) are realized by matrix differential operators. In this realization, the $\mathbb{Z}_2^2$-grading is carried by the matrices which means that if there are nonzero entries in one of the following blocks, the matrix has the indicated $\mathbb{Z}_2^2$-degree:

$$\begin{pmatrix} (0,0) & (1,1) & (1,0) & (0,1) \\ (1,1) & (0,0) & (0,1) & (1,0) \\ (1,0) & (0,1) & (0,0) & (1,1) \\ (0,1) & (1,0) & (1,1) & (0,0) \end{pmatrix}. \tag{4.4}$$

The Clifford algebra $Cl\,(4, 2)$ is generated by $\gamma_i, i = 1, 2, \ldots 6$, which is subject to the relations

$$\{\gamma_i, \gamma_j\} = 2\eta_{ij}, \quad \eta = \mathrm{diag}(1, 1, 1, 1, -1, -1). \tag{4.5}$$

We introduce the anti-commuting matrices X, Y, A and the identity matrix I:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.6}$$

then the real irrep of $Cl\,(4, 2)$ is given by

$$\begin{array}{lll} \gamma_1 = XII,\ (0,0), & \gamma_2 = YII,\ (1,0), & \gamma_3 = AAI,\ (0,1), \\ \gamma_4 = AYA,\ (0,1), & \gamma_5 = AXI,\ (1,0), & \gamma_6 = AYX,\ (0,1), \end{array} \tag{4.7}$$

where a word consisting of these matrices is understood as the tensor product, e.g.,

$$XYA = X \otimes Y \otimes A,$$

and the $\mathbb{Z}_2^2$-degree of γ_i is also indicated. With this eight-dimensional irrep, the $\mathbb{Z}_2^2$-graeded quantum operators are realized as:

$$\varphi_{00} = x_0 I_8, \qquad\qquad p_{00} = -\,i\partial_{x_0} I_8, \tag{4.8}$$

$$\varphi_{11} = x_1\Gamma, \qquad p_{11} = -i\partial_{x_1}\Gamma, \tag{4.9}$$

$$\xi = \frac{i}{2}(\gamma_1\gamma_5 + i\gamma_3\gamma_4\gamma_5), \quad \xi^\dagger = -\frac{i}{2}(\gamma_1\gamma_5 - i\gamma_3\gamma_4\gamma_5), \tag{4.10}$$

$$\eta = \frac{1}{2}(\gamma_3 + i\gamma_4), \qquad \eta^\dagger = \frac{1}{2}(\gamma_3 - i\gamma_4), \tag{4.11}$$

where

$$I_8 = III, \qquad \Gamma = -\gamma_3\gamma_4\gamma_5\gamma_6,$$

and $x_0,\ x_1 \in \mathbb{R}$. The degree $(1,1)$ function W_{11} is also realized by the matrix Γ and the constraints (3.6) read as follows:

$$W_{11} = \tilde{W}_{00}(x_0, x_1)\Gamma, \qquad \partial_{x_0}W_{00} = \partial_{x_1}\tilde{W}_{00}, \qquad \partial_{x_1}W_{00} = \partial_{x_0}\tilde{W}_{00}, \tag{4.12}$$

where $\tilde{W}_{00}$ is a degree $(0,0)$ function. Therefore, we get the two-dimensional or two-particle (same mass) quantum mechanical system in this realization.

The quantized $\mathcal{N} = 2$ supercharges (3.19) are given by

$$Q_{10} = a\xi - b\Gamma\eta, \qquad Q_{10}^\dagger = a^\dagger\xi^\dagger + b^\dagger\Gamma\eta^\dagger, \tag{4.13}$$

$$Q_{01} = a\eta - b\Gamma\xi, \qquad Q_{01}^\dagger = a^\dagger\eta^\dagger + b^\dagger\Gamma\xi^\dagger, \tag{4.14}$$

where

$$a := \frac{1}{\sqrt{2}}(-i\partial_{x_0} - 2iW_{00})$$

and

$$b := \frac{1}{\sqrt{2}}(-i\partial_{x_1} - 2i\tilde{W}_{00}).$$

We introduce the new operators

$$A := \frac{1}{\sqrt{2}}(a + b), \quad B := \frac{1}{\sqrt{2}}(a - b), \tag{4.15}$$

and the unitary matrix which diagonalize the Hamiltonian (3.15)

$$U = \left(\begin{array}{cc:cc|cc:cc} 1 & 0 & & & & & & \\ 0 & 1 & & & & & & \\ \hdashline & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ \hline & & & & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & & \\ & & & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ \hdashline & & & & & & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ & & & & & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right). \tag{4.16}$$

We then have $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-SQM:

$$\tilde{\mathcal{H}} := U^\dagger \mathcal{H} U = \mathrm{diag}(H_1, H_2, H_1, H_2, H_3, H_4, H_3, H_4), \tag{4.17}$$

where

$$
\begin{aligned}
H_1 &= AA^\dagger + B^\dagger B, & H_2 &= A^\dagger A + BB^\dagger, \\
H_3 &= AA^\dagger + BB^\dagger, & H_4 &= A^\dagger A + B^\dagger B
\end{aligned}
\tag{4.18}
$$

with the supercharges

$$
\tilde{Q}_{10} := U^\dagger Q_{10} U = \left(\begin{array}{cc|cc|cc|cc}
 & & & & 0 & A & & \\
 & & & & 0 & iB & & \\
\hline
 & & & & & & 0 & -iA \\
 & & & & & & 0 & B \\
\hline
B & iA & & & & & & \\
0 & 0 & & & & & & \\
\hline
 & & -iB & A & & & & \\
 & & 0 & 0 & & & &
\end{array}\right),
\tag{4.19}
$$

$$
\tilde{Q}_{01} := U^\dagger Q_{01} U = \left(\begin{array}{cc|cc|cc|cc}
 & & & & & & 0 & iA \\
 & & & & & & 0 & B \\
\hline
 & & & & 0 & -A & & \\
 & & & & 0 & iB & & \\
\hline
 & & -B & iA & & & & \\
 & & 0 & 0 & & & & \\
\hline
iB & A & & & & & & \\
0 & 0 & & & & & &
\end{array}\right),
\tag{4.20}
$$

and their hermitian conjugation. Furthermore, we have the nonvanishing degree $(1,1)$ operator (3.30)

$$
\tilde{\mathcal{Z}} := U^\dagger \mathcal{Z} U = \left(\begin{array}{cc|cc|cc|cc}
 & & Z_1 & & & & & \\
 & & & Z_2 & & & & \\
\hline
Z_1^\dagger & & & & & & & \\
 & Z_2^\dagger & & & & & & \\
\hline
 & & & & & & Z_3 & \\
 & & & & & & & Z_4 \\
\hline
 & & & & Z_3^\dagger & & & \\
 & & & & & Z_4^\dagger & &
\end{array}\right),
\tag{4.21}
$$

where

$$
\begin{aligned}
Z_1 &= -AA^\dagger + B^\dagger B = Z_1^\dagger, & Z_2 &= -A^\dagger A + BB^\dagger = Z_2^\dagger, \\
Z_3 &= i(AA^\dagger - BB^\dagger) = -Z_3^\dagger, & Z_4 &= -i(A^\dagger A - B^\dagger B) = -Z_4^\dagger.
\end{aligned}
\tag{4.22}
$$

The products of $A, A^\dagger$ and $B, B^\dagger$ are given by

$$
\begin{aligned}
A^\dagger A &= -\frac{1}{4}(\partial_{x_0} + \partial_{x_1})^2 + W_{00}^2 + \tilde{W}_{00}^2 - \partial_{x_0} W_{00} - \partial_{x_0} \tilde{W}_{00} + 2W_{00}\tilde{W}_{00}, \\
AA^\dagger &= A^\dagger A + 2\partial_{x_0} W_{00} + 2\partial_{x_0} \tilde{W}_{00},
\end{aligned}
\tag{4.23}
$$

and

$$B^\dagger B = -\frac{1}{4}(\partial_{x_0} - \partial_{x_1})^2 + W_{00}^2 + \tilde{W}_{00}^2 - \partial_{x_0} W_{00} + \partial_{x_0} \tilde{W}_{00} - 2W_{00}\tilde{W}_{00},$$
$$BB^\dagger = B^\dagger B + 2\partial_{x_0} W_{00} - 2\partial_{x_0} \tilde{W}_{00}, \tag{4.24}$$

where we used (4.12) to have these formulae. The relations (4.12) are also used to see that the nonvanishing commutation relations among $A^\dagger, A, B^\dagger, B$ are the following:

$$[A, A^\dagger] = 2\partial_{x_0} W_{00} + 2\partial_{x_0} \tilde{W}_{00}, \quad [B, B^\dagger] = 2\partial_{x_0} W_{00} - 2\partial_{x_0} \tilde{W}_{00}. \tag{4.25}$$

It is not difficult to verify that $\tilde{\mathcal{H}}, \tilde{Q}_a, \tilde{Q}_a^\dagger$ and $\tilde{\mathbb{Z}}$ forms the $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetry algebra whose nonvanishing relations are given by

$$\{\tilde{Q}_{10}, \tilde{Q}_{10}^\dagger\} = \{\tilde{Q}_{01}, \tilde{Q}_{01}^\dagger\} = \tilde{\mathcal{H}}, \qquad [\tilde{Q}_{10}, \tilde{Q}_{01}^\dagger] = -[\tilde{Q}_{10}^\dagger, \tilde{Q}_{01}] = \tilde{\mathcal{Z}}. \tag{4.26}$$

It is also immediate that the combined $\mathcal{N} = 1$ supercharges (3.18) satisfy the $\mathcal{N} = 1$ $\mathbb{Z}_2^2$-supersymmetry algebra with vanishing Z:

$$\{Q_{10}, Q_{10}\} = \{Q_{01}, Q_{01}\} = 2\mathcal{H}, \qquad [Q_{10}, Q_{01}] = 0. \tag{4.27}$$

One may also see from (4.17) and (4.21) that $\tilde{\mathcal{Z}}^2 \neq \tilde{\mathcal{H}}^2$. This is the sharp contrast to the $\mathbb{Z}_2^2$-SQMs discussed in the literature [4, 7, 11] where one always observes that $Z^2 = H^2$. The relation $\tilde{\mathcal{Z}}^2 \neq \tilde{\mathcal{H}}^2$ implies that our $\mathbb{Z}_2^2$-SQM is a quantum mechanical realization of an eight-dimensional irrep of $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetry algebra. In [12], it is shown that irreps of the $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetry algebra are four-dimensional if $\tilde{\mathcal{Z}}^2 = \tilde{\mathcal{H}}^2$, but eight-dimensional otherwise. Our $\mathbb{Z}_2^2$-SQM is the first example of the physical realization of eight-dimensional irrep of the $\mathbb{Z}_2^2$-supersymmetry algebra.

The formulae (4.23) and (4.24) suggest the introduction of the light cone coordinates

$$x_+ := x_0 + x_1, \qquad x_- := x_0 - x_1. \tag{4.28}$$

The constraints in (4.12) become

$$\partial_+ W_{00}(x_+, x_-) = \partial_+ \tilde{W}_{00}(x_+, x_-), \tag{4.29}$$
$$\partial_- W_{00}(x_+, x_-) = -\partial_- \tilde{W}_{00}(x_+, x_-), \tag{4.30}$$

and these differential equations may be solved to give the separation of left and right movers

$$W_{00}(x_+, x_-) = \frac{1}{2}\left(W_+'(x_+) + W_-'(x_-)\right), \tag{4.31}$$
$$\tilde{W}_{00}(x_+, x_-) = \frac{1}{2}\left(W_+'(x_+) - W_-'(x_-)\right), \tag{4.32}$$

where the prime stands for the derivative. The operators (4.15) in the light cone coordinates yield the standard ones in the SQM:

$$A = -i\partial_+ - iW_+', \qquad B = -i\partial_- - iW_-', \tag{4.33}$$

which give the following:

$$A^\dagger A = -\partial_+^2 + (W_+')^2 - W_+'', \qquad AA^\dagger = -\partial_+^2 + (W_+')^2 + W_+'',$$
$$B^\dagger B = -\partial_-^2 + (W_-')^2 - W_-'', \qquad BB^\dagger = -\partial_-^2 + (W_-')^2 + W_-''. \tag{4.34}$$

The Hilbert space of our $\mathbb{Z}_2^2$-SQM is $\mathfrak{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^8$, and the space is also $\mathbb{Z}_2^2$-graded:

$$\mathfrak{H} = \mathfrak{H}_{(0,0)} \oplus \mathfrak{H}_{(1,1)} \oplus \mathfrak{H}_{(1,0)} \oplus \mathfrak{H}_{(0,1)}. \tag{4.35}$$

The algebra (4.26) implies that the Hamilotonian $\tilde{\mathcal{H}}$ (4.17) is positive semi-definite. This is also seen from the component Hamiltonian H_k (4.18), all of which are also positive semi-definite. The zero energy ground state Ψ_0 of $\tilde{\mathcal{H}}$ is determined by

$$\tilde{Q}_a \Psi_0 = \tilde{Q}_a^\dagger \Psi_0 = 0. \tag{4.36}$$

This is equivalent to finding the zero energy states of the component Hamiltonian $H_k \psi_0^{(k)} = 0$. More explicitly, $\psi_0^{(k)}$ are solutions of the equations

$$A^\dagger \psi_0^{(1)} = B\psi_0^{(1)} = 0, \qquad A\psi_0^{(2)} = B^\dagger \psi_0^{(2)} = 0,$$
$$A^\dagger \psi_0^{(3)} = B^\dagger \psi_0^{(3)} = 0, \qquad A\psi_0^{(4)} = B\psi_0^{(4)} = 0. \tag{4.37}$$

It is easy to solve these equations:

$$\psi_0^{(1)} = \exp(W_+)\exp(-W_-), \qquad \psi_0^{(2)} = \exp(-W_+)\exp(W_-),$$
$$\psi_0^{(3)} = \exp(W_+)\exp(W_-), \qquad \psi_0^{(4)} = \exp(-W_+)\exp(-W_-). \tag{4.38}$$

It is also easy to see that only one of them is normalizable. For instance, if $\psi_0^{(1)}$ is normalizable, all other functions are not normalizable. Therefore, the possible ground state is one of the following ($c \in \mathbb{C}$ is a constant):

$$\begin{aligned}
(\psi_0^{(1)}, 0, c\,\psi_0^{(1)}, 0, 0, 0, 0, 0) &\in \mathfrak{H}_{(0,0)} \oplus \mathfrak{H}_{(1,1)},\\
(0, \psi_0^{(2)}, 0, c\,\psi_0^{(2)}, 0, 0, 0, 0) &\in \mathfrak{H}_{(0,0)} \oplus \mathfrak{H}_{(1,1)},\\
(0, 0, 0, 0, \psi_0^{(3)}, 0, c\,\psi_0^{(3)}, 0) &\in \mathfrak{H}_{(1,0)} \oplus \mathfrak{H}_{(0,1)},\\
(0, 0, 0, 0, 0, \psi_0^{(4)}, 0, c\,\psi_0^{(4)}) &\in \mathfrak{H}_{(1,0)} \oplus \mathfrak{H}_{(0,1)}.
\end{aligned} \tag{4.39}$$

Therefore, the ground state is either nonexistent or two-fold degenerate and belongs to $\mathfrak{H}_{(0,0)} \oplus \mathfrak{H}_{(1,1)}$ or $\mathfrak{H}_{(1,0)} \oplus \mathfrak{H}_{(0,1)}$.

5. Conclusions

In order to investigate a quantum theory which is relating to the $\mathcal{N} = 1$ $\mathbb{Z}_2^2$-supersymmetric Lagrangian (2.5), we studied the $\mathbb{Z}_2^2$-SQM obtained from the Lagrangian by dimensional reduction. The dimensional reduction increases the supersymmetry from $\mathcal{N} = 1$ to $\mathcal{N} = 2$, and we employed the $\mathbb{Z}_2^2$-extended Dirac-Bargmann method to quantize the system. The $\mathbb{Z}_2^2$-SQM obtained is a two-dimensional or two-particle quantum system in which the right and left movers are separated. It is also a quantum mechanical realization of the eight-dimensional irrep of $\mathcal{N} = 2$ $\mathbb{Z}_2^2$-supersymmetry algebra discussed in [12]. Moreover, it is the first $\mathbb{Z}_2^2$-SQM with $\tilde{\mathcal{Z}}^2 \neq \tilde{\mathcal{H}}^2$.

There is a large freedom of choice of the super potential $W_{\pm}(x_{\pm})$. The simplest but interesting choice is the harmonic oscillator, since we may have a larger symmetry. In [47], it is shown that the largest spectrum generating algebra of the supersymmetric harmonic oscillator is the semidirect sum of $osp(2|2)$ and 1D Heisenberg superalgebra. However, one may easily verify that the operators in the article also close in a $\mathbb{Z}_2^2$-graded Lie superalgebra. If we consider the $\mathbb{Z}_2^2$-supersymmetric harmonic oscillator, then the largest spectrum generating algebra will be higher graded than the $\mathbb{Z}_2^2$-grading. Another interesting choice is the Calogero type potential, which will give a conformal extension of the present $\mathbb{Z}_2^2$-SQM. This potential is also interesting from the viewpoint of representations since it will give a representation of $\mathbb{Z}_2^2$-$osp(1|2)$ [48].

As seen in Section 4, the Hamiltonian of our $\mathbb{Z}_2^2$-SQM is a sum of factorized operators. This implies that the $\mathbb{Z}_2^2$-SQM is related to some solvable potentials. Recall that many solvable potential is a 1D single particle problem, but our Hamiltonian is 2D (or two-particle), so we expect there are some solvable quantum models that have not yet been recognized. The search of such models will be an interesting future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this pape.

References

1. N. Aizawa, R. Ito, T. Tanaka, $\mathbb{Z}_2^2$-graded supersymmetry via superfield on minimal $\mathbb{Z}_2^2$-superspace, *ArXiv*, 2023.
2. A. J. Bruce, On a $\mathbb{Z}_2^n$-graded version of supersymmetry, *Symmetry*, **11** (2019), 116. https://doi.org/10.3390/sym11010116
3. A. J. Bruce, Is the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded sine-Gordon equation integrable? *Nucl. Phys. B*, **971** (2021), 115514. https://doi.org/10.1016/j.nuclphysb.2021.115514
4. A. J. Bruce, S. Duplij, Double-graded supersymmetric quantum mechanics, *J. Math. Phys.*, **61** (2020), 063503. https://doi.org/10.1063/1.5118302
5. F. Toppan, $\mathbb{Z}_2\times\mathbb{Z}_2$-graded parastatics in multiparticle quantum Hamiltonians, *J. Phys. A*, **54** (2021), 115203. https://doi.org/10.1088/1751-8121/abe2f2

6. F. Toppan, Inequivalent quantizations from gradings and $\mathbb{Z}_2\times\mathbb{Z}_2$ parabosons, *J. Phys. A*, **54** (2021), 355202. https://doi.org/10.1088/1751-8121/ac17a5

7. N. Aizawa, K. Amakawa, S. Doi, $\mathcal{N}$-extension of double-graded supersymmetric and superconformal quantum mechanics, *J. Phys. A*, **53** (2020), 065205. https://doi.org/10.1088/1751-8121/ab661c

8. N. Aizawa, K. Amakawa, S. Doi, $\mathbb{Z}_2^n$-graded extensions of supersymmetric quantum mechanics via Clifford algebras, *J. Math. Phys.*, **61** (2020), 052105. https://doi.org/10.1063/1.5144325

9. S. Doi, N. Aizawa, $\mathbb{Z}_2^3$-Graded extensions of Lie superalgebras and superconformal quantum mechanics, *Symmetry Integr. Geom.*, **17** (2021), 071. https://doi.org/10.3842/SIGMA.2021.071

10. N. Aizawa, Z. Kuznetsova, F. Toppan, $\mathbb{Z}_2\times\mathbb{Z}_2$-graded mechanics: the classical theory, *Eur. Phys. J. C*, **80** (2020), 668. https://doi.org/10.1140/epjc/s10052-020-8242-x

11. N. Aizawa, Z. Kuznetsova, F. Toppan, $\mathbb{Z}_2\times\mathbb{Z}_2$-graded mechanics: the quantization, *Nucl. Phys. B*, **967** (2021), 115426. https://doi.org/10.1016/j.nuclphysb.2021.115426

12. N. Aizawa, S. Doi, Irreducible representations of $\mathbb{Z}_2^2$-graded $\mathcal{N}=2$ supersymmetry algebra and $\mathbb{Z}_2^2$-graded supermechanics, *J. Math. Phys.*, **63** (2022), 091704. https://doi.org/10.1063/5.0100182

13. G. Junker, *Supersymmetric methods in quantum and statistical mechanics*, Springer, 1996. https://doi.org/10.1007/978-3-642-61194-0

14. B. Bagchi, *Supersymmetry in quantum and classical mechanics*, Chapman & Hall/CRC, 2001. https://doi.org/10.1201/9780367801670

15. S. H. Dong, *Factorization method in quantum mechanics*, Springer, 2007. https://doi.org/10.1007/978-1-4020-5796-0

16. J. Beckers, N. Debergh, On colour superalgebras in parasupersymmetric quantum mechanics, *J. Phys. A*, **24** (1991), L597. https://doi.org/10.1088/0305-4470/24/11/005

17. N. Aizawa, Z. Kuznetsova, H. Tanaka, F. Toppan, $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie symmetries of the Lévy-Leblond equations, *Prog. Theor. Exp. Phys.*, **2016** (2016), 123A01. https://doi.org/10.1093/ptep/ptw176

18. N. Aizawa, Z. Kuznetsova, H. Tanaka, F. Toppan, Generalized supersymmetry and Lévy-Leblond equation, In: S. Duarte, J. P. Gazeau, S. Faci, T. Micklitz, R. Scherer, F. Toppan, *Physical and mathematical aspects of symmetries*, Springer, 2017. https://doi.org/10.1007/978-3-319-69164-0_11

19. M . A. Vasiliev, de Sitter supergravity with positive cosmological constant and generalised Lie superalgebras, *Class. Quantum Gravity*, **2** (1985), 645. https://doi.org/10.1088/0264-9381/2/5/007

20. P. D. Jarvis, M. Yang, B. G. Wybourne, Generalized quasispin for supergroups, *J. Math. Phys.*, **28** (1987), 1192–1197. https://doi.org/10.1063/1.527566

21. A. A. Zheltukhin, Para-Grassmann extension of the Neveu-Schwartz-Ramond algebra, *Theor. Math. Phys.*, **71** (1987), 491–496. https://doi.org/10.1007/BF01028648

22. V. N. Tolstoy, Once more on parastatistics, *Phys. Part. Nuclei Lett.* **11** (2014), 933–937. https://doi.org/10.1134/S1547477114070449

23. N. I. Stoilova, J. Van der Jeugt, The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{pso}(2m+1|2n)$ and new parastatistics representations, *J. Phys. A*, **51** (2018), 135201. https://doi.org/10.1088/1751-8121/aaae9a
24. N. I. Stoilova, J. van der Jeugt, Parabosons, parafermions and representations of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebras, *J. Phys.*, **1194** (2019), 012102. https://doi.org/10.1088/1742-6596/1194/1/012102
25. N. I. Stoilova, J. van der Jeugt, The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebras $\mathfrak{pso}(2n+1|2n)$ and $\mathfrak{pso}(\infty|\infty)$, and parastatistics Fock spaces, *J. Phys. A*, **55** (2022), 045201. https://doi.org/10.1088/1751-8121/ac451d
26. C. H. Alderete, A. M. Green, N. H. Nguyen, Y. Zhu, B. M. Rodríguez-Lara, N. M. Linke, Experimental realization of para-particle oscillators, *ArXiv*, 2018. https://doi.org/10.48550/arXiv.2108.05471
27. B. L. Roy, $\mathbb{Z}_n^3$-Graded colored supersymmetry, *Czech. J. Phys.*, **47** (1997), 47–54. https://doi.org/10.1023/A:1021491927893
28. L. A. Wills-Toro, (I, q)-graded Lie algebraic extensions of the Poincaré algebra, constraints on I and q, *J. Math. Phys.*, **36** (1995), 2085–2112. https://doi.org/10.1063/1.531102
29. L. A. Wills-Toro, Trefoil symmetries I. Clover extensions beyond Coleman-Mandula theorem, *J. Math. Phys.*, **42** (2001), 3915–3934. https://doi.org/10.1063/1.1383561
30. V. N. Tolstoy, Super-de Sitter and alternative super-Poincaré symmetries, In: V. Dobrev, *Lie theory and its applications in physics*, Tokyo: Springer, 2014. https://doi.org/10.1007/978-4-431-55285-7_26
31. C. Quesne, Minimal bosonization of double-graded quantum mechanics, *Mod. Phys. Lett.*, **A36** (2021), 2150238. https://doi.org/10.1142/S0217732321502382
32. A. J. Bruce, $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supersymmetry: 2-d sigma models, *J. Phys. A*, **53** (2020), 455201. https://doi.org/10.1088/1751-8121/abb47f
33. M. M. Balbino, I. P. de Freitas, R. G. Rana, F. Toppan, Inequivalent $\mathbb{Z}_2^n$-graded brackets, n-bit parastatistics and statistical transmutations of supersymmetric quantum mechanics, *ArXiv*, 2023. https://doi.org/10.48550/arXiv.2309.00965
34. S. Doi, N. Aizawa, Comments of $\mathbb{Z}_2^2$-supersymmetry in superfield formalism, *Nucl. Phys. B*, **974** (2022), 115641. https://doi.org/10.1016/j.nuclphysb.2021.115641
35. N. Aizawa, R. Ito, Z. Kuznetsova, F. Toppan, New aspects of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded 1D superspace: induced strings and 2D relativistic models, *Nuclear Phys. B*, **991** (2023), 116202. https://doi.org/10.1016/j.nuclphysb.2023.116202
36. T. Covolo, J. Grabowski, N. Poncin, The category of $\mathbb{Z}_2^n$-supermanifolds, *J. Math. Phys.*, **57** (2016), 073503. https://doi.org/10.1063/1.4955416
37. N. Poncin, Towards integration on colored supermanifolds, *Banach Cent. Publ.*, **110** (2016), 201–217. https://doi.org/10.4064/bc110-0-14
38. N. Poncin, S. Schouten, *The geometry of supersymmetry / a concise introduction*, *ArXiv*, 2022. https://doi.org/10.48550/arXiv.2207.12974
39. N. Aizawa, R. Ito, Integration on minimal $\mathbb{Z}_2^2$-superspace and emergence of space, *J. Phys. A*, **56** (2023), 485201. https://doi.org/10.1088/1751-8121/ad076e

40. V. Rittenberg, D. Wyler, Generalized superalgebras, *Nucl. Phys. B*, **139** (1978), 189–202. https://doi.org/10.1016/0550-3213(78)90186-4

41. V. Rittenberg, D. Wyler, Sequences of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ graded Lie algebras and superalgebras, *J. Math. Phys.*, **19** (1978), 2193–2200. https://doi.org/10.1063/1.523552

42. R. Ree, Generalized Lie elements, *Canad. J. Math.*, **12** (1960), 493–502. https://doi.org/10.4153/CJM-1960-044-x

43. M. Scheunert, Generalized Lie algebras, *J. Math. Phys.*, **20** (1979), 712–720. https://doi.org/10.1063/1.524113

44. S. Okubo, Real representations of finite Clifford algebras. I. Classification, *J. Math. Phys.*, **32** (1991), 1657–1668. https://doi.org/10.1063/1.529277

45. S. Okubo, Real representations of finite Clifford algebras. II. Explicit construction and pseudo-octonion, *J. Math. Phys.*, **32** (1991), 1669–1673. https://doi.org/10.1063/1.529278

46. H. L. Carrion, M. Rojas, F. Toppan, Quaternionic and octonionic spinors. A classification, *J. High Energy Phys.*, **04** (2003), 040. https://doi.org/10.1088/1126-6708/2003/04/040

47. J. Beckers, V. Hussin, Dynamical supersymmetries of the harmonic oscillator, *Phys. Lett. A*, **118** (1986), 319–321. https://doi.org/10.1016/0375-9601(86)90316-6

48. K. Amakawa, N. Aizawa, A classification of lowest weight irreducible modules over $\mathbb{Z}_2^2$-graded extension of $osp(1|2)$, *J. Math. Phys.*, **62** (2021), 043502. https://doi.org/10.1063/5.0037493

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