AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 10446–10461.
DOI:10.3934/math.2024510
Received: 29 December 2023
Revised: 07 March 2024
Accepted: 12 March 2024
Published: 18 March 2024

Research article

Quantum calculus with respect to another function

Nattapong Kamsrisuk[1], **Donny Passary**[1], **Sotiris K. Ntouyas**[2] **and Jessada Tariboon**[1,*]

[1] Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

[2] Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece

* **Correspondence:** Email: jessada.t@sci.kmutnb.ac.th.

Abstract: In this paper, we studied the generalizations of quantum calculus on finite intervals. We presented the new definitions of the quantum derivative and quantum integral of a function with respect to another function and studied their basic properties. We gave an application of these newly defined quantum calculi by obtaining a new Hermite-Hadamard inequality for a convex function. Moreover, an impulsive boundary value problem involving quantum derivative, with respect to another function, was studied via the Banach contraction mapping principle.

Keywords: quantum calculus; quantum derivative; quantum integral; Hermite-Hadamard inequality; boundary value problem; existence; uniqueness; fixed point theorem
Mathematics Subject Classification: 26D15, 34A08, 34B37

1. Introduction

Quantum calculus, also known as q-calculus, is the study of calculus without the notion of limits, which presents alternative flavors of calculus. More precisely, instead of defining the derivatives and integrals of real value functions by means of limits, quantum calculus makes use of their respective q-analog versions. Quantum calculus is also used to study calculus on discrete or even finite sets. Historically, Euler had obtained some basic formulae in q-calculus in the eighteenth century. However, it is Jackson [1] who established what is now known as q-derivative and q-integral. Interests in the research of q-calculus are currently flourishing due to its several applications in many areas, such as physics, number theory, orthogonal polynomials, hypergeometric functions, and combinatorics [2, 3]. Al-Salam [4] and Agarwal [5] have generalized q-derivatives and q-integrals into orders other than integers, as seen in the monograph [6]. Many interesting results in such areas of research were also presented by several authors in the literature [7–15] and references therein. Tariboon and Ntouyas introduced in [16] the notion of quantum calculus on finite intervals. The interested reader is referred to the monograph [17] for further details.

The basic aid of the present paper is to give new definitions of quantum calculus on finite intervals with respect to another function. We give the new definitions of the quantum derivative and quantum integral and study their basic properties. Also, we apply the new defined quantum calculus with respect to another function and obtain a new Hermite-Hadamard inequality for a convex function. Our results are new and, in special cases, correspond to the existing results in literature. Finally, an impulsive boundary value problem involving the quantum derivative with respect to another function is studied. An existence and uniqueness result is established via the Banach contraction mapping principle.

The rest of the paper is arranged as follows: In Section 2, we present the new results on quantum calculus with respect to another function. We give the new definitions and investigate their properties. In Section 3, we obtain a new Hermite-Hadamard inequality for a convex function by applying the new quantum calculus, while in Section 4 we study an impulsive quantum boundary value problem. An example illustrating the obtained result is also presented.

2. Quantum calculus with respect to another function

Let $q \in (0,1)$ be a quantum number, $a \geq 0$ be a point in real line, and define a q-shifting operator by

$$ {}_a\Phi_q(t) = qt + (1-q)a. \tag{2.1}$$

Let $\psi(t)$ be a strictly increasing function defined on $[a,b]$. We now present a notion of the quantum derivative with respect to ψ.

Definition 2.1. *The q-derivative of a function f with respect to a function ψ on $[a,b]$, is given by*

$$ {}_aD_{q,\psi}f(t) = \frac{f(t)-f({}_a\Phi_q(t))}{\psi(t)-\psi({}_a\Phi_q(t))} = \frac{f(t)-f(qt+(1-q)a)}{\psi(t)-\psi(qt+(1-q)a)}, \quad t \neq a, \tag{2.2}$$

and ${}_aD_{q,\psi}f(a) = \lim_{t\to a}\{{}_aD_{q,\psi}f(t)\}$.

Remark 2.1. (*i*) *If $\psi(t) = t$, then we obtain the Tariboon-Ntouyas q-derivative defined in [16] as*

$$ {}_aD_{q,t}f(t) = \frac{f(t)-f(qt+(1-q)a)}{(1-q)(t-a)}.$$

Furthermore, if $a = 0$, it is reduced to the classical Jackson q-derivative, which was defined in 1910 by Jackson [1] as

$$ {}_0D_{q,t}f(t) = \frac{f(t)-f(qt)}{(1-q)t}.$$

(*ii*) *If $\psi(t) = \log t$, where $\log t = \log_e t$ and $t \in [a,b], a > 0$, then we have*

$$ {}_aD_{q,\log t}f(t) = \frac{f(t)-f(qt+(1-q)a)}{\log t - \log(qt+(1-q)a)}, \tag{2.3}$$

which is the q-analog for the derivative of Hadamard sense. To see this, consider as $q \to 1$ (using the L'Hôpital's rule with respect to q),

$$ \lim_{q\to 1}\frac{f(t)-f(qt+(1-q)a)}{\log t - \log(qt+(1-q)a)} = t\frac{d}{dt}f(t),$$

which is a special case of the Hadamard derivative (see [18, Chapters 2.5 and 2.7]).

Corollary 2.1. *Let $f(t) = \psi^n(t)$ in (2.2), where n is a positive integer, then the following formula holds:*

$$ {}_aD_{q,\psi}\psi^n(t) = \sum_{i=0}^{n-1} \psi^{n-1-i}(t)\psi^i(qt + (1-q)a). \tag{2.4}$$

Proof. From (2.2), we have

$$\begin{aligned} & {}_aD_{q,\psi}\psi^n(t) \\ = \; & \frac{\psi^n(t) - \psi^n({}_a\Phi_q(t))}{\psi(t) - \psi({}_a\Phi_q(t))} \\ = \; & \frac{(\psi(t) - \psi({}_a\Phi_q(t)))(\psi^{n-1}(t) + \psi^{n-2}(t)\psi({}_a\Phi_q(t)) + \cdots + \psi(t)\psi^{n-2}({}_a\Phi_q(t)) + \psi^{n-1}({}_a\Phi_q(t)))}{\psi(t) - \psi({}_a\Phi_q(t))} \\ = \; & \sum_{i=0}^{n-1} \psi^{n-1-i}(t)\psi^i({}_a\Phi_q(t)), \end{aligned}$$

which completes the proof. □

Example 2.1. *Let us consider two examples for computation of the q-derivative with respect to another function.*

(i) *Let $n > m > 0$ be constants and $\psi(t) = (t-a)^m$, then*

$$ {}_aD_{q,(t-a)^m}(t-a)^n = \frac{(t-a)^n - (q(t-a))^n}{(t-a)^m - (q(t-a))^m} = \frac{1-q^n}{1-q^m}(t-a)^{n-m},$$

where ${}_a\Phi_q(t) - a = qt + (1-q)a - a = q(t-a)$. If $m = 1$, it is reduced to the q-derivative of polynomial formula

$$ {}_aD_{q,(t-a)}(t-a)^n = [n]_q(t-a)^{n-1},$$

where $[n]_q = (1-q^n)/(1-q)$ is a q-number of value n. When $a = 0$, $m = 1/2$, and $n = 2$, we have the special case ${}_0D_{q,\sqrt{t}}t^2 = \left(\frac{1-q^2}{1-\sqrt{q}}\right)t^{\frac{3}{2}}$.

(ii) *A special case of (2.4), when $n = 3$, is*

$$ {}_aD_{q,\psi(t)}\psi^3(t) = \psi^2(t) + \psi(t)\psi({}_a\Phi_q(t)) + \psi^2({}_a\Phi_q(t)).$$

If $a = 1$ and $\psi(t) = \log t$, then we get

$$ {}_1D_{q,\log t}\left(\log^3 t\right) = \log^2 t + (\log t)\left(\log(q(t-1)+1)\right) + \log^2(q(t-1)+1).$$

Observe that $\lim_{q\to 1}\left\{{}_1D_{q,\log t}(\log^3 t)\right\} = 3\log^2 t$, which is an analogue to the relation $\left(t\frac{d}{dt}\log^3 t\right)=3\log^2 t$.

Some basic properties of the quantum derivative with respect to another function can be presented as follows.

Lemma 2.1. (i) *Let α, β be constants. The following linear property holds:*

$$ {}_aD_{q,\psi}(\alpha f(t) + \beta g(t)) = \alpha\, {}_aD_{q,\psi}f(t) + \beta\, {}_aD_{q,\psi}g(t).$$

(*ii*) *The quantum derivative of the product of two functions can be expressed as*

$$\begin{aligned} {}_aD_{q,\psi}(fg)(t) &= f(t){}_aD_{q,\psi}g(t) + g(qt+(1-q)a){}_aD_{q,\psi}f(t) \\ &= g(t){}_aD_{q,\psi}f(t) + f(qt+(1-q)a){}_aD_{q,\psi}g(t). \end{aligned}$$

(*iii*) *The quantum derivative of the quotient of two functions can be expressed as*

$${}_aD_{q,\psi}\left(\frac{f}{g}\right)(t) = \frac{g(t){}_aD_{q,\psi}f(t) - f(t){}_aD_{q,\psi}g(t)}{g(t)g(qt+(1-q)a)},$$

where $g(t)g(qt+(1-q)a) \neq 0$ *for all* $t \in [a,b]$.

Proof. To prove (*i*), we see that

$$\begin{aligned} {}_aD_{q,\psi}(\alpha f(t)+\beta g(t)) &= \frac{\alpha f(t)+\beta g(t) - \alpha f({}_a\Phi_q(t)) - \beta g({}_a\Phi_q(t))}{\psi(t)-\psi({}_a\Phi_q(t))} \\ &= \frac{\alpha\left[f(t)-f({}_a\Phi_q(t))\right] + \beta\left[g(t)-g({}_a\Phi_q(t))\right]}{\psi(t)-\psi({}_a\Phi_q(t))} \\ &= \alpha{}_aD_{q,\psi}f(t) + \beta{}_aD_{q,\psi}g(t). \end{aligned}$$

Next, the first equation in (*ii*) follows from

$$\begin{aligned} {}_aD_{q,\psi}(fg)(t) &= \frac{(fg)(t)-(fg)({}_a\Phi_q(t))}{\psi(t)-\psi({}_a\Phi_q(t))} \\ &= \frac{f(t)g(t) - g({}_a\Phi_q(t))f(t) - f({}_a\Phi_q(t))g({}_a\Phi_q(t)) + g({}_a\Phi_q(t))f(t)}{\psi(t)-\psi({}_a\Phi_q(t))} \\ &= f(t){}_aD_{q,\psi}g(t) + g({}_a\Phi_q(t)){}_aD_{q,\psi}f(t). \end{aligned}$$

The second equation can be proved similarly. To claim (*iii*), we have

$$\begin{aligned} {}_aD_{q,\psi}\left(\frac{f}{g}\right)(t) &= \frac{\left(\frac{f}{g}\right)(t) - \left(\frac{f}{g}\right)({}_a\Phi_q(t))}{\psi(t)-\psi({}_a\Phi_q(t))} \\ &= \frac{f(t)g({}_a\Phi_q(t)) + f(t)g(t) - f({}_a\Phi_q(t))g(t) - f(t)g(t)}{g(t)g({}_a\Phi_q(t))\left(\psi(t)-\psi({}_a\Phi_q(t))\right)} \\ &= \frac{g(t){}_aD_{q,\psi}f(t) - f(t){}_aD_{q,\psi}g(t)}{g(t)g({}_a\Phi_q(t))}. \end{aligned}$$

With that, we are done. □

To derive the quantum integral with respect to another function, let us consider the q-shifting operator acting on a function $f(t)$ by

$${}_a\Phi_q f(t) = f({}_a\Phi_q(t)) = f(qt+(1-q)a), \quad \text{with} \quad {}_a\Phi_q^0 f(t) = f(t).$$

We then can see the n-times iteration ${}_a\Phi_q^n f(t) = f({}_a\Phi_q^n(t)) = f(q^n t + (1-q^n)a)$, for all $n \in \mathbb{Z}_0$. To claim this, for $n = 0$, we have ${}_a\Phi_q^0 f(t) = f(q^0 t + (1-q^0)a) = f(t)$. Now, suppose that ${}_a\Phi_q^k f(t) = f(q^k t + (1-q^k)a)$ holds for $n = k$, then we get

$${}_a\Phi_q^{k+1} f(t) = {}_a\Phi_q\left({}_a\Phi_q^k f(t)\right)$$

$$
\begin{aligned}
&= {}_a\Phi_q\left(f(q^k t + (1-q^k)a)\right) \\
&= f(q^k[qt + (1-q)a] + (1-q^k)a) \\
&= f(q^{k+1}t + (1-q^{k+1})a).
\end{aligned}
$$

By mathematical induction, the n-times iteration of the q-shifting operator to a function $f(t)$ is well-defined.

Next, from (2.2), we set

$$
h(t) = \frac{f(t) - f(qt + (1-q)a)}{\psi(t) - \psi(qt + (1-q)a)} = \frac{(1 - {}_a\Phi_q) f(t)}{\psi(t) - \psi(qt + (1-q)a)},
$$

in which we can solve for $f(t)$ and obtain

$$
\begin{aligned}
f(t) &= \frac{1}{1 - {}_a\Phi_q}\left[\psi(t) - \psi(qt + (1-q)a)\right] h(t) \\
&= \sum_{j=0}^{\infty} {}_a\Phi_q^j \left[\psi(t) - \psi(qt + (1-q)a)\right] h(t) \\
&= \sum_{j=0}^{\infty} \left[\psi(q^j t + (1-q^j)a) - \psi(q^{j+1} t + (1-q^{j+1})a)\right] h(q^j t + (1-q^j)a),
\end{aligned}
$$

provided that the righthand side is convergent. With this concept of the quantum antiderivative, we give a new definition of quantum integration with respect to another function.

Definition 2.2. *The definite q-integral of a function f on $[a, b]$ with respect to the function ψ and quantum number q is defined by*

$$
\begin{aligned}
{}_aI_{q,\psi} f(t) &= \int_a^t f(s)\, {}_a d_q^{\psi} s \\
&= \sum_{j=0}^{\infty} \left[\psi(q^j t + (1-q^j)a) - \psi(q^{j+1} t + (1-q^{j+1})a)\right] f(q^j t + (1-q^j)a) \\
&= \sum_{j=0}^{\infty} \left[\psi({}_a\Phi_q^j(t)) - \psi({}_a\Phi_q^{j+1}(t))\right] f({}_a\Phi_q^j(t)), \qquad (2.5)
\end{aligned}
$$

provided that the righthand side exists. In addition, if $c \in (a, b)$, then the definite q-integral can be written as

$$
\begin{aligned}
&\int_c^t f(s)\, {}_a d_q^{\psi} s \\
&= \int_a^t f(s)\, {}_a d_q^{\psi} s - \int_a^c f(s)\, {}_a d_q^{\psi} s \\
&= \sum_{j=0}^{\infty} \left[\psi(q^j t + (1-q^j)a) - \psi(q^{j+1} t + (1-q^{j+1})a)\right] f(q^j t + (1-q^j)a) \\
&\quad - \sum_{j=0}^{\infty} \left[\psi(q^j c + (1-q^j)a) - \psi(q^{j+1} c + (1-q^{j+1})a)\right] f(q^j c + (1-q^j)a).
\end{aligned}
$$

Remark 2.2. (*i*) *If $\psi(t) = t$, then we have the Tariboon-Ntouyas [16] definite q-integral given by*

$$ {}_aI_{q,t}f(t) = (1-q)(t-a)\sum_{j=0}^{\infty} q^j f(q^j t + (1-q^j)a), $$

which, for $a = 0$, gives the Jackson [1] definite q-integral as

$$ {}_0I_{q,t}f(t) = (1-q)t\sum_{j=0}^{\infty} q^j f(q^j t). $$

(*ii*) *For $a > 0$, if $\psi(t) = \log t$, $t \in [a, b]$, then the Hadamard definite q-integral is presented by*

$$ \begin{aligned} {}_aI_{q,\log t}f(t) &= \sum_{j=0}^{\infty} \log\left(\frac{q^j t + (1-q^j)a}{q^{j+1}t + (1-q^{j+1})a}\right) f(q^j t + (1-q^j)a) \\ &= \sum_{j=0}^{\infty} \log\left(\frac{{}_a\Phi_q^j(t)}{{}_a\Phi_q^{j+1}(t)}\right) f({}_a\Phi_q^j). \end{aligned} $$

Remark 2.3. *If $a = 0$ in* (2.5)*, then*

$$ \int_0^t f(s)_0 d_q^{\psi} s = \sum_{j=0}^{\infty}\left[\psi(q^j t) - \psi(q^{j+1}t)\right] f(q^j t), $$

which is Eq (19.9) in [3], although there were no generalizations into the q-integral with respect to other functions as explored in our work.

Example 2.2. *If $\psi(t) = (t-a)^m$, $f(t) = (t-a)^n$, when $m, n > 0$, then we have*

$$ \begin{aligned} \int_a^t (s-a)^n {}_a d_q^{(t-a)^m} s &= \sum_{j=0}^{\infty}\left[\left(q^j(t-a)\right)^m - \left(q^{j+1}(t-a)\right)^m\right]\left(q^j(t-a)\right)^n \\ &= (t-a)^{m+n}\left[\sum_{j=0}^{\infty}(q^{m+n})^j - q^m\sum_{j=0}^{\infty}(q^{m+n})^j\right] \\ &= (t-a)^{m+n}\left(\frac{1-q^m}{1-q^{m+n}}\right), \end{aligned} $$

since ${}_a\Phi_q^j(t) - a = q^j(t-a)$. If $m = 1$, we have the q-integral of the power function formula as

$$ {}_aI_{q,(t-a)}(t-a)^n = \frac{(t-a)^{n+1}}{[n+1]_q}. $$

If $m = 1/2$ and $n = 2$, we have the special case ${}_aI_{q,\sqrt{t-a}}(t-a)^2 = (t-a)^{\frac{5}{2}}\left(\frac{1-\sqrt{q}}{1-q^{\frac{5}{2}}}\right)$.

Remark 2.4. *Since we only have the closed form formula (the geometric series) of infinite series, we can only compute as in Example 2.2 for power function with respect to power function. In addition, it is easy to find that*

$$ {}_aI_{q,\psi}(1)(t) = \psi(t) - \psi(a). \tag{2.6} $$

We now present some q-analog of the fundamental theorem of calculus.

Theorem 2.1. *The following formulas hold:*

(*i*) ${}_aD_{q,\psi}\left({}_aI_{q,\psi}f\right)(t) = f(t)$.
(*ii*) ${}_aI_{q,\psi}\left({}_aD_{q,\psi}f\right)(t) = f(t) - f(a)$, *where* $t \in (a, b]$.

Proof. From the definitions of the q-derivative and q-integral with respect to another function ψ, in (2.2) and (2.5), respectively, we have

$$\begin{aligned}
{}_aD_{q,\psi}\left({}_aI_{q,\psi}f\right)(t) &= {}_aD_{q,\psi}\left\{\sum_{j=0}^{\infty}\left[\psi({}_a\Phi_q^j(t)) - \psi({}_a\Phi_q^{j+1}(t))\right]f({}_a\Phi_q^j(t))\right\} \\
&= \frac{1}{\psi(t) - \psi({}_a\Phi_q(t))}\left\{\sum_{j=0}^{\infty}\left[\psi({}_a\Phi_q^j(t)) - \psi({}_a\Phi_q^{j+1}(t))\right]f({}_a\Phi_q^j(t))\right. \\
&\quad \left. - \sum_{j=0}^{\infty}\left[\psi({}_a\Phi_q^{j+1}(t)) - \psi({}_a\Phi_q^{j+2}(t))\right]f({}_a\Phi_q^{j+1}(t))\right\} \\
&= \frac{1}{\psi(t) - \psi({}_a\Phi_q(t))} \\
&\quad \times\Big[\left(\psi(t) - \psi({}_a\Phi_q(t))\right)f(t) + \left(\psi({}_a\Phi_q(t)) - \psi({}_a\Phi_q^2(t))\right)f({}_a\Phi_q(t)) \\
&\quad + \left(\psi({}_a\Phi_q^2(t)) - \psi({}_a\Phi_q^3(t))\right)f({}_a\Phi_q^2(t)) + \cdots \\
&\quad - \{\left(\psi({}_a\Phi_q(t)) - \psi({}_a\Phi_q^2(t))\right)f({}_a\Phi_q(t)) \\
&\quad + \left(\psi({}_a\Phi_q^2(t)) - \psi({}_a\Phi_q^3(t))\right)f({}_a\Phi_q^2(t)) + \cdots\}\Big] \\
&= f(t),
\end{aligned}$$

which proves (*i*).

Also, we have

$$\begin{aligned}
{}_aI_{q,\psi}\left({}_aD_{q,\psi}f\right)(t) &= I_{q,g}\left\{\frac{f(t) - f({}_a\Phi_q(t))}{\psi(t) - \psi({}_a\Phi_q(t))}\right\} \\
&= \sum_{j=0}^{\infty}\left[\psi({}_a\Phi_q^j(t)) - g({}_a\Phi_q^{+1j}(t))\right]\frac{f({}_a\Phi_q^j(t)) - f({}_a\Phi_q^{j+1}(t))}{\psi({}_a\Phi_q^j(t)) - \psi({}_a\Phi_q^{j+1}(t))} \\
&= f(t) - f({}_a\Phi_q(t)) + \left(f({}_a\Phi_q(t)) - f({}_a\Phi_q^2(t))\right) \\
&\quad + \left(f({}_a\Phi_q^2(t)) - f({}_a\Phi_q^3(t))\right) + \cdots \\
&= f(t) - f(a),
\end{aligned}$$

with $\lim_{u\to\infty} f(q^u t + (1 - q^u)a) = f(a)$. Thus (*ii*) is proved. □

The next theorem shows that the double definite q-integral with respect to another function can be reduced to a single integral.

Theorem 2.2. *The following formula*

$$\int_a^t \int_a^s f(r)\, {}_a d_q^\psi r\, {}_a d_q^\psi s = \int_a^t (\psi(t) - \psi({}_a\Phi_q(s))) f(s)\, {}_a d_q^\psi s, \tag{2.7}$$

holds.

Proof. From Definition 2.5, we have

$$\begin{aligned}
& \int_a^t \int_a^s f(r)\, {}_a d_q^\psi r\, {}_a d_q^\psi s \\
= \;& \int_a^t \sum_{j=0}^{\infty} \left[\psi({}_a\Phi_q^j(s)) - \psi({}_a\Phi_q^{j+1}(s))\right] f({}_a\Phi_q^j(s))\, {}_a d_q^\psi \\
= \;& \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \left[\psi({}_a\Phi_q^m(t)) - \psi({}_a\Phi_q^{m+1}(t))\right] \left\{\psi({}_a\Phi_q^{m+j}(t)) - \psi({}_a\Phi_q^{m+j+1}(t))\right\} \\
& \times f({}_a\Phi_q^{m+j}(t)) \\
= \;& \left[\psi(t) - \psi({}_a\Phi_q(t))\right] \left\{\psi(t) - \psi({}_a\Phi_q(t))\right\} f(t) \\
& + \left[\psi(t) - \psi({}_a\Phi_q(t))\right] \left\{\psi({}_a\Phi_q(t)) - \psi({}_a\Phi_q^2(t))\right\} f({}_a\Phi_q(t)) + \ldots \\
& + \left[\psi({}_a\Phi_q(t)) - \psi({}_a\Phi_q^2(t))\right] \left\{\psi({}_a\Phi_q(t)) - \psi({}_a\Phi_q^2(t))\right\} f({}_a\Phi_q(t)) \\
& + \left[\psi({}_a\Phi_q(t)) - \psi({}_a\Phi_q^2(t))\right] \left\{\psi({}_a\Phi_q^2(t)) - \psi({}_a\Phi_q^3(t))\right\} f({}_a\Phi_q^2(t)) + \cdots \\
& + \left[\psi({}_a\Phi_q^2(t)) - \psi({}_a\Phi_q^3(t))\right] \left\{\psi({}_a\Phi_q^2(t)) - \psi({}_a\Phi_q^3(t))\right\} f({}_a\Phi_q^2(t)) \\
& + \left[\psi({}_a\Phi_q^2(t)) - \psi({}_a\Phi_q^3(t))\right] \left\{\psi({}_a\Phi_q^3(t)) - \psi({}_a\Phi_q^4(t))\right\} f({}_a\Phi_q^3(t)) + \cdots \\
= \;& \left[\psi(t) - \psi({}_a\Phi_q(t))\right] \left\{\psi(t) - \psi({}_a\Phi_q(t))\right\} f(t) \\
& + \left[\psi(t) - \psi({}_a\Phi_q^2(t))\right] \left\{\psi({}_a\Phi_q(t)) - \psi({}_a\Phi_q^2(t))\right\} f({}_a\Phi_q(t)) \\
& + \left[\psi(t) - \psi({}_a\Phi_q^3(t))\right] \left\{\psi({}_a\Phi_q^2(t)) - \psi({}_a\Phi_q^3(t))\right\} f({}_a\Phi_q^2(t)) + \cdots \\
= \;& \sum_{i=0}^{\infty} \left[\psi(t) - \psi({}_a\Phi_q({}_a\Phi_q^i(t)))\right] \left\{\psi({}_a\Phi_q^i(t)) - \psi({}_a\Phi_q^{i+1}(t))\right\} f({}_a\Phi_q^i(t)) \\
= \;& \int_a^t \left(\psi(t) - \psi({}_a\Phi_q(s))\right) f(s)\, {}_a d_q^\psi s.
\end{aligned}$$

The proof is completed. □

3. Quantum Hermite-Hadamard inequality

In this section, we apply quantum calculus with respect to another function to establish a new Hermite-Hadamard inequality for a convex function f on $[a, b]$. Recall that a function f is convex if

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda) f(b),$$

for all $\lambda \in [0, 1]$.

Theorem 3.1. *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex differentiable function on (a,b), then the q-integral of f with respect to $\psi(t) = (t-a)^m$, $m > 0$ satisfies*

$$f\left(\frac{b(1-q^m)+aq^m(1-q)}{1-q^{m+1}}\right) \leq \frac{1}{(b-a)^m}\int_a^b f(s)\,{}_a d_q^{(t-a)^m} s \leq \frac{(1-q^m)f(b)+q^m(1-q)f(a)}{1-q^{m+1}}. \quad (3.1)$$

Proof. Define the point $c := \frac{b(1-q^m)+aq^m(1-q)}{1-q^{m+1}}$, then $c \in [a,b]$. To see this, note that $\lim_{q\to 0} c = b$ and

$$\lim_{q\to 1} \frac{b(1-q^m)+aq^m(1-q)}{1-q^{m+1}} = \frac{mb+a}{m+1} \in (a,b),$$

for all $m > 0$, using the L'Hôpital rule with respect to q.

Since f is a differentiable convex function on (a,b), there exists a tangent line $h(t)$ under the curve of $f(t)$, i.e., for $t \in (a,b)$,

$$h(t) = f(c) + f'(c)(t-c) \leq f(t).$$

To prove the left-hand side of (3.1), by taking the q-integral with respect to a function $\psi(t) = (t-a)^m$, $m > 0$, and applying the formula in Example 2.2 for $n = 0, 1$, we have

$$\begin{aligned}
&\int_a^b h(s)\,{}_a d_q^{(t-a)^m} ds \\
= \ &\int_a^b [f(c) + f'(c)(s-c)]\,{}_a d_q^{(t-a)^m} ds \\
= \ &(b-a)^m f(c) + f'(c)\int_a^b (s-a-(c-a))\,{}_a d_q^{(t-a)^m} ds \\
= \ &(b-a)^m f(c) + f'(c)\left[\frac{(b-a)^{m+1}(1-q^m)}{1-q^{m+1}} - \frac{(b-a)^{m+1}(1-q^m)}{1-q^{m+1}}\right] \\
= \ &(b-a)^m f\left(\frac{b(1-q^m)+aq^m(1-q)}{1-q^{m+1}}\right) \leq \int_a^b f(s)\,{}_a d_q^{(t-a)^m} ds.
\end{aligned}$$

On the other side of the inequality, we set the line $k(t)$ that connects the two points $(a, f(a))$ and $(b, f(b))$, i.e., for $t \in (a,b)$,

$$k(t) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(t-a),$$

which implies that $f(t) \leq k(t)$ for all $t \in [a,b]$ by the convexity of f. Therefore, we obtain

$$\begin{aligned}
\int_a^b k(s)\,{}_a d_q^{(t-a)^m} ds &= \int_a^b \left[f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(s-a)\right]{}_a d_q^{(t-a)^m} ds \\
&= (b-a)^m \left[\frac{(1-q^m)f(b)+q^m(1-q)f(a)}{1-q^{m+1}}\right] \\
&\geq \int_a^b f(s)\,{}_a d_q^{(t-a)^m} ds.
\end{aligned}$$

Combining both cases, we obtain (3.1), and the proof is completed. □

Remark 3.1. *If $m = 1$, then* (3.1) *is presented by*

$$f\left(\frac{b+qa}{1+q}\right) \leq \frac{1}{(b-a)} \int_a^b f(s)\,{}_a d_q s \leq \frac{f(b)+qf(a)}{1+q},$$

which has appeared in [19].

4. Impulsive boundary value problem

Let $t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1}$ be fixed points in $[0, T]$, where $t_0 = 0$, $t_{m+1} = T$, $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a nonlinear function, and $\{c_i\}$, $\{d_i\}$, $i = 1, 2, \ldots, m$, be given sequences of real numbers. Consider the boundary value problem involving the quantum derivative with respect to another function

$$\begin{cases} {}_{t_i}D_{q_i,\psi_i} x(t) = g(t, x(t)), & t \in [t_i, t_{i+1}),\ i = 0, 1, \ldots, m, \\ x(t_i^+) = c_i x(t_i^-) + d_i, & i = 1, 2, \ldots, m, \\ x(0) = \gamma x(T), \end{cases} \tag{4.1}$$

with constants $0 < q_i < 1$, $\psi_i(t)$ a strictly increasing function on $[0, T]$, for all $i = 1, 2, \ldots, m$, $\gamma \in \mathbb{R}$, and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ a given function. Note that if $\gamma = 1$, then (4.1) is reduced to a periodic boundary value problem, and if $\gamma = -1$, it is an anti-periodic boundary value problem.

Our first task is to transform the boundary value problem (4.1) into an integral equation by considering a linear variant of (4.1).

Lemma 4.1. *Let $h \in C([0, T], \mathbb{R})$ and $\Lambda = 1 - \gamma \prod_{i=1}^m c_i \neq 0$, then the linear impulsive boundary value problem*

$$\begin{cases} {}_{t_i}D_{q_i,\psi_i} x(t) = h(t), & t \in [t_i, t_{i+1}),\ i = 0, 1, \ldots, m, \\ x(t_i^+) = c_i x(t_i^-) + d_i, & i = 1, 2, \ldots, m, \\ x(0) = \gamma x(T), \end{cases} \tag{4.2}$$

has the unique solution

$$\begin{aligned} x(t) &= \frac{\gamma}{\Lambda}\left[\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m} c_r \int_{t_{i-1}}^{t_i} h(s)_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s\right) + \sum_{i=1}^{m}\left(\prod_{r=i+1}^{m} c_r d_i\right)\right]\prod_{i=1}^{j} c_i \\ &\quad + \sum_{i=1}^{j}\left(\prod_{r=i}^{j} c_r \int_{t_{i-1}}^{t_i} h(s)_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s\right) + \sum_{i=1}^{j}\left(\prod_{r=i+1}^{j} c_r d_i\right) \\ &\quad + \int_{t_j}^{t} h(s)_{t_j} d_{q_j}^{\psi_j} s, \quad t \in [t_j, t_{j+1}),\ j = 0, 1, \ldots, m. \end{aligned} \tag{4.3}$$

Proof. For the first interval $[0, t_1)$, taking the operator ${}_{t_0}I_{q_0,\psi_0}$ to both sides of the first equation in (4.2), we obtain

$$x(t) = x(0) + \int_{t_0}^{t} h(s)_{t_0} d_{q_0}^{\psi_0} s. \tag{4.4}$$

The second condition for $i = 1$ yields

$$x(t_1^+) = c_1\left(x(0) + \int_{t_0}^{t_1} h(s)_{t_0} d_{q_0}^{\psi_0} s\right) + d_1.$$

The second interval $[t_1, t_2)$ shows

$$x(t) = x(t_1^+) + \int_{t_1}^{t} h(s)_{t_1} d_{q_1}^{\psi_1} s = c_1 \left(x(0) + \int_{t_0}^{t_1} h(s)_{t_0} d_{q_0}^{\psi_0} s \right) + d_1 + \int_{t_1}^{t} h(s)_{t_1} d_{q_1}^{\psi_1} s,$$

by applying the operator ${}_{t_1}I_{q_1,\psi_1}$ to problem (4.2) for $t \in [t_1, t_2)$. Repeating this process for any $t \in [t_j, t_{j+1})$, it follows that

$$x(t) = x(0) \prod_{i=1}^{j} c_i + \sum_{i=1}^{j} \left(\prod_{r=i}^{j} c_r \int_{t_{i-1}}^{t_i} h(s)_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s \right) + \sum_{i=1}^{j} \left(\prod_{r=i+1}^{j} c_r d_i \right) + \int_{t_j}^{t} h(s)_{t_j} d_{q_j}^{\psi_j} s. \tag{4.5}$$

To prove this, we see that (4.5) holds for $j = 0$ by (4.4). For $t \in [t_{j+1}, t_{j+2})$, we have

$$\begin{aligned} x(t) &= x(t_{j+1}^+) + \int_{t_{j+1}}^{t} h(s)_{t_{j+1}} d_{q_{j+1}}^{\psi_{j+1}} s \\ &= c_{j+1} x(t_{j+1}^-) + d_{j+1} + \int_{t_{j+1}}^{t} h(s)_{t_{j+1}} d_{q_{j+1}}^{\psi_{j+1}} s \\ &= c_{j+1} \Bigg[x(0) \prod_{i=1}^{j} c_i + \sum_{i=1}^{j} \left(\prod_{r=i}^{j} c_r \int_{t_{i-1}}^{t_i} h(s)_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s \right) \\ &\quad + \sum_{i=1}^{j} \left(\prod_{r=i+1}^{j} c_r d_i \right) + \int_{t_j}^{t_{j+1}} h(s)_{t_j} d_{q_j}^{\psi_j} s \Bigg] \\ &\quad + d_{j+1} + \int_{t_{j+1}}^{t} h(s)_{t_{j+1}} d_{q_{j+1}}^{\psi_{j+1}} s \\ &= x(0) \prod_{i=1}^{j+1} c_i + \sum_{i=1}^{j+1} \left(\prod_{r=i}^{j+1} c_r \int_{t_{i-1}}^{t_i} h(s)_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s \right) \\ &\quad + \sum_{i=1}^{j+1} \left(\prod_{r=i+1}^{j+1} c_r d_i \right) + \int_{t_{j+1}}^{t} h(s)_{t_{j+1}} d_{q_{j+1}}^{\psi_{j+1}} s, \end{aligned}$$

which shows that (4.5) is true by mathematical induction.

In particular, $t = T$ implies that $j = m$ and

$$x(T) = x(0) \prod_{i=1}^{m} c_i + \sum_{i=1}^{m+1} \left(\prod_{r=i}^{m} c_r \int_{t_{i-1}}^{t_i} h(s)_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s \right) + \sum_{i=1}^{m} \left(\prod_{r=i+1}^{m} c_r d_i \right).$$

Solving the boundary condition of problem (4.2) for a constant $x(0)$, we obtain

$$x(0) = \frac{\gamma}{\Lambda} \left[\sum_{i=1}^{m+1} \left(\prod_{r=i}^{m} c_r \int_{t_{i-1}}^{t_i} h(s)_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s \right) + \sum_{i=1}^{m} \left(\prod_{r=i+1}^{m} c_r d_i \right) \right].$$

Substituting a constant $x(0)$ in (4.5), the nonlinear integral Eq (4.3) is proved.

On the other hand, the converse can be proved by direct computation. The proof is finished. □

To prove the existence and uniqueness of solution to the problem (4.1), we define the space of piecewise continuous functions by the fact that $PC([0,T],\mathbb{R}) = \{x : [0,T] \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_i, such that $x(t_i^+)$ and $x(t_i^-)$ exist and $x(t_i^+) = x(t_i)$ for all $i = 1,2,\dots,m\}$. $PC([0,T],\mathbb{R})$ is a Banach space with norm $\|x\| = \sup\{x(t) : t \in [0,T]\}$.

In view lemma of (4.2), we define an operator $\mathcal{A} : PC([0,T],\mathbb{R}) \to PC([0,T],\mathbb{R})$ by

$$\begin{aligned}\mathcal{A}x(t) &= \frac{\gamma}{\Lambda}\Bigg[\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m} c_r \int_{t_{i-1}}^{t_i} g(s,x(s))_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s\right) + \sum_{i=1}^{m}\left(\prod_{r=i+1}^{m} c_r d_i\right)\Bigg]\prod_{i=1}^{j} c_i \\ &\quad + \sum_{i=1}^{j}\left(\prod_{r=i}^{j} c_r \int_{t_{i-1}}^{t_i} g(s,x(s))_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s\right) + \sum_{i=1}^{j}\left(\prod_{r=i+1}^{j} c_r d_i\right) \\ &\quad + \int_{t_j}^{t} g(s,x(s))_{t_j} d_{q_j}^{\psi_j} s, \quad t \in [t_j, t_{j+1}),\ j = 0,1,\dots,m. \end{aligned} \tag{4.6}$$

By using the Banach contraction principle we will prove the existence and uniqueness of solutions for the impulsive boundary value problem (4.1). For computational convenience we put

$$\begin{aligned}\mathcal{B} &= \sum_{i=1}^{m+1}\left(\prod_{r=i}^{m} |c_r| \left[\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right]\right), \\ C &= \sum_{i=1}^{m}\left(\prod_{r=i+1}^{m} |c_r||d_i|\right), \qquad \mathcal{D} = \prod_{i=1}^{m} |c_i|, \qquad \mathcal{E} = \frac{|\gamma|}{|\Lambda|}\mathcal{D} + 1.\end{aligned}$$

Theorem 4.1. *Suppose that the nonlinear function* $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|g(t,x) - g(t,y)| \le \mathcal{L}|x - y|, \quad \mathcal{L} > 0 \text{ for all } x, y \in \mathbb{R}. \tag{4.7}$$

If $\mathcal{L}\mathcal{B}\mathcal{E} < 1$*, then the boundary value problem* (4.1) *has a unique solution on* $[0,T]$.

Proof. Let us define a ball $B_r = \{x \in PC([0,T],\mathbb{R}) : \|x\| \le r\}$, where a radius satisfies

$$r > \frac{\mathcal{E}(\mathcal{M}\mathcal{B} + C)}{1 - \mathcal{L}\mathcal{B}\mathcal{E}},$$

when $\mathcal{M} := \sup\{|g(t,0)| : t \in [0,T]\}$. For any $x \in B_r$ and $t \in [0,T]$, we have, by (4.7),

$$|g(t,x(t))| \le |g(t,x(t)) - g(t,0)| + |g(t,0)| \le \mathcal{L}|x(t)| + \mathcal{M} \le \mathcal{L}\|x\| + \mathcal{M} \le \mathcal{L}r + \mathcal{M}.$$

For any $x \in B_r$, from (2.6), we have

$$\begin{aligned}|\mathcal{A}x(t)| &\le \frac{|\gamma|}{|\Lambda|}\Bigg[\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m} |c_r| \int_{t_{i-1}}^{t_i} |g(s,x(s))|_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s\right) + \sum_{i=1}^{m}\left(\prod_{r=i+1}^{m} |c_r||d_i|\right)\Bigg]\prod_{i=1}^{j} |c_i| \\ &\quad + \sum_{i=1}^{j}\left(\prod_{r=i}^{j} |c_r| \int_{t_{i-1}}^{t_i} |g(s,x(s))|_{t_{i-1}} d_{q_{i-1}}^{\psi_{i-1}} s\right) + \sum_{i=1}^{j}\left(\prod_{r=i+1}^{j} |c_r||d_i|\right) \\ &\quad + \int_{t_j}^{t} |g(s,x(s))|_{t_j} d_{q_j}^{\psi_j} s\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|\gamma|}{|\Lambda|}\Bigg[(\mathcal{L}r+\mathcal{M})\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m}|c_r|\int_{t_{i-1}}^{t_i}(1)_{t_{i-1}}d_{q_{i-1}}^{\psi_{i-1}}s\right)+\sum_{i=1}^{m}\left(\prod_{r=i+1}^{m}|c_r||d_i|\right)\Bigg]\prod_{i=1}^{m}|c_i| \\
&\quad+(\mathcal{L}r+\mathcal{M})\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m}|c_r|\int_{t_{i-1}}^{t_i}(1)_{t_{i-1}}d_{q_{i-1}}^{\psi_{i-1}}s\right)+\sum_{i=1}^{m}\left(\prod_{r=i+1}^{m}|c_r||d_i|\right) \\
&= \frac{|\gamma|}{|\Lambda|}\Bigg[(\mathcal{L}r+\mathcal{M})\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m}|c_r|\left[\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right]\right) \\
&\quad+\sum_{i=1}^{m}\left(\prod_{r=i+1}^{m}|c_r||d_i|\right)\Bigg]\prod_{i=1}^{m}|c_i|+(\mathcal{L}r+\mathcal{M})\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m}|c_r|\left[\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right]\right) \\
&\quad+\sum_{i=1}^{m}\left(\prod_{r=i+1}^{m}|c_r||d_i|\right) \\
&= \mathcal{E}\left[(\mathcal{L}r+\mathcal{M})\mathcal{B}+C\right]<r,
\end{aligned}
$$

which implies that $\mathcal{A}(B_r)\subset B_r$. Next, we will prove that the operator $\mathcal{A}$ is a contraction. To do this, for any elements $x,y\in B_r$, we have

$$
\begin{aligned}
&|\mathcal{A}x(t)-\mathcal{A}y(t)| \\
&\leq \frac{|\gamma|}{|\Lambda|}\Bigg[\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m}|c_r|\int_{t_{i-1}}^{t_i}|g(s,x)-g(s,y)|_{t_{i-1}}d_{q_{i-1}}^{\psi_{i-1}}s\right)\Bigg]\prod_{i=1}^{j}|c_i| \\
&\quad+\sum_{i=1}^{j}\left(\prod_{r=i}^{j}|c_r|\int_{t_{i-1}}^{t_i}|g(s,x)-g(s,y)|_{t_{i-1}}d_{q_{i-1}}^{\psi_{i-1}}s\right) \\
&\quad+\int_{t_j}^{t}|g(s,x)-g(s,y)|_{t_j}d_{q_j}^{\psi_j}s \\
&\leq \frac{|\gamma|}{|\Lambda|}\Bigg[\mathcal{L}\|x-y\|\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m}|c_r|\int_{t_{i-1}}^{t_i}(1)_{t_{i-1}}d_{q_{i-1}}^{\psi_{i-1}}s\right)\Bigg]\prod_{i=1}^{m}|c_i| \\
&\quad+\mathcal{L}\|x-y\|\sum_{i=1}^{m+1}\left(\prod_{r=i}^{m}|c_r|\int_{t_{i-1}}^{t_i}(1)_{t_{i-1}}d_{q_{i-1}}^{\psi_{i-1}}s\right) \\
&= \mathcal{L}\mathcal{B}\mathcal{E}\|x-y\|,
\end{aligned}
$$

which means that $\|\mathcal{A}x-\mathcal{A}y\|\leq\mathcal{L}\mathcal{B}\mathcal{E}\|x-y\|$. As $\mathcal{L}\mathcal{B}\mathcal{E}<1$, this implies that the operator $\mathcal{A}$ is a contraction on B_r and there exists a unique fixed point in B_r, which is a unique solution of the problem (4.1) on $[0,T]$. Therefore, the theorem is proved. □

Example 4.1. *Consider the following impulsive boundary value problem containing quantum derivative with respect to another function as*

$$
\begin{cases}
{}_{\frac{i}{2}}D_{\frac{i+1}{i+2},\log\left(\frac{i+2}{i+3}+t\right)}x(t) = \dfrac{5}{6(t+7)}\left(\dfrac{x^2(t)+2|x(t)|}{1+|x(t)|}\right)+\dfrac{1}{2}, \quad t \in \left[\dfrac{i}{2}, \dfrac{i+1}{2}\right), \; i = 0,1,2,3,4, \\
x\left(\dfrac{i}{2}^{+}\right) = \left(\dfrac{i+3}{i+4}\right)x\left(\dfrac{i}{2}^{-}\right)+\dfrac{i+4}{i+5}, \quad i = 1,2,3,4, \\
x(0) = \dfrac{3}{2}x\left(\dfrac{5}{2}\right).
\end{cases} \tag{4.8}
$$

From the given problem, we have $q_i = (i+1)/(i+2)$, $\psi_i(t) = \log((i+2)/(i+3)+t)$, $t_i = i/2$, $i = 0,1,2,3,4$, $m = 4$, $t_{m+1} = T = 5/2$, $c_i = (i+3)/(i+4)$, $d_i = (i+4)/(i+5)$, $i = 1,2,3,4$, $\gamma = 3/2$ and $g(t,x(t)) = \frac{5}{6(t+7)}\left(\frac{x^2(t)+2|x(t)|}{1+|x(t)|}\right)+\frac{1}{2}$, $t \in \left[\frac{i}{2}, \frac{i+1}{2}\right)$. This information leads to $\Lambda = 1/4$, $\mathcal{B} \approx 1.005099546$, $\mathcal{C} \approx 2.818204365$, $\mathcal{D} = 1/2$, and $\mathcal{E} = 4$. Also, the nonlinear function g satisfies the Lipschitz condition, as

$$
|g(t,x) - g(t,y)| \le \frac{5}{21}|x-y|,
$$

for all $x, y \in \mathbb{R}$ and $t \in [0, 5/2]$. Setting $\mathcal{L} = 5/21$, the following inequality holds

$$
\mathcal{L}\mathcal{B}\mathcal{E} \approx 0.9572376629 < 1.
$$

Thus, the Theorem 4.1 quarantees that the boundary value problem (4.8) has a unique solution on $[0, 5/2]$.

5. Conclusions

In the present paper, we initiated the study of quantum calculus with respect to another function. The definitions of the quantum derivative and integral with respect to another function were given and their basic properties were investigated. A new Hermite-Hadamard type inequality for a convex function was obtained as an application of these newly defined notions. Moreover, an impulsive boundary value problem involving the quantum derivative with respect to another function was studied via the Banach contraction mapping principle. The results are new and enrich the existing results on quantum calculus.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author was supported by King Mongkut's University of Technology North Bangkok, Contract No. KMUTNB-PHD-63-02.

Conflict of interest

Sotiris K. Ntouyas is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article.

References

1. F. Jackson, q-Difference equations, *Am. J. Math.*, **32** (1910), 305–314. https://doi.org/10.2307/2370183
2. T. Ernst, *A comprehensive treatment of q-calculus*, Basel: Birkhäuser, 2012. https://doi.org/10.1007/978-3-0348-0431-8
3. V. Kac, P. Cheung, *Quantum calculus*, New York: Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7
4. W. Al-Salam, Some fractional q-integrals and q-derivatives, *P. Edinburgh Math. Soc.*, **15** (1966), 135–140. https://doi.org/10.1017/s0013091500011469
5. R. Agarwal, Certain fractional q-integrals and q-derivatives, *Math. Proc. Cambridge*, **66** (1969), 365–370. https://doi.org/10.1017/s0305004100045060
6. M. Annaby, Z. Mansour, *q-Fractional calculus and equations*, Berlin: Springer, 2012. https://doi.org/10.1007/978-3-642-30898-7
7. G. Gasper, M. Rahman, *Basic hypergeometric series*, Cambridge: Cambridge University Press, 1990. https://doi.org/10.1017/cbo9780511526251
8. J. Ma, J. Yang, Existence of solutions for multi-point boundary value problem of fractional q-difference equation, *Electron. J. Qual. Theory Differ. Equ.*, **2011** (2011), 92. https://doi.org/10.14232/ejqtde.2011.1.92
9. C. Yang, Positive solutions for a three-point boundary value problem of fractional q-difference equations, *Symmetry*, **10** (2018), 358. https://doi.org/10.3390/sym10090358
10. C. Guo, J. Guo, S. Kang, H. Li, Existence and uniqueness of positive solutions for nonlinear q-difference equation with integral boundary conditions, *J. Appl. Anal. Comput.*, **10** (2020), 153–164. https://doi.org/10.11948/20190055
11. R. Ouncharoen, N. Patanarapeelert, T. Sitthiwirattham, Nonlocal q-symmetric integral boundary value problem for sequential q-symmetric integrodifference equations, *Mathematics*, **6** (2018), 218. https://doi.org/10.3390/math6110218
12. C. Zhai, J. Ren, Positive and negative solutions of a boundary value problem for a fractional q-difference equation, *Adv. Differ. Equ.*, **2017** (2017), 82. https://doi.org/10.1186/s13662-017-1138-x
13. J. Ren, C. Zhai, Nonlocal q-fractional boundary value problem with Stieltjes integral conditions, *Nonlinear Anal.-Model.*, **24** (2019), 582–602. https://doi.org/10.15388/na.2019.4.6
14. K. Ma, X. Li, S. Sun, Boundary value problems of fractional q-difference equations on the half-line, *Bound. Value Probl.*, **2019** (2019), 46. https://doi.org/10.1186/s13661-019-1159-3
15. A. Wongcharoen, A. Thatsatian, S. K. Ntouyas, J. Tariboon, Nonlinear fractional q-difference equation with fractional Hadamard and quantum integral nonlocal conditions, *J. Funct. Space.*, **2020** (2020), 9831752. https://doi.org/10.1155/2020/9831752
16. J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, *Adv. Differ. Equ.*, **2013** (2013), 282. https://doi.org/10.1186/1687-1847-2013-282

17. B. Ahmad, S. K. Ntouyas, J. Tariboon, *Quantum calculus: new concepts, impulsive IVPs and BVPs, inequalities*, Singapore: World Scientific, 2016. https://doi.org/10.1142/10075

18. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006. https://doi.org/10.1016/s0304-0208(06)x8001-5

19. N. Alp, M. Sarikaya, M. Kunt, İ. İşcan, q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, *J. King Saud Univ. Sci.*, **30** (2018), 193–203. https://doi.org/10.1016/j.jksus.2016.09.007