AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 10416–10445.
DOI:10.3934/math.2024509
Received: 24 December 2023
Revised: 23 February 2024
Accepted: 08 March 2024
Published: 15 March 2024

Research article

Double inertial extrapolations method for solving split generalized equilibrium, fixed point and variational inequity problems

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Abstract: This article proposes an iteration algorithm with double inertial extrapolation steps for approximating a common solution of split equilibrium problem, fixed point problem and variational inequity problem in the framework of Hilbert spaces. Unlike several existing methods, our algorithm is designed such that its implementation does not require the knowledge of the norm of the bounded linear operator and the value of the Lipschitz constant. The proposed algorithm does not depend on any line search rule. The method uses a self-adaptive step size which is allowed to increase from iteration to iteration. Furthermore, using some mild assumptions, we establish a strong convergence theorem for the proposed algorithm. Lastly, we present a numerical experiment to show the efficiency and the applicability of our proposed iterative method in comparison with some well-known methods in the literature. Our results unify, extend and generalize so many results in the literature from the setting of the solution set of one problem to the more general setting common solution set of three problems.

Keywords: double inertial technique; split generalized equilibrium; fixed point and variational inequity problems
Mathematics Subject Classification: 26A33, 34B10, 34B15

1. Introduction

Let K be a nonempty, closed, convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$, and $A : H \to H$ be an operator. The classical variational inequality problem (VIP) was considered independently by Stampacchia [28] and Fichera [9] and it is defined as: Find $s \in K$ such that

$$\langle As, t - s \rangle \geq 0, \ \forall \ t \in K. \tag{1.1}$$

The solution set of (1.1) is denoted by $VI(K, A)$, that is,

$$VI(K, A) = \{s \in K : \langle As, t - s \rangle \geq 0, \ \forall \ t \in K\}. \tag{1.2}$$

The the study of theory of VIP has received massive attention in the last few years as a result of its enormous applications in diverse fields. One of the widely used methods for solving VIP which known as the extragradient method was introduced by Korpelevich [16] as follows:

$$\begin{cases} s_1 \in K, \\ t_m = P_K(s_m - \lambda_m A s_m), \\ s_{m+1} = P_K(t_m - \lambda_m A t_m) \ \forall \ m \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\lambda_m \in (0, \frac{1}{L})$ and L is the Lipschitz constant of A. Under some standard assumptions, it was shown that the sequence $\{s_m\}$ converges to the solution set $VI(K, A)$. The method (1.3) has two major limitations. The first is that the step-size (λ_m) depends on the Lipschitz constant and and the second is that, calculation of two projections are involved. These limitations affect the computational efficiency of the method. In other to avoid these setbacks, many methods have been studied in the last few decades (see, for example, [4, 6, 7, 11, 17, 20–22, 25–28, 34]).

Let $S : K \to K$ be an operator. An element $q \in K$ fulfilling $Sq = q$ is known as the fixed point of S. Let $F(S) = \{q \in K : Sq = q\}$ stand for the set of all fixed points of S. The concept of fixed point can be applied to several areas of engineering and applied sciences problems such as compressed sensing, game theory, approximation theory, mathematical economics and mathematics of fractals. It is important to note that monotone inclusion, convex feasibility, variational inequality, equilibrium, convex optimization and image/signal restoration problems can all be converted into a problem of finding the fixed point of some appropriate nonlinear operator.

Another interesting optimization problem is the equilibrium problem (EP) introduced and studied by Blum and Oettli [2]. Some well-known problems in applied sciences and engineering are special types of the EP. For example, minimization problems, mathematical programming problems, saddle point problems, Nash equilibrium problems, fixed point problems, vector minimization problems, and so on. The EP is defined as finding $s^* \in C$ such that

$$F(s^*, t) \geq 0, \tag{1.4}$$

for all $t \in K$, where $F : K \times K \to \mathbb{R}$ is a bifunction. The solution set of EP is denoted by $EP(F)$. Due to the numerous applications of the theory of EP (1.4), many authors have extended and generalized it in various directions. For instance, the following split equilibrium problem (SEP): Let $K \subseteq H_1$, $Q \subseteq H_2$,

$F_1 : K \times K \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and suppose that $A : H_1 \to H_2$ is a bounded linear operator. The SEP is to find $s^* \in K$ such that

$$F_1(s^*, s) \geq 0 \ \ \forall \ \ s \in K \tag{1.5}$$

and such that

$$t^* = As^* \in Q \ \text{ solves } \ F_2(t^*, y) \geq 0 \ \ \ \forall t \in Q. \tag{1.6}$$

We denote the solution set of (1.5) and (1.6) by $\Gamma = \{s \in EP(F_1) : As \in EP(F_2)\}$. Kazmi and Rizvi studied a Halpern type iterative method to approximate the common solution of SEP, VIP, and fixed point problem (FPP) for a nonexpansive mapping T on K. The iterative method is given as follows:

$$\begin{cases} u_m = T_{r_m}^{F_1}(s_m + \gamma A^*(T_{r_m}^{F_2} - I)As_m), \\ t_m = P_K(u_m - \lambda_m f u_m), \\ s_{m+1} = \alpha_m v + \beta_m s_m + \gamma_m T t_m, \end{cases} \tag{1.7}$$

where $r_m \subset (0, \infty), \lambda_m \in (0, 2\tau)$ and $\gamma \in (0, \frac{1}{\|A\|^2})$, $\{\alpha_m\}, \{\beta_m\}$, and $\{\gamma_m\}$ are sequences in $(0, 1)$. They established that under some standard conditions, the sequence derived by the iterative algorithm (1.7) converges strongly to the common solution of SEP, VIP, and FPP for a nonexpansive mapping. Obviously, the above iterative algorithm have some drawbacks. For example, the way the step sizes $\{\lambda_m\}$, and γ are defined, and the cost operator (f) of the VIP is inversely strongly monotone. In the light of this, it will be interesting if one can further modify the above iterative method. The notion of SEP was generalized by the concept of generalized split equilibrium problem (GSEP). This problem is defined as: Find

$$s^* \in K \text{ such that } F_1(s^*, s) + \phi_1(s, s^*) - \phi_1(s^*, s^*) \geq 0, \ \forall \ s \in K, \tag{1.8}$$

and such that

$$t = As^* \in Q \text{ solves } F_2(s^*, s) + \phi_1(s, s^*) - \phi_1(s^*, s^*) \geq 0, \ \forall \ t \in Q, \tag{1.9}$$

where $\phi_1 : K \times K \to \mathbb{R}$ and $\phi_2 : Q \times Q \to \mathbb{R}$ are nonlinear mappings. It is easy to see that (1.8) is the generalized equilibrium problem (GEP), and we denote its solution set by Sol (GEP (1.8)). The concept of GSEP generalizes the notion of multiple -sets split feasibility problems. It is also well-known that the split variational inequality problem is a special case of the GSEP [3, 5, 19]. Several authors have introduced and studied different iterative algorithms for problems (1.8) and (1.9). For details, see [1, 14, 15, 31] and the references therein. In particular, Farid and Kazmi [10] introduced and studied an iterative algorithm for approximating the solution of a fixed point problem, variational inequality problem, and (1.8) and (1.9). They gave the following iterative algorithm:

$$\begin{cases} u_m = T_{r_m}^{(F_1,\phi_1)}(s_m + \delta A^*(T_{r_m}^{(F_2,\phi_2)} - I)As_m), \\ t_m = P_K(u_m - \lambda_m B u_m), \\ s_{m+1} = \alpha_m \gamma f(K_m s_m) v + \beta_m s_m + ((1 - \beta_m)I - \alpha_m D)K_m T t_m, \end{cases} \tag{1.10}$$

where D is a strongly positive bounded linear operator, B is τ-inverse strongly monotone mapping, $\lambda_m \in (0, 2\tau), \delta \in (0, \frac{1}{L})$, and K_m is a finite family of nonexpansive mapping. It is easy to see that the above iterative algorithm has a lot of drawbacks. Thus, it is natural to ask if one can develop a better version of the above iterative algorithm.

The potential use of such a common solution problem to mathematical models whose constraints can be described as GSEP, FPP, and VIP is one of our motivations for exploring it. Signal processing, network resource allocation, and image recovery are examples of practical problems where this occurs (see [13, 18, 33] and the references therein).

Inspired by the above results and the ongoing research interest in this direction, in this study, we propose an iterative method with double inertial extrapolation steps for approximating the common solution of GSEP, FPP, and VIP in the framework of Hilbert spaces. Unlike several existing methods, our algorithm is designed such that its implementation does not require the knowledge of the norm of the bounded linear operator and the value of the Lipschitz constant. The proposed method does not depend on any line search rule. The suggested algorithm uses a self-adaptive step size, which is allowed to increase from iteration to iteration. Furthermore, using some mild assumptions, we establish a strong convergence theorem for the proposed algorithm. Lastly, we present a numerical experiment to show the efficiency and the applicability of our proposed iterative method in comparison with some well-known methods in the literature. Unlike the results obtained in [14], estimation of spectral radius of the bounded linear operator and its adjoint is not required in our results. Furthermore, the class of demimetric mappings, which is embedded in our method, is more general than the classes of nonexpansive, quasi-nonexpansive, strictly pseudocontraction, and demicontraction mappings, which have been studied by many authors (see, for example, [36, 37] and the references therein). Our results unify, extend, and generalize so many results in the literature, from the setting of the solution set of one problem to the more general setting common solution set of three problems.

The remaining part of this article is arranged as follows: In Section 2, we give some useful results and definitions in this study. In Section 3, we establish the strong convergence results of the suggested method. In Section 4, we present a numerical experiment to show the efficiency and applicability of our method. Lastly, in Section 5, we give a conclusion of our study.

2. Preliminaries

In this section, we present some relevant results which are needed in the sequel. Let H be a real Hilbert space. The weak and strong convergence of the sequence $\{s_m\}$ to s as $m \to \infty$ is denoted by "$\to$" and "$\rightharpoonup$", respectively. For any $s, t \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$||s - t||^2 = ||s||^2 - 2\langle s, t\rangle + ||t||^2. \tag{2.1}$$

$$||s + t||^2 = ||s||^2 + 2\langle s, t\rangle + ||t||^2. \tag{2.2}$$

$$||s - t||^2 \leq ||s||^2 + 2\langle t, s - t\rangle. \tag{2.3}$$

$$||\alpha s + (1 - \alpha)t||^2 = \alpha||s||^2 + (1 - \alpha)||t||^2 - \alpha(1 - \alpha)||s - t||^2. \tag{2.4}$$

$$||\alpha s + \beta t + \gamma z||^2 = \alpha||s||^2 + \beta||t||^2 + \gamma||z||^2 - \alpha\beta||s - t||^2 - \alpha\gamma||s - z||^2 - \beta\gamma||t - z||^2. \tag{2.5}$$

Definition 2.1. *Let T be a self mapping defined on H, then the operator T is said to be*

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Ts - Tt\| \leq L\|s - t\|,$$

for all $s, t \in H$. If $L = 1$, then T is called nonexpansive;

(b) monotone if

$$\langle Ts - Tt, s - t\rangle \geq 0, \quad \forall s, t \in H;$$

(c) pseudomonotone if

$$\langle Ts, t - s\rangle \geq 0 \Rightarrow \langle Tt, t - s\rangle \geq 0, \quad \forall s, t \in H;$$

(d) β- strongly monotone if there exists $\alpha > 0$, such that

$$\langle Ts - Tt, s - t\rangle \geq \beta\|s - t\|^2, \quad \forall\ s, t \in H;$$

(e) firmly nonexpansive

$$\|Ts - Tt\|^2 \leq \langle Ts - Tt, s - t\rangle \ \ \forall\ s, t \in H;$$

or equivalently

$$\|Ts - Tt\|^2 \leq \|s - t\|^2 - \|(I - T)s - (I - T)t\|^2 \ \ \forall\ s, t \in H;$$

(f) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\langle s - t, Ts - Tt\rangle \geq \|Ts - p\|^2 \ \ \forall\ s \in H \ \ \text{and} \ \ p \in F(T);$$

(g) sequentially weakly continuous if for each sequence $\{s_m\}$, we obtain that $\{s_m\}$ converges weakly to s, implies that Ts_m converges weakly to Ts.

Let K stand for a closed, nonempty, and convex subset of H. For every $h \in H$, a unique point $P_K h \in K$ exists such that

$$\|h - P_K h\| \leq \|h - y\| \ \ \forall y \in K.$$

P_K is known as the metric projection of H onto K. The operator P_C is a nonexpansive and it satisfies

$$\langle s - y, P_K s - P_K y\rangle \geq \|P_K s - P_K t\|^2, \tag{2.6}$$

for all $s, t \in H$. Moreover, P_K is characterized by the feature

$$\|s - t\|^2 \geq \|s - P_K t\|^2 + \|t - P_K s\|^2$$

and for all $s \in H$ and $t \in K$.

Lemma 2.1. *[29] Let K be a nonempty, convex, and closed subset of H and $A : K \to H$ a continuous and monotone operator, then for $s \in K$, we have*

$$s \in VI(K, A) \quad \text{if and only if} \quad \langle At, t - s\rangle \geq 0 \quad \forall t \in K.$$

Lemma 2.2. *[32] Let K be nonempty, convex and closed subset of a real Hilbert space H. For every $s \in H$ and $z \in K$, one has $z = P_K s$ if and only if $\langle s - z, z - t \rangle \leq 0 \; \forall \; t \in K$.*

For any nonexpansive mapping T, it is well known the set of fixed points of T is convex and closed. Also, T fulfills the following inequality

$$\langle (s - Ts) - (t - Tt), Tt - Ts \rangle \leq \frac{1}{2} \|(Ts - s) - (Tt - t)\|^2, \; \forall \; s, t \in H. \tag{2.7}$$

Thus, for all $s \in H$ and $s^* \in F(T)$, we get

$$\langle s - Ts, s^* - Ts \rangle \leq \frac{1}{2} \|Ts - s\|^2, \; \forall \; s, t \in H. \tag{2.8}$$

Lemma 2.3. *[35] Let T be a self mapping defined on H, then the following statements are equivalent:*

(1) T is directed;
(2) the following relation holds

$$\|s - Ts\|^2 \leq \langle s - q, s - Ts \rangle \quad \forall q \in F(T), \; s \in H; \tag{2.9}$$

(3) the following relation holds

$$\|Ts - q\|^2 \leq \|s - q\|^2 - \|s - Ts\|^2 \quad \forall q \in F(T), \; s \in H. \tag{2.10}$$

Lemma 2.4. *[8] Let $F : K \times K \to \mathbb{R}$ and $\phi : K \times K \to \mathbb{R}$ be bifunctions satisfying the following assumptions:*

Assumption 2.1. *(1) $F(s, s) = 0$ for all $s \in K$;*
(2) F is monotone. That is $F(s, t) + F(t, s) \leq 0$ for all $s \in K$;
(3) for each $s, t, z \in K, \limsup_{g \to 0^+} F(gz + (1 - g)s, t) \leq F(s, t)$;
(4) for every $s \in K, t \mapsto F(s, t)$ is convex and lower semi-continuous;
(5) $\phi(\cdot, \cdot)$ is weakly continuous and $\phi(\cdot, y)$ is convex;
(6) ϕ is skew-symmetric, that is,

$$\phi(s, s) - \phi(s, t) + \phi(t, t) - \phi(t, s) \geq 0, \quad \forall \; s, t \in K.$$

Next, we define $T_r^{(F,\phi)} : H \to K$ as follows:

$$T_r^{(F,\phi)}(z) = \{s \in K : F(s, t) + \phi(t, s) - \phi(s, s) + \frac{1}{r} \langle t - s, s - z \rangle \geq 0, \quad \forall \; t \in K\}, \tag{2.11}$$

where r is a positive real number.

Assume that $F, \phi : K \times K \to \mathbb{R}$ satisfies Assumption 2.1 and for any $r > 0$ and $x \in H$, define $T_r^F : H \to K$ as in (2.11), then the following hold:

(1) $T_r^{(F,\phi)}$ is nonempty and single valued;
(2) $T_r^{(F,\phi)}$ is firmly nonexpansvie;
(3) $T_r^{(F,\phi)} = Sol(GEP)(1.8)$;
(4) $Sol(GEP)$ is closed and convex.

Lemma 2.5. *[23] Let $\{a_m\}$ be a sequence of positive real numbers, $\{\alpha_m\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{m=1}^{\infty}\alpha_m = \infty$, and $\{d_m\}$ be a sequence of real numbers. Suppose that*

$$a_{m+1} \leq (1-\alpha_m)a_m + \alpha_m d_m, m \geq 1.$$

If $\limsup_{k\to\infty} d_{m_k} \leq 0$ for all subsequences $\{a_{m_k}\}$ of $\{a_m\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{m_k+1} - a_{m_k}\} \geq 0,$$

then, $\lim_{k\to\infty} a_m = 0$.

Lemma 2.6. *[30, 32] Let K be a nonempty, closed, and convex subset of a Hilbert space H and let $T : K \to H$ be a ρ-demimetric operator with $\rho \in (-\infty, 1)$ and $F(T) \neq \emptyset$. Let ψ be a real number with $0 < \psi < 1-\rho$ and let $D = (1-\psi)I + \psi T$, then, D is a quasi-nonexpansive operator.*

Lemma 2.7. *[30] Let K be a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space and $F : K \to K$ be a nonexpansive mapping. For each $s \in K$ and the Cesaro mean $T_m s = \frac{1}{N}\sum_{i=0}^{N-1} T_i s$, then $\limsup_{m\to\infty} \|T_m s - F(T_m s)\| = 0$.*

Definition 2.2. *A function $k : H \to \mathbb{R}$ is said to be Gateaux differentiable at $s \in H$, if there exists an element denoted by $k'(s) \in H$ such that*

$$\lim_{h\to 0}\frac{k(s+hy) - k(s)}{h} = \langle y, k'(s)\rangle$$

for all $y \in H, h \in [0,1]$, where $k'(s)$ is called the Gateaux differential of k at s. We also recall that if for each $s \in H$, k is Gateaux differentiable at s, then k is Gateaux differentiable on H.

Definition 2.3. *Let H be a real Hilbert space. A function $k : H \to \mathbb{R} \cup \infty$ is said to be weakly lower semicontinuous at $s \in H$ if*

$$\lim_{m\to\infty} k(s_m) \geq k(s)$$

holds for any arbitrary sequence $\{s_m\}$ in H, satisfying the fact that $\{s_m\}$ converges weakly to s.

Definition 2.4. *A convex set $h : H \to \mathbb{R}$ is said to be subdifferentiable at a point $s \in H$ if the set*

$$\partial h(s) = \{p \in H : h(y) \geq h(p) + \langle p, y-s\rangle \quad \forall y \in H\} \tag{2.12}$$

is nonempty. Each element in $\partial h(s)$ is called a subgradient of h at s. We note that if h is subdifferentiable at each $s \in H$ then h is subdifferentiable on H. It is also known that if h is Gateaux differentiable at s, then h is subdifferential at s and $\partial h(s) = \{h'(s)\}$.

Lemma 2.8. *[12] Let $VI(K, A)$ be the solution set for the VIP (1.1) such that $VI(K, A) \neq \emptyset$ and K is defined as $K := \{s \in H : h(s) \leq 0\}$, where $h : H \to \mathbb{R}$ is a continuously differentiable convex function. Let $p \in K$, and $p \in VI(K, A)$ if and only if,*

(1) $A(p) = 0$ or
(2) $p \in \partial K$ and there exists $\tau > 0$ such that $A(p) = \tau h'(p)$, where ∂K denotes the boundary of K.

3. Main results

In this section, we present our proposed method and also establish its convergence results.

Assumption 3.1. Condition A. *Suppose*

(1) Let H_1 and H_2 be two real Hilbert spaces.
(2) Let K and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively such that K is defined as

$$K = \{s \in H_1 : q(s) \leq 0\},$$

where $q : H_1 \to \mathbb{R}$ and satisfies the following conditions:

(a) q is a continuously differentiable convex function such that $q'(.)$ is L_2-Lipschitz continuous and q is weakly lower semicontinuous on H_1.

(3) Let $A : H_1 \to H_2$ be a bounded linear operator and $f : K \to K$ be a contraction with contraction constant $k \in [0, 1)$.
(4) Let $F_1 : K \times K \to \mathbb{R}, \phi_1 : K \times K \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$, and $\phi_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.1. F_2 is upper semi-continuous in the first argument.
(5) $B : H_1 \to H_1$ is a monotone, L_1- Lipschitz continuous operator (Lipschitz constant need not to be known).
(6) For all $i \in \{1, 2, \cdots, N\}$, $S_i : H_1 \to H_1$ is a finite family of ρ-demimetric operators with $\rho \in (-\infty, 1)$, such that $I - S_i$ is demiclosed at zero.
(7) The solution set $\Omega = \Big(VI(K, B) \cap_{i=1}^{N} F(S_i)\Big) \cap \Gamma \neq \emptyset$, where $\Gamma = \{s \in K : s \in EP(F_1, \phi_1)$ and $As \in EP(F_2, \phi_2)\}$.

Condition B. *Suppose that $\{\alpha_m\}, \{\psi_m\}, \{\beta_m\}, \{\epsilon_m\}$ and $\{\eta_m\}$ are positive sequences such that*

(1) $\{\psi_m\}, \{\alpha_m\} \subset (0, 1), \{\eta_m\}, \{\beta_m\} \subset [a, b] \subset (0, 1), \lim_{m\to\infty} \alpha_m = 0$, such that $\alpha_m + \beta_m + \eta_m = 1$ and $\sum_{m=1}^{\infty} \alpha_m = \infty, \lim_{m\to\infty} \frac{\epsilon_m}{\alpha_m} = 0.$, where $\{\epsilon_m\}$ is a positive sequence.
(2) $0 < \omega_m < 1 - \rho, \lambda_1 > 0, \tau, \mu \in (0, 1)$ $0 \leq \delta_m \leq \bar{\delta}_m \leq \theta_m \leq \bar{\theta}_m < 1$.

Now, we propose iterative Algorithm 3.1.

Algorithm 3.1. Initialization Step. *Take $s_0, s_1 \in H_1$, given the iterates s_{m-1} and s_m for all $m \in \mathbb{N}$.*

$$\bar{\delta}_m = \begin{cases} \min\{\delta, \frac{\epsilon_m}{\|s_m - s_{m-1}\|}\}, & \text{if } s_m \neq s_{m-1}, \\ \delta, & \text{otherwise}. \end{cases} \tag{3.1}$$

$$\bar{\theta}_m = \begin{cases} \min\{\theta, \frac{\epsilon_m}{\|s_m - s_{m-1}\|}\}, & \text{if } s_m \neq s_{m-1}, \\ \theta, & \text{otherwise}. \end{cases} \tag{3.2}$$

Step 1. *Compute*

$$z_m = \alpha_m s_m + (1-\alpha_m)(s_m + \delta_m(s_m - s_{m-1})), \tag{3.3}$$
$$w_m = \psi_m s_m + (1-\psi_m)(s_m + \theta_m(s_m - s_{m-1})), \tag{3.4}$$

$$y_m = T_{r_m}^{(F_1,\phi_1)}(w_m + \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m),$$

where

$$\gamma_m \in \left(\epsilon, \frac{\|(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2}{\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2} - \epsilon\right). \tag{3.5}$$

Step 2. *Compute*

$$u_m = P_{K_m}(y_m - \lambda_m B y_m),$$
$$v_m = (1-\tau)y_m + \tau u_m + \tau\lambda_m(By_m - Bu_m),$$

where

$$K_m = \{s \in \mathcal{H}_1 \mid q(y_m) + \langle q'(y_m), s - y_m\rangle \leq 0\}$$

and

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu\|y_m - u_m\|}{\sqrt{\|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2}}, \lambda_m\right\}, & \text{if } \|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2 \neq 0, \\ \lambda_m, & \text{otherwise.} \end{cases} \tag{3.6}$$

Step 3. *Compute*

$$s_{m+1} = \alpha_m f(z_m) + \beta_m v_m + \eta_m S_m v_m, \tag{3.7}$$

where $S_m := \frac{1}{N}\sum_{i=0}^{N-1}((1-\omega_m)I + \omega_m S_i)$.

Remark 3.1. *We note that* $S_m := \frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_m)I + \psi_m S_i)$ *is quasi-nonexpansive mapping. To see this, let* $p \in \Omega$, *and using Lemma 2.6, we have*

$$\begin{aligned} \|S_m s - p\| &= \|\frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_m)I + \psi_m S_i)s - p\| \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|((1-\psi_m)I + \psi_m S_i)s - p\| \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|s - p\| \\ &= \|s - p\|. \end{aligned} \tag{3.8}$$

Thus, T_m *is quasi-nonexpansive.*

Lemma 3.1. *The step size $\{\gamma_m\}$ defined by (3.5) is well-defined.*

Proof. Assume $p \in \Gamma$, then $Ap \in (T_{r_m}^{(F_2,\phi_2)})$, since $(T_{r_m}^{(F_2,\phi_2)})$ is firmly nonexpansive and $F(T_{r_m}^{(F_2,\phi_2)}) \neq \emptyset$. Using (2.9), we obtain

$$\begin{aligned}\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|\|w_m - p\| &\geq \langle A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m, w_m - p\rangle \\ &= \langle((T_{r_n}^{(F_2,\phi_2)} - I)Aw_m, Aw_m - Ap\rangle \\ &\geq \|(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2. \end{aligned} \tag{3.9}$$

Since $T_{r_m}^{(F_2,\phi_2)}Aw_m \neq Aw_m$, $\|(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\| > 0$, then $\|w_m - p\|\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\| > 0$, hence, $\|A^*(T_{r_n}^{(F_2,\phi_2)} - I)Aw_m\| \neq 0$. Thus, γ_m is well defined. □

Lemma 3.2. *Suppose $\{s_m\}$ is a sequence generated by Algorithm 3.1. Under Assumption 3.1, $\{s_m\}$ is bounded.*

Proof. Let $p \in \Omega$. Using (3.3) and the fact that $0 \leq \theta_m \leq \bar{\theta}_m$, we have

$$\theta_m\|s_m - s_{m-1}\| \leq \bar{\theta}_m\|s_m - s_{m-1}\| \leq \epsilon_m.$$

Therefore, it follows from $\lim_{m\to\infty} \frac{\epsilon_m}{\alpha_m} = 0$, that

$$\lim_{m\to\infty} \frac{\theta_m}{\alpha_m}\|s_m - s_{m-1}\| \leq \lim_{m\to\infty} \frac{\epsilon_m}{\alpha_m} = 0. \tag{3.10}$$

It follows that the sequence $\{\frac{\theta_m}{\alpha_m}\|s_m - s_{m-1}\|\}$ is bounded. Hence, there exists $N_2 > 0$ such that $\frac{\theta_m}{\alpha_m}\|s_m - s_{m-1}\| \leq N_2$, for all $m \in \mathbb{N}$. By Algorithm 3.1, we have

$$\begin{aligned}\|w_m - p\| &= \|\psi_m s_m + (1 - \psi_m)(s_m + \theta_m(s_m - s_{m-1})) - p\| \\ &\leq \psi_m\|s_m - p\| + (1 - \psi_m)\|s_m - p\| + \theta_m(1 - \psi_m)\|s_m - s_{m-1}\| \\ &\leq \|s_m - p\| + \theta_m\|s_m - s_{m-1}\| \\ &= \|s_m - p\| + \alpha_m\frac{\theta_m}{\alpha_m}\|s_m - s_{m-1}\| \\ &\leq \|s_m - p\| + \alpha_m N_2. \end{aligned} \tag{3.11}$$

Using a similar argument as in (3.11), we have

$$\begin{aligned}\|z_m - p\| &= \|\alpha_m s_m + (1 - \alpha_m)(s_m + \delta_m(s_m - s_{m-1})) - p\| \\ &\leq \alpha_m\|s_m - p\| + (1 - \alpha_m)\|s_m - p\| + \delta_m(1 - \alpha_m)\|s_m - s_{m-1}\| \\ &\leq \|s_m - p\| + \delta_m\|s_m - s_{m-1}\| \\ &= \|s_m - p\| + \alpha_m\frac{\delta_m}{\alpha_m}\|s_m - s_{m-1}\| \\ &\leq \|s_m - p\| + \alpha_m N_1. \end{aligned} \tag{3.12}$$

Also, using the fact that $T_{r_m}^{(F_1,\phi_1)}, T_{r_m}^{(F_2,\phi_2)}$ is firmly nonexpansive, (2.2), $F(T_{r_m}^{(F_2,\phi_2)}) \neq \emptyset$, (2.9), and the step size of γ_m in (3.5), we have

$$\|y_m - p\|^2 \leq \|T_{r_m}^{(F_2,\phi_1)}[w_m + \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m] - T_{r_m}^{(F_2,\phi_1)}p\|^2$$

$$
\begin{aligned}
&\leq ||w_m + \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m - p||^2 \\
&= ||w_m - p||^2 + \gamma_m^2 ||A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m||^2 + 2\gamma_m \langle w_m - p, A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle \\
&= ||w_m - p||^2 + \gamma_n^2 ||A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m||^2 + 2\gamma_m \langle A(w_m - p), (T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle \\
&\leq ||w_m - p||^2 + \gamma_m^2 ||A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m||^2 - \gamma_m ||(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m||^2 \\
&= ||w_m - p||^2 - \gamma_m (||(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m||^2 - \gamma_m ||A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m||^2) \\
&\leq ||w_m - p||^2 - \gamma_m \varepsilon ||A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m||^2 \\
&\leq ||w_m - p||^2. \qquad (3.13)
\end{aligned}
$$

This implies

$$||y_m - p|| \leq ||w_m - p||. \qquad (3.14)$$

Furthermore, since, $u_m = P_{K_m}(y_m - \lambda_m B y_m)$, and using the characteristics of the metric projection, we have

$$\langle y_m - \lambda_m B y_m - u_m, u_m - p\rangle \geq 0, \qquad (3.15)$$

that is

$$2\langle y_m - u_m, u_m - p\rangle - 2\lambda_m \langle By_m - Bu_m, u_m - p\rangle - 2\lambda_m \langle Bu_m, u_m - p\rangle \geq 0. \qquad (3.16)$$

Now, observe that

$$2\langle y_m - u_m, u_m - p\rangle = ||y_m - p||^2 - ||y_m - u_m||^2 - ||u_m - p||^2, \qquad (3.17)$$

thus, we have that (3.16) becomes

$$||y_m - p||^2 - ||y_m - u_m||^2 - ||u_m - p||^2 - 2\lambda_m \langle By_m - Bu_m, u_m - p\rangle - 2\lambda_m \langle Bu_m, u_m - p\rangle \geq 0. \qquad (3.18)$$

In addition, using the fact that B is monotone. It is easy to see that

$$\langle Bu, u_m - p\rangle = \langle Bu_m - Bp, u_m - p\rangle + \langle Bp, u_m - p\rangle \geq \langle Bp, u_m - p\rangle. \qquad (3.19)$$

Thus, (3.18) becomes

$$||u_m - p||^2 \leq ||y_m - p||^2 - ||y_m - u_m||^2 - 2\lambda_m \langle By_m - Bu_m, u_m - p\rangle + 2\lambda_m \langle Bp, p - u_m\rangle. \qquad (3.20)$$

Additionally, using Algorithm 3.1 and (3.20), we get

$$
\begin{aligned}
&||v_m - p||^2 \\
&= ||(1-\tau)y_m + \tau u_m + \tau\lambda_m(By_m - Bu_m) - p||^2 \\
&= ||(1-\tau)(y_m - p) + \tau(u_m - p) + \tau\lambda_m(By_m - Bu_m)||^2 \\
&= (1-\tau)^2||y_m - p||^2 + \tau^2||u_m - p||^2 + \tau^2\lambda_m^2||By_m - Bu_m||^2 \\
&+ 2\tau(1-\tau)\langle y_m - p, u_m - p\rangle + 2\lambda_n\tau(1-\tau)\langle y_m - p, By_n - Bu_n\rangle \\
&+ 2\lambda_n\tau^2\langle u_m - p, By_m - Bu_n\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\tau)^2\|y_m - p\|^2 + \tau^2\|u_m - p\|^2 + \tau^2\lambda_m^2\|By_m - Bu_m\|^2 + \tau(1-\tau)[\|y_m - p\|^2 \\
&+ \|u_m - p\|^2 - \|y_m - u_m\|^2] + 2\lambda_m\tau(1-\tau)\langle y_m - p, By_m - Bu_m\rangle \\
&+ 2\lambda_m\tau^2\langle u_m - p, By_m - Bu_m\rangle \\
&= (1-\tau)\|y_m - p\|^2 + \tau\|u_m - p\|^2 + \tau^2\lambda_m^2\|By_m - Bu_m\|^2 - \tau(1-\tau)\|y_m - u_m\|^2 \\
&+ 2\lambda_m\tau(1-\tau)\langle y_m - p, By_m - Bu_m\rangle + 2\lambda_m\tau^2\langle u_m - p, By_m - Bu_m\rangle \\
&\le (1-\tau)\|y_m - p\|^2 + \tau[\|y_m - p\|^2 - \|y_m - u_m\|^2 - 2\lambda_m\langle By_m - Bu_m, u_m - p\rangle + 2\lambda_m\langle Bp, p - u_m\rangle] \\
&+ \tau^2\lambda_m^2\|By_m - Bu_m\|^2 - \tau(1-\tau)\|y_m - u_m\|^2 \\
&+ 2\lambda_m\tau(1-\tau)\langle y_n - p, By_m - Bu_m\rangle + 2\lambda_m\tau^2\langle u_m - p, By_m - Bu_m\rangle \\
&= \|y_m - p\|^2 - \tau(2-\tau)\|y_m - u_m\|^2 + \tau^2\lambda_m^2\|By_m - Bu_m\|^2 + 2\lambda_m\tau(1-\tau)\langle y_m - u_m, By_m - Bu_m\rangle \\
&+ 2\lambda_m\tau\langle Bp, p - u_m\rangle. \qquad (3.21)
\end{aligned}
$$

To conclude the estimation for (3.21), we need to consider two cases: Case I, when $Bp \neq 0$, and Case II, when $Bp = 0$.

For Case I ($Bp \neq 0$), we have $p \in \partial K$, and there exists $l > 0$ such that $Bp = -lq'(p)$. Note that $q(p) = 0$ since $p \in \partial K$. Using (2.12), we have

$$
q(u_m) \ge q(p) + \langle q'(p), u_m - p\rangle = \frac{-1}{l}\langle Bp, u_m - p\rangle,
$$

that is

$$
-\langle Bp, u_m - p\rangle = \langle Bp, p - u_m\rangle \le lq(u_m). \qquad (3.22)
$$

Since, $u_m \in K_m$, we obtain

$$
q(y_m) + \langle q'(y_m), u_m - y_m\rangle \le 0. \qquad (3.23)
$$

Also, using (2.12), we have

$$
q(u_m) + \langle q'(u_m), y_m - u_m\rangle - q(y_m) \le 0. \qquad (3.24)
$$

Adding (3.23) and (3.24), we get

$$
q(u_m) \le \langle q'(u_m) - q'(y_m), u_m - y_m\rangle. \qquad (3.25)
$$

Thus, (3.22) becomes

$$
\langle Bp, p - u_m\rangle \le lq(u_m) \le l\langle q'(u_m) - q'(y_m), u_m - y_m\rangle. \qquad (3.26)
$$

Thus, we have that (3.21) becomes

$$
\begin{aligned}
\|v_m - p\|^2 &\le \|y_m - p\|^2 - \tau(2-\tau)\|y_m - u_m\|^2 + \tau^2\lambda_m^2\|By_m - Bu_m\|^2 + 2\lambda_m\tau(1-\tau)\langle y_m - u_m, By_m - Bu_m\rangle \\
&+ 2\lambda_m\tau l\langle q'(u_m) - q'(y_m), u_m - y_m\rangle \\
&\le \|y_m - p\|^2 - \tau(2-\tau)\|y_m - u_m\|^2 + \tau\lambda_m^2\|By_m - Bu_m\|^2 + 2\lambda_m\tau(1-\tau)\langle y_m - u_m, By_m - Bu_m\rangle \\
&+ \lambda_m^2\tau\|q'(u_m) - q'(y_m)\|^2 + l^2\tau\|u_m - y_m\|^2
\end{aligned}
$$

$$= ||y_m - p||^2 - \tau(2-\tau)||y_m - u_m||^2 + \tau\lambda_m^2[||q'(u_m) - q'(y_m)||^2 + ||By_m - Bu_m||^2]$$
$$+ 2\lambda_m\tau(1-\tau)\langle y_m - u_m, By_m - Bu_m\rangle + l^2\tau||u_m - y_m||^2$$
$$\leq ||y_m - p||^2 - \tau(2-\tau)||y_m - u_m||^2 + \tau\frac{\lambda_m^2\mu^2}{\lambda_{m+1}^2}||y_m - u_m||^2$$
$$+ 2\lambda_m\tau(1-\tau)||y_m - u_m||||By_m - Bu_m|| + l^2\tau||u_m - y_m||^2. \tag{3.27}$$

It is easy to see from (3.6) that

$$\lambda_{m+1}^2||By_m - Bu_m||^2 \leq \mu^2||y_m - u_m||^2 - \lambda_{m+1}^2||q'(u_m) - q'(y_m)||^2 \leq \mu^2||y_m - u_m||^2. \tag{3.28}$$

Thus, we have

$$||By_m - Bu_m|| \leq \frac{\mu}{\lambda_{m+1}}||y_m - u_m||. \tag{3.29}$$

Therefore, (3.27) becomes

$$||v_m - p||^2 \leq ||y_m - p||^2 - \tau(2-\tau)||y_m - u_m||^2 + \tau\frac{\lambda_m^2\mu^2}{\lambda_{m+1}^2}||y_m - u_m||^2$$
$$+ 2\lambda_m\tau(1-\tau)\frac{\mu}{\lambda_{m+1}}||y_m - u_m||^2 + l^2\tau||u_m - y_m||^2$$
$$= ||y_m - p||^2 - \tau\Big[2 - \tau - \frac{\lambda_m^2\mu^2}{\lambda_{m+1}^2} - 2(1-\tau)\mu\frac{\lambda_m}{\lambda_{m+1}} - l^2\Big]||y_m - u_m||^2. \tag{3.30}$$

Using the fact that $\lim_{m\to\infty}\lambda_m = \lambda_{m+1} > 0, \mu, l, \tau \in (0,1)$, we have $\lim_{m\to\infty}\tau\Big[2-\tau-\frac{\lambda_m^2\mu^2}{\lambda_{m+1}^2}-2(1-\tau)\mu\frac{\lambda_m}{\lambda_{m+1}}-l^2\Big] = \tau\Big[2-\tau-\mu^2-2(1-\tau)\mu - l^2\Big] > 0$. Thus, we have that (3.30) becomes

$$||v_m - p|| \leq ||y_m - p||. \tag{3.31}$$

For Case II ($Bp = 0$), using a similar approach as in Case I, we have

$$||v_m - p||^2 \leq ||y_m - p||^2 - \tau(2-\tau)||y_m - u_m||^2 + \tau^2\lambda_n^2||By_m - Bu_m||^2 + 2\lambda_m\tau(1-\tau)\langle y_m - u_m, By_m - Bu_m\rangle$$
$$\leq ||y_m - p||^2 - \tau(2-\tau)||y_m - u_m||^2 + \tau\frac{\lambda_m^2\mu^2}{\lambda_{m+1}^2}||y_m - u_m||^2 + 2\lambda_m\tau(1-\tau)\frac{\mu\lambda_m}{\lambda_{m+1}}||y_m - u_m||^2$$
$$= ||y_m - p||^2 - \tau\Big[2 - \tau - \frac{\lambda_m^2\mu^2}{\lambda_{m+1}^2} - 2(1-\tau)\mu\frac{\lambda_m}{\lambda_{m+1}}\Big]||y_m - u_m||^2$$
$$\leq ||y_m - p||^2. \tag{3.32}$$

Thus, we have that (3.32) becomes

$$||v_m - p|| \leq ||y_m - p||. \tag{3.33}$$

Finally, using Algorithm 3.1, (3.11)–(3.13) and (3.33), we get

$$||s_{m+1} - p|| = ||\alpha_m f(z_m) + \beta_m v_m + \eta_m S_m v_m - p||$$

$$
\begin{aligned}
&= \|\alpha_m(f(z_m)-p)+\beta_m(v_m-p)+\eta_m(S_m v_m-p)\| \\
&\le \alpha_m\|f(z_m)-f(p)\|+\alpha_m\|f(p)-p\|+\beta_m\|v_m-p\|+\eta_m\|S_m v_m-p\| \\
&\le \alpha_m k\|z_m-p\|+\alpha_m\|f(p)-p\|+\beta_m\|v_m-p\|+\eta_m\|v_m-p\| \\
&\le \alpha_m k\|z_m-p\|+\alpha_m\|f(p)-p\|+(1-\alpha_m)\|v_m-p\| \\
&\le \alpha_m k\|s_m-p\|+\alpha_m N_1+\alpha_m\|f(p)-p\|+(1-\alpha_m)\|s_m-p\|+\alpha_m N_2 \\
&= (1-\alpha_m(1-k))\|s_m-p\|+\alpha_m N_3+\alpha_m\|f(p)-p\| \\
&\le (1-\alpha_m(1-k))\|s_m-p\|+\alpha_m(1-k)\frac{N_3+\|f(p)-p\|}{(1-k)} \\
&\le \max\left\{\|s_m-p\|,\frac{N_3+\|f(p)-p\|}{1-k}\right\} \\
&\le \\
&\vdots \\
&\le \max\left\{\|s_0-p\|,\frac{N_3+\|f(p)-p\|}{1-k}\right\}, \qquad (3.34)
\end{aligned}
$$

where $N_3 = N_1 + N_2$. Thus, $\{s_m\}$ is bounded. □

Lemma 3.3. *Let $\{s_m\}$ be the sequence generated by iterative Algorithm 3.1 under Assumption 3.1. Suppose that the subsequence $\{s_{m_k}\}$ of $\{s_m\}$ converges weakly to a point s^*, and $\lim_{k\to\infty}\|u_{m_k}-y_{n_k}\| = 0 = \lim_{k\to\infty}\|y_{m_k}-w_{m_k}\|$, then, $s^*\in VI(K,A)$ and $As^*\in EP(F_2,\phi_2)$.*

Proof. Let $\{s_{m_k}\}$ be a subsequence of $\{s_m\}$ which converges weakly to $s^*\in H_1$. It is easy to see that

$$\lim_{k\to\infty}\|w_{m_k}-s_{m_k}\| \le \lim_{k\to\infty}\alpha_{m_k}\cdot\frac{\theta_{m_k}}{\alpha_{m_k}}\|s_{m_k}-s_{m_k-1}\| = 0. \qquad (3.35)$$

Using the hypothesis in the lemma, we get

$$\|y_{m_k}-s_{m_k}\| \le \|y_{m_k}-w_{m_k}\|+\|w_{m_k}-s_{m_k}\| \to 0 \text{ as } k\to\infty. \qquad (3.36)$$

Since A is a bounded linear operator, it follows from (3.35) that $\{Aw_{m_k}\}$ converges weakly to As^*. Also, by (3.36), we have that y_{m_k} converges weakly to s^*. Thus, we have

$$\|u_{m_k}-s_{m_k}\| \le \|u_{m_k}-y_{m_k}\|+\|y_{m_k}-s_{m_k}\| \to 0 \text{ as } k\to\infty. \qquad (3.37)$$

From (3.13), we have

$$
\begin{aligned}
\|y_{m_k}-p\|^2 &\le \|w_{m_k}-p\|^2-\gamma_m\varepsilon\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)}-I)Aw_{m_k}\|^2 \\
&\le \|w_{m_k}-p\|^2-\varepsilon^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)}-I)Aw_{m_k}\|^2, \qquad (3.38)
\end{aligned}
$$

which implies that

$$\varepsilon^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)}-I)Aw_{m_k}\|^2 \le \|w_{m_k}-p\|^2-\|y_{m_k}-p\|^2 \le \|w_{m_k}-y_{m_k}\|^2+2\|y_{m_k}-p\|\|w_{m_k}-y_{m_k}\|,$$

therefore, we have

$$\lim_{k\to\infty} \|A^*(T_{r_{m_k}}^{(F_2,\phi_2} - I)Aw_{m_k}\| = 0. \tag{3.39}$$

In addition, from (3.13)

$$\begin{aligned} \|y_{m_k} - p\|^2 &\le \|w_{m_k} - p\|^2 - \gamma_{m_k}\|(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 + \gamma_{m_k}^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 \\ &\le \|w_{m_k} - p\|^2 - \varepsilon\|(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 + \gamma_{m_k}^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2, \end{aligned} \tag{3.40}$$

which implies that

$$\begin{aligned} \varepsilon\|(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Bw_m\|^2 &\le \|w_{m_k} - p\|^2 - \|y_{m_k} - p\|^2 + \gamma_{m_k}^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 \\ &\le (\|w_{m_k} - y_{m_k}\| + \|y_{m_k} - p\|)^2 - \|y_{m_k} - p\|^2 + \gamma_{m_k}^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 \\ &\le \|w_{m_k} - y_{m_k}\|^2 + 2\|y_{m_k} - p\|\|w_{m_k} - y_{m_k}\| + \gamma_m^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2, \end{aligned} \tag{3.41}$$

thus, we have

$$\lim_{k\to\infty} \|(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\| = 0. \tag{3.42}$$

Using the demiclosedness principle and (3.42), we have

$$As^* \in F(T_{r_{m_k}}^{(F_2,\phi_2)}) \Rightarrow As^* \in EP(F_2, \phi_2). \tag{3.43}$$

Since, the subsequence $\{y_{m_k}\}$ of $\{y_m\}$ is weakly convergent to a point s^*, it follows from (3.37) that u_{m_k} converges to s^*. Also, since $u_{m_k} \in K_{m_k}$ and using the definition of K_m, we obtain

$$q(y_{m_k}) + \langle q'(y_{m_k}), u_{m_k} - y_{m_k}\rangle \le 0. \tag{3.44}$$

Using the Cauchy Schwartz inequality, we obtain

$$q(y_{m_k}) \le \|q'(y_{m_k})\|\|u_{m_k} - y_{m_k}\|. \tag{3.45}$$

Furthermore, since $q'(.)$ is Lipschitz continuous, we have that, it is bounded on any subset of H_1. Thus, using the boundedness of $\{y_m\}$, there exists a constant $N > 0$ such that

$$\|q'(y_{m_k})\| \le N,$$

for all $k \in \mathbb{N}$. Consequently, (3.45) becomes

$$q(y_{m_k}) \le N\|u_{m_k} - y_{m_k}\|, \tag{3.46}$$

and using our hypothesis, we obtain

$$\lim_{k\to\infty} q(y_{m_k}) = 0. \tag{3.47}$$

In addition, using the fact that q is weakly lower semicontinuous, we obtain $q(s^*)\leq \liminf_{k\to\infty} q(y_{m_k})\leq 0$. This implies that $s^* \in K$. Since $u_{m_k} = P_K(y_{m_k} - \lambda_{m_k} B y_{m_k})$, from the characteristic of the metric projection, we have

$$\langle y_{m_k} - \lambda_{m_k} B y_{m_k} - u_{m_k}, s - u_{m_k}\rangle \leq 0 \ \ \forall \quad s \in K \subset K_m. \tag{3.48}$$

Using the fact that B is monotone, we have

$$\begin{aligned}
0 &\leq \langle u_{m_k} - y_{m_k}, s - u_{m_k}\rangle + \lambda_{m_k}\langle B y_{m_k}, s - u_{m_k}\rangle \\
&= \langle u_{m_k} - y_{m_k}, s - u_{m_k}\rangle + \lambda_{n_k}\langle B y_{m_k}, s - y_{m_k}\rangle + \lambda_{m_k}\langle B y_{m_k}, y_{m_k} - u_{m_k}\rangle \\
&= \langle u_{m_k} - y_{m_k}, s - u_{m_k}\rangle + \lambda_{m_k}\langle B s, s - y_{m_k}\rangle + \lambda_{m_k}\langle B y_{m_k} - B s, s - y_{m_k}\rangle + \lambda_{m_k}\langle B y_{m_k}, y_{m_k} - u_{m_k}\rangle \\
&\leq \langle u_{m_k} - y_{m_k}, s - u_{m_k}\rangle + \lambda_{m_k}\langle B s, s - y_{m_k}\rangle + \lambda_{m_k}\langle B y_{m_k}, y_{m_k} - u_{m_k}\rangle,
\end{aligned}$$

and using our hypothesis and the fact that $\lim_{k\to\infty} \lambda_{m_k} > 0$, we have

$$\langle B s, s - s^*\rangle \geq 0.$$

Using Lemma 2.1, we obtain $s^* \in VI(B, K)$. □

Theorem 3.1. *If $\{s_m\}$ is a sequence generated by Algorithm 3.1. Then, under the Assumption 3.1, $\{s_m\}$ converges strongly to $p \in \Omega$.*

Proof. Let $p \in \Omega$, and using Algorithm 3.1 and (2.4), we get

$$\begin{aligned}
\|z_m - p\|^2 &= \|\alpha_m s_m + (1 - \alpha_m)(s_m + \delta_m(s_m - s_{m-1})) - p\|^2 \\
&= \|\alpha_m(s_m - p) + (1 - \alpha_m)(s_m + \delta_m(s_m - s_{m-1}) - p)\|^2 \\
&\leq \alpha_m\|s_m - p\|^2 + (1 - \alpha_m)\|(s_m - p) + \delta_m(s_m - s_{m-1})\|^2 \\
&\leq \alpha_m\|s_m - p\|^2 + (1 - \alpha_m)\|s_m - p\|^2 + \delta_m^2\|s_m - s_{m-1}\|^2 + 2\delta_m\langle s_m - p, s_m - s_{m-1}\rangle \\
&\leq \alpha_m\|s_m - p\|^2 + (1 - \alpha_m)\|s_m - p\|^2 + \delta_m^2\|s_m - s_{m-1}\|^2 + 2\delta_m\|s_m - p\|\|s_m - s_{m-1}\| \\
&= \|s_m - p\|^2 + \delta_m\|s_m - s_{m-1}\|[\delta_m\|s_m - s_{m-1}\| + 2\|s_m - p\|] \\
&\leq \|s_m - p\|^2 + \delta_m\|s_m - s_{m-1}\|N_4,
\end{aligned} \tag{3.49}$$

where $N_4 = \sup_{m\in\mathbb{N}}\{2\|s_m - p\|, \delta_m\|s_m - s_{m-1}\|\}$. Using a similar arguement as in (3.49), we have

$$\begin{aligned}
\|w_m - p\|^2 &\leq \|s_m - p\|^2 + \delta_m\|s_m - s_{m-1}\|[\theta_m\|s_m - s_{m-1}\| + 2\|s_m - p\|] \\
&\leq \|s_m - p\|^2 + \theta_m\|s_m - s_{m-1}\|N_5,
\end{aligned} \tag{3.50}$$

where $N_5 = \sup_{m\in\mathbb{N}}\{2\|s_m - p\|, \theta_m\|s_m - s_{m-1}\|\}$.

In addition, we obtain

$$\begin{aligned}
&\|s_{m+1} - p\|^2 \\
&= \|\alpha_m f(z_m) + \beta_m v_m + \eta_m S_m v_m - p\|^2 \\
&= \|\alpha_m(f(z_m) - f(p)) + \beta_m(v_m - p) + \eta_m(S_m v_m - p) + \alpha_m(f(p) - p)\|^2 \\
&\leq \|\alpha_m(f(z_m) - f(p)) + \beta_m(v_m - p) + \eta_m(S_m v_m - p)\|^2 + 2\alpha_m\langle f(p) - p, s_{m+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_m\|f(z_m)-f(p)\|^2+\beta_m\|v_m-p\|^2+\eta_m\|S_mv_m-p\|^2+2\alpha_m\langle f(p)-p,s_{m+1}-p\rangle\\
&\le \alpha_m k^2\|z_m-p\|^2+\beta_m\|v_m-p\|^2+\eta_m\|v_m-p\|^2+2\alpha_m\langle f(p)-p,s_{m+1}-p\rangle\\
&\le \alpha_m k\|z_m-p\|^2+(1-\alpha_m)\|v_m-p\|^2+2\alpha_m\langle f(p)-p,s_{m+1}-p\rangle\\
&\le \alpha_m k[\|s_m-p\|^2+\delta_m\|s_m-s_{m-1}\|N_4]+(1-\alpha_m)\|y_m-p\|^2+2\alpha_m\langle f(p)-p,s_{m+1}-p\rangle\\
&\le \alpha_m k[\|s_m-p\|^2+\delta_m\|s_m-s_{m-1}\|N_4]+(1-\alpha_m)\|w_m-p\|^2+2\alpha_n\langle f(p)-p,s_{m+1}-p\rangle\\
&\le \alpha_m k\|s_m-p\|^2+\delta_m\|s_m-s_{m-1}\|N_4+(1-\alpha_m)[\|s_m-p\|^2+\theta_m\|s_m-s_{m-1}\|N_5]+2\alpha_m\langle f(p)-p,s_{m+1}-p\rangle\\
&\le (1-\alpha_m(1-k))\|s_m-p\|^2+\alpha_m(1-k)\Big[\frac{\delta_m\|s_m-s_{m-1}\|N_4}{\alpha_m(1-k)}+\frac{\theta_m\|s_m-s_{m-1}\|N_5}{\alpha_m(1-k)}+\frac{2}{(1-k)}\langle f(p)-p,s_{m+1}-p\rangle\Big]\\
&=(1-\alpha_m(1-k))\|s_m-p\|^2+\alpha_m(1-k)\Psi_m, \qquad (3.51)
\end{aligned}
$$

where $\Psi_m = \Big[\frac{\delta_m\|s_m-s_{m-1}\|N_4}{\alpha_m(1-k)}+\frac{\theta_m\|s_m-s_{m-1}\|N_5}{\alpha_m(1-k)}+\frac{2}{(1-k)}\langle f(p)-p,s_{m+1}-p\rangle\Big]$. According to Lemma 2.5, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\Psi_{m_k}\le 0$ for every subsequence $\{\|s_{m_k}-p\|\}$ of $\{\|s_m-p\|\}$ satisfying the condition:

$$
\liminf_{k\to\infty}\{\|s_{m_k+1}-p\|-\|s_{m_k}-p\|\}\ge 0. \qquad (3.52)
$$

From (3.32), (3.49) and (3.51), we get

$$
\begin{aligned}
&\|s_{m_k+1}-p\|^2\\
&\le \alpha_{m_k}\|z_{m_k}-p\|^2+(1-\alpha_{m_k})\|v_{m_k}-p\|^2+2\alpha_{m_k}\langle f(p)-p,s_{m_k+1}-p\rangle\\
&\le \alpha_{m_k}[\|s_{m_k}-p\|^2+\delta_{m_k}\|s_{m_k}-s_{m_k-1}\|N_4]+(1-\alpha_{m_k})[\|w_{m_k}-p\|^2\\
&-\tau\Big[2-\tau-\frac{\lambda_{m_k}^2\mu^2}{\lambda_{m_k+1}^2}-2(1-\tau)\mu\frac{\lambda_{m_k}}{\lambda_{m_k+1}}-l^2\Big]\|y_{m_k}-u_{m_k}\|^2]+2\alpha_{m_k}\langle f(p)-p,s_{m_k+1}-p\rangle\\
&\le \alpha_{m_k}\|s_{m_k}-p\|^2+\delta_{m_k}\|s_{m_k}-s_{m_k-1}\|N_4+(1-\alpha_{m_k})[\|s_{m_k}-p\|^2+\theta_{m_k}\|s_{m_k}-s_{m_k-1}\|N_5]\\
&-(1-\alpha_{m_k})\tau\Big[2-\tau-\frac{\lambda_{m_k}^2\mu^2}{\lambda_{m_k+1}^2}-2(1-\tau)\mu\frac{\lambda_{m_k}}{\lambda_{m_k+1}}-l^2\Big]\|y_{m_k}-u_{m_k}\|^2+2\alpha_{m_k}\langle f(p)-p,s_{m_k+1}-p\rangle\\
&\le \alpha_{m_k}\|s_{m_k}-p\|^2+\delta_{m_k}\|s_{m_k}-s_{m_k-1}\|N_4+\theta_{m_k}\|s_{m_k}-s_{m_k-1}\|N_5\\
&-(1-\alpha_{m_k})\tau\Big[2-\tau-\frac{\lambda_{m_k}^2\mu^2}{\lambda_{m_k+1}^2}-2(1-\tau)\mu\frac{\lambda_{m_k}}{\lambda_{m_K+1}}-l^2\Big]\|y_{m_k}-u_{m_k}\|^2+2\alpha_{m_k}\langle f(p)-p,s_{m_k+1}-p\rangle, \qquad (3.53)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\Big((1-\alpha_{m_k})\tau\Big[2-\tau-\frac{\lambda_{m_k}^2\mu^2}{\lambda_{m_k+1}^2}-2(1-\tau)\mu\frac{\lambda_{m_k}}{\lambda_{m_k+1}}-l^2\Big]\|y_{m_k}-u_{m_k}\|^2\Big)\\
&\le \limsup_{k\to\infty}\Big[\|s_{m_k}-p\|^2+\alpha_{m_k}\frac{\theta_{m_k}}{\alpha_{m_k}}\|s_{m_k}-s_{m_k-1}\|N_5+\alpha_{m_k}\frac{\delta_{m_k}}{\alpha_{m_k}}\|s_{m_k}-s_{m_k-1}\|N_4\\
&+2\alpha_{m_k}\langle f(p)-p,s_{m+1}-p\rangle\ -\|s_{m_k+1}-p\|^2\Big]\\
&\le -\liminf_{k\to\infty}[\|s_{m_k+1}-p\|^2-\|s_{m_k}-p\|^2]\le 0.
\end{aligned}
$$

Thus, we have

$$\lim_{k\to\infty} \|y_{m_k} - u_{m_k}\| = 0. \tag{3.54}$$

Using Algorithm 3.1 and Step 2, we get

$$\begin{aligned}\|v_{m_k} - y_{m_k}\| &= \|(1-\tau)(y_{m_k} - y_{m_k}) + \tau(u_{m_k} - y_{m_k}) + \tau\lambda_{m_k}(By_{m_k} - Bu_{m_k})\| \\ &\leq \tau\|u_{m_k} - y_{m_k}\| + \tau\lambda_{m_k}\|By_{m_k} - Bu_{m_k}\| \\ &\leq \tau\|u_{m_k} - y_{m_k}\| + \frac{\tau\lambda_{m_k}\mu}{\lambda_{m_k+1}}\|y_{m_k} - u_{m_k}\|.\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty} \|v_{m_k} - y_{m_k}\| = 0. \tag{3.55}$$

Also, using (3.54) and (3.55), we obtain

$$\lim_{k\to\infty} \|u_{m_k} - v_{m_k}\| \leq \lim_{k\to\infty} \|u_{m_k} - y_{m_k}\| + \lim_{k\to\infty} \|y_{m_k} - v_{m_k}\| = 0. \tag{3.56}$$

From (3.13), (3.49), and (3.51), we get

$$\begin{aligned}&\|s_{m_k+1} - p\|^2 \\ &\leq \alpha_{m_k} k[\|s_{m_k} - p\|^2 + \delta_{m_k}\|s_{m_k} - s_{m_k-1}\|N_4] + (1-\alpha_{m_k})\|y_{m_k} - p\|^2 + 2\alpha_{m_k}\langle f(p) - p, s_{m_k+1} - p\rangle \\ &\leq \alpha_{m_k}\|s_{m_k} - p\|^2 + \delta_{m_k}\|s_{m_k} - s_{m_k-1}\|N_4 + (1-\alpha_{m_k})[\|w_{m_k} - p\|^2 - \varepsilon^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2] \\ &+ 2\alpha_m\langle f(p) - p, s_{m_k+1} - p\rangle \\ &= \alpha_{m_k}\|s_{m_k} - p\|^2 + \delta_{m_k}\|s_{m_k} - s_{m_k-1}\|N_4 + (1-\alpha_{m_k})\|w_{m_k} - p\|^2 - (1-\alpha_{m_k})\varepsilon^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 \\ &+ 2\alpha_{m_k}\langle f(p) - p, s_{m_k+1} - p\rangle \\ &\leq \alpha_{m_k}\|s_{m_k} - p\|^2 + \delta_{m_k}\|s_{m_k} - s_{m_k-1}\|N_4 + (1-\alpha_{m_k})[\|s_{m_k} - p\|^2 + \theta_{m_k}\|s_{m_k} - s_{m_k-1}\|N_5] \\ &+ 2 - (1-\alpha_{m_k})\varepsilon^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 + \alpha_{m_k}\langle f(p) - p, s_{m_k+1} - p\rangle \\ &\leq \|s_{m_k} - p\|^2 + \delta_{m_k}\|s_{m_k} - s_{m_k-1}\|N_4 + \theta_m\|s_{m_k} - s_{m_k-1}\|N_5 - (1-\alpha_m)\varepsilon^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2 \\ &+ 2\alpha_{m_k}\langle f(p) - p, s_{m_k+1} - p\rangle,\end{aligned} \tag{3.57}$$

which implies that

$$\begin{aligned}&\limsup_{k\to\infty}\Big((1-\alpha_{m_k})\varepsilon^2\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\|^2\Big) \\ &\leq \limsup_{k\to\infty}\Big[\|s_{m_k} - p\|^2 + \alpha_{m_k}\frac{\theta_{m_k}}{\alpha_{m_k}}\|s_{m_k} - s_{m_k-1}\|N_5 + \alpha_{m_k}\frac{\delta_{m_k}}{\alpha_{m_k}}\|s_{m_k} - s_{m_k-1}\|N_4 \\ &+ 2\alpha_{m_k}\langle f(p) - p, s_{m_k+1} - p\rangle - \|s_{m_k+1} - p\|^2\Big] \\ &\leq -\liminf_{k\to\infty}[\|s_{m_k+1} - p\|^2 - \|s_{m_k} - p\|^2] \leq 0.\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty} \|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\| = 0. \tag{3.58}$$

Using (3.9), (3.58) and the boundedness of $\{w_m\}$, we obtain

$$\lim_{k\to\infty}\|(T_{r_{m_k}}^{(F_2,\phi_2)}-I)Aw_{m_k}\|=0. \tag{3.59}$$

In addition, using Algorithm 3.1 and (2.5), we have

$$\begin{aligned}
&\|s_{m+1}-p\|^2\\
&=\|\alpha_m(f(z_m)-p)+\beta_m(v_m-p)+\eta_m(S_mv_m-p)\|^2\\
&\le \alpha_m\|f(z_m)-p\|^2+\beta_m\|v_m-p\|^2+\eta_m\|S_mv_m-p\|^2-\beta_m\eta_m\|v_m-S_mv_m\|^2\\
&\le \alpha_m[\|f(z_m)-f(p)\|+\|f(p)-p\|]^2+(1-\alpha_m)\|v_m-p\|^2-\beta_m\eta_m\|v_m-S_mv_m\|^2\\
&\le \alpha_m[k\|z_m-p\|+\|f(p)-p\|]^2+(1-\alpha_m)\|v_m-p\|^2-\beta_m\eta_m\|v_m-S_mv_m\|^2\\
&\le \alpha_m[k^2\|z_m-p\|^2+\|f(p)-p\|^2+2k\|f(p)-p\|\|z_m-p\|]+(1-\alpha_m)\|w_m-p\|^2-\beta_m\eta_m\|v_m-S_mv_m\|^2\\
&\le \alpha_m\|z_m-p\|^2+\alpha_m\|f(p)-p\|^2+2k\alpha_m\|f(p)-p\|\|z_m-p\|+(1-\alpha_m)\|w_m-p\|^2-\beta_m\eta_m\|v_m-S_mv_m\|^2\\
&\le \alpha_m[\|s_m-p\|^2+\delta_m\|s_m-s_{m-1}\|N_4]+\alpha_m\|f(p)-p\|^2+2k\alpha_m\|f(p)-p\|\|z_m-p\|\\
&+(1-\alpha_m)[\|s_m-p\|^2+\theta_m\|s_m-s_{m-1}\|N_5]-\beta_m\eta_m\|v_m-S_mv_m\|^2\\
&\le \|s_m-p\|^2+\alpha_m\delta_m\|s_m-s_{m-1}\|N_4+\alpha_m\|f(p)-p\|^2+2k\alpha_m\|f(p)-p\|\|z_m-p\|\\
&+\theta_m\|s_m-s_{m-1}\|N_5-\beta_m\eta_m\|v_m-S_mv_m\|^2,
\end{aligned} \tag{3.60}$$

which implies

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\beta_{m_k}\eta_{m_k}\|v_{m_k}-S_{m_k}v_{m_k}\|^2\right)\\
&\le \limsup_{k\to\infty}\Big[\|s_{m_k}-p\|^2+\alpha_{m_k}\frac{\theta_{m_k}}{\alpha_{m_k}}\|s_{m_k}-s_{m_k-1}\|N_5\\
&+\alpha_{m_k}\delta_{m_k}\|s_{m_k}-s_{m_k-1}\|N_4+\alpha_{m_k}\|f(p)-p\|^2+2k\alpha_{m_k}\|f(p)-p\|\|z_{m_k}-p\| -\|s_{m_k+1}-p\|^2\Big]\\
&\le -\liminf_{k\to\infty}[\|s_{m_k+1}-p\|^2-\|s_{m_k}-p\|^2]\le 0.
\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\|v_{m_k}-S_{m_k}v_{m_k}\|=0. \tag{3.61}$$

Now, using the fact that $T_{r_m}^{(F_1,\phi_1)}$ is firmly nonexpansive, we get

$$\begin{aligned}
&\|y_m-p\|^2\\
&=\|T_{r_m}^{(F_1,\phi_1)}[w_m+\gamma_mA^*(T_{r_m}^{(F_2,\phi_2)}-I)Aw_m]-p\|^2\\
&=\|T_{r_m}^{(F_1,\phi_1)}[w_m+\gamma_mA^*(T_{r_m}^{(F_2,\phi_2)}-I)Aw_m]-T_{r_m}^{(F_1,\phi_1)}p\|^2\\
&\le \langle y_m-p,w_m+\gamma_mA^*(T_{r_m}^{(F_2,\phi_2)}-I)Aw_m\rangle\\
&=\frac{1}{2}[\|y_m-p\|^2+\|w_m+\gamma_mA^*(T_{r_m}^{(F_2,\phi_2)}-I)Aw_m-p\|^2-\|y_m-p-(w_m+\gamma_mA^*(T_{r_m}^{(F_2,\phi_2)}-I)Aw_m-p)\|^2]\\
&=\frac{1}{2}[\|y_m-p\|^2+\|w_m-p+\gamma_mA^*(T_{r_m}^{(F_2,\phi_2)}-I)Aw_m\|^2-\|y_m-w_m-\gamma_mA^*(T_{r_m}^{(F_2,\phi_2)}-I)Aw_m\|^2]
\end{aligned}$$

$$= \frac{1}{2}[\|y_m - p\|^2 + \|w_m - p\|^2 + \gamma_m^2\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2 + 2\langle w_m - p, \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle$$
$$- \|y_m - w_m\| - \gamma_m\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2 + 2\langle y_m - w_m, \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle]$$
$$= \frac{1}{2}[\|y_m - p\|^2 + \|w_m - p\|^2 - \|y_m - w_m\| + 2\langle y_m - p, \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle], \quad (3.62)$$

which implies that

$$\|y_m - p\|^2 \leq \|w_m - p\|^2 - \|y_m - w_m\| + 2\langle y_m - p, \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle. \quad (3.63)$$

Furthermore, using (3.63), we get

$$\|s_{m+1} - p\|^2$$
$$\leq \alpha_m[\|s_m - p\|^2 + \delta_m\|s_m - s_{m-1}\|N_4] + (1 - \alpha_m)\|y_m - p\|^2 + 2\alpha_m\langle f(p) - p, s_{m+1} - p\rangle$$
$$\leq \alpha_m[\|s_m - p\|^2 + \delta_m\|s_m - s_{m-1}\|N_4] + (1 - \alpha_m)[\|w_m - p\|^2 - \|y_m - w_m\|$$
$$+ 2\langle y_m - p, \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle] + 2\alpha_m\langle f(p) - p, s_{m+1} - p\rangle$$
$$\leq \alpha_m\|s_m - p\|^2 + \delta_m\|s_m - s_{m-1}\|N_4 + (1 - \alpha_m)\|s_m - p\|^2 + \theta_m\|s_m - s_{m-1}\|N_5 - (1 - \alpha_m)\|y_m - w_m\|$$
$$+ 2\langle y_m - p, \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\rangle + 2\alpha_m\langle f(p) - p, s_{m+1} - p\rangle$$
$$\leq \|s_m - p\|^2 + \delta_m\|s_m - s_{m-1}\|N_4 + \theta_m\|s_m - s_{m-1}\|N_5 - (1 - \alpha_m)\|y_m - w_m\|$$
$$+ 2\gamma_m\|y_m - p\|\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\| + 2\alpha_m\langle f(p) - p, s_{m+1} - p\rangle, \quad (3.64)$$

which implies that

$$\limsup_{k\to\infty}\Big((1 - \alpha_{m_k})\|y_{m_k} - w_{m_k}\|\Big)$$
$$\leq \limsup_{k\to\infty}\Big[\|s_{m_k} - p\|^2 + \alpha_{m_k}\frac{\theta_{m_k}}{\alpha_{m_k}}\|s_{m_k} - s_{m_k-1}\|N_5 + \alpha_{m_k}\frac{\delta_{m_k}}{\alpha_{m_k}}\|s_{m_k} - s_{m_k-1}\|N_4 + 2\alpha_m\langle f(p) - p, s_{m_k+1} - p\rangle$$
$$+ 2\gamma_{m_k}\|y_{m_k} - p\|\|A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\| \; - \|s_{m_k+1} - p\|^2\Big]$$
$$\leq -\liminf_{k\to\infty}[\|s_{m_k+1} - p\|^2 - \|s_{m_k} - p\|^2] \leq 0.$$

We get

$$\lim_{k\to\infty}\|y_{m_k} - w_{m_k}\| = 0. \quad (3.65)$$

Using Algorithm 3.1, we have

$$\lim_{k\to\infty}\|z_{m_k} - s_{m_k}\| \leq \lim_{k\to\infty}\alpha_{m_k}\cdot\frac{\delta_{m_k}}{\alpha_{m_k}}\|s_{m_k} - s_{m_k-1}\| = 0, \quad (3.66)$$

and

$$\lim_{k\to\infty}\|w_{m_k} - s_{m_k}\| \leq \lim_{k\to\infty}\alpha_{m_k}\cdot\frac{\theta_{m_k}}{\alpha_{m_k}}\|s_{m_k} - s_{m_k-1}\| = 0. \quad (3.67)$$

Using (3.54), (3.61), and (3.65)–(3.67), we get

$$\lim_{k\to\infty} \|y_{m_k} - s_{m_k}\| \le \lim_{k\to\infty} \|y_{m_k} - w_{m_k}\| + \lim_{k\to\infty} \|w_{m_k} - s_{m_k}\| = 0, \tag{3.68}$$

$$\lim_{k\to\infty} \|u_{m_k} - s_{m_k}\| \le \lim_{k\to\infty} \|u_{m_k} - y_{m_k}\| + \lim_{k\to\infty} \|y_{m_k} - s_{m_k}\| = 0, \tag{3.69}$$

$$\lim_{k\to\infty} \|v_{m_k} - s_{m_k}\| \le \lim_{k\to\infty} \|v_{m_k} - y_{m_k}\| + \lim_{k\to\infty} \|y_{m_k} - s_{m_k}\| = 0, \tag{3.70}$$

$$\lim_{k\to\infty} \|s_{m_k+1} - v_{m_k}\| \le \lim_{k\to\infty} \alpha_{m_k}\|f(z_{m_k}) - v_{m_k}\| + \lim_{k\to\infty} \beta_{m_k}\|v_{m_k} - v_{m_k}\| + \lim_{k\to\infty} \eta_{m_k}\|S_{m_k} v_{m_k} - v_{m_k}\| = 0, \tag{3.71}$$

$$\lim_{k\to\infty} \|s_{m_k+1} - s_{m_k}\| \le \lim_{k\to\infty} \|s_{m_k+1} - v_{m_k}\| + \lim_{k\to\infty} \|v_{m_k} - s_{m_k}\| = 0. \tag{3.72}$$

Now, since $\{s_{m_k}\}$ is bounded, there exists a subsequence $\{s_{m_{k_j}}\}$ of $\{s_{m_k}\}$ such that $\{s_{m_{k_j}}\}$ converges weakly to $s^* \in H_1$. From (3.69), it follows that $\{u_{m_{k_j}}\}$ converges weakly to s^*. We now establish that $s^* \in EP(F_1, \phi_1)$. Using the definition of $y_{m_k} = T_{r_{m_k}}^{(F_1,\phi_1)}[w_{m_k} + \gamma_{m_k} A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}]$, we have

$$F_1(y_{m_k}, y) + \phi_1(y, y_{m_k}) - \phi_1(y_{m_k}, y_{m_k}) + \frac{1}{r_{m_k}}\langle y - y_{m_k}, y_{m_k} - w_{m_k} + \gamma_m A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\rangle \ge 0 \quad \forall \quad y \in K.$$

Using Assumption 2.1(2) (using monotonocity of F_1), we have

$$\phi_1(y, y_{m_k}) - \phi_1(y_{m_k}, y_{m_k}) + \frac{1}{r_{m_k}}\langle y - y_{m_k}, y_{m_k} - w_{m_k} + \gamma_{m_k} A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}\rangle \ge F_1(y, y_{m_k}) \quad \forall \quad y \in K,$$

$$F_1(y_{m_k}, y) \le \phi_1(y, y_{m_k}) - \phi_1(y_{m_k}, y_{m_k}) + \left\langle y - y_{m_k}, \frac{y_{m_k} - w_{m_k} - \gamma_m A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}}{r_{m_k}}\right\rangle \forall\ y \in K. \tag{3.73}$$

For any $0 < t \le 1$ and $y \in K$, suppose that $y_t = ty + (1-t)s^*$. It is easy to see that $y_t \in K$, since $y \in K$ and $s^* \in K$; thus, $y_t \in K$. By (3.73), we have

$$0 \le F_1(y_{m_k}, y_t) - \phi_1(y_t, y_{m_k}) + \phi_1(y_{m_k}, y_{m_k}) - \left\langle y_t - y_{m_k}, \frac{y_{m_k} - w_{m_k} - \gamma_{m_k} A^*(T_{r_{m_k}}^{(F_2,\phi_2)} - I)Aw_{m_k}}{r_{m_k}}\right\rangle \quad \forall \quad y \in K. \tag{3.74}$$

Using the fact that $\liminf_{k\to\infty} r_{m_k} > 0$, (3.65) and (3.58), we have

$$\phi_1(y_t, s^*) - \phi_1(s^*, s^*) \le F_1(s^*, y_t).$$

For $t > 0$,

$$\begin{aligned} 0 = F(y_t, y_t) &= tF(y_t, y) + (1-t)F(y_t, s^*) \\ &\ge tF(y_t, y) + (1-t)[\phi_1(y_t, s^*) - \phi_1(s^*, s^*)] \end{aligned}$$

$$\geq tF(y_t, y) + (1-t)t[\phi_1(y, x^*) - \phi_1(s^*, s^*)]$$
$$\geq tF(y_t, y) + (1-t)[\phi_1(y, s^*) - \phi_1(s^*, s^*)]. \quad (3.75)$$

Taking $t \to \infty$, we have

$$\phi_1(y, s^*) - \phi_1(s^*, s^*) + F_1(s^*, y) \geq 0 \quad \forall\, y \in K.$$

Thus, we have $s^* \in EP(F_1, \phi_1)$. Additionally, since A is a bounded linear operator $Aw_{m_k} \rightharpoonup As^*$, suppose that

$$d_{m_k} = Aw_{m_k} - T_{r_{m_k}}^{(F_2,\phi_2)} Aw_{m_k}.$$

It follows from (3.59) that $\liminf_{k\to\infty} d_{m_k} = 0$ and $Aw_{m_k} - d_{m_k} = T_{r_{m_k}}^{(F_2,\phi_2)} Aw_{m_k}$. Using Lemma 2.4, we have

$$F_2(Aw_{m_k} - d_{m_k}, y) + \phi_2(y, y_{m_k}) - \phi_1(y_{m_k}, y_{m_k}) + \frac{1}{r_{m_k}}\langle y - (Aw_{m_k} - d_{m_k}), (Aw_{m_k} - d_{m_k}, Aw_{m_k})\rangle \geq 0, \quad \forall y \in Q.$$

Since F_2 is upper semi-continuous in the first argument, taking limit superior of the above inequality as $k \to \infty$ and using the condition, we get

$$F_2(As^*, y) + \phi_2(y, s^*) - \phi_1(s^*, s^*) \geq 0. \quad (3.76)$$

Hence, we obtain that $As^* \in EP(F_2, \phi_2)$. Furthermore, since p is a solution of Ω, we get

$$\limsup_{k\to\infty}\langle f(p) - p, s_{m_k} - p\rangle = \limsup_{j\to\infty}\langle f(p) - p, s_{m_{k_j}} - p\rangle = \langle f(p) - p . s^* - p\rangle \leq 0, \quad (3.77)$$

Using (3.72) and (3.77), we have

$$\begin{aligned}
&\limsup_{k\to\infty}\langle f(p) - p, s_{m_k+1} - p\rangle \\
&= \limsup_{k\to\infty}\langle f(p) - p, s_{m_k+1} - s_{m_k}\rangle + \limsup_{k\to\infty}\langle f(p) - p, s_{m_k} - p\rangle \\
&= \langle f(p) - p, s^* - p\rangle \leq 0, \quad (3.78)
\end{aligned}$$

which implies that

$$\limsup_{k\to\infty}\langle f(p) - p, s_{m_k+1} - p\rangle \leq 0. \quad (3.79)$$

Using our Assumption 3.1 and the above inequality, we have that

$$\limsup_{k\to\infty}\Psi_{m_k} = \limsup_{k\to\infty}\left[\frac{\delta_m\|s_m - s_{m-1}\|N_4}{\alpha_m(1-k)} + \frac{\theta_m\|s_m - s_{m-1}\|N_5}{\alpha_m(1-k)} + \frac{2}{(1-k)}\langle f(p) - p, s_{m+1} - p\rangle\right] \leq 0.$$

Thus, by Lemma 2.5, we have $\lim_{m\to\infty}\|s_m - p\| = 0$. Thus, $\{s_m\}$ converges strongly to $p \in \Omega$. □

From our main results, the following corollaries follow immediately:

Corollary 3.1. *Suppose Assumption 3.1 holds, where ϕ_1 and ϕ_2 are identity mapping. The sequence $\{s_n\}$ is the sequence defined by Algorithm 3.2.*

Algorithm 3.2. Initialization Step. *Take $s_0, s_1 \in H_1$, given the iterates s_{m-1} and s_m for all $m \in \mathbb{N}$.*

$$\bar{\delta}_m = \begin{cases} \min\{\delta, \frac{\epsilon_m}{\|s_m - s_{m-1}\|}\}, & \text{if } s_m \neq s_{m-1}, \\ \delta, & \text{otherwise}. \end{cases} \tag{3.80}$$

$$\bar{\theta}_m = \begin{cases} \min\{\theta, \frac{\epsilon_m}{\|s_m - s_{m-1}\|}\}, & \text{if } s_m \neq s_{m-1}, \\ \theta, & \text{otherwise}. \end{cases} \tag{3.81}$$

Step 1. *Compute*

$$z_m = \alpha_m s_m + (1 - \alpha_m)(s_m + \delta_m(s_m - s_{m-1})), \tag{3.82}$$

$$w_m = \psi_m s_m + (1 - \psi_m)(s_m + \theta_m(s_m - s_{m-1})), \tag{3.83}$$

$$y_m = T^{F_1}_{r_m}(w_m + \gamma_m A^*(T^{F_2}_{r_m} - I)Aw_m),$$

where

$$\gamma_n \in \left(\epsilon, \frac{\|(T^{F_2}_{r_m} - I)Aw_m\|^2}{\|A^*(T^{F_2}_{r_m} - I)Aw_m\|^2} - \epsilon\right). \tag{3.84}$$

Step 2. *Compute*

$$u_m = P_{K_m}(y_m - \lambda_m By_m),$$
$$v_m = (1 - \tau)y_m + \tau u_m + \tau\lambda_m(By_m - Bu_m),$$

where

$$K_m = \{s \in \mathcal{H}_1 \mid q(y_m) + \langle q'(y_m), s - y_m\rangle \leq 0\}$$

and

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu\|y_m - u_m\|}{\sqrt{\|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2}}, \lambda_m\right\}, & \text{if } \|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2 \neq 0, \\ \lambda_m, & \text{otherwise}. \end{cases} \tag{3.85}$$

Step 3. *Compute*

$$s_{m+1} = \alpha_m f(z_m) + \beta_m v_m + \eta_m S_m v_m, \tag{3.86}$$

where $S_m := \frac{1}{N}\sum_{i=0}^{N-1}((1 - \omega_m)I + \omega_m S_i)$.

Then, $\{s_m\}$ converges strongly to $p \in \Omega$.

Corollary 3.2. *Suppose Assumption 3.1 holds and $\{s_n\}$ is the sequence defined by Algorithm 3.3.*

Algorithm 3.3.

$$\bar{\delta}_m = \begin{cases} \min\{\delta, \frac{\epsilon_m}{\|s_m - s_{m-1}\|}\}, & \text{if } s_m \neq s_{m-1}, \\ \\ \delta, & \text{otherwise}. \end{cases} \tag{3.87}$$

$$\bar{\theta}_m = \begin{cases} \min\{\theta, \frac{\epsilon_m}{\|s_m - s_{m-1}\|}\}, & \text{if } s_m \neq s_{m-1}, \\ \\ \theta, & \text{otherwise}. \end{cases} \tag{3.88}$$

Step 1. *Compute*

$$z_m = s_m + \delta_m(s_m - s_{m-1}), \tag{3.89}$$
$$w_m = s_m + \theta_m(s_m - s_{m-1}), \tag{3.90}$$

$$y_m = T_{r_m}^{(F_1,\phi_1)}(w_m + \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m),$$

where

$$\gamma_m \in \left(\epsilon, \frac{\|(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2}{\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2} - \epsilon\right). \tag{3.91}$$

Step 2. *Compute*

$$u_m = P_{K_m}(y_m - \lambda_m By_m),$$
$$v_m = (1-\tau)y_m + \tau u_m + \tau\lambda_m(By_m - Bu_m),$$

where

$$P_m = \{s \in \mathcal{H}_1 \mid q(y_m) + \langle q'(y_m), s - y_m\rangle \leq 0\}$$

and

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu\|y_m - u_m\|}{\sqrt{\|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2}}, \lambda_m\right\}, & \text{if } \|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2 \neq 0, \\ \lambda_n, & \text{otherwise}. \end{cases} \tag{3.92}$$

Step 3. *Compute*

$$s_{m+1} = \alpha_m f(z_m) + \beta_m v_m + \eta_m S_m v_m, \tag{3.93}$$

where $S_m := \frac{1}{N}\sum_{i=0}^{N-1}((1-\omega_m)I + \omega_m S_i)$.

Then, $\{s_m\}$ *converges strongly to* $p \in \left(VI(K,B) \cap_{i=1}^{N} F(S_i)\right)$, *where* $\Gamma = \{p \in K : p \in EP(F_1)$ *and* $Ap \in EP(F_2)\}$.

Corollary 3.3. *Suppose Assumption 3.1 holds and $\{s_m\}$ is the sequence defined by Algorithm 3.4.*

Algorithm 3.4. Initialization Step: *Take $s_0, s_1 \in H_1$, given the iterates s_{m-1} and s_m for all $m \in \mathbb{N}$.*

$$\bar{\theta}_m = \begin{cases} \min\{\theta, \frac{\epsilon_m}{\|s_m - s_{m-1}\|}\}, & \text{if } s_m \neq s_{m-1}, \\ \theta, & \text{otherwise}. \end{cases} \tag{3.94}$$

Step 1. *Compute*

$$w_m = s_m + \theta_m(s_m - s_{m-1}), \tag{3.95}$$

$$y_m = T_{r_m}^{(F_1,\phi_1)}(w_m + \gamma_m A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m),$$

where

$$\gamma_m \in \left(\epsilon, \frac{\|(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2}{\|A^*(T_{r_m}^{(F_2,\phi_2)} - I)Aw_m\|^2} - \epsilon\right). \tag{3.96}$$

Step 2. *Compute*

$$\begin{aligned} u_m &= P_{K_m}(y_m - \lambda_m B y_m), \\ v_m &= (1-\tau)y_m + \tau u_m + \tau\lambda_m(By_m - Bu_m), \end{aligned}$$

where

$$K_m = \{s \in \mathcal{H}_1 \mid q(y_m) + \langle q'(y_m), s - y_m\rangle \leq 0\}$$

and

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu\|y_m - u_m\|}{\sqrt{\|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2}}, \lambda_m\right\}, & \text{if } \|By_m - Bu_m\|^2 + \|q'(y_m) - q'(u_m)\|^2 \neq 0, \\ \lambda_m, & \text{otherwise}. \end{cases} \tag{3.97}$$

Step 3. *Compute*

$$s_{m+1} = \alpha_m f(s_m) + \beta_m v_m + \eta_m S v_m, \tag{3.98}$$

where S is a nonexpansive mapping.

Then, $\{s_m\}$ converges strongly to $p \in \Big(VI(K,B) \cap F(S)\Big) \cap \Gamma$, where $\Gamma = \{p \in K : p \in EP(F_1)$ and $Ap \in EP(F_2)\}$.

4. Numerical example

In this section we provide a numerical example that will be used to test the computational advantage of our proposed method with some well-known methods in the existing literature.

Example 4.1. Let $H_1 = (\mathbb{R}^k, \|x\|_2)$, $H_2 = (\mathbb{R}^k, \|s\|_2)$, $K = \{s \in \mathbb{R}^n : -1 \leq s_i \leq 5, i = 1, \cdots, n\}$ and $Q = \{s \in \mathbb{R}^k : -2 \leq s_i \leq 5, i = 1, \cdots, n\}$. Let the bifunctions $F_1, \phi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $F_1(s, y) = \phi_1(s, y) = \langle Ps + Gy + p, y - s\rangle$, where p is a vector in $\mathbb{R}^m$, P and G are two order m matrices with G being a symmetric positive semi-definite and $G-P$ is negative semi-definite. Let the bifunctions $F_2, \phi_2 : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ be defined by $F_2(s, y) = \phi_2(s, y) = h(y) - h(s)$, where $h(s) = \frac{1}{2}s^T Ns + q^T s$, such that $q \in \mathbb{R}^k$ and N being a symmetric positive definite matrix of order k. Next, we consider the operator $A : \mathbb{R}^k \to \mathbb{R}^k$ is defined by a matrix of size $k \times m$. Note that the solution to the GSEP in this case is $s^* = 0$, that is, $\Gamma = \{0\}$. For more details, see, [24, 36] and the references in them.

Now, suppose $B : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $V(s) = \Psi s + e$, where $e \in \mathbb{R}^n$ and $\Psi = JJ^T + D + E$, where E is a $n \times n$ diagonal matrix whose diagonal terms are nonnegative (hence Ψ is positive symmetric definite), D is a $n\times n$ skew-symmetric and J is a $n\times n$ matrix. Under the feasible set K, it is not hard to see that the mapping V is monotone and L-Lipschitz continuous with $L = \|\Psi\|$. For $e = 0$, the solution set $VI(K, B) = \{0\}$. On the other hand, let $S_i u = \frac{u}{2i}$, then it is not hard to see that S_i is a finite family of demimetric mappings. Clearly, the common fixed point of S_i is 0, i.e., $F(S) = \bigcap_{i=1}^N S_i = \{0\}$. Hence, $\Omega = \Big(VI(C, B) \cap_{i=1}^N F(S_i)\Big) \cap \Gamma = \{0\}$.

Since the feasible set K is a box in $\mathbb{R}^n$, the projection of a point $s \in \mathbb{R}^m$ onto K and can be evaluated as follows:

$$[P_K(s)]_i = \begin{cases} s_i, & \text{if } s_i \in [-1, 5], \\ -1, & \text{if } s_i < -1, \\ 5, & \text{if } s_i > 5. \end{cases}$$

Also, since the feasible set Q is a box in $\mathbb{R}^k$, the projection of a point $s \in \mathbb{R}^k$ onto Q and can be evaluated as follows:

$$[P_Q(s)]_i = \begin{cases} s_i, & \text{if } s_i \in [-2, 5], \\ -2, & \text{if } s_i < -2, \\ 5, & \text{if } s_i > 5. \end{cases}$$

In this numerical experiment, we compare the computational advantage of Algorithm 3.1 (shortly, Alg. 3.2), Algorithm 3.3 (shortly, Alg. 3.10) and Algorithm 3.4 (shortly, Alg. 3.13) with Algorithm 3 of Farid et al. [10] (shortly, Alg. 1.10). For Alg. 3.2, Alg. 3.11 and AK Alg. 3.13, we choose the following parameters: $f(s) = \frac{s}{3}$, $\epsilon_m = \frac{1}{(2m+1)^3}$, $\alpha_m = \frac{1}{(2m+1)}$, $\beta_m = \eta_m = \frac{1}{2}(1 - \alpha_m)$, $\delta = 0.8$, $\theta = 0.98$, $\mu = 0.6$, $\tau = 0.4$, $\lambda_1 = 2.5$, $\rho = 0.5$ and $\omega_m = 0.75$. FRD Alg. 3.1, we choose $\alpha_m = \beta_m = r_m = 0.5$, $D(s) = \frac{s}{4}$, $\gamma = \frac{1}{4}$, $f(s) = \frac{s}{3}$ and $K_m(s) = \frac{s}{2}$. To obtain the nonempty solution set for the problem and to reach all steps of the algorithms, we take two vectors r and t to equal zero vectors in $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. We consider the following cases for parameters n and k: Case 1: $(n = 30, k = 10)$, Case 2: $(n = 60, k = 20)$, Case 3: $(n = 120, k = 40)$ and Case 4: $(n = 240, k = 80)$. Now, for all algorithms, we consider the starting point $s_0 = (1, 1, \cdots, 1) \in \mathbb{R}^n$ and stopping criterion $E_m = \|s_{m+1} - s_m\| < 10^{-5}$.

Remark 4.1. *It is easy to see from the numerical experiment as presented in Table 1 and Figure 1 that our proposed iterative method performs better when compared with others.*

Table 1. Results of the numerical simulations for different dimensions.

Numerical Results for various cases in Example 4.1								
	Alg. 3.2		Alg. 3.11		Alg. 3.13		Alg. 1.10	
Case	Iter	CPU time (sec.)	Iter	CPU time (sec.)	Iter	CPU time (sec.)	Iter	CPU time (sec.)
Case 1	7	0.0012	12	0.0019	14	0.0035	17	0.0060
Case 2	9	0.0014	12	0.0021	15	0.0068	18	0.0198
Case 3	8	0.0014	12	0.0027	1	0.0075	19	0.0207
Case 4	8	0.0015	13	0.0038	14	0.0108	24	0.0297

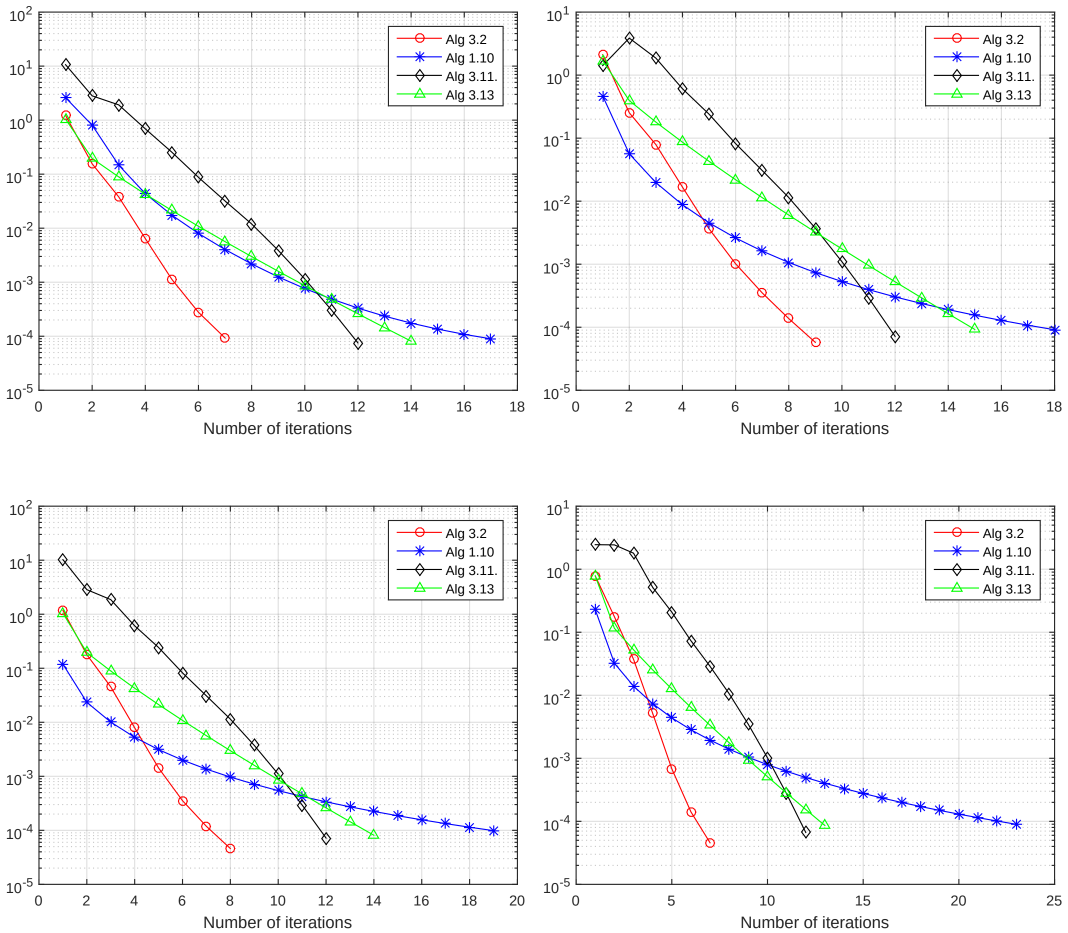

Figure 1. Example 4.1, Top Left: Case 1; Top Right: Case 2; Bottom Left: Case 3; Bottom Right: Case 4. Note that the y-axis is the error.

5. Conclusions

In this work, we have introduced an efficient iterative method for solving GSEPs, FPPs and VIPs. The new method is embedded with double inertial extrapolations steps and the viscosity technique, which speeds up the convergence rate of the proposed method. We used some mild conditions in obtaining our convergence results. The class of mappings studied in this work is more general than those studied in several articles in the literature. We used a nontrivial numerical example to show that our method outperforms some well-known iterative methods with single or no inertial terms.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this work through the projection number (PSAU/2023/01/8980).

Conflict of interest

The authors declare no conflict of interests.

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