Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 10363–10385.
DOI:10.3934/math.2024507
Received: 28 January 2024
Revised: 23 February 2024
Accepted: 27 February 2024
Published: 15 March 2024

Research article

Finite soft-open sets: characterizations, operators and continuity

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Abstract: In this paper, we present a novel family of soft sets named "finite soft-open sets". The purpose of investigating this kind of soft sets is to offer a new tool to structure topological concepts that are stronger than their existing counterparts produced by soft-open sets and their well-known extensions, as well as to provide an environment that preserves some topological characteristics that have been lost in the structures generated by celebrated extensions of soft-open sets, such as the distributive property of a soft union and intersection for soft closure and interior operators, respectively. We delve into a study of the properties of this family and explore its connections with other known generalizations of soft-open sets. We demonstrate that this family strictly lies between the families of soft-clopen and soft-open sets and derive under which conditions they are equivalent. One of the unique features of this family that we introduce is that it constitutes an infra soft topology and fails to be a supra soft topology. Then, we make use of this family to exhibit some operators in soft settings, i.e., soft fo-interior, fo-closure, fo-boundary, and fo-derived. In addition, we formulate three types of soft continuity and look at their main properties and how they behave under decomposition theorems. Transition of these types between realms of soft topologies and classical topologies is examined with the help of counterexamples. On this point, we bring to light the role of extended soft topologies to validate the properties of soft topologies by exploring them for classical topologies and vice-versa.

Keywords: finite soft-open set; extended soft topology; soft fo-interior operator; fo-closure operator; soft continuity
Mathematics Subject Classification: 03E72, 54A40, 54C99

1. Introduction

With the development of human activities, uncertainty and suspicion have become ubiquitous in real-life issues, ranging from daily life decisions to social affairs, national policies, and other aspects. To transact with uncertainty, Molodtsov [55] came up with a fresh paradigm named soft sets in 1999. The advantage of this paradigm is that it is free from the inherent restrictions of the foregoing approaches since it does not require previous procedures such as the use of membership functions in the theory of fuzzy sets and equivalence relations in the theory of rough sets. Molodtsov [55] succeeded in presenting some applications of soft sets in numerous disciplines. Then, many authors applied soft sets to solve the imperfect knowledge, vagueness and inconsistency existing in daily practical problems in various areas, including information theory [6, 18], economics [19, 45], graph theory [46], medical science [40, 50] and differential equations [24]. The main principles and concepts of soft set theory have been detailed and discussed by several researchers [7, 44, 52]. Various authors have proposed new definitions of some concepts to get rid of the drawbacks of their previous ones that limit their applications and often contradict their ontology; for more details, see [10, 24].

In 2011, the structure of soft topologies was introduced by Shabir and Naz [60] with the same postulations of classical topology. They provided the essential ideas of soft topologies, such as soft operators of interior and closure, soft relative topologies, and soft axioms of separation. Further investigation into soft separation axioms was conducted by Min [54] who demonstrated that a soft T_3-space is soft T_2. Later on, a lot of articles described separation axioms and topological operators in the environment of soft topologies; recent examples are [1, 3, 4, 21, 38, 43, 48]. The concepts of soft compact and Lindelöf spaces were defined by Aygünoğlu and Aygün [37]. Generalizations of these spaces were discussed by Hida [47] in terms of belonging relations and by other authors who focused on generalizations of soft-open sets [11, 18, 33, 34]. Also, the notions of soft connectedness, soft hyperconnectedness, soft extremally disconnectedness, maximal soft connectedness, and other types were introduced and probed by several topological researchers [36, 58, 62]. The definitions of basis, functions, Menger spaces, expandable spaces, continuity and homeomorphism were also created for soft topology by the authors of [2, 5, 27, 28, 49, 51, 59, 64]. The need to produce several forms of soft topological concepts results in the introduction of many generalizations of soft-open sets by following similar techniques as those applied in classical topologies. With the introduction of these types of soft subsets, many of the key characteristics of soft-open subsets have been extended to a broader family in a soft topological space. In 2013, soft semi-open sets and their characterizations were studied by Chen [39]. Then, the classes of α-open, somewhere dense, soft parametric somewhat-open, Q-sets, and minimal sets were implemented in soft settings by the authors of [8, 9, 12, 30]. Some of these generalizations were applied to handle practical problems as pointed out in [20]. Moreover, Al-shami, with his coauthors, suggested a new approach to create generalizations of soft-open sets that are inspired by the classical topologies associated with an original soft topology, namely, weakly soft β-open [22], weakly soft pre-open [25], weakly soft b-open [26], weakly soft α-open [31], weakly soft semi-open [32], and weakly soft somewhat open sets [35]. These generalizations make up a weaker form of their analogous existing counterparts in soft topologies and promote the production of novel notions for soft topologies with unique properties.

To link classical topologies with soft topologies, it has been proven that the following formula generates classical topologies from a soft topology:

$$\tau_{a} = \{\mathcal{S}(a) : (\mathcal{S}, \mho) \in \tau\}. \tag{1.1}$$

It is conspicuous that a soft topology is considered to be a classical topology when the parameters set is a singleton. As expected, some properties and relationships existing in classical topologies are not valid in the realms of soft topologies; for example,the systematic relation T_4 implies T_3, and the compact subsets of Hausdorff space are closed. To shed light on these divergences, the manuscripts [13, 61] were written. As a matter of fact, these divergences prompt topologists to look at how topological principles behave between classical and soft topological structures, which provides a tool to discover the properties of one space by using another well-known one. In this regard, Al-shami and Kočinac [29] concluded the sufficient terms that guarantee the navigation of most of the topological properties between two diverse structures (crisp and soft). They first showed the correspondence between special types of soft topologies, namely extended and enriched soft topologies, and then they proved that this type of soft topology leads to the equality between soft interior and closure operators of soft set and their classical counterparts that are computed for each component of this soft set. That is, if $(\mathcal{S}, \mho)$ is a soft subset of an extended soft topological space $(Z, \tau, \mho)$, then $(\boldsymbol{i}(S), \mho) = \boldsymbol{I}(\mathcal{S}, \mho)$ and $(\boldsymbol{c}(S), \mho) = \boldsymbol{C}(\mathcal{S}, \mho)$, where $\boldsymbol{i}$ and $\boldsymbol{c}$ respectively denote the interior and closure operators in classical topology, and $\boldsymbol{I}$ and $\boldsymbol{C}$ respectively denote the soft interior and closure operators in soft topology.

This article is laid out as follows. The basic concepts that will be prerequisite to understanding this content are recalled in the next section. In Section 3, we introduce the concept of "finite soft-open sets" as soft-open sets with the finite region of a soft boundary. Then, in Section 4, we define the soft fo-interior, fo-closure, fo-boundary, and fo-derived operators. We derive some properties of these operators and reveal the relationships between them. Section 5 introduces three types of soft continuity and investigates their transition from a soft topology to crisp topologies that are generated by using (1.1). To illustrate this matter, we present some counterexamples. Finally, we analyze the obtained results and propound some directions for future work in Section 6.

2. Fundamentals of soft sets and soft topologies

In this section, we mention some essential concepts and outcomes that will be used in the sequel.

2.1. Soft set theory

Definition 2.1. *([55]) A pair $(\mathcal{S}, \mho)$ is stated as a soft set (in short, s-set) over $Z \neq \emptyset$ (known as the universal set) with $\mho \neq \emptyset$ (known as a set of parameters) provided that $\mathcal{S} : \mho \to 2^{Z}$ is a function, where 2^{Z} is the power set of Z.*

The mathematical representation of an s-set is given by

$$(\mathcal{S}, \mho) = \{(a, \mathcal{S}(a)) : a \in \mho \text{ and } \mathcal{S}(a) \in 2^{Z}\}.$$

The components of an s-set $(\mathcal{S}, \mho)$ are the images of each parameter under a function $\mathcal{S}$.

Through this content, the collection of all s-sets initiated with respect to Z and $\mho$ is denoted by $\Sigma(Z, \mho)$.

Definition 2.2. *([10]) The complement of* $(\mathcal{S}, \mho)$*, indicated by* $(\mathcal{S}, \mho)^c$ *or* $(\mathcal{S}^c, \mho)$*, is given by*

$$\mathcal{S}^c(\mathrm{a}) = \mathrm{Z} - \mathcal{S}(\mathrm{a}) \textit{ for each } \mathrm{a} \in \mho.$$

Definition 2.3. *([42, 53, 56]) An absolute s-set* $(\mathcal{S}, \mho)$ *is defined by* $\mathcal{S}(\mathrm{a}) = \mathrm{Z}$ *for each* $\mathrm{a} \in \mho$*, whereas its complement is named a null s-set. They are denoted by* $\widetilde{\mathrm{Z}}$ *and* Φ*, respectively. If* $(\mathcal{S}, \mho)$ *is given by* $\mathcal{S}(\mathrm{a}) = \mathrm{z} \in \mathrm{Z}$ *and, for each* $\mathrm{b} \in \mho - \{\mathrm{a}\}$*, we have* $\mathcal{S}(\mathrm{b}) = \emptyset$*, then we call* $(\mathcal{S}, \mho)$ *a soft point and it is denoted by* z_{a}*. An s-set* $(\mathcal{S}, \mho)$ *is referred to as infinite (resp., uncountable) if* $\mathcal{S}(\mathrm{a})$ *is infinite (resp., uncountable) for some* $\mathrm{a} \in \mho$*; otherwise it is called finite (resp., countable). An s-set with the property* $\mathcal{S}(\mathrm{a}) = \emptyset$ *or* $\mathcal{S}(\mathrm{a}) = \mathrm{Z}$ *for each* $\mathrm{a} \in \mho$ *is known as a pseudo-constant s-set. We refer to* $(\mathcal{S}, \mho)$ *as a stable s-set if* $\mathcal{S}(\mathrm{a}) = A \subseteq \mathrm{Z}$ *for each* $\mathrm{a} \in \mho$*. Note that the absolute and null s-sets are stable and pseudo-constant s-sets.*

Definition 2.4. *Let* $(\mathcal{S}, \mho)$ *and* $(\mathcal{K}, \mho)$ *be s-sets. Then we write*

(i) $(\mathcal{S}, \mho) \sqsubseteq (\mathcal{K}, \mho)$ if, for every $\mathrm{a} \in \mho$, we have that $\mathcal{S}(\mathrm{a}) \subseteq \mathcal{K}(\mathrm{a})$ [44].

(ii) $(\mathcal{S}, \mho) \sqcup (\mathcal{K}, \mho) = (\mathcal{U}, \mho)$ if, for every $\mathrm{a} \in \mho$, we have that $\mathcal{U}(\mathrm{a}) = \mathcal{S}(\mathrm{a}) \bigcup \mathcal{K}(\mathrm{a})$ [53].

(iii) $(\mathcal{S}, \mho) \sqcap (\mathcal{K}, \mho) = (\mathcal{U}, \mho)$ if, for every $\mathrm{a} \in \mho$, we have that $\mathcal{U}(\mathrm{a}) = \mathcal{S}(\mathrm{a}) \bigcap \mathcal{K}(\mathrm{a})$ [10].

(iv) $(\mathcal{S}, \mho) \triangle (\mathcal{K}, \mho) = (\mathcal{U}, \mho)$ if, for every $\mathrm{a} \in \mho$, we have that $\mathcal{U}(\mathrm{a}) = \mathcal{S}(\mathrm{a}) \backslash \mathcal{K}(\mathrm{a})$ [10].

(v) $(\mathcal{S}, \mho) \times (\mathcal{K}, \mho) = (\mathcal{U}, \mho \times \mho)$ if, for every $(\mathrm{a}, \mathrm{b}) \in \mho \times \mho$, we have that $\mathcal{U}(\mathrm{a}, \mathrm{b}) = \mathcal{S}(\mathrm{a}) \times \mathcal{K}(\mathrm{b})$.

Definition 2.5. *([42, 60, 63]) Let* $\mathrm{z} \in \mathrm{Z}$ *and* $(\mathcal{S}, \mho)$ *be an s-set. Then, we say that*

(i) $\mathrm{z}_{\mathrm{a}} \in (\mathcal{S}, \mho)$ whenever $\mathrm{z} \in \mathcal{S}(\mathrm{a})$.

(ii) $\mathrm{z} \in (\mathcal{S}, \mho)$ (resp., $\mathrm{z} \Subset (\mathcal{S}, \mho)$) if $\mathrm{z} \in \mathcal{S}(\mathrm{a})$ for every (resp., some) $\mathrm{a} \in \mho$.

We provide the negation of the above-mentioned relations as follows.

(i) $\mathrm{z}_{\mathrm{a}} \notin (\mathcal{S}, \mho)$ whenever $\mathrm{z} \notin \mathcal{S}(\mathrm{a})$.

(ii) $\mathrm{z} \notin (\mathcal{S}, \mho)$ (resp., $\mathrm{z} \not\Subset (\mathcal{S}, \mho)$) if $\mathrm{z} \notin \mathcal{S}(\mathrm{a})$ for some (resp., every) $\mathrm{a} \in \mho$.

Definition 2.6. *([15]) A soft function* $\mathrm{E}_{\xi} : \Sigma(\mathrm{Z}, \mho) \to \Sigma(\mathrm{X}, Q)$*, where* $\mathrm{E} : \mathrm{Z} \to \mathrm{X}$ *and* $\xi : \mho \to Q$ *are crisp functions, is a relation that associates each* $\mathrm{z}_{\mathrm{a}} \in \Sigma(\mathrm{Z}, \mho)$ *with one and only one* $\mathtt{x}_{\mathrm{q}} \in \Sigma(\mathrm{X}, Q)$ *such that*

$$\mathrm{E}_{\xi}(\mathrm{z}_{\mathrm{a}}) = \mathrm{E}(\mathrm{z})_{\xi(\mathrm{a})} \text{ for every } \mathrm{z} \in \mathrm{Z} \text{ and } \mathrm{a} \in \mho.$$

In addition, $\mathrm{E}_{\xi}^{-1}(\mathtt{x}_q) = \bigsqcup_{\substack{\mathrm{z} \in \mathrm{E}^{-1}(\mathtt{x}) \\ \mathrm{a} \in \xi^{-1}(q)}} \mathrm{z}_{\mathrm{a}}$ for every $\mathtt{x} \in \mathrm{X}$ and $q \in \mathrm{Q}$.

2.2. Soft topology

Definition 2.7. *([60]) A subfamily τ of $\Sigma(\mathrm{Z}, \mho)$ forms a soft topology over Z with $\mho$ provided that the absolute and null s-sets are elements of it, and that it is closed under finite soft intersections and arbitrary soft unions.*

In this case, the triplet $(\mathrm{Z}, \tau, \mho)$ is called a soft topological space (briefly, ST-space). Every element of τ is called soft-open (in short, s-open) and its complement is called soft-closed (in short, s-closed).

We term an ST-space $(\mathrm{Z}, \tau, \mho)$ soft locally indiscrete if $(\mathcal{S}, \mho)$ is s-closed whenever it is s-open.

Definition 2.8. *([60]) For an s-subset $(\mathcal{S}, \mho)$ of an ST-space $(\mathrm{Z}, \tau, \mho)$, the soft union of all s-open subsets of $(\mathcal{S}, \mho)$ is referred to as a soft-interior of $(\mathcal{S}, \mho)$, and the soft intersection of all s-closed supersets of $(\mathcal{S}, \mho)$ is referred to as a soft-closure of $(\mathcal{S}, \mho)$. These operators are respectively denoted by $\boldsymbol{I}(\mathcal{S}, \mho)$ and $\boldsymbol{C}(\mathcal{S}, \mho)$. The soft boundary of an s-subset $(\mathcal{S}, \mho)$, denoted by $\boldsymbol{B}(\mathcal{S}, \mho)$, is given by $\boldsymbol{B}(\mathcal{S}, \mho) = \boldsymbol{C}(\mathcal{S}, \mho)\Delta\boldsymbol{I}(\mathcal{S}, \mho)$.*

Definition 2.9. *([36, 58, 62])*

(i) If the only soft-clopen subsets of an ST-space $(\mathrm{Z}, \tau, \mho)$ are null and absolute s-sets, then $(\mathrm{Z}, \tau, \mho)$ is referred to as soft connected.

(ii) If the soft-closure of every s-open subset of an ST-space $(\mathrm{Z}, \tau, \mho)$ is s-open, then $(\mathrm{Z}, \tau, \mho)$ is referred to as soft extremally disconnected.

(iii) If every s-dense subset of an ST-space $(\mathrm{Z}, \tau, \mho)$ is s-open, then $(\mathrm{Z}, \tau, \mho)$ is referred to as soft submaximal.

Definition 2.10. *([8, 12, 39]) An s-subset $(\mathcal{S}, \mho)$ of $(\mathrm{Z}, \tau, \mho)$ is called soft α-open, soft pre-open, soft semi-open, soft b-open, soft β-open, or soft somewhere dense subset if the following conditions are respectively satisfied:*

$$(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{I}(\boldsymbol{C}(\boldsymbol{I}(\mathcal{S}, \mho))),$$

$$(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{I}(\boldsymbol{C}(\mathcal{S}, \mho)),$$

$$(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{C}(\boldsymbol{I}(\mathcal{S}, \mho)),$$

$$(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{I}(\boldsymbol{C}(\mathcal{S}, \mho)) \sqcup \boldsymbol{C}(\boldsymbol{I}(\mathcal{S}, \mho)),$$

$$(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{C}(\boldsymbol{I}(\boldsymbol{C}(\mathcal{S}, \mho))), \text{ and}$$

$$(\mathcal{S}, \mho) = \Phi \text{ or } \boldsymbol{I}(\boldsymbol{C}(\mathcal{S}, \mho)) \neq \Phi.$$

The complements of the above-mentioned s-subsets are respectively termed soft soft α-closed, soft pre-closed, soft semi-closed, soft b-closed, soft β-closed, and soft cs-dense sets.

The next formula explains how to inherit crisp topologies from a soft topology.

Proposition 2.11. *([60]) Let $(\mathrm{Z}, \tau, \mho)$ be an ST-space. Then, for each $\mathrm{a} \in \mho$, the class*

$$\tau_a = \{\mathcal{S}(a) : (\mathcal{S}, \mho) \in \tau\}$$

represents a crisp topology on Z. Since this class is produced for each parameter, we shall name a parametric topology (in short, p-topology).

Definition 2.12. *([29]) Let $(\mathcal{S}, \mho)$ be an s-subset of an ST-space $(\mathrm{Z}, \tau, \mho)$. Then $(\boldsymbol{I}(\mathcal{S}), \mho)$ and $(\boldsymbol{C}(\mathcal{S}), \mho)$ are given by $\boldsymbol{I}(\mathcal{S})(\mathrm{a}) = \boldsymbol{I}(\mathcal{S}(\mathrm{a}))$ and $\boldsymbol{C}(\mathcal{S})(\mathrm{a}) = \boldsymbol{C}(\mathcal{S}(\mathrm{a}))$, respectively, where $\boldsymbol{I}(\mathcal{S}(\mathrm{a}))$ and $\boldsymbol{C}(\mathcal{S}(\mathrm{a}))$ are the interior and closure operators of $\mathcal{S}(\mathrm{a})$ in a p-topological space $(\mathrm{Z}, \tau_{\mathrm{a}})$, respectively.*

Proposition 2.13. *([60]) Let $(\mathcal{S}, \mho)$ be an s-subset of an ST-space $(\mathrm{Z}, \tau, \mho)$. Then, $(\boldsymbol{C}(\mathcal{S}), \mho) \sqsubseteq (\boldsymbol{C}(\mathcal{S}), \mho)$.*

Definition 2.14. *([56, 57]) A soft topology τ is referred to as follows:*

(i) an enriched soft topology if every pseudo-constant s-set is a member of τ;

(ii) an extended soft topology if $(\mathcal{S}, \mho) \in \tau$ iff $\mathcal{S}(\mathrm{a}) \in \tau_{\mathrm{a}}$ for each $\mathrm{a} \in \mho$.

The soft topologies referenced above were explored rigorously by Al-shami and Kočinac [29]. In the structures produced by these soft topologies, the interchangeable property for the crisp and soft interior and crisp and soft closure topological operators was proved, which is key in the depiction of how topological notions behave from the perspective of the spaces of a soft topology and its p-topologies.

Theorem 2.15. *([29]) An s-subset of an ST-space $(\mathrm{Z}, \tau, \mho)$ is extended if and only if $(\boldsymbol{I}(F), \mho) = \boldsymbol{I}(\mathcal{S}, \mho)$ and $(\boldsymbol{C}(F), \mho) = \boldsymbol{C}(\mathcal{S}, \mho)$ for any s-subset $(\mathcal{S}, \mho)$.*

Theorem 2.16. *([29]) If $\mathrm{E}_{\xi} : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \theta, \mho)$ is soft-continuous, then $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{a}}) \to (\mathrm{X}, \theta_{\xi(\mathrm{a})})$ is continuous for all $\mathrm{a} \in \mho$.*

3. Finite s-open sets

This section has been designed to introduce the concept of finite s-open sets as a fresh collection of s-subsets of an ST-space. In contrast to the celebrated extensions of s-open sets, we find that this class constitutes an infra soft topology and fails to be a supra soft topology. Also, this class maintains some topological characteristics that are lost in the structures generated by well-known extensions of s-open sets, to name a few, the distributive property of the soft union and intersection for soft closure and interior operators, respectively. The relationships between this class and some of the previous ones are showed with the aid of some counterexamples.

Definition 3.1. *An s-subset $(\mathcal{S}, \mho)$ of an ST-space $(\mathrm{Z}, \tau, \mho)$ is said to be a finite s-open set provided that $(\mathcal{S}, \mho)$ is s-open and $\boldsymbol{C}(\mathcal{S}, \mho) \triangle (\mathcal{S}, \mho)$ is finite. That is, $(\mathcal{S}, \mho)$ is stated to be a finite s-open set provided that it is s-open and its soft boundary is a finite s-set. We refer to the complement of a finite s-open set as a finite s-closed set.*

The analogous characteristic of a finite s-closed set is provided in the following proposition.

Proposition 3.2. *Let $(\mathcal{S}, \mho)$ be an s-subset of an ST-space $(\mathrm{Z}, \tau, \mho)$. Then, $(\mathcal{S}, \mho)$ is finite s-closed if and only if it is s-closed and $(\mathcal{S}, \mho) \triangle \boldsymbol{I}(\mathcal{S}, \mho)$ is finite.*

Proof. " $\Rightarrow$ ": Assume that $(\mathcal{S}, \mho)$ is finite s-closed. Directly, we obtain that $(\mathcal{S}, \mho)$ is s-closed and $\boldsymbol{C}(\mathcal{S}^c, \mho) \vartriangle (\mathcal{S}^c, \mho)$ is finite. The next equality relation is obvious:

$$\boldsymbol{C}(\mathcal{S}^c, \mho) \vartriangle (\mathcal{S}^c, \mho)$$

$$= [\boldsymbol{I}(\mathcal{S}, \mho)]^c \cap (\mathcal{S}, \mho)$$

$$= (\mathcal{S}, \mho) \vartriangle \boldsymbol{I}(\mathcal{S}, \mho).$$

Subsequently, $(\mathcal{S}, \mho) \vartriangle \boldsymbol{I}(\mathcal{S}, \mho)$ is finite.
" $\Leftarrow$ ": Let the sufficient conditions hold. Then, $(\mathcal{S}^c, \mho)$ is s-open. By the above equivalence, we get that $\boldsymbol{C}(\mathcal{S}^c, \mho) \vartriangle (\mathcal{S}^c, \mho)$ is finite. Thus, $(\mathcal{S}^c, \mho)$ is finite s-open, which proves the required result.

Before we discover the main properties of these types of s-sets, we shall display the next proposition, which we need to understand the sequels of this manuscript.

Proposition 3.3. *The following properties hold for every s-subset* $(\mathcal{S}, \mho)$ *of an ST-space* $(\mathrm{Z}, \tau, \mho)$*:*

(i) $\boldsymbol{B}(\mathcal{S}, \mho) = \boldsymbol{C}(\mathcal{S}, \mho) \sqcap \boldsymbol{C}(\mathcal{S}^c, \mho)$.

(ii) $\boldsymbol{B}(\mathcal{S}, \mho)$ is s-closed.

(iii) $(\mathcal{S}, \mho)$ is s-closed (resp., s-open) iff $\boldsymbol{B}(\mathcal{S}, \mho) \sqsubseteq (\mathcal{S}, \mho)$ (resp., $\boldsymbol{B}(\mathcal{S}, \mho) \sqsubseteq (\mathcal{S}^c, \mho)$).

(iv) $(\mathcal{S}, \mho)$ is soft-clopen iff $\boldsymbol{B}(\mathcal{S}, \mho) = \Phi$.

Proof. It is analogous to the proof given in the classical topology.

Remark 3.4. **(i)** *An s-open set* $(\mathcal{S}, \mho)$ *is finite s-open provided that* $\boldsymbol{B}(\mathcal{S}, \mho)$ *is finite.*

(ii) *A finite s-set is finite s-open provided that* $\boldsymbol{B}(\mathcal{S}, \mho) \sqsubseteq (\mathcal{S}^c, \mho)$.

Proposition 3.5. *The inverse image of a finite s-open set is preserved under injective soft continuity.*

Proof. Let $(\mathcal{S}, \mho)$ be a finite s-open subset of $(\mathrm{W}, \theta, \mho)$ and $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{W}, \theta, \mho)$ be a soft continuous function. We directly obtain that $\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)$ is an s-open set and

$$\boldsymbol{C}[\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)] \vartriangle \mathrm{E}_\xi^{-1}(\mathcal{S}, \mho) \sqsubseteq$$

$$\mathrm{E}_\xi^{-1}[\boldsymbol{C}(\mathcal{S}, \mho)] \vartriangle \mathrm{E}_\xi^{-1}(\mathcal{S}, \mho) =$$

$$\mathrm{E}_\xi^{-1}[\boldsymbol{C}(\mathcal{S}, \mho) \vartriangle (\mathcal{S}, \mho)].$$

Since $\boldsymbol{C}(\mathcal{S}, \mho) \vartriangle (\mathcal{S}, \mho)$ is finite, then, by the injectiveness of E_ξ we find that $\mathrm{E}_\xi^{-1}[\boldsymbol{C}(\mathcal{S}, \mho) \vartriangle (\mathcal{S}, \mho)]$ is finite. Hence, $\boldsymbol{C}[\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)] \vartriangle \mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)$ is finite. This completes the proof.

Corollary 3.6. *Finite s-open sets possess the invariant property.*

The condition of the injectiveness provided in the aforementioned proposition is dispensable, as the below example points out.

Example 3.7. *Suppose that we have ST-spaces* $(\mathbb{R}, \upsilon, \mho)$ *and* $(\mathrm{Z}, \tau, \mho)$ *such that* $\mathbb{R}$ *is the set of real numbers,* $\mathrm{Z} = \{\mathrm{y}, \mathrm{z}\}$*,* $\mho = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$*, and the soft topologies* υ *over* $\mathbb{R}$ *with* $\mho$ *and* τ *over* Z *with* $\mho$ *are given by*

$$\upsilon = \{\widetilde{\mathbb{R}}, (\mathcal{S}, \mho) \sqsubseteq \widetilde{\mathbb{R}} : 1 \in (\mathcal{S}, \mho)\} \cup \{\Phi\}, \text{ and}$$
$$\tau = \{\Phi, \widetilde{Z}, (\mathcal{U}, \mho) = \{(a, \{z\}), (b, \{z\}), (c, \{z\})\}\},$$

where a soft function $E_\xi : (\mathbb{R}, \upsilon, \mho) \to (Z, \tau, \mho)$ is defined as follows:

$$\xi \text{ is the identity function and}$$

$$E(1) = z \text{ , and for each } r \neq 1, \text{ we have that } E(r) = y.$$

It is obvious that E_ξ is not injective. Now, $(\mathcal{U}, \mho)$ is a finite s-open subset of $(Z, \tau, \mho)$, but $E_\xi^{-1}(\mathcal{U}, \mho) = \{(a, \{1\}), (b, \{1\}), (c, \{1\})\}$ is not finite s-open in $(\mathbb{R}, \upsilon, \mho)$ in spite of E_ξ being soft-continuous.

Proposition 3.8. *The family of finite s-open subsets is closed under a finite Cartesian product.*

Proof. Let $(\mathcal{S}, \mho)$ and $(\mathcal{U}, \mho)$ be finite s-open sets. As we know, the product of finite numbers of s-open sets is s-open; also, we know that $\boldsymbol{C}[(\mathcal{S}, \mho) \times (\mathcal{U}, \mho)] = \boldsymbol{C}(\mathcal{S}, \mho) \times \boldsymbol{C}(\mathcal{U}, \mho)$. By the hypothesis, the following is a finite s-set:

$$[\boldsymbol{C}(\mathcal{S}, \mho) \vartriangle (\mathcal{S}, \mho)] \times [\boldsymbol{C}(\mathcal{U}, \mho) \vartriangle (\mathcal{U}, \mho)]$$

$$= [\boldsymbol{C}(\mathcal{S}, \mho) \times \boldsymbol{C}(\mathcal{U}, \mho)] \vartriangle [(\mathcal{S}, \mho) \times (\mathcal{U}, \mho)]$$
$$= \boldsymbol{C}[(\mathcal{S}, \mho) \times (\mathcal{U}, \mho)] \vartriangle [(\mathcal{S}, \mho) \times (\mathcal{U}, \mho)],$$

which ends the proof that $(\mathcal{S}, \mho) \times (\mathcal{U}, \mho)$ is a finite s-open set.

Proposition 3.9. *Let $(Z, \tau_1, \mho)$ and $(Z, \tau_2, \mho)$ be ST-spaces such that $\tau_1 \subseteq \tau_2$. If $(\mathcal{S}, \mho)$ is a finite s-open subset of τ_1, then it is also a finite s-open subset of τ_2.*

Proof. Let $(\mathcal{S}, \mho)$ be a finite s-open subset of $(Z, \tau_1, \mho)$. Since $\tau_1 \subseteq \tau_2$, $(\mathcal{S}, \mho) \in \tau_2$ and $\boldsymbol{C}_{\tau_2}(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{C}_{\tau_1}(\mathcal{S}, \mho)$, it follows that $\boldsymbol{B}_{\tau_2}(\mathcal{S}, \mho)$ is finite. Hence, we have finished the proof.

One can see that the converse of Proposition 3.9 is false by taking a member of τ_2 which is not a member of τ_1.

Proposition 3.10. *Let $(Z, \tau, \mho)$ be an ST-space and $(\mathcal{S}_j, \mho)$ be s-subsets of it for each $j \in J$. Then, $\boldsymbol{B}[\sqcup_{j \in J}(\mathcal{S}_j, \mho)] \sqsubseteq \sqcup_{j \in J} \boldsymbol{B}(\mathcal{S}_j, \mho)$.*

Proof. We provide the proof for two s-subsets. To do this, let $z_a \in \boldsymbol{B}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)]$. Then,

$$z_a \in \boldsymbol{C}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)] \text{ and } z_a \notin \boldsymbol{I}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)]$$

$$\Longrightarrow z_a \in \boldsymbol{C}(\mathcal{S}, \mho) \text{ or } \boldsymbol{C}(\mathcal{U}, \mho) \text{ and } z_a \notin \boldsymbol{I}(\mathcal{S}, \mho) \text{ and } z_a \notin \boldsymbol{I}(\mathcal{U}, \mho)$$
$$\Longrightarrow z_a \in \boldsymbol{C}(\mathcal{S}, \mho) \vartriangle \boldsymbol{I}(\mathcal{S}, \mho) \text{ or } z_a \in \boldsymbol{C}(\mathcal{U}, \mho) \vartriangle \boldsymbol{I}(\mathcal{U}, \mho)$$
$$\Longrightarrow z_a \in \boldsymbol{B}(\mathcal{S}, \mho) \sqcup \boldsymbol{B}(\mathcal{U}, \mho).$$

Hence, we obtain $\boldsymbol{B}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)] \sqsubseteq \boldsymbol{B}(\mathcal{S}, \mho) \sqcup \boldsymbol{B}(\mathcal{U}, \mho)$, as required. One can obtain the proof for an arbitrary number of s-subsets by using the mathematical induction proof.

Corollary 3.11. *The finite numbers of soft unions (resp., soft intersections) of finite s-open (resp., finite s-closed) subsets of an ST-space* $(Z, \tau, \mho)$ *is finite s-open (resp., finite s-closed).*

Corollary 3.12. *The family of finite s-open sets structures an infra soft topology.*

Proof. It is obvious that the absolute and null *s*-sets are finite *s*-open. By Corollary 3.11, this family is closed under a finite soft intersection. Thus, the required finding is obtained.

One may prove the next result by following similar arguments given in the proof of Proposition 3.10.

Proposition 3.13. *Let* $(Z, \tau, \mho)$ *be an ST-space and* $(\mathcal{S}_j, \mho)$ *be s-subsets of it for each* $j \in J$*. Then,* $\boldsymbol{B}[\sqcap_{j\in J}(\mathcal{S}_j, \mho)] \sqsubseteq \sqcup_{j\in J}\boldsymbol{B}(\mathcal{S}_j, \mho)$.

Corollary 3.14. *The finite numbers of soft unions (resp., soft intersections) of finite s-closed (resp., finite s-open) subsets of an ST-space* $(Z, \tau, \mho)$ *is finite s-closed (resp., finite s-open).*

In the next example, we show that the converse of Proposition 3.10 is not always true, as well as reveal that both classes of finite *s*-open and finite *s*-closed subsets are not always closed under arbitrary numbers of soft unions and intersections.

Example 3.15. *Suppose that we have an ST-space* $(\mathbb{R}, \tau, \mho)$ *such that* $\mathbb{R}$ *is the real numbers set,* $\mho = \{a, b\}$*, and* τ *is the usual soft topology; that is,* τ *is generated by the soft basis* $\{(\mathcal{S}, \mho) : \mathcal{S}(a)$ *and* $\mathcal{S}(b)$ *are open intervals in the forms of* $(i, j)\}$*. By taking the following s-sets:*

$$(\mathcal{S}, \mho) = \{(a, (3, 4)), (b, (3, 4))\}, \text{ and}$$
$$(\mathcal{U}, \mho) = \{(a, [4, 5)), (b, [4, 5))\}.$$

We find that

$$\boldsymbol{B}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)] = \{(a, \{3, 5\}), (b, \{3, 5\})\} \text{ , whereas}$$
$$\boldsymbol{B}(\mathcal{S}, \mho) \sqcup \boldsymbol{B}(\mathcal{S}, \mho) = \{(a, \{3, 4, 5\}), (b, \{3, 4, 5\})\}.$$

Therefore, $\boldsymbol{B}[\sqcup_{j\in J}(\mathcal{S}_j, \mho)]$ is a proper *s*-subset of $\sqcup_{j\in J}\boldsymbol{B}(\mathcal{S}_j, \mho)$.

Also, by taking the *s*-sets of the form $(\mathcal{S}_i, \mho) = \{(a, (i, i + 1)), (b, (i, i + 1))\}$, we get that $(\mathcal{S}_i, \mho)$ is finite *s*-open for each i in the set of natural numbers $\mathbb{N}$. In contrast, the *s*-set $\sqcup_{i\in\mathbb{N}}(\mathcal{S}_i, \mho)$ is not finite *s*-open because

$$\begin{aligned}&\boldsymbol{C}[\sqcup_{i\in\mathbb{N}}(\mathcal{S}_i, \mho)]\triangle \sqcup_{i\in\mathbb{N}} (\mathcal{S}_i, \mho)\\ &= \boldsymbol{C}[\{(a, [1, \infty)\backslash\mathbb{N}), (b, [1, \infty)\backslash\mathbb{N})\}]\triangle\{(a, [1, \infty)\backslash\mathbb{N}), (b, [1, \infty)\backslash\mathbb{N})\}\\ &= \{(a, [1, \infty)), (b, [1, \infty))\}]\triangle\{(a, [1, \infty)\backslash\mathbb{N}), (b, [1, \infty)\backslash\mathbb{N})\}\\ &= \{(a, \mathbb{N}), (b, \mathbb{N})\},\end{aligned}$$

which is an infinite *s*-set. So, the class of finite *s*-open sets is not closed under an arbitrary number of soft unions.

Moreover, by taking the *s*-sets of the form $(\mathcal{S}_i, \mho) = \{(a, (\frac{-1}{i}, \frac{1}{i})), (b, (\frac{-1}{i}, \frac{1}{i}))\}$, we get that $(\mathcal{S}_i, \mho)$ is finite *s*-open for each i in the set of natural numbers $\mathbb{N}$. But the *s*-set $\sqcap_{i\in\mathbb{N}}(\mathcal{S}_i, \mho) = \{(a, \{0\}), (b, \{0\})\}$ is not *s*-open; thus, it is not finite *s*-open. Hence, the class of finite *s*-open sets is not closed under an arbitrary numbers of soft intersections.

By taking the complement of the second and third computations, we also infer that the class of finite *s*-closed sets is not closed under an arbitrary number of soft unions and intersections.

Proposition 3.16. *Let* $(\mathrm{Z}, \tau, \mho)$ *be an* ST*-space and* $(\mathcal{S}, \mho)$ *and* $(\mathcal{U}, \mho)$ *be s-subsets of it such that* $\boldsymbol{C}(\mathcal{S}, \mho)$ *and* $\boldsymbol{C}(\mathcal{U}, \mho)$ *are disjoint. Then,* $\boldsymbol{B}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)] = \boldsymbol{B}(\mathcal{S}, \mho) \sqcup \boldsymbol{B}(\mathcal{U}, \mho)$.

In what follows, we will discuss the connections of this kind of s-set with s-open sets and their known generalizations.

Proposition 3.17. **(i)** *Every soft-clopen set is finite s-open.*

(ii) *Every finite s-open set is s-open (soft* ζ*-open), where* $\zeta \in \{\alpha, pre, semi, b, \beta, somewhere\ dense\}$.

Proof. Since $\boldsymbol{B}(\mathcal{S}, \mho)$ is the null s-set for every soft-clopen subset $(\mathcal{S}, \mho)$, the proof of (i) follows. The proof of (ii) is obvious by Definition 3.1 for the case of s-open sets, and by the previous relationships that are well known in the published articles for the case of between brackets.

By Example 3.15, it can be seen that the s-set $(\mathcal{S}, \mho) = \{(\mathrm{a}, (4, 5]), (\mathrm{b}, [2, 3))\}$ is soft ζ-open for $\zeta \in \{semi, b, \beta, somewhere\ dense\}$ but not finite s-open. Also, $\{(\mathrm{a}, [1, \infty)\backslash\mathbb{N}), (\mathrm{b}, [1, \infty)\backslash\mathbb{N})\}$ is an s-open set that is not finite s-open. On the other hand, in a soft topology consisting of three elements over the finite universal set we get a finite s-open set that is not soft-clopen. Thus, the converse of Proposition 3.17 is false, in general.

Proposition 3.18. *If all proper s-closed subsets of an infinite* ST*-space are finite, then an s-set is s-open if and only if it is finite s-open.*

Proof. To prove the necessary side, let $(\mathcal{S}, \mho)$ be s-open. Suppose, to the contrary, that $\boldsymbol{B}(\mathcal{S}, \mho)$ is infinite. Then, $\boldsymbol{C}(\mathcal{S}, \mho)$ is infinite. By the hypothesis, $\boldsymbol{C}(\mathcal{S}, \mho)$ must equal the absolute s-set. Therefore, $\boldsymbol{B}(\mathcal{S}, \mho)$ is the null s-set. But, this contradicts the idea that $\boldsymbol{B}(\mathcal{S}, \mho)$ is infinite, so $(\mathcal{S}, \mho)$ is finite s-open. The sufficient side is obvious by (ii) of Proposition 3.17.

Example 3.15 elucidates the necessity of the condition that all proper s-closed subsets are finite to satisfy the equality in Proposition 3.18. That is, $\{(\mathrm{a}, [1, \infty)\backslash\mathbb{N}), (\mathrm{b}, [1, \infty)\backslash\mathbb{N})\}$ is an s-open subset of an ST-space $(\mathbb{R}, \tau, \mho)$ defined in Example 3.15 that is not finite s-open.

The following lemma will be useful to demonstrate some of the upcoming results.

Proposition 3.19. *The families of s-open and finite s-open subsets of an* ST*-space* $(\mathrm{Z}, \tau, \mho)$ *are identical provided that one of the following conditions holds:*

(i) The absolute s-set is finite.

(ii) The soft topology is soft-clopen.

(iii) The soft topology is locally indiscrete.

Proof. (i): It is obvious.
(ii): It follows from (iv) of Proposition 3.3.
(iii): It is sufficient to show that $\boldsymbol{B}(\mathcal{S}, \mho)$ is null for any s-open subset of a soft locally indiscrete topological space. To demonstrate this, let $(\mathcal{S}, \mho)$ be s-open. By the hypothesis, $(\mathcal{S}, \mho)$ is expressed as a soft union of soft-clopen subsets, i.e.,

$$(\mathcal{S}, \mho) = \sqcup_{j \in J}(\mathcal{U}_j, \mho) \text{ where } (\mathcal{U}_j, \mho) \text{ is an element of soft basis for each } j. \tag{3.1}$$

By Proposition 3.10, we have the following inclusion

$$\boldsymbol{B}[\sqcup_{j\in J}(\mathcal{U}_j, \mho)] \sqsubseteq \sqcup_{j\in J}\boldsymbol{B}(\mathcal{U}_j, \mho).$$

Since every $(\mathcal{U}_j, \mho)$ is soft-clopen, $\boldsymbol{B}(\mathcal{U}_j, \mho) = \Phi$. This implies that $\boldsymbol{B}[\sqcup_{j\in J}(\mathcal{U}_j, \mho)] = \Phi$. From 3.1 we prove that $(\mathcal{S}, \mho)$ is finite s-open.

Corollary 3.20. *The closeness of finite s-open sets for arbitrary soft unions is satisfied provided that one of the following conditions holds:*

(i) The absolute s-set is finite.

(ii) The soft topology is soft-clopen.

(iii) The soft topology is locally indiscrete.

Proposition 3.21. *Let* $(\mathcal{S}, \mho)$ *be a finite s-open subset of a soft extremally disconnected space* $(\mathrm{Z}, \tau, \mho)$. *Then* $\boldsymbol{C}(\mathcal{S}, \mho)$ *is finite s-open.*

Proof. Since $\boldsymbol{C}(\mathcal{S}, \mho)$ is s-open and $\boldsymbol{B}[\boldsymbol{C}(\mathcal{S}, \mho)] \sqsubseteq \boldsymbol{C}(\mathcal{S}, \mho)$, the proof follows.

Recall that $(\mathcal{S}, \mho)$ is called a discrete s-subset of an ST-space $(\mathrm{Z}, \tau, \mho)$ if every soft point $\mathrm{z_a} \in (\mathcal{S}, \mho)$ has an s-neighborhood $(\mathcal{U}, \mho)$ such that $(\mathcal{S}, \mho) \sqcap (\mathcal{U}, \mho) = \mathrm{z_a}$.

Lemma 3.22. *An ST-space* $(\mathrm{Z}, \tau, \mho)$ *is soft-submaximal if and only if* $\boldsymbol{C}(\mathcal{S}, \mho)\Delta\boldsymbol{I}(\mathcal{S}, \mho)$ *is a discrete s-set.*

Proof. It follows by a similar argument as that given in the proof of Theorem 3.3 of [41].

Theorem 3.23. *Let* $(\mathcal{S}, \mho)$ *be an s-subset of a soft compact and soft submaximal space* $(\mathrm{Z}, \tau, \mho)$. *Then,* $\boldsymbol{C}(\mathcal{S}, \mho)\Delta\boldsymbol{I}(\mathcal{S}, \mho)$ *is a finite s-set.*

Proof. Since $\boldsymbol{C}(\mathcal{S}, \mho)\Delta\boldsymbol{I}(\mathcal{S}, \mho)$ is an s-closed set, it follows from Lemma 3.22 that it is a discrete s-set. By the hypothesis of soft compactness, we obtain that $\boldsymbol{C}(\mathcal{S}, \mho)\Delta\boldsymbol{I}(\mathcal{S}, \mho)$ is also a soft compact set. Hence, $\boldsymbol{C}(\mathcal{S}, \mho)\Delta\boldsymbol{I}(\mathcal{S}, \mho)$ must be a finite s-set.

Corollary 3.24. *If* $(\mathrm{Z}, \tau, \mho)$ *is a soft compact and soft submaximal space, then the families of s-open and finite s-open sets are identical.*

We conclude this section by looking at the transition of the feature of being a finite s-open set between soft and classical realms.

Proposition 3.25. *If* $(\mathcal{S}, \mho)$ *is a finite s-open subset of* $(\mathrm{Z}, \tau, \mho)$*, then* $\mathcal{S}(\mathrm{a})$ *is a finite open subset of* (Z, τ_a) *for every* $\mathrm{a} \in \mho$.

Proof. It is obvious that $\mathcal{S}(\mathrm{a})$ is an open subset of (Z, τ_a) for every $\mathrm{a} \in \mho$ when $(\mathcal{S}, \mho)$ is a finite s-open subset of $(\mathrm{Z}, \tau, \mho)$. It remains to be shown that $\boldsymbol{B}(\mathcal{S}(\mathrm{a}))$ is finite. From Proposition 2.13, $(\boldsymbol{C}(\mathcal{S}), \mho) \sqsubseteq \boldsymbol{C}(\mathcal{S}, \mho)$, which automatically means that $(\boldsymbol{B}(\mathcal{S}), \mho)$ is finite. Hence, $\boldsymbol{B}(\mathcal{S}(\mathrm{a}))$ is finite, as required.

The following counterexample is presented to clarify that the inverse direction of Proposition 3.25 need not be correct, in general.

Example 3.26. *Let the real numbers and natural numbers sets (respectively denoted by $\mathbb{R}$ and $\mathbb{N}$) be the universal and parameter sets, respectively. Define an ST τ over $\mathbb{R}$ with $\mathbb{N}$ as follows:*

$$\tau = \{\Phi, \widetilde{\mathbb{R}}, (\mathcal{S}_i, \mathbb{N}) : \mathcal{S}_i(n) = \mathbb{R} \text{ for all but finitely many } n \in \mathbb{N}\}.$$

Now, one can remark that an s-set $(\mathcal{U}, \mathbb{N})$ defined by

$$\mathcal{U}(1) = \{1\} \text{ and for } n \neq 1 : \mathcal{U}(n) = \mathbb{R}$$

is not finite s-open in spite of all components being finite open sets.

Proposition 3.27. *Let τ be extended and $(\mathcal{S}, \mho)$ be an s-subset of $(\mathrm{Z}, \tau, \mho)$. Then, $(\mathcal{S}, \mho)$ is a finite s-open set if and only if $\mathcal{S}(\mathrm{a})$ is a finite open subset of (Z, τ_a) for every $\mathrm{a} \in \mho$.*

Proof. The necessary part has been shown in the previous proposition. To prove the sufficient part, let $\mathcal{S}(\mathrm{a})$ be a finite open subset of (Z, τ_a) for every $\mathrm{a} \in \mho$. Since τ is extended, it respectively follows from Theorem 2.15 and Definition 2.14 that the equality relation $\boldsymbol{C}(\mathcal{S}, \mho) = (\boldsymbol{C}(\mathcal{S}), \mho)$ holds, and an s-set $(\mathcal{S}), \mho) \in \tau$ if and only if $\mathcal{S}(\mathrm{a}) \in \tau_a$ for every $\mathrm{a} \in \mho$. Therefore, an s-set consists of $\mathcal{S}(\mathrm{a})$ for all $\mathrm{a} \in \mho$, i.e., $(\mathcal{S}, \mho)$, is s-open, and $(\boldsymbol{C}(\mathcal{S}), \mho)\triangle(\mathcal{S}, \mho)$ is finite. By the following equalities, we find that $\boldsymbol{B}(\mathcal{S}, \mho)$ is finite, which ends the proof.

$$\begin{aligned}&\boldsymbol{B}(\mathcal{S}, \mho)\\ &= \boldsymbol{C}(\mathcal{S}, \mho)\triangle(\mathcal{S}, \mho)\\ &= (\boldsymbol{C}(\mathcal{S}), \mho)\triangle(\mathcal{S}, \mho).\end{aligned}$$

4. Soft interior and closure operators for finite s-open sets

In this section, we put forward some operators that have been inspired by finite s-open and finite s-closed sets. We document that some topological properties of these operators are missing, to name a few, the equality between an s-subset and its soft fo-interior (resp., soft fo-closure) operator does not imply that this s-subset is finite s-open (resp., finite soft closed). We offer some formulas that describe the interrelations between these operators and build some illustrative examples.

Definition 4.1. *Let $(\mathcal{S}, \mho)$ be an s-set in $(\mathrm{Z}, \tau, \mho)$; the union of all finite s-open subsets which are contained in $(\mathcal{S}, \mho)$, is defined as the fo-interior operator of $(\mathcal{S}, \mho)$, and it will be denoted by $\boldsymbol{I}_{fo}(\mathcal{S}, \mho)$.*

The next result follows directly from the above definition. The validity of its converse is false in general, as revealed by the example given after this proposition.

Proposition 4.2. *If $(\mathcal{S}, \mho)$ is a finite s-open subset of $(\mathrm{Z}, \tau, \mho)$, then $(\mathcal{S}, \mho) = \boldsymbol{I}_{fo}(\mathcal{S}, \mho)$.*

Example 4.3. *In Example 3.15, we showed that $\{(\mathrm{a}, [1, \infty)\backslash\mathbb{N}), (\mathrm{b}, [1, \infty)\backslash\mathbb{N})\}$ is not finite s-open. On the other hand, notice that $\{(\mathrm{a}, [1, \infty)\backslash\mathbb{N}), (\mathrm{b}, [1, \infty)\backslash\mathbb{N})\} = \sqcup_{i\in\mathbb{N}}\{(\mathrm{a}, (i, i+1)), (\mathrm{b}, (i, i+1))\}$, where $\{(\mathrm{a}, (i, i+1)), (\mathrm{b}, (i, i+1))\}$ is finite s-open for each $i \in \mathbb{N}$. Hence, $\{(\mathrm{a}, [1, \infty)\backslash\mathbb{N}), (\mathrm{b}, [1, \infty)\backslash\mathbb{N})\} = \boldsymbol{I}_{fo}(\{(\mathrm{a}, [1, \infty)\backslash\mathbb{N}), (\mathrm{b}, [1, \infty)\backslash\mathbb{N})\})$.*

Proposition 4.2 says that the fo-interior operator does not fulfill a main feature of soft interior operator; that is, if, for every soft-point z_a belonging to $(\mathcal{S}, \mho)$ there exists a finite s-open set $(\mathcal{U}, \mho)$ such that $z_a \in (\mathcal{U}, \mho) \sqsubseteq (\mathcal{S}, \mho)$, then $(\mathcal{S}, \mho)$ need not be finite s-open.

The following properties of fo-interior operators can be proved easily, so we omit their proofs.

Proposition 4.4. *For all s-subsets* $(\mathcal{S}, \mho)$, $(\mathcal{U}, \mho)$ *of* $(Z, \tau, \mho)$*, we have*

$$\boldsymbol{I}_{fo}(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{I}_{fo}(\mathcal{U}, \mho) \textit{ when } (\mathcal{S}, \mho) \sqsubseteq (\mathcal{U}, \mho).$$

Corollary 4.5. *Let* $(\mathcal{S}, \mho)$ *and* $(\mathcal{U}, \mho)$ *be s-subsets of* $(Z, \tau, \mho)$*. Then* $\boldsymbol{I}_{fo}(\mathcal{S}, \mho) \sqcup \boldsymbol{I}_{fo}(\mathcal{U}, \mho) \sqsubseteq \boldsymbol{I}_{fo}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)]$.

The relations of inclusion deduced in the aforementioned results (Proposition 4.4 and Corollary 4.5) cannot be replaced by equalities, as clarified by the following counterexamples.

Example 4.6. *Let* $(Z, \tau, \mho)$ *be an ST-space, where* $Z = \{x, y, z\}$, $\mho = \{a, b, c\}$*, and the members of* τ *are the null and absolute s-sets and the following s-sets:*

$$(\mathcal{S}, \mho) = \{(a, Z), (b, \emptyset), (c, \emptyset)\} \text{ and } (\mathcal{U}, \mho) = \{(a, \emptyset), (b, Z), (c, Z)\}.$$

Take the s-sets as $(\mathcal{H}, \mho) = \{(a, \emptyset), (b, Z), (c, \emptyset)\}$ and $(\mathcal{K}, \mho) = \{(a, \emptyset), (b, \emptyset), (c, Z)\}$. Then, $\boldsymbol{I}_{fo}(\mathcal{H}, \mho) = \boldsymbol{I}_{fo}(\mathcal{K}, \mho) = \Phi$ in spite of $(\mathcal{H}, \mho)$ and $(\mathcal{K}, \mho)$ being independent of each other with respect to the inclusion relation. Moreover,

$$\boldsymbol{I}_{fo}(\mathcal{H}, \mho) \sqcup \boldsymbol{I}_{fo}(\mathcal{K}, \mho) = \Phi,$$

whereas

$$\boldsymbol{I}_{fo}[(\mathcal{H}, \mho) \sqcup (\mathcal{K}, \mho)] = (\mathcal{U}, \mho).$$

Proposition 4.7. *Let* $(\mathcal{S}, \mho)$ *and* $(\mathcal{U}, \mho)$ *be s-subsets of* $(Z, \tau, \mho)$*. Then* $\boldsymbol{I}_{fo}[(\mathcal{S}, \mho) \sqcap (\mathcal{U}, \mho)] = \boldsymbol{I}_{fo}(\mathcal{S}, \mho) \sqcap \boldsymbol{I}_{fo}(\mathcal{U}, \mho)$.

Proof. The side $\boldsymbol{I}_{fo}[(\mathcal{S}, \mho) \sqcap (\mathcal{U}, \mho)] \sqsubseteq \boldsymbol{I}_{fo}(\mathcal{S}, \mho) \sqcap \boldsymbol{I}_{fo}(\mathcal{U}, \mho)$ follows from Proposition 4.4. The converse side can be obtained from Corollary 3.11.

Now, we go into fo-closure operators, which are considered as the dual of fo-interior operators.

Definition 4.8. *Let* $(\mathcal{S}, \mho)$ *be an s-set in* $(Z, \tau, \mho)$*; the intersection of all finite s-closed subsets containing* $(\mathcal{S}, \mho)$ *is defined as the fo-closure operator of* $(\mathcal{S}, \mho)$*, and it will be denoted by* $\boldsymbol{C}_{fo}(\mathcal{S}, \mho)$.

Proposition 4.9. *Suppose that there exists an s-subset* $(\mathcal{S}, \mho)$ *of* $(Z, \tau, \mho)$ *and* $z_a \in \widetilde{Z}$*. Then,* $z_a \in \boldsymbol{C}_{fo}(\mathcal{S}, \mho)$ *if and only if* $(\mathcal{U}, \mho) \sqcap (\mathcal{S}, \mho) \neq \Phi$ *for each finite s-open set* $(\mathcal{U}, \mho)$ *containing* z_a.

Proof. Necessity: Let $z_a \in \boldsymbol{C}_{fo}(\mathcal{S}, \mho)$ and $(\mathcal{U}, \mho)$ be a finite s-open set such that $z_a \in (\mathcal{U}, \mho)$. Suppose that $(\mathcal{U}, \mho) \sqcap (\mathcal{S}, \mho) = \Phi$. Now, we have that $(\mathcal{S}, \mho) \sqsubseteq (\mathcal{U}^c, \mho)$. It follows that $\boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqsubseteq (\mathcal{U}^c, \mho)$. But, this contradicts that $z_a \in \boldsymbol{C}_{fo}(\mathcal{S}, \mho)$. Hence, the intersection of $(\mathcal{U}, \mho)$ and $(\mathcal{S}, \mho)$ must be the non-null s-set.
Sufficiency: Let us consider that the sufficient part holds. Suppose, to the contrary, that $z_a \notin \boldsymbol{C}_{fo}(\mathcal{S}, \mho)$. This means that we can find a finite s-closed set $(\mathcal{H}, \mho)$ containing $(\mathcal{U}, \mho)$ such that $z_a \notin (\mathcal{H}, \mho)$. Accordingly, we have $(\mathcal{H}^c, \mho)$ as a finite s-open set containing z_a and its intersection with $(\mathcal{S}, \mho)$ is the null s-set. This is a contradiction. Hence, we demonstrate that $z_a \in \boldsymbol{C}_{fo}(\mathcal{S}, \mho)$.

Corollary 4.10. *Let $(\mathcal{S}, \mho)$ be a finite s-open set and $(\mathcal{U}, \mho)$ an s-set in $(Z, \tau, \mho)$. Then, $(\mathcal{S}, \mho) \sqcap (\mathcal{U}, \mho) = \Phi$ implies that $(\mathcal{S}, \mho) \sqcap \boldsymbol{C}_{fo}(\mathcal{U}, \mho) = \Phi$. Moreover, if $(\mathcal{S}, \mho)$ and $(\mathcal{U}, \mho)$ are finite s-open sets, then $\boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqcap \boldsymbol{C}_{fo}(\mathcal{U}, \mho) = \Phi$.*

Proposition 4.11. *If $(\mathcal{S}, \mho)$ is an s-subset of $(Z, \tau, \mho)$; then, the following holds:*

(i) If $(\mathcal{S}, \mho)$ is a finite s-closed set, then $(\mathcal{S}, \mho) = \boldsymbol{C}_{fo}(\mathcal{S}, \mho)$.

(ii) $[\boldsymbol{I}_{fo}(\mathcal{S}, \mho)]^c = \boldsymbol{C}_{fo}(\mathcal{S}^c, \mho)$ and $[\boldsymbol{C}_{fo}(\mathcal{S}, \mho)]^c = \boldsymbol{I}_{fo}(\mathcal{S}^c, \mho)$.

Proof. (i): It is straightforward.
(ii):

$$\begin{aligned} & [\boldsymbol{I}_{fo}(\mathcal{S}, \mho)]^c \\ &= [\sqcup_{j \in J}(\mathcal{H}_j, \mho) : (\mathcal{H}_j, \mho) \sqsubseteq (\mathcal{S}, \mho) \text{ where } (\mathcal{H}_j, \mho) \text{ is a finite } s\text{-open set}]^c \\ &= [\sqcap_{j \in J}(\mathcal{H}_j^c, \mho) : (\mathcal{S}^c, \mho) \sqsubseteq (\mathcal{H}_j^c, \mho) \text{ where } (\mathcal{H}_j^c, \mho) \text{ is a finite } s\text{-closed set}] \\ &= \boldsymbol{C}_{fo}(\mathcal{S}^c, \mho). \end{aligned}$$

Proposition 4.12. *Let $(\mathcal{S}, \mho)$ be an s-subset of $(Z, \tau, \mho)$. Then, $\boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{C}_{fo}(\mathcal{U}, \mho)$ when $(\mathcal{S}, \mho) \sqsubseteq (\mathcal{U}, \mho)$.*

Proof. Owing to the fact that every finite s-closed set contains $(\mathcal{U}, \mho)$ is also contains $(\mathcal{S}, \mho)$, the proof is completed.

Corollary 4.13. *Let $(\mathcal{S}, \mho)$ and $(\mathcal{U}, \mho)$ be s-subsets $(Z, \tau, \mho)$. Then, $\boldsymbol{C}_{fo}[(\mathcal{S}, \mho) \sqcap (\mathcal{U}, \mho)] \sqsubseteq \boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqcap \boldsymbol{C}_{fo}(\mathcal{U}, \mho)$.*

By Example 4.6, it can be confirmed that the inclusion relations deduced in Proposition 4.12 and Corollary 4.13 cannot be replaced by equalities.

Proposition 4.14. *Let $(\mathcal{S}, \mho)$ and $(\mathcal{U}, \mho)$ be s-subsets of $(Z, \tau, \mho)$. Then, $\boldsymbol{C}_{fo}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)] = \boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqcup \boldsymbol{C}_{fo}(\mathcal{U}, \mho)$.*

Proof. The side $\boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqcup \boldsymbol{C}_{fo}(\mathcal{U}, \mho) \sqsubseteq \boldsymbol{C}_{fo}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)]$ follows from Proposition 4.12. The converse side can be obtained from Corollary 3.11.

Definition 4.15. *Let $(\mathcal{S}, \mho)$ be an s-subset of $(Z, \tau, \mho)$. The complement of $\boldsymbol{I}_{fo}(\mathcal{S}, \mho) \sqcup \boldsymbol{I}_{fo}(\mathcal{S}^c, \mho)$ is said to be an fo-boundary of $(\mathcal{S}, \mho)$; it will be denoted by $\boldsymbol{B}_{fo}(\mathcal{S}, \mho)$.*

Proposition 4.16. *Let $(\mathcal{S}, \mho)$ be an s-subset of $(Z, \tau, \mho)$. Then,*

$$\boldsymbol{B}_{fo}(\mathcal{S}, \mho) = \boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqcap \boldsymbol{C}_{fo}(\mathcal{S}^c, \mho).$$

Proof.
$$\begin{aligned} \boldsymbol{B}_{fo}(\mathcal{S}, \mho) &= [\boldsymbol{I}_{fo}(\mathcal{S}, \mho) \sqcup \boldsymbol{I}_{fo}(\mathcal{S}^c, \mho)]^c \\ &= [\boldsymbol{I}_{fo}(\mathcal{S}, \mho)]^c \sqcap [\boldsymbol{I}_{fo}(\mathcal{S}^c, \mho)]^c \quad \text{(De Morgan's law)} \\ &= \boldsymbol{C}_{fo}(\mathcal{S}^c, \mho) \sqcap \boldsymbol{C}_{fo}(\mathcal{S}, \mho). \quad \text{(Proposition 4.11 (ii))} \end{aligned}$$

Corollary 4.17. *Let $(\mathcal{S}, \mho)$ be an s-subset of $(Z, \tau, \mho)$. Then, the following holds:*

(i) $\boldsymbol{B}_{fo}(\mathcal{S},\mho) = \boldsymbol{B}_{fo}(\mathcal{S}^c,\mho)$.

(ii) $\boldsymbol{B}_{fo}(\mathcal{S},\mho) = \boldsymbol{C}_{fo}(\mathcal{S},\mho)\Delta\, \boldsymbol{I}_{fo}(\mathcal{S},\mho)$.

(iii) $\boldsymbol{C}_{fo}(\mathcal{S},\mho) = \boldsymbol{I}_{fo}(\mathcal{S},\mho) \sqcup\, \boldsymbol{B}_{fo}(\mathcal{S},\mho)$.

(iv) $\boldsymbol{I}_{fo}(\mathcal{S},\mho) = (\mathcal{S},\mho)\Delta\, \boldsymbol{B}_{fo}(\mathcal{S},\mho)$.

Proof. (i): It is straightforward.
(ii): $\boldsymbol{B}_{fo}(\mathcal{S},\mho) = \boldsymbol{C}_{fo}(\mathcal{S},\mho) \sqcap \boldsymbol{C}_{fo}(\mathcal{S}^c,\mho) = \boldsymbol{C}_{fo}(\mathcal{S},\mho)\Delta\, [\boldsymbol{C}_{fo}(\mathcal{S}^c,\mho)]^c$. By (ii) of Proposition 4.11, the required relation is obtained.
(iii): $\boldsymbol{I}_{fo}(\mathcal{S},\mho) \sqcup\, \boldsymbol{B}_{fo}(\mathcal{S},\mho) = \boldsymbol{I}_{fo}(\mathcal{S},\mho) \sqcup\, [\boldsymbol{C}_{fo}(\mathcal{S},\mho)\Delta\, \boldsymbol{I}_{fo}(\mathcal{S},\mho)] = \boldsymbol{C}_{fo}(\mathcal{S},\mho)$.
(iv):
$$\begin{aligned}(\mathcal{S},\mho)\Delta\, \boldsymbol{B}_{fo}(\mathcal{S},\mho) &= (\mathcal{S},\mho)\Delta\, [\boldsymbol{C}_{fo}(\mathcal{S},\mho)\Delta\, \boldsymbol{I}_{fo}(\mathcal{S},\mho)]\\ &= (\mathcal{S},\mho) \sqcap [\boldsymbol{C}_{fo}(\mathcal{S},\mho) \sqcap (\boldsymbol{I}_{fo}(\mathcal{S},\mho))^c]^c\\ &= (\mathcal{S},\mho) \sqcap [(\boldsymbol{C}_{fo}(\mathcal{S},\mho))^c \sqcup\, \boldsymbol{I}_{fo}(\mathcal{S},\mho)]\\ &= [(\mathcal{S},\mho) \sqcap (\boldsymbol{C}_{fo}(\mathcal{S},\mho))^c] \sqcup [(\mathcal{S},\mho) \sqcap\, \boldsymbol{I}_{fo}(\mathcal{S},\mho)]\\ &= \boldsymbol{I}_{fo}(\mathcal{S},\mho).\end{aligned}$$

Note that $\boldsymbol{B}_{fo}(\mathcal{S},\mho)$ need not be a finite s-closed set.

From (ii) of Corollary 4.17, the proof of the next proposition follows.

Proposition 4.18. *The next formula holds for any s-subset* $(\mathcal{S},\mho)$ *of* $(\mathrm{Z},\tau,\mho)$*:*

$$\boldsymbol{B}_{fo}(\boldsymbol{I}_{fo}(\mathcal{S},\mho)) \sqcup \boldsymbol{B}_{fo}(\boldsymbol{C}_{fo}(\mathcal{S},\mho)) \sqsubseteq \boldsymbol{B}_{fo}(\mathcal{S},\mho).$$

The inclusion relation of Proposition 4.18 cannot be replaced by an equality, as demonstrated by the following example.

Example 4.19. *Let* $(\mathcal{S},\mho) = \{(\mathrm{a},\mathbb{Q}),(\mathrm{b},\mathbb{Q})\}$ *be an s-subset an* ST*-space* $(\mathrm{Z},\tau,\mho)$ *provided in Example 3.15. Then,* $\boldsymbol{B}_{fo}(\boldsymbol{I}_{fo}(\mathcal{S},\mho)) \sqcup \boldsymbol{B}_{fo}(\boldsymbol{C}_{fo}(\mathcal{S},\mho)) = \Phi$*, whereas* $\boldsymbol{B}_{fo}(\mathcal{S},\mho) = \{(\mathrm{a},\mathbb{R}),(\mathrm{b},\mathbb{R})\}$.

Proposition 4.20. *Let* $(\mathcal{S},\mho)$ *be an s-subset of* $(\mathrm{Z},\tau,\mho)$.

(i) If $(\mathcal{S},\mho)$ is a finite s-open set, then $\boldsymbol{B}_{fo}(\mathcal{S},\mho) \sqcap (\mathcal{S},\mho) = \Phi$.

(ii) If $(\mathcal{S},\mho)$ of $(\mathrm{Z},\tau,\mho)$ is a finite s-closed set, then $\boldsymbol{B}_{fo}(\mathcal{S},\mho) \sqsubseteq (\mathcal{S},\mho)$.

Proof. (i). Since $(\mathcal{S},\mho)$ is finite s-open, then by (iv) of Corollary 4.17, we get that $\boldsymbol{I}_{fo}(\mathcal{S},\mho) = (\mathcal{S},\mho) = (\mathcal{S},\mho)\Delta\boldsymbol{B}_{fo}(\mathcal{S},\mho)$. Thus, $\boldsymbol{B}_{fo}(\mathcal{S},\mho) \sqcap (\mathcal{S},\mho) = \Phi$.
(ii). Obviously, $\boldsymbol{B}_{fo}(\mathcal{S},\mho) = \boldsymbol{C}_{fo}(\mathcal{S},\mho) \sqcap \boldsymbol{C}_{fo}(\mathcal{S}^c,\mho) \sqsubseteq \boldsymbol{C}_{fo}(\mathcal{S},\mho)$. Since $(\mathcal{S},\mho)$ is finite s-closed, we obtain the proof.

Corollary 4.21. *If* $(\mathcal{S},\mho)$ *is a finite soft-clopen subset of an* ST*-space, then* $\boldsymbol{B}_{fo}(\mathcal{S},\mho) = \Phi$.

Definition 4.22. *Let* $(\mathcal{S},\mho)$ *be an s-subset of an* ST*-space* $(\mathrm{Z},\tau,\mho)$*; the union of soft points* $\mathrm{z_a}$ *satisfying that*

$$[(\mathcal{U},\mho)\backslash \mathrm{z_a}] \sqcup (\mathcal{S},\mho) \neq \Phi \text{ for each finite } s\text{-open set containing } \mathrm{z_a}$$

is said to be an fo-derived operator of $(\mathcal{S},\mho)$; it will be denoted by $\boldsymbol{L}_{fo}(\mathcal{S},\mho)$.

The following properties of fo-derived operators can be proved easily, so we omit their proofs.

Proposition 4.23. *Let* $(\mathcal{S}, \mho)$ *and* $(\mathcal{U}, \mho)$ *be s-subsets of an* ST*-space* $(\mathrm{Z}, \tau, \mho)$*. Then* $(\mathcal{S}, \mho) \sqsubseteq (\mathcal{U}, \mho)$ *implies that* $\boldsymbol{L}_{fo}(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{L}_{fo}(\mathcal{U}, \mho)$*.*

Corollary 4.24. *Let* $(\mathcal{S}, \mho)$ *and* $(\mathcal{U}, \mho)$ *be s-subsets of an* ST*-space* $(\mathrm{Z}, \tau, \mho)$*. Then* $\boldsymbol{L}_{fo}[(\mathcal{S}, \mho) \sqcap (\mathcal{U}, \mho)] \sqsubseteq \boldsymbol{L}_{fo}(\mathcal{S}, \mho) \sqcap \boldsymbol{L}_{fo}(\mathcal{U}, \mho)$*.*

Proposition 4.25. *Let* $(\mathcal{S}, \mho)$ *and* $(\mathcal{U}, \mho)$ *be s-subsets of* $(\mathrm{Z}, \tau, \mho)$*. Then* $\boldsymbol{L}_{fo}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)] = \boldsymbol{L}_{fo}(\mathcal{S}, \mho) \sqcup \boldsymbol{L}_{fo}(\mathcal{U}, \mho)$*.*

Proof. The side $\boldsymbol{L}_{fo}[(\mathcal{S}, \mho) \sqcup (\mathcal{U}, \mho)] \sqsubseteq \boldsymbol{L}_{fo}(\mathcal{S}, \mho) \sqcup \boldsymbol{L}_{fo}(\mathcal{U}, \mho)$ follows from Proposition 4.23. The converse side can be obtained from Corollary 3.11.

Theorem 4.26. *Let* $(\mathcal{S}, \mho)$ *be an s-subset of an* ST*-space* $(\mathrm{Z}, \tau, \mho)$*.*

(i) If $(\mathcal{S}, \mho)$ is a finite s-closed set, then $\boldsymbol{L}_{fo}(\mathcal{S}, \mho) \sqsubseteq (\mathcal{S}, \mho)$.

(ii) $\boldsymbol{C}_{fo}(\mathcal{S}, \mho) = (\mathcal{S}, \mho) \sqcup \boldsymbol{L}_{fo}(\mathcal{S}, \mho)$.

Proof. (i): Assume that $(\mathcal{S}, \mho)$ is a finite s-closed set and $\mathrm{z_a} \notin (\mathcal{S}, \mho)$. Then, $\mathrm{z_a} \in (\mathcal{S}, \mho)^c$ which is a finite s-open set. From the fact that $(\mathcal{S}, \mho)^c \sqcap (\mathcal{S}, \mho) = \Phi$, we obtain that $\mathrm{z_a} \notin \boldsymbol{L}_{fo}(\mathcal{S}, \mho)$. Thus, $\boldsymbol{L}_{fo}(\mathcal{S}, \mho) \sqsubseteq (\mathcal{S}, \mho)$.
(ii): Let $\mathrm{z_a} \notin [(\mathcal{S}, \mho) \sqcup \boldsymbol{L}_{fo}(\mathcal{S}, \mho)]$. Then $\mathrm{z_a} \notin (\mathcal{S}, \mho)$ and $\mathrm{z_a} \notin \boldsymbol{L}_{fo}(\mathcal{S}, \mho)$. Therefore, there is a finite s-open set $(\mathcal{U}, \mho)$ containing $\mathrm{z_a}$ with $(\mathcal{U}, \mho) \sqcap (\mathcal{S}, \mho) = \Phi$. Thus, $\boldsymbol{C}_{fo}(\mathcal{S}, \mho) \sqsubseteq (\mathcal{S}, \mho) \sqcup \boldsymbol{L}_{fo}(\mathcal{S}, \mho)$. On the other hand, it is well known that $(\mathcal{S}, \mho) \sqcup \boldsymbol{L}_{fo}(\mathcal{S}, \mho) \sqsubseteq \boldsymbol{C}_{fo}(\mathcal{S}, \mho)$.

5. Soft fo-continuous functions

Here, we will discuss novel kinds of soft continuous functions and establish their fundamentals. Main characterizations will be provided and the behaviors under decomposition theorems will be scrutinized. With the help of counterexamples, we will describe the transition of these types between realms of soft topologies and classical topologies as well as explain the role of extended soft topologies to guarantee this movement between these realms.

Definition 5.1. *A soft function* $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \mu, \mho)$ *is said to be soft* ff*-continuous (resp., soft* of*-continuous, soft* fo*-continuous) if the inverse image of each finite s-open (resp., finite s-open, s-open) subset is finite s-open (resp., s-open, finite s-open).*

Proposition 5.2. **(i)** *A soft* fo*-continuous function is soft* ff*-continuous.*

(ii) *A soft* ff*-continuous function is soft* of*-continuous.*

It is not necessary for the above proposition's converse to be true. The example which follows illustrates it.

Example 5.3. *Suppose that we have* ST*-spaces* $(\mathbb{R}, \upsilon, \mho)$*,* $(\mathbb{R}, \mu, \mho)$ *and* $(\mathrm{Z}, \tau, \mho)$ *such that* $\mathbb{R}$ *is the set of real numbers,* $\mathrm{Z} = \{\mathrm{y}, \mathrm{z}\}$*,* $\mho = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ *and the soft topologies* υ*,* μ *over* $\mathbb{R}$ *with* $\mho$ *and* τ *over* Z *with* $\mho$ *are given by*

$$\upsilon = \{\widetilde{\mathbb{R}}, (\mathcal{S}, \mho) \sqsubseteq \widetilde{\mathbb{R}} : 1 \in (\mathcal{S}, \mho)\} \cup \{\Phi\},$$

$$\mu = \{\Phi, \widetilde{\mathbb{R}}, (\mathcal{U}, \mho) = \{(\mathrm{a}, \{1\}), (\mathrm{b}, \{1\}), (\mathrm{c}, \{1\})\}\}, \text{ and}$$

$$\tau = \{\Phi, \widetilde{\mathrm{Z}}, (\mathcal{U}, \mho) = \{(\mathrm{a}, \{\mathrm{z}\}), (\mathrm{b}, \{\mathrm{z}\}), (\mathrm{c}, \{\mathrm{z}\})\}\}.$$

Take soft functions $\mathrm{C}_\xi : (\mathbb{R}, \upsilon, \mho) \to (\mathbb{R}, \mu, \mho)$ and $\mathrm{E}_\xi : (\mathbb{R}, \mu, \mho) \to (\mathrm{Z}, \tau, \mho)$ such that the functions $\mathrm{C} : \mathbb{R} \to \mathbb{R}$ and $\xi : \mho \to \mho$ are identities and $\mathrm{E} : \mathbb{R} \to \mathrm{Z}$ is given by

$$\mathrm{E}(1) = \mathrm{z} \text{ and for each } r \neq 1 \text{ we have } \mathrm{E}(r) = \mathrm{y}.$$

Then, C_ξ and E_ξ are soft ff-continuous and soft of-continuous, respectively. In contrast, neither C_ξ is soft fo-continuous nor E_ξ is soft ff-continuous.

Proposition 5.4. *The soft ff-continuity and soft of-continuity are identical under an injective condition.*

Proof. By Proposition 3.5, we get the identity between them.

Some descriptions for these sorts of soft continuity are presented in the following proposition.

Proposition 5.5. *Let* $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \mu, \mho)$ *be a soft function. Then, the following holds:*

(i) If E_ξ is soft ff-continuous, then, for every $\mathrm{z_a} \in \mathrm{Z}$ and every finite s-open subset $(\mathcal{S}, \mho)$ containing $\mathrm{E}_\xi(\mathrm{z_a})$, there is a finite s-open set $(\mathcal{U}, \mho)$ containing $\mathrm{z_a}$ with $\mathrm{E}_\xi(\mathcal{U}, \mho) \sqsubseteq (\mathcal{S}, \mho)$.

(ii) If E_ξ is soft of-continuous, then, for every $\mathrm{z_a} \in \mathrm{Z}$ and every finite s-open set $(\mathcal{S}, \mho)$ containing $\mathrm{E}_\xi(\mathrm{z_a})$, there is an s-open set $(\mathcal{U}, \mho)$ containing $\mathrm{z_a}$ with $\mathrm{E}_\xi(\mathcal{U}, \mho) \sqsubseteq (\mathcal{S}, \mho)$.

(iii) If E_ξ is soft fo-continuous, then, for every $\mathrm{z_a} \in \mathrm{Z}$ and every s-open set $(\mathcal{S}, \mho)$ containing $\mathrm{E}_\xi(\mathrm{z_a})$, there is a finite s-open set $(\mathcal{U}, \mho)$ containing $\mathrm{z_a}$ with $\mathrm{E}_\xi(\mathcal{U}, \mho) \sqsubseteq (\mathcal{S}, \mho)$.

Proof. We prove (i) and one can prove the other cases by following similar arguments.
Let $\mathrm{z_a} \in \mathrm{Z}$ and $(\mathcal{S}, \mho)$ be a finite s-open set containing $\mathrm{E}_\xi(\mathrm{z_a})$. Since $\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)$ is a finite s-open set containing $\mathrm{z_a}$, there is a finite s-open set $(\mathcal{U}, \mho)$ containing $\mathrm{z_a}$ with $\mathrm{E}_\xi(\mathcal{U}, \mho) \sqsubseteq (\mathcal{S}, \mho)$.

The composition theorem for these soft continuity types are exhibited in the following result.

Proposition 5.6. *Let* $\mathrm{C}_\Phi : (\mathrm{X}, \upsilon, \mho) \to (\mathrm{Y}, \omega, \mho)$ *and* $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \mu, \mho)$ *be soft functions. Then,*

(i) If E_ξ and C_Φ are soft ff-continuous (resp., soft fo-continuous), then $\mathrm{C}_\Phi \circ \mathrm{E}_\xi$ is soft ff-continuous (resp., soft fo-continuous), too.

(ii) If E_ξ is soft ff-continuous and C_Φ is soft fo-continuous, then $\mathrm{C}_\Phi \circ \mathrm{E}_\xi$ is soft fo-continuous.

(iii) If E_ξ is soft fo-continuous (resp., soft of-continuous) and C_Φ is soft of-continuous (resp., soft fo-continuous), then $\mathrm{C}_\Phi \circ \mathrm{E}_\xi$ is soft ff-continuous (resp., soft continuous).

Proof. It is straightforward.

Note that the decomposition of soft of-continuous functions need not be a soft of-continuous function.

Now, we will derive some characterizations of soft ff-continuity, soft fo-continuity, and soft of-continuity.

Proposition 5.7. *Let* $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \mu, \mho)$ *be a soft function. Then,* E_ξ *is soft* ff*-continuous (resp., soft* fo*-continuous, soft* of*-continuous) if and only if the pre-image of every finite s-closed (resp., s-closed, finite s-closed) subset is finite s-closed (resp., finite s-closed, s-closed).*

The next theorems and counterexamples will describe the transitions of these soft continuity types in the realms of soft and crisp topologies.

Theorem 5.8. *If* $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \mu, \mho)$ *is a soft* fo*-continuous function, then* $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{a}}) \to (\mathrm{X}, \mu_{\xi(\mathrm{a})})$ *is an* fo*-continuous function.*

Proof. Let U be an open subset of $(\mathrm{X}, \mu_{\xi(\mathrm{a})})$. Then, there is an s-open subset $(\mathcal{S}, \mho)$ of $(\mathrm{X}, \mu, \mho)$ such that $\mathcal{S}(\xi(\mathrm{a})) = U$. By the hypothesis, E_ξ is soft fo-continuous, so $\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)$ is a finite s-open set. It follows from Proposition 3.25 that each component of $\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)$ is a finite open set. We remark that an s-subset $\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho) = (\mathcal{K}, \mho)$ of $(\mathrm{Z}, \tau, \mho)$ is calculated by applying $\mathcal{K}(\mathrm{a}) = \mathrm{E}^{-1}(\mathcal{S}(\xi(\mathrm{a})))$ for each $\mathrm{a} \in \mho$. Accordingly, we obtain that $\mathrm{E}^{-1}(\mathcal{S}(\xi(\mathrm{a}))) = \mathrm{E}^{-1}(U)$ is finite open. Hence, we show the fo-continuity of $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{a}}) \to (\mathrm{X}, \mu_{\xi(\mathrm{a})})$.

In general, the converse of Theorem 5.8 fails, as the next counterexample illustrates.

Example 5.9. *Let* $\mho = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ *and* $\tau = \{\widetilde{\Phi}, \widetilde{\mathrm{Z}}, (\mathcal{S}, \mho)\}$, $\mu = \{\widetilde{\Phi}, \widetilde{\mathrm{Z}}, (\mathcal{U}, \mho)\}$ *be soft topologies on* $\mathrm{Z} = \{\mathrm{y}, \mathrm{z}\}$*, where*

$$(\mathcal{S}, \mho) = \{(\mathrm{a}, \{\mathrm{z}\}), (\mathrm{b}, \mathrm{Z}), (\mathrm{c}, \{\mathrm{z}\})\}, \text{ and}$$

$$(\mathcal{U}, \mho) = \{(\mathrm{a}, \emptyset), (\mathrm{b}, \mathrm{Z}), (\mathrm{c}, \{\mathrm{z}\})\}.$$

Consider the following identity functions $\xi : \mho \to \mho$ and $\mathrm{E} : \mathrm{Z} \to \mathrm{Z}$. Then, $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{a}}) \to (\mathrm{Z}, \mu_{\xi(\mathrm{a})=\mathrm{a}})$, $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{b}}) \to (\mathrm{X}, \mu_{\xi(\mathrm{b})=\mathrm{b}})$ and $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{c}}) \to (\mathrm{Z}, \mu_{\xi(\mathrm{c})=\mathrm{c}})$ are fo-continuous. On the other hand, $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{Z}, \mu, \mho)$ is not a soft fo-continuous function because $\mathrm{E}_\xi^{-1}(\mathcal{U}, \mho) = (\mathcal{U}, \mho) \notin \tau$.

Theorem 5.10. *Let* τ *be an extended soft topology on* Z *with* $\mho$*. Then,* $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \mu, \mho)$ *is soft* fo*-continuous if and only if* $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{a}}) \to (\mathrm{X}, \mu_{\xi(\mathrm{a})})$ *is* fo*-continuous for every parameter of* $\mho$*.*

Proof. In Theorem 5.8, we proved the necessary part.

To clarify the sufficiency, consider $(\mathcal{S}, \mho)$ as an s-open subset of $(\mathrm{X}, \mu, \mho)$. Clearly, a soft subset $\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho) = (\mathcal{K}, \mho)$ of $(\mathrm{Z}, \tau, \mho)$ is given by $\mathcal{K}(\mathrm{a}) = \mathrm{E}^{-1}(\mathcal{S}(\xi(\mathrm{a})))$ for each $\mathrm{a} \in \mho$. Since E is fo-continuous, $\mathrm{E}^{-1}(\mathcal{S}(\xi(\mathrm{a})))$ is a finite open subset of $(\mathrm{Z}, \tau_{\mathrm{a}})$. Since a soft topology τ is extended, so we obtain that $\mathrm{E}_\xi^{-1}(\mathcal{S}, \mho)$ is a finite s-open subset of $(\mathrm{Z}, \tau, \mho)$ by Proposition 3.27. This ends the proof that E_ξ is soft fo-continuous.

Theorem 5.11. *Let* τ *be an extended soft topology on* Z *with* $\mho$*. Then,* $\mathrm{E}_\xi : (\mathrm{Z}, \tau, \mho) \to (\mathrm{X}, \mu, \mho)$ *is soft* ff*-continuous (resp., soft* of*-continuous) if and only if* $\mathrm{E} : (\mathrm{Z}, \tau_{\mathrm{a}}) \to (\mathrm{X}, \mu_{\xi(\mathrm{a})})$ *is* ff*-continuous (resp.,* of*-continuous) for every parameter of* $\mho$*.*

Proof. We prove the theorem in the case of soft ff-continuity and ff-continuity. One may follow a similar technique to demonstrate the case between brackets.

Necessity: Let U be a finite open subset of $(X, \mu_{\xi(a)})$. Then by Proposition 3.27, there is a finite s-open subset $(\mathcal{S}, \mho)$ of $(X, \mu, \mho)$ such that $\mathcal{S}(\xi(a)) = U$. By the hypothesis, E_ξ is soft ff-continuous, so $E_\xi^{-1}(\mathcal{S}, \mho)$ is a finite s-open set. Again by Proposition 3.25, we get that each component of $E_\xi^{-1}(\mathcal{S}, \mho)$ is a finite open set. Accordingly, we obtain that $E^{-1}(\mathcal{S}(\xi(a))) = E^{-1}(U)$ is finite open. Hence, $E : (Z, \tau_a) \to (X, \mu_{\xi(a)})$ is ff-continuous, as required.

Sufficiency: Take an arbitrary finite s-open subset $(\mathcal{S}, \mho)$ of $(X, \mu, \mho)$. Clearly, a soft subset $E_\xi^{-1}(\mathcal{S}, \mho) = (\mathcal{K}, \mho)$ of $(Z, \tau, \mho)$ is given by $\mathcal{K}(a) = E^{-1}(\mathcal{S}(\xi(a)))$ for each $a \in \mho$. Since E is ff-continuous, $E^{-1}(\mathcal{S}(\xi(a)))$ is a finite open subset of (Z, τ_a). Since a soft topology τ is extended, so by Proposition 3.27, $E_\xi^{-1}(\mathcal{S}, \mho)$ is a finite s-open subset of $(Z, \tau, \mho)$. Thus, $E_\xi : (Z, \tau, \mho) \to (X, \mu, \mho)$ is soft ff-continuous.

6. Conclusions and future work

Topology and its uncertain versions like soft topology are vital tools to address many impediments that we face in different situations of our life [16–20]. The s-open sets are the unit of building soft topology, so expanding or restricting this unit creates novel frames of study and is sometimes useful to deal with the phenomena under consideration. So, different forms of s-open sets deserve further and deeper investigation.

In the current work, we have introduced the concept of finite s-open sets as a subclass that strictly lies between the classes of soft-clopen and s-open sets. We have demonstrated that this class has unique features that differ from the well-known extensions of s-open sets; for instance, it constitutes an infra soft topology and fails to be a supra soft topology. We have elucidated the connections between the current class and other known generalizations of s-open sets and determined under which conditions they are equivalent. Also, we have defined the concepts of soft fo-interior and fo-closure operators and showed that these operators maintain the distributive properties of soft unions and intersections, respectively. Ultimately, we have discussed some types of soft continuity that have been inspired by finite s-open sets and described their main characterizations. The sufficient conditions that are required to navigate these types of soft continuity from soft topologies to classical topologies and vice versa have been provided. We have presented numerous interesting examples that illustrate the implications of the obtained findings and interrelations.

Last but not least, as it is well known that topological operators and generalizations of open sets are alternative tools to describe the approximation operators of subsets and heighten their accuracy measures, one of the main focuses of future work is to look at how the proposed class can be applied to information systems to select the optimal choice and make an accurate decision. Also, we plan to familiarize other topological concepts using the class of finite s-open sets, such as soft regularity and normality spaces, soft covering properties, soft connectedness, etc. Moreover, we shall study the ideas discussed here within the context of infra soft topologies [14] and supra soft topologies [23].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number "NBU-FFR-2024-2727-02".

The authors are also grateful to Middle East University, Amman, Jordan for the financial support granted to cover the publication fee of this research article.

Conflict of interest

The authors declare that they have no competing interest.

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