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Research article

On the convergence of a new fourth-order method for finding a zero of a derivative

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Abstract: On the basis of Wang's method, a new fourth-order method for finding a zero of a derivative was presented. Under the hypotheses that the third and fourth order derivatives of nonlinear function were bounded, the local convergence of a new fourth-order method was studied. The error estimate, the order of convergence, and uniqueness of the solution were also discussed. In particular, Herzberger's matrix method was used to obtain the convergence order of the new method to four. By comparing the new method with Wang's method and the same order method, numerical illustrations showed that the new method has a higher order of convergence and accuracy.

Keywords: iterative method; convergence ball; error estimates; Fibonacci sequence; estimate of radius
Mathematics Subject Classification: 65B99, 65H05

1. Introduction

In this paper, we are concerned with the problem of approximating a solution s_* of the equation

$$p'(x) = 0, \tag{1.1}$$

where the differentiable function p is defined in a convex subset N in real space $\mathbb{R}$.

The above issue plays an important role in many applications, particularly in function optimization [1–4]. The K-T (Kuhn-Tucker) condition [5] for the no restriction optimal problem

$$\min p(x), \tag{1.2}$$

claims that if p is a sufficiently differentiable function, the optimal solution for (1.2) must be a solution of (1.1). The solutions of Eq (1.1) can only be not found in the closed form in certain cases. Therefore, iterative methods can often be used to solve such problems.

The majority of iterative methods for solving (1.1) are Newton-like methods [6–8]. Iterative methods have many different properties [9, 10]. Newton's method is defined by

$$s_{n+1} = s_n - \frac{p'(s_{n+1})}{p''(s_n)} \quad (n \geq 0)\ (s_0 \in N). \tag{1.3}$$

Newton's method has quadratic convergence. Under the Lipschitz condition,

$$|p'(s_*)^{-1}(p'(s) - p'(t))| \leq K|s-t| \quad (s, t \in N)(K > 0),$$

the radius of the convergence ball of Newton's method is $r = \frac{2}{3K}$ for finding a zero of function [11]. However, it needs to compute the second-order derivative, and this is frequently difficult to calculate in some cases. To avoid this, one can use the secant method instead

$$s_{n+1} = s_n - \frac{s_n - s_{n-1}}{p'(s_n) - p'(s_{n-1})} p'(s_n) \quad (n \geq 0)\ (s_0, s_{-1} \in N). \tag{1.4}$$

The order of convergence of the secant method is 1.618.... Under the Lipschitz condition,

$$|p''(s_*)^{-1}(p''(s) - p''(t))| \leq K|s-t| \quad (s, t \in N)(K > 0),$$

the radius of the convergence ball of the secant method is $r = \frac{2}{3K}$, at least for finding a zero of derivatives [12]. In addition, M*ü*ller's method is a generalization of the secant method [13]. Under the conditions

$$|p'(s_*)^{-1}p''(s)| \leq F \quad (s \in N)(F > 0), |p'(s_*)^{-1}p'''(s)| \leq M \quad (s \in N)(M > 0), 1215F^2 \leq 32M,$$

the radius of the convergence ball of M*ü*ller's method is $r = \sqrt{\frac{6}{5M}}$, at least for finding a zero of functions.

In order to avoid calculating second-order derivatives and maintaining the order of convergence as two, Wang [14] proposed an iterative method. Wang's method is defined by

$$s_{n+1} = s_n - \frac{p'(s_n)}{\delta(p; s_n, s_{n-1})} \quad (n \geq 0)\ (s_0, s_{-1} \in N), \tag{1.5}$$

where

$$\delta(p; s, t) = \frac{1}{s-t}(4p'(s) - 6\frac{p(s) - p(t)}{s-t} + 2p'(t)). \tag{1.6}$$

The convergence analysis for Wang's method was given under different conditions. Under the conditions,

$$|p''(s_*)^{-1}p'''(s)| \leq X \quad (s \in N, X > 0), \tag{1.7}$$

and

$$|p''(s_*)^{-1}p^{(IV)}(s)| \leq Y \quad (s \in N, Y > 0). \tag{1.8}$$

The radius of the convergence ball of Wang's method is $r = \frac{12}{\sqrt{81X^2+96Y}+9X}$, at least for finding a zero of derivatives [15].

In this paper, based on Wang's method, a new four-order method is presented for analyzing a zero of a derivative. This new method is defined for $s_0, s_{-1} \in N$, and all $n = 0, 1, 2, \ldots$ as follows:

$$\begin{cases} t_n = s_n - \dfrac{p'(s_n)}{\delta(p; s_n, s_{n-1})}, \\ s_{n+1} = t_n - \dfrac{p'(t_n)}{\delta(p; s_n, t_n)}. \end{cases} \tag{1.9}$$

Under the conditions (1.7) and (1.8), the radius of the convergence ball of the new method (1.9) is studied.

The rest part of this paper is laid out as follows: Section 2 is devoted to the convergence ball and error analysis of the new method (1.9) under assumptions that the third and fourth order derivatives of function p are bounded. Also, Herzberger's matrix method is used to obtain the convergence order of the new method to four. In Section 3, two examples are given. By comparing the new method with Wang's method, numerical illustrations show that the new method has a higher order of convergence and accuracy. Finally, conclusions appear in Section 4.

2. Local convergence

This section deals with the convergence ball, error analysis, and the convergence order for the new method (1.9).

Theorem 2.1. Suppose s_* is a solution of Eq (1.1), $p''(s_*) \neq 0$, and the conditions (1.7) and (1.8) hold. Denote

$$R = \frac{12}{\sqrt{81X^2 + 96Y} + 9X}.$$

If $B(s_*, R) \subseteq N$, starting from any initial points $s_0, s_{-1} \in B(s_*, R)$, our method (1.9) generates the sequence $\{s_n\}$, which is well-defined, and converges to its unique solution s_* in $B(s_*, \frac{2}{X}) \cap N$. $B(s_*, R)$ remains in $B(s_*, \frac{2}{X}) \cap N$. Furthermore, we get the following error estimate

$$|s_* - s_n| \leq R(\frac{|s_* - s_1|}{R})^{2F_n}(\frac{|s_* - s_0|}{R})^{F_{n-1}}, (n \geq 1), \tag{2.1}$$

where F_n is the Fibonacci sequence, which is defined by $F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}(n \geq 1)$.

Proof. According to the conditions (1.7) and (1.8) and initial points $s_0, s_{-1} \in B(s_*, R)$, we can obtain $|s_* - s_0| < R, |s_* - s_{-1}| < R$. Suppose that $s_n, s_{n-1}, t_n \in B(s_*, R)$ are defined by our method (1.9).

By (1.6) and Lemma 2 in [16], for $n \geq 0$, we obtain

$$\delta(p; s_n, s_{n-1}) = p''(s_n) - (s_n - s_{n-1})^2 \int_0^1 p^{(IV)}(s_{n-1} + x(s_n - s_{n-1}))x^2(1 - x)dx, \tag{2.2}$$

and

$$\delta(p; s_n, t_n) = p''(s_n) - (s_n - t_n)^2 \int_0^1 p^{(IV)}(t_n + x(s_n - t_n))x^2(1 - x)dx. \tag{2.3}$$

According to the properties of divided differences (see [17]), we have

$$\begin{aligned} (s_* - s_n)p''(s_n) + p'(s_n) &= (s_* - s_n)p''(s_n) + p'(s_n) - p'(s_*) \\ &= (s_* - s_n)p'[s_n, s_n] + (s_* - s_n)p'[s_n, s_*] = -(s_* - s_n)^2 p'[s_n, s_n, s_*], \end{aligned} \tag{2.4}$$

where $p[.,.]$ and $p[.,.,.]$ are first-order and second-order divided differences.

Using (1.9), (2.2), and (2.4), we have

$$\begin{aligned}
s_* - t_n &= s_* - s_n + \frac{p'(s_n)}{\delta(p; s_n, s_{n-1})} \\
&= \frac{(s_* - s_n)p''(s_n) - (s_* - s_n)(s_n - s_{n-1})^2 \int_0^1 f^{(IV)}(t_n + x(s_n - t_n))x^2(1-x)dx + p'(s_n)}{\delta(p; s_n, s_{n-1})} \\
&= \frac{-(s_* - s_n)^2 p'[s_n, s_n, s_*] - (s_* - s_n)(s_n - s_{n-1})^2 \int_0^1 f^{(IV)}(t_n + x(s_n - t_n))x^2(1-x)dx}{\delta(p; s_n, s_{n-1})}.
\end{aligned} \tag{2.5}$$

Using (1.4) and (2.2), we have

$$\begin{aligned}
&|1 - p''(s_*)^{-1}\delta(p; s_n, s_{n-1})| \\
&= |p''(s_*)^{-1}(p''(s_*) - p''(s_n) + (s_n - s_{n-1})^2 \int_0^1 f^{(IV)}(s_{n-1} + x(s_n - s_{n-1}))x^2(1-x)dx)| \\
&= |p''(s_*)^{-1}((s_* - s_n)p''[s_*, s_n] + (s_n - s_{n-1})^2 \int_0^1 f^{(IV)}(s_{n-1} + x(s_n - s_{n-1}))x^2(1-x)dx)|.
\end{aligned} \tag{2.6}$$

Using (1.9), (2.1), and (2.2), we have

$$\begin{aligned}
s_* - s_{n+1} &= s_* - t_n + \frac{p'(t_n)}{\delta(p; s_n, t_n)} \\
&= \frac{(s_* - t_n)p''(t_n) - (s_* - t_n)(s_n - t_n)^2 \int_0^1 f^{(IV)}(t_n + x(s_n - t_n))x^2(1-x)dx + p'(t_n)}{\delta(p; s_n, t_n)} \\
&= \frac{-(s_* - t_n)^2 p'[t_n, t_n, s_*] - (s_* - t_n)(s_n - t_n)^2 \int_0^1 f^{(IV)}(t_n + x(s_n - t_n))x^2(1-x)dx}{\delta(p; s_n, t_n)}.
\end{aligned} \tag{2.7}$$

Using (1.4) and (2.2), we have

$$\begin{aligned}
&|1 - p''(s_*)^{-1}\delta(p; s_n, t_n)| \\
&= |p''(s_*)^{-1}(p''(s_*) - p''(s_n) + (s_n - t_n)^2 \int_0^1 f^{(IV)}(t_n + x(s_n - t_n))x^2(1-x)dx)| \\
&= |p''(s_*)^{-1}((s_* - s_n)p''[s_*, s_n] + (s_n - t_n)^2 \int_0^1 f^{(IV)}(t_n + x(s_n - t_n))x^2(1-x)dx)|.
\end{aligned} \tag{2.8}$$

According to the definition of R, $R = \frac{12}{\sqrt{81X^2+96Y}+9X}$ is easily proved that it is the unique positive solution of the equation

$$\frac{XR}{2} + \frac{YR^2}{3} - (1 - (XR + \frac{Y}{3}R^2)) = 0,$$

which ensures that all iteration points hold in the convergence ball. Under the conditions (1.7) and (1.8), $p''(s_*) \neq 0$, the definition of R, and (2.4), we obtain

$$\begin{aligned}
|1 - p''(s_*)^{-1}\delta(p; s_n, s_{n-1})| &\leq X|s_* - s_n| + \frac{Y}{12}|s_n - s_{n-1}|^2 \\
&< XR + \frac{Y}{3}R^2 < 1.
\end{aligned} \tag{2.9}$$

Thus, we see that $\delta(p;s_n,s_{n-1})\neq 0$ by the Banach lemma, and s_{n+1} is defined. In addition, we have

$$|\delta(p;s_n,s_{n-1})^{-1}p''(s_*)|\leq \frac{1}{1-(X|s_*-s_n|+\frac{Y}{12}|s_n-s_{n-1}|^2)}<\frac{1}{1-(XR+\frac{Y}{3}R^2)}. \tag{2.10}$$

Using (2.5), (2.10), and the definition of R, we have

$$|s_*-t_n|\leq \frac{\frac{X}{2}|s_*-s_n|^2+\frac{Y}{12}|s_*-s_n||s_n-s_{n-1}|^2}{1-(X|s_*-s_n|+\frac{Y}{12}|s_n-s_{n-1}|^2)}\leq \frac{\frac{XR}{2}+\frac{YR^2}{3}}{1-(XR+\frac{Y}{3}R^2)}|s_*-s_n|=|s_*-s_n|<R. \tag{2.11}$$

So, we obtain $t_n\in B(s_*,R)$.

By the similar method to that of (2.10), we have

$$|\delta(p;s_n,t_n)^{-1}p''(s_*)|\leq \frac{1}{1-(X|s_*-t_n|+\frac{Y}{12}|s_n-t_n|^2)}<\frac{1}{1-(XR+\frac{Y}{3}R^2)}. \tag{2.12}$$

Using (2.7), (2.12), and the definition of R, we have

$$|s_*-s_{n+1}|\leq \frac{\frac{X}{2}|s_*-t_n|^2+\frac{Y}{12}|s_*-t_n||s_n-t_n|^2}{1-(X|s_*-t_n|+\frac{Y}{12}|s_n-t_n|^2)}\leq \frac{\frac{XR}{2}+\frac{YR^2}{3}}{1-(XR+\frac{Y}{3}R^2)}|s_*-t_n|=|s_*-t_n|<R. \tag{2.13}$$

So, we obtain $s_{n+1}\in B(s_*,R)$. Thus, by induction hypotheses, starting from any initial points $s_0,s_{-1}\in B(s_*,R)$, the sequence $\{s_n\}$ generated by our method (1.9) is defined, $s_n\in B(s_*,R)$, so

$$|s_*-s_n|<R,(n\geq 0). \tag{2.14}$$

Denote $\epsilon_n=s_*-s_n,\epsilon_{n,t}=s_*-t_n,(n\geq 0)$. By (2.11), we have

$$|\epsilon_{n,t}|\leq \frac{\frac{X}{2}|\epsilon_n|^2+\frac{Y}{12}|\epsilon_n||\epsilon_n-\epsilon_{n-1}|^2}{1-(X|\epsilon_n|+\frac{Y}{12}|\epsilon_n-\epsilon_{n-1}|^2)}. \tag{2.15}$$

$$|\epsilon_{n+1}|\leq \frac{\frac{X}{2}|\epsilon_n|^2+\frac{Y}{12}|\epsilon_n||\epsilon_n-\epsilon_{n,t}|^2}{1-(X|\epsilon_n|+\frac{Y}{12}|\epsilon_n-\epsilon_{n,t}|^2)}. \tag{2.16}$$

So, we have

$$|\epsilon_n|\leq|\epsilon_{n-1}|,|\epsilon_{n+1}|\leq|\epsilon_{n,t}|,(n\geq 1). \tag{2.17}$$

Using (2.15) and (2.17), we get

$$\begin{aligned}|\epsilon_{n,t}|&\leq \frac{\frac{X}{2}|\epsilon_n|^2+\frac{Y}{12}|\epsilon_n||\epsilon_n-\epsilon_{n-1}|^2}{1-(X|\epsilon_n|+\frac{Y}{12}|\epsilon_n-\epsilon_{n-1}|^2)}\leq\frac{\frac{X}{2}|\epsilon_n||\epsilon_{n-1}|+\frac{YR}{3}|\epsilon_n||\epsilon_{n-1}|}{1-(XR+\frac{Y}{3}R^2)}\\&\leq \frac{\frac{XR}{2}+\frac{YR}{3}}{1-XR-\frac{Y}{3}R^2}|\epsilon_n||\epsilon_{n-1}|,(n\geq 1).\end{aligned} \tag{2.18}$$

Using (2.16) and (2.17), we get

$$\begin{aligned}|\epsilon_{n+1}|&\leq \frac{\frac{X}{2}|\epsilon_n|^2+\frac{Y}{12}|\epsilon_n||\epsilon_n-\epsilon_{n,t}|^2}{1-(X|\epsilon_n|+\frac{Y}{12}|\epsilon_n-\epsilon_{n,t}|^2)}\leq\frac{\frac{X}{2}|\epsilon_n||\epsilon_{n,t}|+\frac{YR}{3}|\epsilon_n||\epsilon_{n,t}|}{1-(XR+\frac{Y}{3}R^2)}\\&\leq \frac{\frac{XR}{2}+\frac{YR}{3}}{1-XR-\frac{Y}{3}R^2}|\epsilon_n||\epsilon_{n,t}|\leq\frac{(\frac{XR}{2}+\frac{YR}{3})^2}{(1-XR-\frac{Y}{3}R^2)^2}|\epsilon_n|^2|\epsilon_{n-1}|,(n\geq 1).\end{aligned} \tag{2.19}$$

Denote $\rho_n = \frac{|s_* - s_n|}{R} (n \geq 0)$. Using the definition of R, we have

$$\frac{|\epsilon_{n,t}|}{R} \leq \frac{(\frac{XR}{2} + \frac{YR}{3})R}{1 - XR - \frac{Y}{3}R^2} \frac{|\epsilon_n|}{R} \frac{|\epsilon_{n-1}|}{R}, (n \geq 1), \tag{2.20}$$

and

$$\frac{|\epsilon_{n+1}|}{R} \leq \frac{|\epsilon_n|}{R} \frac{|\epsilon_{n,t}|}{R} \leq \frac{|\epsilon_n|^2}{R^2} \frac{|\epsilon_{n-1}|}{R}, (n \geq 1). \tag{2.21}$$

Then, we obtain

$$\rho_{n+1} \leq \rho_n^2 \rho_{n-1}, (n \geq 1). \tag{2.22}$$

Using (2.22), by induction hypotheses, we easily obtain that

$$\rho_n \leq \rho_1^{2F_n} \rho_0^{F_{n-1}}, (n \geq 1), \tag{2.23}$$

where F_n is the Fibonacci sequence, and $F_0 = F_1 = 1, F_{n+1} = F_n + F_{n+1} (n \geq 1)$. By (2.23), we can prove that the error Eq (2.1) holds.

Finally, to prove the uniqueness of the solution s_*, suppose there exists a second solution $y_* \in B(s_*, \frac{2}{X})$, then $p(y_*) = 0$. Denote $Q = p'[y_*, s_*]$. Since $Q(y_* - s_*) = p'(y_*) - p'(s_*) = 0$, if Q is invertible, then $y_* = s_*$. In fact, according to $|p''(s_*)^{-1} p'''(s)| \leq X (s \in N)$, we obtain

$$\begin{aligned} |1 - p''(s_*)^{-1} Q| &= |p''(s_*)^{-1} \int_0^1 (p''(s_*) - p''(x s_* + (1-x) y_*)) dx| \\ &\leq X \int_0^1 (1-x) |s_* - y_*| dx = \frac{X}{2} |s_* - y_*| < 1. \end{aligned} \tag{2.24}$$

Thus, we ensure that the operator Q is invertible by the Banach Lemma. According to the definition of R, we deduce that the ball $B(s_*, \frac{2}{X}) \bigcap N$ is larger than the convergence ball $B(s_*, R)$.

Theorem 2.2. The new method (1.9) has convergence order of at least $\rho(A^{(2)}) = 4$, where $\rho(A^{(2)})$ is the spectral radius of the matrix

$$A^{(2)} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

Proof. We will use Herzberger's matrix method [18] to analyze convergence order. Denote the lower bound of order of a single step s-point method $s_k = G(s_{k-1}, s_{k-2}, \ldots, s_{k-s})$ as the spectral radius of the matrix $A^{(s)} = (a_{ij})$, related to this method with the elements:

$$\begin{aligned} &a_{1,j} = \text{amount of information required at point } s_{k-j} (j = 1, 2, \ldots, s), \\ &a_{i,i-1} = 1 (i = 2, 3, \ldots, s), \\ &a_{i,j} = 0 \text{ otherwise.} \end{aligned} \tag{2.25}$$

Additionally, the lower bound of order of an s-step method $G = G_1 \circ G_2 \circ \cdots \circ G_s$ is the spectral radius of the product of matrices

$$A^{(s)} = A_1 \cdot A_2 \cdots A_s.$$

According to our method (1.9), the respective matrices are formed,

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

Therefore,

$$A^{(2)} = A_1 \cdot A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

The characteristic polynomial of the matrix $A^{(2)}$ is

$$P_2(\lambda) = \lambda^2 - 5\lambda + 4. \tag{2.26}$$

So, its roots are 1, 4; thus, the spectral radius of the matrix $A^{(2)}$ is $\rho(A^{(2)}) = 4$, which gives the lower bound of order of our method.

Remark 2.3. Whether the error estimation of our method (1.9) matches its convergence order remains to be further studied.

Remark 2.4. Whether the estimate radius R of the convergence ball of our method (1.9) is optimal remains to be further studied.

3. Numerical examples

In this section, we apply the following two numerical examples to compute the above convergence ball result, then our method (1.9) is compared with Wang's method (1.5) and fourth-order method (3.1) by numerical experiments.

Wang et al. in [19] proposed the following fourth-order method:

$$\begin{cases} t_n = s_n - \dfrac{1}{B} f(s_n), \\ s_{n+1} = t_n - (3 - \dfrac{2}{B} f[s_n, t_n]) \dfrac{1}{B} f(t_n), \end{cases} \tag{3.1}$$

where $B = f[w_n, v_n], w_n = s_n + f(s_n), v_n = s_n - f(s_n)$.

Example 3.1. Let $N = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Define the function p_1 on N by

$$p_1(x) = sin(x) - x^2 - x. \tag{3.2}$$

Additionally, a root of $p_1'(x) = 0$ is $s_* = 0$ in N. Since

$$p_1'(x) = cos(x) - 2x - 1, \;\; p_1''(x) = -sin(x) - 2, \;\; p_1'''(x) = -cos(x), \;\; p_1^{(IV)}(x) = sin(x), \;\; p_1''(s_*) = -2, \tag{3.3}$$

for all $x \in N$, we have

$$|p_1''(s_*)^{-1} p_1'''(x)| \le \frac{1}{2}, \;\; |p_1''(s_*)^{-1} p_1^{(IV)}(x)| \le \frac{1}{2}. \tag{3.4}$$

So, $X = \frac{1}{2}, Y = \frac{1}{2}$. By applying Theorem 2.1, the radius of the convergence ball of our method is

$$R = \frac{12}{\sqrt{81X^2 + 96Y} + 9X} \approx 0.9403.$$

Example 3.2. Let $N = [-1, 1]$. Define the function p_2 on N by

$$p_2(x) = e^x + x^2 - x. \tag{3.5}$$

Additionally, a root of $p_2'(x) = 0$ is $s_* = 0$ in N. Since

$$p_2'(x) = e^x + 2x - 1,\ \ p_2''(x) = e^x + 2,\ \ p_2'''(x) = e^x,\ \ p_2^{(IV)}(x) = e^x,\ \ p_2''(s_*) = 3, \tag{3.6}$$

for all $x \in N$, we have

$$|p_2''(s_*)^{-1} p_2'''(x)| \leq \frac{e}{3},\ \ |p_2''(s_*)^{-1} p_2^{(IV)}(x)| \leq \frac{e}{3}. \tag{3.7}$$

So, $X = \frac{e}{3}$, $Y = \frac{e}{3}$. By applying Theorem 2.1, the radius of the convergence ball of our method is

$$R = \frac{12}{\sqrt{81X^2 + 96Y} + 9X} \approx 0.5841.$$

Applying the above two functions, Wang's method (1.5), fourth-order method (3.1), and our method (1.9) are compared by numerical experiments. In Table 1, the absolute errors $|s_n - s_*|$ and initial point s_0 are shown. The number of iterations for methods (1.5), (3.1), and (1.9) are five. CPU is the computational time and ρ is the order of computational convergence.

Table 1. Numerical results of methods (1.5), (3.1), and (1.9).

$Method$	$Function$	s_0	$iter$	$\lvert s_n - s_*\rvert$	CPU	ρ
(1.5)	f_1	0.3	5	$3.81105e - 80$	0.3125	2.0
(3.1)	f_1	0.3	5	$2.43616e - 957$	0.5000	4.0
(1.9)	f_1	0.3	5	$4.84331e - 1179$	0.5938	4.0
(1.5)	f_2	0.5	5	$8.08287e - 85$	0.3438	2.0
(3.1)	f_2	0.5	5	$4.40115e - 749$	0.5313	4.0
(1.9)	f_2	0.5	5	$6.63462e - 1202$	0.5313	4.0

In Table 1, our method (1.9) has the same initial point and the number of iterations as methods (1.5) and (3.1). Our method (1.9) has roughly the same CPU time as the same order method (3.1). However, our method has a higher order of convergence and higher accuracy.

4. Conclusions

In this paper, the convergence ball of a new fourth-order method for finding a zero of a derivative was studied by using hypotheses that the third-order and fourth-order derivatives of function p were bounded. The error estimate, order of convergence, and uniqueness of the solution were also discussed. In addition, Herzberger's matrix method was used to obtain the convergence order of our method to four. Finally, the convergence criteria was verified by two numerical examples, and our method was compared with Wang's method and the fourth-order method by numerical experiments. The experimental results showed that our method has the higher order of convergence and higher accuracy, so our method is finer.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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