AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 10333–10352.
DOI: 10.3934/math.2024505
Received: 08 January 2024
Revised: 23 February 2024
Accepted: 07 March 2024
Published: 14 March 2024

Research article

Reliability analysis for two populations Nadarajah-Haghighi distribution under Joint progressive type-II censoring

Mustafa M. Hasaballah[1,*], Yusra A. Tashkandy[2], Oluwafemi Samson Balogun[3] and M. E. Bakr[2]

[1] Marg Higher Institute of Engineering and Modern Technology, Cairo 11721, Egypt

[2] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

[3] Department of Computing, University of Eastern Finland, Kuopio FI-70211, Finland

[*] **Correspondence:** Email: mustafamath7@yahoo.com.

Abstract: In order to evaluate the competitive advantages and dependability of two products in a competitive environment, comparative lifespan testing becomes essential. We examine the inference problems that occur when two product lines follow the Nadarajah-Haghighighi distribution in the setting of joint type-II censoring. In the present study, we derived the maximum likelihood estimates for the Nadarajah-Haghighi population parameters. Additionally, a Fisher information matrix was constructed based on these maximum likelihood estimations. Furthermore, Bayesian estimators and their corresponding posterior risks were calculated, considering both gamma and non-informative priors under symmetric and asymmetric loss functions. To assess the performance of the overall parameter estimators, we conducted a Monte Carlo simulation using numerical methods. Lastly, a real data analysis was carried out to validate the accuracy of the models and methods discussed.

Keywords: Nadarajah-Haghighighi distributions; Bayesian estimation; Joint progressive censoring scheme; MCMC method
Mathematics Subject Classification: 62N05, 62F10

1. Introduction

1.1. *Joint progressive type-II censoring scheme*

Applying censoring techniques to a single population entails specific limitations. While progressive type-II censoring permits the exclusion of specific data, obtaining a sufficient number of observations remains a costly endeavor. Moreover, if our emphasis is on understanding the interactions and interdependencies among populations, experiments conducted solely on a single population may not provide conclusive evidence.

The joint progressive type-II censoring scheme (JPT-II-CS) provides notable benefits for comparing the lifespan distributions of products produced by different units within the same facility. JPT-II-CS has attracted considerable attention in the research community, with numerous authors exploring JPT-II-CS and related inference methods in the literature. For instance, Rasouli and Balakrishnan [32], Doostparast et al. [18], Balakrishnan et al. [15], Mondal and Kundu [26], Krishna and Goel [24], Goel and Krishna [21], and Goel and Krishna [22] have contributed to this body of work.

Recently, a multitude of researchers have investigated a range of strategies and different lifetime models in various applications. For more detailed information, please consult the publications by Pandey and Srivastava [29], Qiao and Gui [31], Abdel-Aty et al. [1], Ferreira and Silva [20], Celik and Guloksuz [16], Chiou and Chen [17], Panahi and Lone [28], Yan et al. [34], Asadi et al. [4, 7–12].

Within the JPT-II-CS framework, two samples, one from Population-A (Pop-A) and the other from Population-B (Pop-B), each comprising m and n units, are amalgamated for a life-testing experiment. Let k be the total number of observed failures in this experiment. Moreover, let $R_1, R_2, ..., R_k$ signify the number of units removed, such that $\sum_{i=1}^{k}(R_i + 1) = m + n$, where $R_i = S_i + T_i$, and R_i is the sum of units removed from Pop-A (S_i) and Pop-B (T_i) at the i-th stage. Considering the joint sample upon the first failure event denoted as W_1, $R_1 = S_1 + T_1$ units are randomly chosen from the remaining pool of $m + n - 1$ surviving units. Here S_1 and T_1 represent the units removed from Population (A) and Population (B) respectively. In a similar fashion, at the second stage, $R_2 = S_2 + T_2$ units are randomly selected from the remaining pool of $m + n - 2 - R_1$ surviving units and so forth. Finally, at the k-th failure, all the remaining $R_k = n + m - k - \sum_{i=1}^{k} R_i$ surviving units are removed. It is crucial to highlight that when $R_1 = R_2 = ... = R_k = 0$, it implies that $S_i = T_i = 0$ for all $i = 1, 2, ..., k$, simplifying the JPT-II-CS to the scenario of complete samples. Furthermore, if $R_1 = R_2 = ... = R_{k-1} = 0$, resulting in $R_k = n + m - k$, then the censoring scheme transforms back to a conventional Type-II censoring scheme for two distinct samples.

In the context of the JPT-II-CS framework, the observed data comprises (W, Z, S), where W is depicted as $(W_1, W_2, ..., W_k)$, where k is a specified integer such that $1 \leqslant k < m + n$. Z is denoted as $(Z_1, Z_2, ..., Z_k)$, where

$$Z_i = \begin{cases} 1 & \text{if } W_i \text{ drawn from the X-sample,} \\ 0 & \text{if } W_i \text{ drawn from the Y-sample .} \end{cases}$$

Moreover, S is depicted as $(S_1, S_2, ..., S_k)$. We employ k_1 to indicate the number of failed units from Pop-A, and k_2 to signify the count of failed units from Pop-B, where $k_1 = \sum_{i=1}^{k} Z_i$ and $k_2 = k - \sum_{i=1}^{k} Z_i$, respectively.

1.2. Nadarajah-Haghighi distribution

The exponential distribution is renowned for its constant failure rate and memoryless property. However, when exploring phenomena in lifetime and reliability studies, opting for the exponential model may not be conducive, as it lacks the ability to exhibit both monotone (increasing and decreasing) and non-monotone (bathtub and upside-down bathtub) failure rate behaviors. To address this limitation and provide flexibility, alternative generalizations of the exponential distribution have been proposed, such as the Nadarajah–Haghighi distribution (NHD). Introduced by Nadarajah and

Haghighi [27], this distribution serves as a widespread statistical extension of the conventional exponential distribution and has been recently named the NHD, abbreviating the authors' names. The cumulative distribution function (CDF) of the NHD is expressed as follows (see Figure 1):

$$F(x) = 1 - e^{\left[1-(1+\alpha x)^{\gamma}\right]}, \qquad x > 0, \alpha, \gamma > 0. \tag{1.1}$$

Additionally, its corresponding probability density function (PDF) is given by:

$$f(x) = \alpha\gamma(1 + \alpha x)^{\gamma-1} e^{\left[1-(1+\alpha x)^{\gamma}\right]}, \qquad x > 0, \alpha, \gamma > 0. \tag{1.2}$$

In this context, γ and α denote the shape and scale parameters, respectively. By setting $\gamma = 1$ for the variable in Eq (1.1), the inverted exponential distribution is presented as a particular case. Nadarajah and Haghighi [27] demonstrated that the density of the NHD may exhibit a decreasing trend. Furthermore, its shapes, both unimodal and the hazard rate function, can resemble the increasing, decreasing, or constant patterns observed in gamma, Weibull, and generalized-exponential distributions.

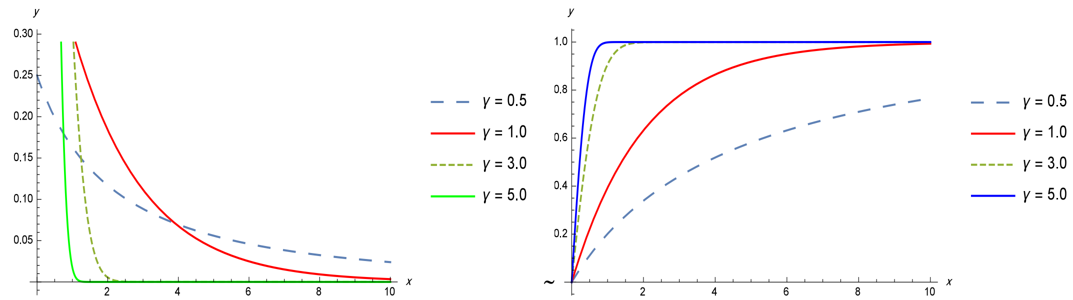

Figure 1. PDF and CDF of the NHD for different values of parameters.

Several researchers have delved into the estimation challenges associated with the NHD. For instance, Selim [33] investigated the estimation and prediction of the NHD based on record values. Inferences and ideal censoring techniques for the progressively first-failure censored NHD were covered by Ashour et al. [5]. Elshahhat et al. [19] used type-II adaptive progressive hybrid censoring with applications to investigate inferences for Nadarajah-Haghighighi parameters. The half logistic inverted NHD was studied by Alotaibi et al. [2] under ranked set Sampling with applications. With applications to COVID-19 mortality data and cancer data, Azimi and Esmailian [6] examined a novel generalisation of the NHD.

In this publication, we tackle the challenges associated with estimating the NHD. We present both Bayesian and maximum likelihood estimators (MLEs) for the NHD using the JPT-II-CS, accompanied by their corresponding confidence intervals. We derive Bayes estimators for the squared error (SE) and linear exponential (LINEX) loss functions, assuming independent gamma priors. Monte Carlo simulations are conducted to assess the performance of various estimators, evaluating them based on mean squared error (MSE) and average values. Additionally, we scrutinize the average confidence lengths of the 95% two-sided interval estimations to gauge their effectiveness. Finally, we illustrate the practical application of our approach using real-world datasets.

The following outlines the order of the remaining sections in the paper following this introduction: Section 2 examines the concept of sampling significance. Section 3 explains the process of obtaining maximum probability estimates for the unknown parameters. The construction of the Fisher information matrix (FIM) using MLEs is done in Section 4. In Section 5, Bayes estimation is performed with various loss functions using gamma and non-informative priors. Section 6 presents a Monte Carlo simulation analysis along with the results, serving as model validation. Two real data analyses to demonstrate each of the suggested methodologies are carried out in Section 7. Section 8 provides a concise summary of the main points.

2. Model description

In a particular context, we possess independent and identically distributed (iid) lifetimes, denoted as $X_1, X_2, ..., X_m$, originating from Pop-A. These lifetimes adhere to a NHD characterized by the CDF F(x) and the PDF f(x). In the same vein, we possess $Y_1, Y_2, ..., Y_n$ as iid lifetimes from Pop-B, also adhering to a NHD defined by the CDF G(y) and the PDF g(y). Given a JPT-II-CS $(m, n); R_1, R_2, ..., R_k$, we can define $B = (W, Z, S)$ as the JPT-II-CS of size k obtained from Pop-A and Pop-B. It comprises $(w_1, z_1, s_1), (w_2, z_2, s_2), ..., (w_k, z_k, s_k)$. The likelihood function based on the observed JPT-II-CS can be expressed as follows:

$$L(\alpha_1, \alpha_2, \gamma_1, \gamma_2|data) = C \prod_{i=1}^{r} \Big[[f(w_i)]^{z_i}[g(w_i)]^{1-z_i}\Big][\bar{F}(w_i)]^{s_i}[\bar{G}(w_i)]^{t_i}, \tag{2.1}$$

where $w_1 \leq w_2 \leq \ldots \leq w_k$, $\bar{F} = 1 - F$, $\bar{G} = 1 - G$, $\sum_{i=1}^{k} s_i + \sum_{i=1}^{k} t_i = \sum_{i=1}^{k} R_i$ and $C = B_1 B_2$ with

$$B_1 = \prod_{j=1}^{k} \left\{ \left(m - \sum_{i=1}^{j-1} z_i - \sum_{i=1}^{j-1} s_i \right) z_j + \left(n - \sum_{i=1}^{j-1} (1 - z_i) - \sum_{i=1}^{j-1} (R_i - s_i) \right) (1 - z_j) \right\},$$

$$B_2 = \prod_{j=1}^{k} \left\{ \frac{\binom{m - \sum_{i=1}^{j-1} z_i - \sum_{i=1}^{j-1} s_i}{s_i} \binom{n - \sum_{i=1}^{j-1}(1 - z_i) - \sum_{i=1}^{j-1}(R_i - s_i)}{t_i}}{\binom{m + n - j - \sum_{i=1}^{j-1} R_i}{R_i}} \right\}.$$

3. Maximum likelihood estimation

MLE stands as a powerful statistical method widely employed in the field of data analysis and parameter estimation. In this manuscript, we employ MLE to estimate the parameters of the NHD. The main principle of MLE is to find the parameter values that maximize the likelihood function, which quantifies the probability of observing the given data under a specific set of parameter values. By maximizing the likelihood, we aim to identify the parameter values that make the observed data most plausible. The MLE estimates are obtained by solving an optimization problem, either analytically or numerically, to find the parameter values that maximize the likelihood function. These estimates are known to possess desirable properties, such as efficiency and consistency, which means that they converge to the true parameter values as the sample size increases. MLE is a versatile and widely applicable method that plays a crucial role in statistical inference and parameter estimation.

The following outcome is obtained by applying the CDF and PDF from Eqs (1.1) and (1.2) to the likelihood equation given in (2.1):

$$\begin{aligned} L(\alpha_1, \alpha_2, \gamma_1, \gamma_2|data) =& \alpha_1^{k_1}\gamma_1^{k_1}\alpha_2^{k_2}\gamma_2^{k_2} e^{(\gamma_1-1)\sum_{i=1}^{k} z_i \ln(1+\alpha_1 w_i)} e^{\sum_{i=1}^{k} z_i[1-(1+\alpha_1 w_i)^{\gamma_1}]} \\ &\times e^{(\gamma_2-1)\sum_{i=1}^{k}(1-z_i)\ln(1+\alpha_2 w_i)} e^{\sum_{i=1}^{k}(1-z_i)[1-(1+\alpha_2 w_i)^{\gamma_2}]} \\ &\times e^{\sum_{i=1}^{k} s_i[1-(1+\alpha_1 w_i)^{\gamma_1}]} e^{\sum_{i=1}^{k} t_i[1-(1+\alpha_2 w_i)^{\gamma_2}]}. \end{aligned} \tag{3.1}$$

Given that the log-likelihood function exhibits the same monotonic behavior as the likelihood function, it is expressed as follows:

$$\begin{aligned} \ell(\alpha_1, \alpha_2, \gamma_1, \gamma_2|data) =& k_1 \ln \alpha_1 + k_1 \ln \gamma_1 + k_2 \ln \alpha_2 + k_2 \ln \gamma_2 + (\gamma_1 - 1)\sum_{i=1}^{k} z_i \ln(1 + \alpha_1 w_i) \\ &+ \sum_{i=1}^{k} z_i[1 - (1 + \alpha_1 w_i)^{\gamma_1}] + (\gamma_2 - 1)\sum_{i=1}^{k}(1 - z_i)\ln(1 + \alpha_2 w_i) \\ &+ \sum_{i=1}^{k}(1 - z_i)[1 - (1 + \alpha_2 w_i)^{\gamma_2}] + \sum_{i=1}^{k} s_i[1 - (1 + \alpha_1 w_i)^{\gamma_1}] \\ &+ \sum_{i=1}^{k} t_i[1 - (1 + \alpha_2 w_i)^{\gamma_2}]. \end{aligned} \tag{3.2}$$

By partially differentiating Eq (3.2) with respect to $\alpha_1, \alpha_2, \gamma_1$ and γ_2 and setting the derivatives equal to zero, the resulting equations are as follows.

$$\frac{k_1}{\alpha_1} + (\gamma_1 - 1)\sum_{i=1}^{k}\frac{z_i w_i}{(1 + \alpha_1 w_i)} - \sum_{i=1}^{k} z_i w_i \gamma_1 (1 + \alpha_1 w_i)^{\gamma_1 - 1} - \sum_{i=1}^{k} s_i w_i \gamma_1 (1 + \alpha_1 w_i)^{\gamma_1 - 1} = 0, \tag{3.3}$$

$$\frac{k_2}{\alpha_2} + (\gamma_2 - 1)\sum_{i=1}^{k}\frac{(1 - z_i) w_i}{(1 + \alpha_2 w_i)} - \sum_{i=1}^{k}(1 - z_i) w_i \gamma_2 (1 + \alpha_2 w_i)^{\gamma_2 - 1} - \sum_{i=1}^{k} t_i w_i \gamma_2 (1 + \alpha_2 w_i)^{\gamma_2 - 1} = 0, \tag{3.4}$$

$$\frac{k_1}{\gamma_1} + \sum_{i=1}^{k} z_i \ln(1 + \alpha_1 w_i) - \sum_{i=1}^{k} z_i (1 + \alpha_1 w_i)^{\gamma_1}\ln(1 + \alpha_1 w_i) - \sum_{i=1}^{k} s_i (1 + \alpha_1 w_i)^{\gamma_1}\ln(1 + \alpha_1 w_i) = 0, \tag{3.5}$$

$$\frac{k_2}{\gamma_2} + \sum_{i=1}^{k}(1 - z_i)\ln(1 + \alpha_2 w_i) - \sum_{i=1}^{k}(1 - z_i)(1 + \alpha_2 w_i)^{\gamma_2}\ln(1 + \alpha_2 w_i) - \sum_{i=1}^{k} t_i (1 + \alpha_2 w_i)^{\gamma_2}\ln(1 + \alpha_2 w_i) = 0. \tag{3.6}$$

It's evident, as indicated in Eqs (3.3)–(3.6), that explicit expressions for the MLEs $\alpha_1, \alpha_2, \gamma_1$, and γ_2 are not available. Hence, we recommend using numerical iterative methods, like the Newton-Raphson procedure, to derive the values of $\alpha_1, \alpha_2, \gamma_1$, and γ_2.

4. Fisher information matrix

We investigate approximate confidence intervals in this context for the unknown parameters $(\alpha_1, \alpha_2, \gamma_1, \gamma_2)$, using large sample approximations of the MLEs, which are often known as asymptotic theory. We do this by using the observed FIM to calculate the MLEs' asymptotic variance for the unknown parameters. Next, the observed FIM is represented as:

$$I^{-1}(\alpha_1, \alpha_2, \gamma_1, \gamma_2) = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha_1^2} & -\frac{\partial^2 \ell}{\partial \alpha_1 \partial \alpha_2} & -\frac{\partial^2 \ell}{\partial \alpha_1 \partial \gamma_1} & -\frac{\partial^2 \ell}{\partial \alpha_1 \partial \gamma_1} \\ -\frac{\partial^2 \ell}{\partial \alpha_2 \partial \alpha_1} & -\frac{\partial^2 \ell}{\partial \alpha_2^2} & -\frac{\partial^2 \ell}{\partial \alpha_2 \partial \gamma_1} & -\frac{\partial^2 \ell}{\partial \alpha_2 \partial \gamma_2} \\ -\frac{\partial^2 \ell}{\partial \gamma_1 \partial \alpha_1} & -\frac{\partial^2 \ell}{\partial \gamma_1 \partial \alpha_2} & -\frac{\partial^2 \ell}{\partial \gamma_1^2} & -\frac{\partial^2 \ell}{\partial \gamma_1 \partial \gamma_2} \\ -\frac{\partial^2 \ell}{\partial \gamma_2 \partial \alpha_1} & -\frac{\partial^2 \ell}{\partial \gamma_2 \partial \alpha_2} & -\frac{\partial^2 \ell}{\partial \gamma_2 \partial \gamma_1} & -\frac{\partial^2 \ell}{\partial \gamma_2^2} \end{pmatrix}^{-1}, \tag{4.1}$$

$$I^{-1}(\alpha_1, \alpha_2, \gamma_1, \gamma_2) = \begin{pmatrix} \widehat{var}(\hat{\alpha_1}) & cov(\hat{\alpha_1}, \hat{\alpha_2}) & cov(\hat{\alpha_1}, \hat{\gamma_1}) & cov(\hat{\alpha_1}, \hat{\gamma_2}) \\ cov(\hat{\alpha_2}, \hat{\alpha_1}) & \widehat{var}(\hat{\alpha_2}) & cov(\hat{\alpha_2}, \hat{\gamma_1}) & cov(\hat{\alpha_2}, \hat{\gamma_2}) \\ cov(\hat{\gamma_1}, \hat{\alpha_1}) & cov(\hat{\gamma_1}, \hat{\alpha_2}) & \widehat{var}(\hat{\gamma_1}) & cov(\hat{\gamma_1}, \hat{\gamma_2}) \\ cov(\hat{\gamma_2}, \hat{\alpha_1}) & cov(\hat{\gamma_2}, \hat{\alpha_2}) & cov(\hat{\gamma_2}, \hat{\gamma_1}) & \widehat{var}(\hat{\gamma_2}) \end{pmatrix}. \tag{4.2}$$

Derived from the log-likelihood function in (3.2), the following is evident:

$$\begin{aligned} &\frac{-k_1}{\alpha_1^2} + (\gamma_1 - 1)\sum_{i=1}^{k} \frac{-2 z_i w_i^2 (1+\alpha_1 w_i)}{(1+\alpha_1 w_i)^2} - \sum_{i=1}^{k} z_i w_i^2 \gamma_1(\gamma_1 - 1)(1+\alpha_1 w_i)^{\gamma_1 - 2} \\ &\quad - \sum_{i=1}^{k} s_i w_i^2 \gamma_1(\gamma_1 - 1)(1+\alpha_1 w_i)^{\gamma_1 - 2} = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\frac{-k_2}{\alpha_2^2} + (\gamma_2 - 1)\sum_{i=1}^{k} \frac{-2(1 - z_i) w_i^2 (1+\alpha_2 w_i)}{(1+\alpha_2 w_i)^2} - \sum_{i=1}^{k} (1 - z_i) w_i^2 \gamma_2(\gamma_2 - 1)(1+\alpha_2 w_i)^{\gamma_2 - 2} \\ &\quad - \sum_{i=1}^{k} t_i w_i^2 \gamma_2(\gamma_2 - 1)(1+\alpha_2 w_i)^{\gamma_2 - 2} = 0, \end{aligned} \tag{4.4}$$

$$\frac{-k_1}{\gamma_1^2} - \sum_{i=1}^{k} z_i (1+\alpha_1 w_i)^{\alpha_1} [\ln(1+\alpha_1 w_i)]^2 - \sum_{i=1}^{k} s_i (1+\alpha_1 w_i)^{\gamma_1} [\ln(1+\alpha_1 w_i)]^2 = 0, \tag{4.5}$$

$$\frac{-k_2}{\gamma_2^2} - \sum_{i=1}^{k} (1 - z_i)(1+\alpha_2 w_i)^{\alpha_2} [\ln(1+\alpha_2 w_i)]^2 - \sum_{i=1}^{k} t_i (1+\alpha_2 w_i)^{\gamma_2} [\ln(1+\alpha_2 w_i)]^2 = 0. \tag{4.6}$$

Calculating the asymptotic confidence intervals (ACIs) for the parameters $\alpha_1, \alpha_2, \gamma_1$, and γ_2 is feasible by leveraging the asymptotic normality of the maximum likelihood estimators (MLEs). This enables us to estimate the $(1-\mu)100\%$ ACIs for $\alpha_1, \alpha_2, \gamma_1$, and γ_2 in an approximate manner

$$\left(\hat{\alpha_1} \pm Z_{\mu/2}\sqrt{\widehat{var}(\hat{\alpha_1})}\right), \left(\hat{\alpha_2} \pm Z_{\mu/2}\sqrt{\widehat{var}(\hat{\alpha_2})}\right), \left(\hat{\gamma_1} \pm Z_{\mu/2}\sqrt{\widehat{var}(\hat{\gamma_1})}\right) \text{and} \left(\hat{\gamma_2} \pm Z_{\mu/2}\sqrt{\widehat{var}(\hat{\gamma_2})}\right). \tag{4.7}$$

Here, $Z_{\mu/2}$ represents the upper $\mu/2$th percentile of the standard normal distribution.

5. Bayesian inference

In the field of reliability analysis, there are instances where classical estimation using the MLE approach may encounter challenges, particularly when the available data lacks sufficient sampling details. To address this issue, incorporating prior information in conjunction with Bayesian analysis proves beneficial. This section focuses on the Bayesian approach for estimating unknown parameters and deriving the corresponding credible intervals (CRIs).

5.1. Prior distribution

In Bayesian statistical inference, the role of the prior distribution is pivotal as it represents our existing knowledge or beliefs regarding the parameters, facilitating a more accurate estimation of the posterior distribution. The selection of an appropriate prior distribution is crucial as it can influence the ultimate results of the inference. The gamma distribution, recognized for its flexibility and favorable properties, is a frequently chosen continuous probability distribution for Bayesian parameter priors. The parameters of the gamma distribution can be adjusted to accommodate diverse prior beliefs. Additionally, the gamma distribution exhibits conjugacy, signifying that when employed as a prior distribution, its product with the likelihood function remains a gamma distribution, simplifying posterior distribution calculations.

The choice of hyperparameter values, like any other parameter selection in Bayesian analysis, depends on various factors including prior knowledge, the characteristics of the data, and modeling considerations. Without specific details about the context of your simulation, I can provide some general considerations:

- Prior Knowledge: If you have prior information about the hyperparameters, such as from previous studies or domain expertise, you might set them based on that knowledge. The values could reflect your beliefs about the likely range or distribution of the true hyperparameters.
- Empirical Bayes: If you don't have strong prior knowledge, you might consider using an empirical Bayes approach. This involves estimating the hyperparameters from the data itself. In such cases, the hyperparameters are determined based on the observed data distribution.
- Conjugate Priors: If you choose hyperparameters that make your prior distribution conjugate to the likelihood, it can simplify computations. However, this choice might not always reflect your beliefs about the parameters.
- Non-Informative Priors: A common choice for hyperparameters, especially in the absence of strong prior information, is to set them to values that make the prior distribution non-informative or weakly informative. For instance, setting hyperparameters to achieve a flat or diffuse prior.

In this study, we assert that c_i and d_i are both greater than 0, where i takes values 1, 2, 3, and 4. This results in the following relationships:

$$\pi_1(\alpha_1) \propto \alpha_1^{c_1-1} e^{-d_1\alpha_1}, \qquad \alpha_1 > 0, c_1, d_1 > 0,$$

$$\pi_2(\alpha_2) \propto \alpha_2^{c_2-1} e^{-d_2\alpha_2}, \qquad \alpha_2 > 0, c_2, d_2 > 0,$$

$$\pi_3(\gamma_1) \propto \gamma_1^{c_3-1} e^{-d_3\gamma_1}, \qquad \gamma_1 > 0, c_3, d_3 > 0.$$

$$\pi_4(\gamma_2) \propto \gamma_2^{c_4-1} e^{-d_4\gamma_2}, \qquad \gamma_2 > 0, c_4, d_4 > 0.$$

In this scenario, c_i and d_i where $i = 1, 2, 3, 4$, are introduced to integrate prior information concerning the unidentified parameters. Consequently, the joint prior density function for α_1, α_2, γ_1, and γ_2 is structured as follows:

$$\pi(\alpha_1, \alpha_2, \gamma_1, \gamma_2) \propto \alpha_1^{c_1-1}\alpha_2^{c_2-1}\gamma_1^{c_3-1}\gamma_2^{c_4-1}e^{-d_1\alpha_1-d_2\alpha_2-d_3\gamma_1-d_4\gamma_2}. \tag{5.1}$$

5.2. Posterior distribution

The posterior distribution is a fundamental concept in Bayesian statistics, playing a crucial role in the inference process. It represents the updated belief or knowledge about a parameter or set of parameters after taking into account both prior information and observed data. In Bayesian analysis, the posterior distribution is obtained by combining the prior distribution, which encapsulates our initial beliefs, with the likelihood function, which quantifies the probability of observing the data given the parameter values. The posterior distribution provides a comprehensive summary of uncertainty, allowing us to estimate parameters, make predictions, and conduct various analyses. It serves as a bridge between prior knowledge and observed evidence, enabling us to make informed decisions and update our understanding of the underlying phenomenon.

By merging Eqs (3.1) and (5.1), it becomes viable to represent the joint posterior density function for α_1, α_2, γ_1 and γ_2 as following

$$\pi^*(\alpha_1, \alpha_2, \gamma_1, \gamma_2|data) = \frac{\pi(\alpha_1, \alpha_2, \gamma_1, \gamma_2)L(\alpha_1, \alpha_2, \gamma_1, \gamma_2|data)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha_1, \alpha_2, \gamma_1, \gamma_2)L(\alpha_1, \alpha_2, \gamma_1, \gamma_2|data)d\alpha_1 d\alpha_2 d\gamma_1 d\gamma_2}. \tag{5.2}$$

The joint posterior density function for α_1, α_2, γ_1 and γ_2 can be formulated as:

$$\begin{aligned}\pi^*(\alpha_1, \alpha_2, \gamma_1, \gamma_2|data) \propto\ & \alpha_1^{k_1+c_1-1}\gamma_1^{k_1+c_2-1}\alpha_2^{k_2+c_3-1}\gamma_2^{k_2+c_4-1}e^{(\gamma_1-1)\sum_{i=1}^k z_i\ln(1+\alpha_1 w_i)} \\ & \times e^{\sum_{i=1}^k z_i[1-(1+\alpha_1 w_i)]^{\gamma_1}}e^{(\gamma_2-1)\sum_{i=1}^k(1-z_i)\ln(1+\alpha_2 w_i)}e^{\sum_{i=1}^k(1-z_i)[1-(1+\alpha_2 w_i)]^{\gamma_2}} \\ & \times e^{\sum_{i=1}^k s_i[1-(1+\alpha_1 w_i)]^{\gamma_1}}e^{\sum_{i=1}^k t_i[1-(1+\alpha_2 w_i)]^{\gamma_2}}e^{-d_1\alpha_1-d_2\alpha_2-d_3\gamma_1-d_4\gamma_2}.\end{aligned} \tag{5.3}$$

Evidently, owing to the nonlinear form of (5.3), there is no closed-form solution for the Bayes estimators of α_1, α_2, γ_1, or γ_2 when employing the SE and LINEX loss functions. Consequently, we recommend employing the Markov Chain Monte Carlo (MCMC) approach to acquire the Bayes estimates and establish the corresponding CRIs.

5.3. MCMC approach

To generate samples using the MCMC approach, we must first determine the conditional posterior distributions of the unknown NHD parameters α_1, α_2, γ_1 and γ_2. The conditions are given by

$$\begin{aligned}\pi_1^*(\alpha_1|\alpha_2, \gamma_1, \gamma_2) \propto\ & \alpha_1^{k_1+c_1-1}e^{-d_1\alpha_1-\sum_{i=1}^k z_i\ln(1+\alpha_1 w_i)}e^{\sum_{i=1}^k z_i[1-(1+\alpha_1 w_i)]^{\gamma_1}} \\ & \times e^{\sum_{i=1}^k s_i[1-(1+\alpha_1 w_i)]^{\gamma_1}},\end{aligned} \tag{5.4}$$

$$\begin{aligned}\pi_2^*(\alpha_2|\alpha_1, \gamma_1, \gamma_2) \propto\ & \alpha_2^{k_2+c_2-1}e^{-d_2\alpha_2-\sum_{i=1}^k(1-z_i)\ln(1+\alpha_2 w_i)}e^{\sum_{i=1}^k(1-z_i)[1-(1+\alpha_2 w_i)]^{\gamma_2}} \\ & \times e^{\sum_{i=1}^k t_i[1-(1+\alpha_2 w_i)]^{\gamma_2}},\end{aligned} \tag{5.5}$$

$$\pi_3^*(\gamma_1|\alpha_1, \alpha_2, \gamma_2) \propto \gamma_1^{k_1+c_3-1} e^{-\gamma_1\left[d_3+\sum_{i=1}^{k} z_i \ln(1+\alpha_1 w_i)\right]}, \tag{5.6}$$

$$\pi_4^*(\gamma_2|\alpha_1, \alpha_2, \gamma_1) \propto \gamma_2^{k_2+c_4-1} e^{-\gamma_2\left[d_4+\sum_{i=1}^{k}(1-z_i) \ln(1+\alpha_2 w_i)\right]}. \tag{5.7}$$

From Eqs (5.6) and (5.7), it's apparent that the posterior densities of γ_1 and γ_2 follow the gamma distribution, given that

$$\pi_3^*(\gamma_1|\alpha_1, \alpha_2, \gamma_2) \sim Gamma\Big(k_1 + c_3, d_3 + \sum_{i=1}^{k} z_i \ln(1 + \alpha_1 w_i)\Big), \tag{5.8}$$

and

$$\pi_4^*(\gamma_2|\alpha_1, \alpha_2, \gamma_1) \sim Gamma\Big(k_2 + c_4, d_4 + \sum_{i=1}^{k}(1 - z_i) \ln(1 + \alpha_2 w_i)\Big). \tag{5.9}$$

It's evident that gamma densities can be used to generate random samples of γ_1 and γ_2. However, it's not feasible to algebraically simplify the density functions of $\pi_1^*(\alpha_1|\alpha_2, \gamma_1, \gamma_2)$ and $\pi_2^*(\alpha_2|\alpha_1, \gamma_1, \gamma_2)$ to well-known distributions. Therefore, obtaining samples directly through conventional procedures is not practical. Hence, we turn to the Markov Chain Monte Carlo (MCMC) approach and generate a sample using Gibbs sampling with the Metropolis-Hastings (M-H) algorithm (see Metropolis et al. [25] and Hastings [23]), employing a standard proposal, as described below.

- **Step 1.** Begin by employing the initial values $(\alpha_1^{(0)}, \alpha_2^{(0)}, \gamma_1^{(0)}, \gamma_2^{(0)})$.
- **Step 2.** Set $j = 1$.
- **Step 3.** Generate $\gamma_1^{(j)}$ from Gamma $\Big(k_1 + c_3, d_3 + \sum_{i=1}^{k} z_i \ln(1 + \alpha_1 w_i)\Big)$.
- **Step 4.** Generate $\gamma_2^{(j)}$ from Gamma $\Big(k_2 + c_4, d_4 + \sum_{i=1}^{k}(1 - z_i) \ln(1 + \alpha_2 w_i)\Big)$.
- **Step 5.** Equations (5.4) and (5.5) can be employed to generate $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$ using the Metropolis-Hastings (M-H) algorithm. The recommended normal distributions to use are $N(\alpha_1^{(j-1)}, \mathrm{var}(\alpha_1))$ and $N(\alpha_2^{(j-1)}, \mathrm{var}(\alpha_2))$. In this instance, the major diagonal of the inverted FIM can be utilized to calculate $\mathrm{var}(\alpha_1)$ and $\mathrm{var}(\alpha_2)$.
(I) Produce proposed values α_1^* and α_2^* from the corresponding normal distributions.
(II) Determine the acceptance probabilities using the following procedure:

$$r_1 = \min\left[1, \frac{\pi_1^*(\alpha_1^*|\alpha_2^{(j-1)}, \gamma_1^j, \gamma_2^j, data)}{\pi_1^*(\alpha_1^{(j-1)}|\alpha_2^{(j-1)}, \gamma_1^j, \gamma_2^j, data}\right],$$

$$r_2 = \min\left[1, \frac{\pi_2^*(\alpha_2^*|\alpha_1^j, \gamma_1^j, \gamma_2^j, data)}{\pi_2^*(\alpha_2^{(j-1)}|\alpha_1^j, \gamma_1^j, \gamma_2^j, data)}\right].$$

(III) Produce a random value u from a uniform distribution within the range of $(0, 1)$.
(IV) If $u \leq r_1$, approve the proposal and assign $\alpha_1^{(j)}$ as α_1^*; otherwise, maintain $\alpha_1^{(j)}$ as $\alpha_1^{(j-1)}$.
(V) If $u \leq r_2$, approve the proposal and assign $\alpha_2^{(j)}$ as α_2^*; otherwise, maintain $\alpha_2^{(j)}$ as $\alpha_2^{(j-1)}$.
- **Step 7.** Set j = j + 1.

- **Step 8.** Repeat 2–7, for V times. Therefore, the estimated posterior means of $(\alpha_1, \alpha_2, \gamma_1, \gamma_2)$ represented by λ under the squared error loss function, can be determined as:

$$\hat{\lambda}_{BS} = E[\lambda|\underline{x}] = \frac{1}{V-N}\sum_{i=N+1}^{V} \lambda^{(j)}. \tag{5.10}$$

Lastly, calculate the Bayesian estimates of λ utilizing the LINEX loss function:

$$\hat{\lambda}_{BL} = -\frac{1}{a}\ln\left[\frac{1}{V-N}\sum_{i=N+1}^{V} e^{-a\lambda^{(j)}}\right]. \tag{5.11}$$

Here, N denotes the burn-in period.

6. Numerical study

This section's goal is to assess the potency of the various estimation techniques covered in earlier sections. We demonstrate this by examining an actual dataset and conducting a simulated experiment to evaluate the statistical performance of the estimators under the JPT-II-CS. The computations were carried out using Mathematica ver. 10 software.

6.1. Simulation study

In this section, we conduct simulation studies to assess the performance of the estimation methods developed in the preceding sections. We explore various sample sizes for the two populations, including $(m, n) = (10, 20), (20, 30), (40, 50)$, and different numbers of failures for each sample size, such as $(15, 20, 30), (35, 40, 50), (60, 70, 90)$, respectively. The parameter values for the two populations are set as $(\alpha_1, \alpha_2, \gamma_1, \gamma_2) = (1.5, 1.3, 0.4, 0.3)$.

We computed MLEs along with 95% CIs for the parameters $(\alpha_1, \alpha_2, \gamma_1, \gamma_2)$ across all specified scenarios. This process was repeated 1000 times, and we calculated the mean values of MLEs and their respective lengths. The results are presented in Tables 1–4. Additionally, we employed informative gamma priors for for α_1, α_2, γ_1, and γ_2 in the context of Bayesian estimation under SE and LINEX loss functions. The hyperparameters were set as $c_i = 0.8$ and $d_i = 2.5$, where $i = 1, 2, 3, 4$, with $a = 3$ denoting overestimation and $a = -3$ representing underestimation. We employed the(MCMC approach with $11,000$ samples to derive Bayesian estimates for α_1, α_2, γ_1, and γ_2, along with 95% CRIs, through 1000 simulations. The initial 1000 values were excluded due to "burn-in". MSE was utilized to evaluate the performance of the generated estimators for α_1, α_2, γ_1, and γ_2. Following 1000 repetitions of this process, we computed the mean values of MLEs and their respective lengths. Tables 1–4 present the results.

Table 1. Average value, length and corresponding MSE (in parentheses) of estimates for the parameter α_1.

(m, n)	r	Scheme	Non-Bayesian		Bayesian			
			MLE	Length	SE	LINEX		Length
						$a=-3$	$a=3$	
(10, 20)	15	$(0_{(14)}, 15)$	1.0089	7.7486	1.3852	1.3856	1.3849	0.0611
			(0.2412)		(0.0132)	(0.0131)	(0.0130)	
	15	$(15, 0_{(14)})$	1.9192	9.5216	2.5732	2.5736	2.5728	0.0604
			(0.1757)		(1.1517)	(1.1526)	(1.1509)	
	20	$(0_{(19)}, 10)$	1.4528	8.1424	1.8099	1.8104	1.8095	0.0633
			(0.0992)		(0.0961)	(0.0963)	(0.0958)	
	20	$(10, 0_{(19)})$	2.0877	9.2575	1.8322	1.8343	1.8302	0.1077
			(0.3454)		(0.1104)	(0.1118)	(0.1090)	
	25	$(0_{(7)}, 2, 0_{(4)}, 1, 0_{(3)}, 2, 0_{(8)})$	1.6978	6.5167	2.3954	2.3955	2.3952	0.0363
			(0.0391)		(0.0175)	(0.0191)	(0.0144)	
	30	$(0_{(30)})$	0.9524	4.0016	1.5637	1.5639	1.5635	0.0398
			(0.2999)		(0.0041)	(0.0041)	(0.0040)	
(20, 30)	35	$(0_{(34)}, 15)$	1.4461	5.6789	1.7213	1.7214	1.7213	0.0272
			(0.0029)		(0.0490)	(0.0490)	(0.0490)	
	35	$(15, 0_{(34)})$	1.2765	3.9130	1.8908	1.8909	1.8908	0.0223
			(0.1500)		(0.0527)	(0.0527)	(0.0527)	
	40	$(0_{(39)}, 10)$	1.9292	5.5891	1.2361	1.2366	1.2356	0.0540
			(0.1843)		(0.0697)	(0.0685)	(0.0672)	
	40	$(10, 0_{(39)})$	1.9814	5.9534	1.8851	1.8855	1.8847	0.0601
			(0.2318)		(0.1483)	(0.1486)	(0.1480)	
	45	$(0_{(5)}, 3, 0_{(33)}, 2, 0_{(5)})$	1.3855	4.2144	0.7017	0.7025	0.7008	0.0709
			(0.2131)		(0.0637)	(0.0534)	(0.0544)	
	50	$(0_{(50)})$	1.7681	4.4709	1.8890	1.8890	1.8890	0.0261
			(0.0719)		(0.0513)	(0.0514)	(0.0512)	
(40, 50)	60	$(0_{(59)}, 30)$	2.0131	6.0591	2.8808	2.8806	2.8802	0.0311
			(0.2633)		(0.0684)	(0.0688)	(0.0679)	
	60	$(30, 0_{(59)})$	0.8016	2.2477	0.5408	0.5407	0.5405	0.0130
			(0.4878)		(0.0400)	(0.0321)	(0.201)	
	70	$(0_{(69)}, 20)$	0.8469	1.9784	0.6599	0.6599	0.6598	0.0164
			(0.4265)		(0.1058)	(0.1057)	(0.1052)	
	70	$(20, 0_{(69)})$	1.0524	2.3534	1.2987	1.2987	1.2986	0.0140
			(0.2003)		(0.0405)	(0.0402)	(0.0401)	
	80	$(4, 0_{(8)}, 3, 0_{(60)}, 3, 0_{(9)})$	2.4501	5.8484	4.4654	4.4656	4.4651	0.0423
			(0.9027)		(0.7933)	(0.7922)	(0.7824)	
	90	$(0_{(90)})$	1.8013	3.3309	1.6733	1.6734	1.6731	0.0265
			(0.0908)		(0.0765)	(0.0768)	(0.0763)	

Table 2. Average value, length and corresponding MSE (in parentheses) of estimates for the parameter α_2.

(m,n)	r	Scheme	Non-Bayesian		Bayesian			
			MLE	Length	SE	LINEX		Length
						$a=-3$	$a=3$	
(10, 20)	15	$(0_{(14)},15)$	0.9247	7.4446	1.3417	1.3422	1.3413	0.0636
			(0.1409)		(0.0017)	(0.0018)	(0.0017)	
	15	$(15,0_{(14)})$	0.1539	0.9332	0.1811	0.1811	0.1811	0.0097
			(0.3134)		(0.2519)	(0.2519)	(0.2519)	
	20	$(0_{(19)},10)$	0.6347	3.2751	0.5126	0.5127	0.5126	0.0150
			(0.4426)		(0.6200)	(0.6210)	(0.6120)	
	20	$(10,0_{(19)})$	2.6140	8.4100	2.8049	2.8055	2.8043	0.0712
			(0.7267)		(0.2647)	(0.2664)	(0.2629)	
	25	$(0_{(7)},2,0_{(4)},1,0_{(3)},2,0_{(8)})$	0.6021	2.473	0.8028	0.8029	0.8028	0.0268
			(0.4871)		(0.2472)	(0.2471)	(0.2470)	
	30	$(0_{(30)})$	1.7301	5.7329	2.7105	2.7107	2.7103	0.0599
			(0.1850)		(0.1795)	(0.1702)	(0.1601)	
(20, 30)	35	$(0_{(34)},15)$	0.3714	1.8375	1.3453	1.3453	1.3453	0.0082
			(0.1623)		(0.0021)	(0.0021)	(0.0021)	
	35	$(15,0_{(34)})$	1.8284	5.0672	1.4205	1.4208	1.4202	0.0544
			(0.2792)		(0.0145)	(0.0146)	(0.0144)	
	40	$(0_{(39)},10)$	2.0993	5.2002	0.9454	0.9455	0.9453	0.0287
			(0.6390)		(0.1257)	(0.1250)	(0.1248)	
	40	$(10,0_{(39)})$	2.5850	8.5699	2.4874	2.4876	2.4872	0.0414
			(0.6512)		(0.4099)	(0.4095)	(0.4090)	
	45	$(0_{(5)},3,0_{(33)},2,0_{(5)})$	2.649	7.6263	1.4086	1.4088	1.4083	0.0468
			(0.8198)		(0.0118)	(0.0118)	(0.0117)	
	50	$(0_{(50)})$	2.1930	5.6644	1.8755	1.8482	1.8655	0.0350
			(0.7975)		(0.3469)	(0.3471)	(0.3468)	
(40, 50)	60	$(0_{(59)},30)$	1.0377	3.2544	1.5236	1.5236	1.5236	0.0207
			(0.0688)		(0.0500)	(0.0500)	(0.0500)	
	60	$(30,0_{(59)})$	2.1438	4.4834	2.4051	2.4052	2.4050	0.0370
			(0.7120)		(0.2213)	(0.2217)	(0.2210)	
	70	$(0_{(69)},20)$	0.969	2.4503	1.7391	1.7391	1.7391	0.0141
			(0.1096)		(0.0928)	(0.0924)	(0.0920)	
	70	$(20,0_{(69)})$	1.5667	3.193	1.7013	1.7012	1.7010	0.0154
			(0.0712)		(0.0161)	(0.0162)	(0.0158)	
	80	$(4,0_{(8)},3,0_{(60)},3,0_{(9)})$	1.3662	2.709	1.6230	1.6222	1.6220	0.0167
			(0.1244)		(0.1043)	(0.1041)	(0.1032)	
	90	$(0_{(90)})$	1.4794	2.6736	0.8824	0.8820	0.8811	0.0112
			(0.0322)		(0.0174)	(0.0171)	(0.0160)	

Table 3. Average value, length and corresponding MSE (in parentheses) of estimates for the parameter γ_1.

(m, n)	r	Scheme	Non-Bayesian		Bayesian			
			MLE	Length	SE	LINEX		Length
						$a = -3$	$a = 3$	
(10, 20)	15	$(0_{(14)}, 15)$	0.4500	2.0736	1.0201	1.2450	0.8765	1.3259
			(0.0025)		(0.0023)	(0.0020)	(0.0020)	
	15	$(15, 0_{(14)})$	0.3766	0.8338	0.5666	0.7000	0.4843	1.0085
			(0.0201)		(0.0270)	(0.0278)	(0.0268)	
	20	$(0_{(19)}, 10)$	0.3205	0.8781	0.3128	0.2280	0.2623	0.8047
			(0.1963)		(0.1108)	(0.1832)	(0.0688)	
	20	$(10, 0_{(19)})$	0.4035	0.7331	0.6691	0.7739	0.5954	0.9246
			(0.0854)		(0.0724)	(0.0724)	(0.0382)	
	25	$(0_{(7)}, 2, 0_{(4)}, 1, 0_{(3)}, 2, 0_{(8)})$	0.5471	1.0630	0.8072	0.9125	0.7279	0.9644
			(0.0216)		(0.0165)	(0.0127)	(0.0107)	
	30	$(0_{(30)})$	0.6514	1.3684	0.745	0.8343	0.6765	0.8887
			(0.1632)		(0.1191)	(0.1286)	(0.0765)	
(20, 30)	35	$(0_{(34)}, 15)$	0.7796	0.7379	0.7796	0.8391	0.7307	0.7379
			(0.1291)		(0.1241)	(0.1028)	(0.1094)	
	35	$(15, 0_{(34)})$	0.4192	0.5150	0.5787	0.6148	0.5477	0.5777
			(0.1010)		(0.0320)	(0.0461)	(0.0461)	
	40	$(0_{(39)}, 10)$	0.2902	0.3452	0.8947	0.9753	0.8295	0.8497
			(0.2144)		(0.0125)	(0.0122)	(0.0121)	
	40	$(10, 0_{(39)})$	0.4189	0.4923	0.4864	0.4291	0.4499	0.2145
			(0.0930)		(0.0820)	(0.0724)	(0.0620)	
	45	$(0_{(5)}, 3, 0_{(33)}, 2, 0_{(5)})$	0.3938	0.5113	0.4844	0.4782	0.4653	0.3813
			(0.434)		(0.0354)	(0.0352)	(0.0344)	
	50	$(0_{(50)})$	0.3731	0.3433	0.4990	0.4265	0.4744	0.3129
			(0.0898)		(0.0396)	(0.0394)	(0.0390)	
(40, 50)	60	$(0_{(59)}, 30)$	0.2999	0.4200	0.5524	0.4799	0.5277	0.3152
			(0.0954)		(0.0779)	(0.0462)	(0.0563)	
	60	$(30, 0_{(59)})$	0.6208	0.8487	0.5244	0.5301	0.5030	0.4250
			(0.0488)		(0.0244)	(0.0223)	(0.0222)	
	70	$(0_{(69)}, 20)$	0.4048	0.4479	0.3692	0.3196	0.3245	0.1888
			(0.4478)		(0.1177)	(0.3900)	(0.3548)	
	70	$(20, 0_{(69)})$	0.6135	0.6612	0.6751	0.6168	0.6376	0.4304
			(0.0456)		(0.0258)	(0.0671)	(0.0915)	
	80	$(4, 0_{(8)}, 3, 0_{(60)}, 3, 0_{(9)})$	0.3866	0.3552	0.5170	0.5274	0.5070	0.3215
			(0.0212)		(0.0137)	(0.0162)	(0.0115)	
	90	$(0_{(90)})$	0.3746	0.2523	0.4336	0.4443	0.4234	0.2224
			(0.1064)		(0.0178)	(0.0208)	(0.0152)	

Table 4. Average value, length and corresponding MSE (in parentheses) of estimates for the parameter γ_2.

(m,n)	r	Scheme	Non-Bayesian		Bayesian			
			MLE	Length	SE	LINEX		Length
						$a=-3$	$a=3$	
(10, 20)	15	$(0_{(14)}, 15)$	0.4110	2.0336	1.0052	1.2843	0.8484	1.4092
			(0.0123)		(0.0119)	(0.0112)	(0.0101)	
	15	$(15, 0_{(14)})$	0.8193	2.8032	1.0506	1.2216	0.9307	1.2030
			(0.2696)		(0.3634)	(0.4493)	(0.3978)	
	20	$(0_{(19)}, 10)$	0.5335	1.5963	1.3381	1.6395	1.1490	1.5159
			(0.0545)		(0.0777)	(0.7941)	(0.7208)	
	20	$(10, 0_{(19)})$	0.2255	0.2141	0.3820	0.3992	0.3666	0.4006
			(0.0085)		(0.0067)	(0.0078)	(0.0044)	
	25	$(0_{(7)}, 2, 0_{(4)}, 1, 0_{(3)}, 2, 0_{(8)})$	0.4095	0.6813	0.6038	0.6416	0.5715	0.5963
			(0.0120)		(0.0092)	(0.0081)	(0.0073)	
	30	$(0_{(30)})$	0.2772	0.2753	0.3940	0.4056	0.3833	0.3367
			(0.0188)		(0.0088)	(0.0075)	(0.0069)	
(20, 30)	35	$(0_{(34)}, 15)$	0.7089	2.2453	0.8012	0.8529	0.7567	0.6981
			(0.1672)		(0.1512)	(0.1057)	(0.1086)	
	35	$(15, 0_{(34)})$	0.2850	0.2466	0.5229	0.5426	0.5049	0.4333
			(0.1491)		(0.0497)	(0.0589)	(0.0420)	
	40	$(0_{(39)}, 10)$	0.3109	0.3064	0.3965	0.3623	0.3410	0.2909
			(0.5851)		(0.4866)	(0.4851)	(0.4840)	
	40	$(10, 0_{(39)})$	0.3604	0.2478	0.3265	0.3382	0.3156	0.2302
			(0.1458)		(0.0160)	(0.0191)	(0.0134)	
	45	$(0_{(5)}, 3, 0_{(33)}, 2, 0_{(5)})$	0.6419	0.6039	0.5232	0.5379	0.5095	0.3788
			(0.3430)		(0.0498)	(0.0566)	(0.0439)	
	50	$(0_{(50)})$	0.2571	0.1927	0.2427	0.2527	0.2332	0.1163
			(0.1144)		(0.0204)	(0.0233)	(0.0177)	
(40, 50)	60	$(0_{(59)}, 30)$	0.4642	0.7702	0.9308	0.9680	0.8969	0.6030
			(0.0270)		(0.0179)	(0.0171)	(0.0165)	
	60	$(30, 0_{(59)})$	0.2766	0.2781	0.2510	0.2591	0.2430	0.1898
			(0.1222)		(0.0228)	(0.0254)	(0.0204)	
	70	$(0_{(69)}, 20)$	0.3746	0.4258	0.3499	0.3682	0.3329	0.2235
			(0.4441)		(0.1225)	(0.1356)	(0.1108)	
	70	$(20, 0_{(69)})$	0.2747	0.1723	0.4435	0.4506	0.4368	0.2654
			(0.1324)		(0.0206)	(0.0227)	(0.0187)	
	80	$(4, 0_{(8)}, 3, 0_{(60)}, 3, 0_{(9)})$	0.6604	0.5647	0.5858	0.5983	0.5741	0.3527
			(0.1204)		(0.0817)	(0.0890)	(0.0751)	
	90	$(0_{(90)})$	0.2699	0.1466	0.2436	0.2524	0.2351	0.1170
			(0.1100)		(0.0593)	(0.0637)	(0.0553)	

Based on the aforementioned data, several conclusions can be drawn:

(1) Tables 1 to 4 reveal that, in most cases, Bayesian estimates outperform MLEs in terms of MSEs.

(2) Examining Tables 1 to 4, it becomes apparent that CRIs exhibit the shortest average length of intervals compared to the average length of CIs, indicating the superiority of Bayesian estimators over MLEs.

(3) Notably, Bayesian estimation using the LINEX loss function at $a = 3$ surpasses Bayesian estimators using the LINEX loss function at $a = -3$ and the SE loss function in terms of MSEs.

(4) The results indicate that both MLEs and Bayesian estimators yield favorable outcomes for two sample lines, whether they have the same or different population numbers, suggesting the model's suitability for various sample situations.

6.2. Data analysis

The data represent the air-conditioning system failure times (in hours) for planes 7913 and 7914, originally sourced from Proschan [30]. The assumption is made that the two datasets are independent, and within each dataset, the failure times are also considered independent. The data is provided below.
Data 1. (Plane 7913): 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.
Data 2. (Plane 7914): 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210.

Table 5 displays the outcomes of the Kolmogorov-Smirnov (K-S) test, employed to assess the data's adherence to the NHD.

Table 5. K-S test and P-value.

Data set	Size (n)	K-S (Calculated)	K-S (5% Significance)	P-value
I	27	0.1011	0.2544	0.9192
II	24	0.0870	0.2693	0.9858

Taking into account the details in Table 5, it is apparent that the computed K-S values for the data are lower than the corresponding values expected at a significance level of 5%. Additionally, we have noted notably high P-values. Consequently, we can reasonably conclude that the NHD serves as a well-fitted model for the data. Moreover, we have generated empirical $S(x)$ and fitted $S(x)$ for each dataset, as depicted in Figures 2 and 3, respectively. These plots provide additional confirmation that the NHD model offers a superior fit to the data.

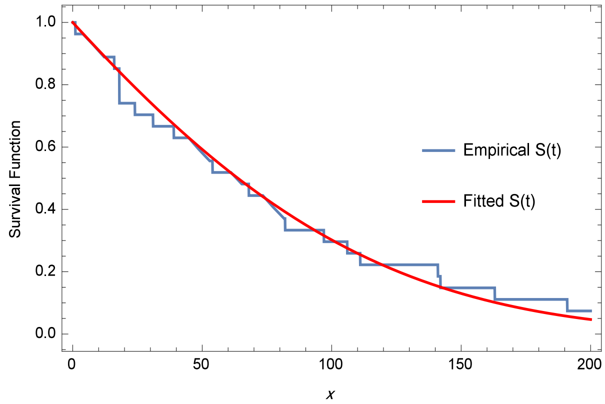

Figure 2. Plots of fitted functions of the NHD for data set I.

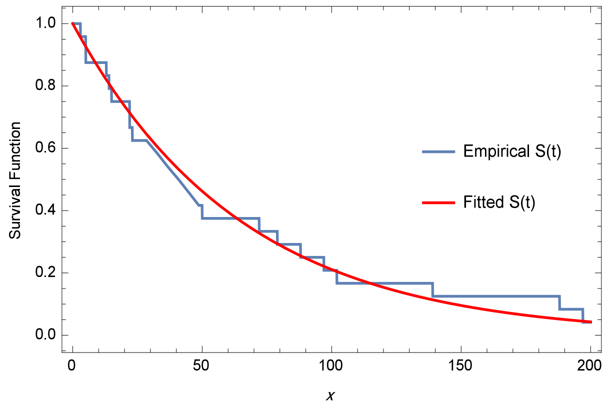

Figure 3. Plots of fitted functions of the NHD for data set II.

We generated a JPT-IISC sample from the aforementioned datasets using the following censoring scheme. For the first sample, set $m = 27$, and for the second sample, set $n = 24$, implementing JPT-IISC with $r = 20$. The censoring vectors are defined as:

$$\begin{aligned} S &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,16), \\ R &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,31), \\ T &= (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,15). \end{aligned}$$

Here are the generated datasets:

$$\begin{aligned} w &= (1,3,4,5,5,11,13,14,15,16,18,18,18,22,22,23,24,30,31,36), \\ z &= (1,0,1,0,0,1,0,0,0,1,1,1,1,0,0,0,1,0,1,0). \end{aligned}$$

We obtained estimates for $\alpha_1, \alpha_2, \gamma_1$, and γ_2 using the MLE method, depending on the data type used in this study. The corresponding results are displayed in Table 6, whereas Tables 7 and 8 present the 95% ACIs for $\alpha_1, \alpha_2, \gamma_1$, and γ_2. We utilized the MCMC method for Bayesian estimation conducting 20000 iterations to ensure convergence we excluded the initial 5000 iterations as 'burn in' We have selected hyperparameters c_i and d_i as 0.0001, approaching values near zero for the prior distributions.

Bayesian estimates for $\alpha_1, \alpha_2, \gamma_1$, and γ_2 were derived using both the SE loss and LINEX loss functions, with the corresponding results detailed in Table 6. Additionally, the 95% CRIs for $\alpha_1, \alpha_2, \gamma_1$, and γ_2 are provided in Tables 7 and 8.

Table 6. Different point estimates of $(\alpha_1, \alpha_2, \gamma_1, \gamma_2)$.

Parameters	MLE	MCMC			
		SE	LINEX		
			$a = -3.0$	$a = 10^{-4}$	$a = 3.0$
α_1	0.0211	0.0612	0.0612	0.0612	0.0612
α_2	0.0112	0.0406	0.0406	0.0406	0.0406
γ_1	0.6548	1.5915	2.2824	1.5915	1.2740
γ_2	1.2882	2.0091	2.8235	2.0091	1.6094

Table 7. 95% CIs and CIRs for (α_1, α_2).

Method	α_1			α_2		
	Lower	Upper	Length	Lower	Upper	Length
CI	-0.0801	0.1223	0.2024	-0.0392	0.0615	0.1008
CRI	0.0606	0.0621	0.0014	0.0401	0.0411	0.0010

Table 8. 95% CIs and CIRs for (γ_1, γ_2).

Method	γ_1			γ_2		
	Lower	Upper	Length	Lower	Upper	Length
CI	-1.7852	3.0948	4.8801	-3.6776	6.2541	9.9317
CRI	0.7213	2.8120	2.0906	1.0133	3.3537	2.3404

7. Conclusions

In this paper, we explored statistical inference for two populations characterized by NHD. These distributions feature distinct shape and scale parameters, and our analysis focused on a JPT-II-CS. Assuming life distribution for both populations, we derived maximum likelihood estimates for unknown parameters and performed Bayesian estimation under gamma and non-information priors. Two loss functions, namely the SE and LINEX loss functions, were employed. To assess the effectiveness of the proposed estimates, we conducted Monte Carlo simulation experiments, revealing that Bayes estimates, along with their associated CRIs, outperform other estimators. Finally, we presented a numerical example to illustrate the inferential results established in this study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research project was supported by the Researchers Supporting Project Number (RSP2024R488), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflict of interest.

References

1. Y. Abdel-Aty, M. Kayid, G. Alomani, Generalized Bayes estimation based on a Joint type-II censored sample from K-Exponential populations, *Mathematics*, **11** (2023), 2190. https://doi.org/10.3390/math11092190
2. N. Alotaibi, A. S. Al-Moisheer, I. Elbatal, M. Shrahili, M. Elgarhy, E. M. Almetwally, Half logistic inverted Nadarajah-Haghighi distribution under ranked set sampling with applications, *Mathematics*, **11** (2023), 1693. https://doi.org/10.3390/math11071693
3. K. Andrzejczak, L. Bukowski, A method for estimating the probability distribution of the lifetime for new technical equipment based on expert judgement, *Eksploat. Niezawodn.*, **23** (2021), 757–769. https://doi.org/10.17531/ein.2021.4.18
4. S. Asadi, H. Panahi, S. Anwar, S. A. Lone, Reliability estimation of Burr type III distribution under improved adaptive progressive censoring with application to surface coating, *Eksploat. Niezawodn.*, **25** (2023), https://doi.org/10.17531/ein/163054
5. S. K. Ashour, A. A. El-Sheikh, A. Elshahhat, Inferences and optimal censoring schemes for progressively first-failure censored Nadarajah-Haghighi distribution, *Sankhya A*, 2020, 1–39.
6. R. Azimi, M. Esmailian, A new generalization of Nadarajah-Haghighi distribution with application to cancer and COVID-19 deaths data, *Math. Slovaca*, **73** (2023), 221–244. https://doi.org/10.1515/ms-2023-0020
7. B. Long, Z. F. Jiang, Estimation and prediction for two-parameter Pareto distribution based on progressively double type-II hybrid censored data, *AIMS Math.*, **8** (2023), 15332–15351. https://doi.org/10.3934/math.2023784
8. H. S. Migdadi, N. M. Al-Olaimat, M. Mohiuddin, O. Meqdadi, Statistical inference for the Power Rayleigh distribution based on adaptive progressive type-II censored data, *AIMS Math.*, **8** (2023), 22553–22576. https://doi.org/10.3934/math.20231149
9. M. S. Eliwa, E. A. Ahmed, Reliability analysis of constant partially accelerated life tests under progressive first failure type-II censored data from Lomax model: EM and MCMC algorithms, *AIMS Math.*, **8** (2023), 29–60. https://doi.org/10.3934/math.2023002
10. H. P. Ren, X. Hu, Estimation for inverse Weibull distribution under progressive type-II censoring scheme, *AIMS Math.*, **8** (2023), 22808–22829. https://doi.org/10.3934/math.20231162
11. A. Elshahhat, V. K. Sharma, H. S. Mohammed, Statistical analysis of progressively first-failure-censored data via beta-binomial removals, *AIMS Math.*, **8** (2023), 22419–22446. https://doi.org/10.3934/math.20231144

12. X. Hu, H. P. Ren, Statistical inference of the stress-strength reliability for inverse Weibull distribution under an adaptive progressive type-II censored sample, *AIMS Math.*, **8** (2023), 28465–28487. https://doi.org/10.3934/math.20231457

13. N. Balakrishnan, M. Burkschat, E. Cramer, G. Hofmann, Fisher information based progressive censoring plans, *Comput. Stat. Data Anal.*, **53** (2008), 366–380. https://doi.org/10.1016/j.csda.2008.07.038

14. N. Balakrishnan, E. Cramer, *The art of progressive censoring: Applications to reliability and quality*, Springer: New York, NY, USA, 2014.

15. N. Balakrishnan, S. Feng, L. Kin-yat, Exact likelihood inference for k exponential populations under joint progressive type-II censoring, *Commun. Stat.-Simul. C.*, **44** (2015), 902–923. https://doi.org/10.1080/03610918.2013.795594

16. N. Celik, C. T. Guloksuz, A new lifetime distribution, *Eksploat. Niezawodn.*, **19** (2017), 634–639, https://doi.org/10.17531/ein.2017.4.18

17. K. C. Chiou, K. S. Chen, Lifetime performance evaluation model based on quick response thinking, *Eksploat. Niezawodn.*, **24** (2022), 1–6. http://doi.org/10.17531/ein.2022.1.1

18. M. Doostparast, M. V. Ahmadi, J. Ahmadi, Bayes estimation based on joint progressive type-II censored data under LINEX loss function, *Commun. Stat.-Simul. C.,* **42** (2013), 1865–1886.

19. A. Elshahhat, R. Alotaibi, M. Nassar, Inferences for Nadarajah-Haghighi Parameters via type-II adaptive progressive hybrid censoring with applications, *Mathematics*, **10** (2022), 3775. https://doi.org/10.3390/math10203775

20. L. A. Ferreira, J. L. Silva, Parameter estimation for Weibull distribution with right censored data using EM algorithm, *Eksploat. Niezawodn.*, **19** (2017), 310–315. https://doi.org/10.17531/ein.2017.2.20

21. R. Goel, H. Krishna, Likelihood and Bayesian inference for k Lindley populations under joint type-II censoring scheme, *Commun. Stat.-Simul. C.*, **52** (2021), 3475–3490. https://doi.org/10.1080/03610918.2021.1937648

22. R. Goel, H. Krishna, Statistical inference for two Lindley populations under balanced joint progressive type-II censoring scheme, *Comput. Stat.*, **37** (2021), 263–286. https://doi.org/10.1007/s00180-021-01122-2

23. W. K. Hastings, Monte Carlo sampling methods using Markov chains and their applications, *Biometrika*, **57** (1970), 97–109. https://doi.org/10.1093/biomet/57.1.97

24. H. Krishna, R. Goel, Inferences for two Lindley populations based on joint progressive type-II censored data, *Commun. Stat.-Simul. C.*, **51** (2020), 4919–4936. https://doi.org/10.1080/03610918.2020.1751851

25. N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, E. Teller, Equation of state calculations by fast computing machines, *J. Chem. Phys.*, **21** (1953), 1087–1092. https://doi.org/10.1063/1.1699114

26. S. Mondal, D. Kundu, Point and interval estimation of Weibull parameters based on joint progressively censored data, *Sankhya B*, **81** (2019), 1–25.

27. S. Nadarajah, F. Haghighi, An extension of the exponential distribution, *Statistics*, **45** (2011), 543–558. https://doi.org/10.1080/02331881003678678
28. H. Panahi, S. A. Lone, Estimation procedures for partially accelerated life test model based on unified hybrid censored sample from the Gompertz distribution, *Eksploat. Niezawodn.*, **24** (2022), 427–436. http://doi.org/10.17531/ ein.2022.3.4
29. R. Pandey, P. Srivastava, Bayesian inference for two log-logistic populations under joint progressive type II censoring schemes, *Int. J. Syst. Assur. Eng. Manag.*, **13** (2022), 2981–2991. https://doi.org/10.1007/s13198-022-01769-0
30. F. Proschan, Theoretical explanation of observed decreasing failure rate, *Technometrics*, **15** (1963), 375–383.
31. Y. Qiao, W. Gui, Statistical inference of weighted exponential distribution under Joint progressive type-II censoring, *Symmetry*, **14** (2022), 2031. https://doi.org/10.3390/sym14102031
32. A. Rasouli, N. Balakrishnan, Exact likelihood inference for two exponential populations under joint progressive type-II censoring, *Commun. Stat.-Theor. M.*, **39** (2010), 2172–2191. https://doi.org/10.1080/03610920903009418
33. M. A. Selim, Estimation and prediction for Nadarajah-Haghighi distribution based on record values, *Pak. J. Stat.*, **34** (2018), 77–90.
34. Y. Yan, J. Zhou, Y. Yin, H. Nie, X. Wei, T. Liang, Reliability estimation of retraction mechanism kinematic accuracy under small sample, *Eksploat. Niezawodn.*, 2023. https://doi.org/10.17531/ein/174777