AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 10304–10332.
DOI: 10.3934/math.2024504
Received: 26 November 2023
Revised: 07 January 2024
Accepted: 17 January 2024
Published: 14 March 2024

Research article

Estimation methods based on ranked set sampling for the arctan uniform distribution with application

Salem A. Alyami[1], **Amal S. Hassan**[2,∗], **Ibrahim Elbatal**[1], **Naif Alotaibi**[1], **Ahmed M. Gemeay**[3] **and Mohammed Elgarhy**[4,5,∗]

[1] Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, 11432, Saudi Arabia

[2] Faculty of Graduate Studies for Statistical Research, Cairo University, 5 Dr. Ahmed Zewail Street, Giza, 12613, Egypt

[3] Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt

[4] Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef, 62521, Egypt

[5] Department of Basic Sciences, Higher Institute for Administrative Sciences, Belbeis, AlSharkia, Egypt

∗ **Correspondence:** Email: m_elgarhy85@sva.edu.eg, amal52_soliman@cu.edu.eg.

Abstract: The arctan uniform distribution (AUD) is a brand-new bounded distribution that may be used for modeling a variety of existing bounded real-world datasets. Ranked set sampling (RSS) is a useful technique for parameter estimation when accurate measurement of the observation is challenging and/or expensive. In the current study, the parameter estimator of the AUD is addressed based on RSS and simple random sampling (SRS) techniques. Some of the popular conventional estimating techniques are considered. The efficiency of the produced estimates is compared using a Monte Carlo simulation. It appears that the maximum product spacing method has an advantage in assessing the quality of proposed estimates based on the outcomes of our simulations for both the SRS and RSS datasets. In comparison to estimates produced from the SRS datasets, it can be seen that those from the RSS datasets are more reliable. This implies that RSS is a more effective sampling technique in terms of generating estimates with a smaller mean squared error. The benefit of the RSS design over the SRS design is further supported by real data results.

Keywords: ranked set sampling; arctan uniform distribution; mean absolute relative error
Mathematics Subject Classification: 62F15, 62G20, 65C60

1. Introduction

Continuous data that strictly falls in the open interval (0, 1) is something we see rather frequently. Practitioners must represent this using the proper distributions, management, etc. This type of analysis includes studying ratios, percentages, etc. The beta distribution, which is utilized in a variety of situations, is one of the most versatile such distributions. However, there is a disadvantage to utilizing the beta distribution, as it is insufficient for some real-world scenarios, such as hydrological data. Considering this, the Topp-Leone distribution [1] and Kumaraswamy's distribution [2] merit consideration as alternatives to the beta model that have similar structure. The distribution and quantile functions may be represented in closed forms, which is a benefit in this case. To model datasets in the fields of biology, engineering, actuarial science, economics, and financial risk management, among others, several unit distributions have been created. Some of these significant, well-known distributions include the unit logistic distribution [3], unit Gompertz and unit Birnbaum-Saunders distributions [4,5], extended reduced Kies distribution [6], unit-Weibull distributions [7], unit generalized half normal distribution [8], unit Lindley distribution [9], unit Burr XII distribution [10], power unit Burr-XII distribution [11], unit gamma-Gompertz distribution [12], unit Teissier distribution [13], and generalized unit half-logistic geometric distribution [14], etc.

Recently, Kharazmi et al. [15] proposed a new one-parameter unit distribution based on the definition of the arctan function. The new bounded distribution is called the arctan uniform distribution (AUD). The probability density function (PDF) and cumulative distribution function (CDF) of the AUD are given, respectively, by:

$$f(z) = \frac{\delta}{\tan^{-1}(\delta) + \delta^2 z^2 \tan^{-1}(\delta)}, \quad 0 < z < 1, \delta > 0, \tag{1.1}$$

and

$$F(z) = \frac{\tan^{-1}(\delta z)}{\tan^{-1}(\delta)}, \quad 0 < z < 1, \delta > 0, \tag{1.2}$$

where δ is the scale parameter. We depict the PDF (1.1) in Figure 1 for a few different choices of the parameter δ to examine the effect of δ on the PDF behavior. It can be concluded that the AUD has asymmetric shapes. Kharazmi et al. [15] also provided the moments of this distribution.

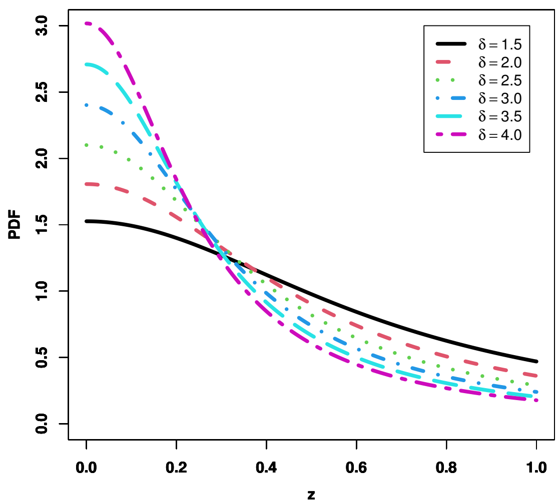

Figure 1. The PDF plots of the AUD.

Cost-effective sampling is a major issue in some research, especially when measuring the relevant feature is costly, inconvenient, or time-consuming. It is possible to give the sample items that are gathered additional structure using ranked set sampling (RSS) and to leverage this structure to create effective inferential processes. It is possible to rank tiny groups of units exactly, even without true quantification. The ranking might be carried out using eye examination, preliminary data, expert judgment, prior sampling episodes, or other imprecise techniques without the need for real measurement. The RSS method is a great instrument for attaining observational economy since it increases the accuracy attained per unit of the sample. This method of data collection was initially put forth by McIntyre [16] as an alternative to the widely used simple random sample (SRS) methodology for improving the effectiveness of the sample mean. It is used extensively in the fields of agriculture, biology, engineering, quality control, and environmental studies (see [17–25]). The procedures below should be followed to implement the RSS of s observations from a population:

(1) Choose (s) SRS with size (s) each, with s to be a low number.

(2) In each sample, order the units from smallest to greatest. Without actually measuring the units about the variable of interest, ranking is performed.

(3) Only the a_1th greatest unit in the a_1th sample, $a_1 = 1, .., s$ is used for actual measurements. As a result, the RSS associated with this cycle will be $Z_{1(1:m^{\circ\circ})}, Z_{2(2:m^{\circ\circ})}, ..., Z_{m^{\circ\circ}(m^{\circ\circ}:m^{\circ\circ})}$. Note that $Z_{a_1(a_1:m^{\circ\circ})}$ stands for the a_1th order statistic from the a_1th row.

(4) Carry out the preceding steps v times (cycles) to obtain sample size $m^{\circ\circ} = sv$, where s is the set size and v is the cycle number. As a result, the observed RSS for this v cycle will be $Z_{a_1(a_1:m^{\circ\circ})a_2}$, $a_1 = 1, .., s$, $a_2 = 1, .., v$, where s is the set size and v is the cycle count. Hence for simplified form, $Z_{a_1 a_2}$ will be used for the rest of the article instead of $Z_{a_1(a_1:m^{\circ\circ})a_2}$.

Statistical inference relies heavily on the parametric estimate approach employing the sampling design strategy. Numerous studies have examined various estimation techniques for estimating parameters based on RSS designs and their extensions. Reference [26] investigated how to estimate the location-scale family distributions' parameters. Some examples included the normal, exponential, and gamma distributions [27], the half logistic distribution [28], the Gumbel distribution [29], the generalized Rayleigh distribution [30], the Pareto distribution [31, 32], the x-gamma distribution [33], the new Weibull-Pareto distribution [34], the extended inverted Topp-Leone distribution [35], the generalized quasi-Lindley distribution [36], and the inverse Kumaraswamy distribution [37]. For more, see [38–42].

Statistical inference relies heavily on the parametric estimating approach and the sampling design strategy. The statistical literature frequently proposes different estimation approaches, since parameter estimation is important in practice. Generally, maximum likelihood (ML) estimation is the first step in the estimation process. This method's easy-to-understand formulation is the reason for its popularity. The estimators that are produced using this approach, for instance, are normally distributed and asymptotically consistent. Other, more widely used estimating techniques are available in the literature. These techniques include maximum product of spacing (MPS), least squares (LS), weighted LS (WLS), Cramér-von Mises (CM), Anderson-Darling (AD), minimum spacing absolute-log distance (MSALD), right-tail AD (RAD), left-tail AD (LAD), minimum spacing absolute distance (MSAD), percentile (PS), and a few more.

This study's objective is to present an in-depth assessment of several frequentist approaches to the AUD. The wide range of fields in which the RSS approach is used served as the inspiration for this concentration. In addition, the RSS design offers, for a fixed sample size, more efficient estimators than the SRS design. We use certain significant traditional estimating techniques based on the following procedures: RSS and SRS. The following estimation methods are taken into consideration: MPS, LS, ML, WLS, CM, AD, MSALD, RAD, LAD, MSAD, and PS. A simulation task is then used to compare the suggested estimates based on the RSS design to those offered by the SRS approach for the same sample size. Some precision metrics are used in comparison studies. The novelty of this study stems from the lack of prior research evaluating all of these estimating techniques for the AUD based on RSS. For illustration reasons, an insurance data set is investigated as well. Therefore, the study will serve as a guide for selecting the most appropriate estimating technique for the AUD, which we believe applied statisticians would find fascinating.

The following describes how this article is organized: Section 2 addresses the ML estimate (MLE) of the AUD parameter based on the RSS and SRS approaches. A few essential minimum distances of estimation for the proposed AUD are discussed in Section 3. In Section 4, several maximum and minimum product of spacing estimation are covered. The WLS, LS, and PS estimation techniques of the AUD are presented in Section 5. The effectiveness of the supplied estimating techniques is compared and evaluated in Section 6 using a Monte Carlo simulation. In Sections 7 and 8, respectively, an analysis of an insurance dataset is provided, followed by a conclusion.

2. Maximum likelihood estimator

In this section, the MLE of parameter δ of the AUD is considered based on RSS and SRS. At first, assume that $Z_{a_1a_2} = \{Z_{a_1a_2},\ a_1 = 1, ..., s,\ a_2 = 1, ..., v\}$, is an RSS of size $m^{\circ\circ}$ with PDF (1.1) and CDF (1.2), where v is the cycles count and s is the set size. It can be seen from the structure of RSS that the data are all mutually independent, and, in addition, for each $a_1 = 1, ..., s$, the data are identically distributed. It should be noted that, if the judgment ranking is perfect, the PDF of a_1th order statistics $Z_{a_1a_2}$ is as below:

$$f_{Z_{a_1a_2}}(z) = \frac{m^{\circ\circ}!}{(a_1-1)!(m^{\circ\circ}-a_1)}[f(z)]^{a_1-1}[1-F(z)]^{m^{\circ\circ}-a_1}. \tag{2.1}$$

The likelihood function (LF) of the AUD, based on RSS, is given by:

$$L(\delta) = \prod_{a_1=1}^{s}\prod_{a_2=1}^{v}\frac{m^{\circ\circ}!}{(a_1-1)!(m^{\circ\circ}-a_1)}\left[\frac{\delta}{\tan^{-1}(\delta)\left(1+\delta^2 z_{a_1a_2}^2\right)}\right]^{a_1-1}\left[1-\frac{\tan^{-1}(\delta z_{a_1a_2})}{\tan^{-1}(\delta)}\right]^{m^{\circ\circ}-a_1}. \tag{2.2}$$

The log-LF of (2.2), denoted by ℓ^*, is as follows:

$$\ell^* \propto \sum_{a_1=1}^{s}\sum_{a_2=1}^{v}(a_1-1)\left[\ln\delta - \ln\left(\tan^{-1}(\delta)\right) - \ln\left(1+\delta^2 z_{a_1a_2}^2\right)\right] + (m^{\circ\circ}-a_1)\ln\left[\mathbb{C}(\delta)\right], \tag{2.3}$$

where $\mathbb{C}(\delta) = \left[1-\frac{\tan^{-1}(\delta z_{a_1a_2})}{\tan^{-1}(\delta)}\right]$. The MLE of δ says $\hat{\delta}_1$ is obtained by maximizing (2.3), which can be computed as the solution of the following nonlinear equation:

$$\frac{\partial\ell^*}{\partial\delta} = \sum_{a_1=1}^{s}\sum_{a_2=1}^{v}\left[\frac{(a_1-1)}{\delta} - \frac{2\delta(a_1-1)z_{a_1a_2}^2}{\left(1+\delta^2 z_{a_1a_2}^2\right)} - \frac{(a_1-1)}{(1+\delta^2)\tan^{-1}(\delta)}\right] + \sum_{a_1=1}^{s}\sum_{a_2=1}^{v}\frac{(m^{\circ\circ}-a_1)\,\mathbb{C}'(\delta)}{\mathbb{C}(\delta)}, \tag{2.4}$$

where $\mathbb{C}'(\delta) = \frac{\tan^{-1}(\delta z_{a_1 a_2})}{\left[\tan^{-1}(\delta)\right]^2\left(1+\delta^2\right)} - \frac{1}{\left(1+\delta^2 z_{a_1 a_2}^2\right)\tan^{-1}(\delta)}$.
Setting (2.4) to zero and solving numerically, we get the MLE $\hat{\delta}_1$ of δ.

Additionally, the MLE of the AUD parameter under SRS is the subject of the following discussion. We assume that $z_1, z_2, ..., z_{m^{\circ\circ}}$ is an observed SRS of size $m^{\circ\circ}$ from the AUD with PDF (1.1). The log-LF, say ℓ_1^*, based on SRS, is given by

$$\ell_1^* = m^{\circ\circ}\ln(\delta) - m^{\circ\circ}\ln\left[\tan^{-1}(\delta)\right] - \sum_{i=1}^{m^{\circ\circ}}\ln\left(1+\delta^2 z_i^2\right).$$

The MLE $\ddddot{\delta}_1$, of δ, is provided as the solution of the following non-linear equation after setting with zero:

$$\frac{\partial \ell_1^*}{\partial \delta} = \frac{m^{\circ\circ}}{\delta} - \sum_{i=0}^{m^{\circ\circ}}\frac{m^{\circ\circ}}{(1+\delta^2)\tan^{-1}(\delta)} - \sum_{i=0}^{m^{\circ\circ}}\frac{2\delta z_i}{(1+\delta^2 z_i^2)}. \tag{2.5}$$

Then, $\ddddot{\delta}_1$ is provided from (2.5) after setting with zero and using the numerical technique.

3. Minimum distances estimators

This section illustrates four estimation methods that minimize goodness-of-fit statistics, including AD, RAD, LAD, and CM. This series of estimating methods was created based on the discrepancy between the estimated CDF and the actual distribution function. This section presents the estimates of AUD parameters for SRS and RSS using the mentioned methodologies.

3.1. Anderson-Darling estimators

Reference [43] introduced the AD test as an alternative to traditional statistical procedures to identify sample distribution deviations from the presumed distribution. Here, six estimators of parameter δ are produced based on the RSS and SRS.

Suppose that the ordered items $Z_{(1:m^{\circ\circ})}, Z_{(2:m^{\circ\circ})}, ..., Z_{(m^{\circ\circ}:m^{\circ\circ})}$ are an RSS drawn from the AUD with sample size $m^{\circ\circ} = sv$, where s is set size and v is the cycle count. By minimizing the following equation, the AD estimate (ADE) $\hat{\delta}_2$ of δ for the AUD is generated.

$$\vartheta_1 = -m^{\circ\circ} - \frac{1}{m^{\circ\circ}}\sum_{k=1}^{m^{\circ\circ}}(2k-1)\left\{\log F\left(z_{(k:m^{\circ\circ})}\,|\delta\right) + \log \bar{F}\left(z_{(m^{\circ\circ}-k+1:m^{\circ\circ})}\,|\delta\right)\right\}. \tag{3.1}$$

Instead of using (3.1), the ADE $\hat{\delta}_2$ of the AUD may be calculated by solving the nonlinear equation illustrated below:

$$\sum_{k=1}^{m^{\circ\circ}}(2k-1)\left\{\frac{\varphi_1\left(z_{(k:m^{\circ\circ})}\,|\delta\right)}{F\left(z_{(k:m^{\circ\circ})}\,|\delta\right)} - \frac{\varphi_2\left(z_{(m^{\circ\circ}-k+1:m^{\circ\circ})}\,|\delta\right)}{\bar{F}\left(z_{(m^{\circ\circ}-k+1:m^{\circ\circ})}\,|\delta\right)}\right\} = 0,$$

where,

$$\varphi_1\left(z_{(k:m^{\circ\circ})}\,|\delta\right) = \frac{\tan^{-1}(\delta z_{(k:m^{\circ\circ})})}{\left[\tan^{-1}(\delta)\right]^2(1+\delta^2)} - \frac{z_{(k:m^{\circ\circ})}}{\left(1+\delta^2 z_{(k:m^{\circ\circ})}^2\right)\tan^{-1}(\delta)}, \tag{3.2}$$

and

$$\varphi_2\left(z_{(m^{\circ\circ}-k+1:m^{\circ\circ})}\,|\delta\right) = \frac{\tan^{-1}(\delta z_{(m^{\circ\circ}-k+1:m^{\circ\circ})})}{\left[\tan^{-1}(\delta)\right]^2(1+\delta^2)} - \frac{z_{(m^{\circ\circ}-k+1:m^{\circ\circ})}}{\left(1+\delta^2 z^2_{(m^{\circ\circ}-k+1:m^{\circ\circ})}\right)\tan^{-1}(\delta)}. \tag{3.3}$$

The following function is used to provide the RAD estimate (RADE) $\hat{\delta}_3$ for δ of the AUD:

$$\vartheta_2 = \frac{m^{\circ\circ}}{2} - 2\sum_{k=1}^{m^{\circ\circ}} F\left(z_{(k:m^{\circ\circ})}\,|\delta\right) - \frac{1}{m^{\circ\circ}}\sum_{k=1}^{m^{\circ\circ}}(2k-1)\log\bar{F}\left(z_{(m^{\circ\circ}+1-k:m^{\circ\circ})}\,|\delta\right). \tag{3.4}$$

Instead of using (3.4), the RADE $\hat{\delta}_3$ of the AUD may be calculated by solving the nonlinear equation illustrated below:

$$-2\sum_{k=1}^{m^{\circ\circ}}\varphi_1\left(z_{(k:m^{\circ\circ})}\,|\delta\right) + \frac{1}{m^{\circ\circ}}\sum_{k=1}^{m^{\circ\circ}}\frac{(2k-1)\varphi_2\left(z_{(m^{\circ\circ}-k+1:m^{\circ\circ})}\,|\delta\right)}{\bar{F}\left(z_{(m^{\circ\circ}-k+1:m^{\circ\circ})}\,|\delta\right)} = 0,$$

where $\varphi_1(.)$ and $\varphi_2(.)$ are defined in (3.2) and (3.3).

The following function is used to provide the LAD estimate (LADE) $\hat{\delta}_4$ for δ of the AUD:

$$\vartheta_3 = \frac{-3m^{\circ\circ}}{2} + 2\sum_{k=1}^{m^{\circ\circ}} F\left(z_{(k:m^{\circ\circ})}\,|\delta\right) - \frac{1}{m^{\circ\circ}}\sum_{k=1}^{m^{\circ\circ}}(2k-1)\log F\left(z_{(k:m^{\circ\circ})}\,|\delta\right).$$

To obtain the LADE $\hat{\delta}_4$ of the AUD, the following nonlinear equation may be solved:

$$2\sum_{k=1}^{m^{\circ\circ}}\varphi_1\left(z_{(k:m^{\circ\circ})}\,|\delta\right) - \frac{1}{m^{\circ\circ}}\sum_{k=1}^{m^{\circ\circ}}\frac{(2k-1)\varphi_1\left(z_{(k:m^{\circ\circ})}\,|\delta\right)}{F\left(z_{(k:m^{\circ\circ})}\,|\delta\right)} = 0,$$

where, $\varphi_1(.)$ is defined in (3.2).

Next, let us consider the scenario in which the ordered items $Z_{(1)}, Z_{(2)}, ..., Z_{(m^{\circ\circ})}$ are SRS seen from AUD with sample size $m^{\circ\circ}$. The following function is used to provide the ADE $\ddddot{\delta}_2$, for δ of the AUD:

$$\vartheta_1^{\bullet} = -m^{\circ\circ} - \frac{1}{m^{\circ\circ}}\sum_{l=1}^{m^{\circ\circ}}(2l-1)\left\{\log F\left(z_{(l)}\,|\delta\right) + \log\bar{F}\left(z_{(m^{\circ\circ}-l+1)}\,|\delta\right)\right\}, \tag{3.5}$$

with respect to δ. The following equation which is equivalent to (3.5) may be solved numerically to provide $\ddddot{\delta}_2$

$$\sum_{l=1}^{m^{\circ\circ}}(2l-1)\left\{\frac{\varphi'_1\left(z_{(l)}\,|\delta\right)}{F\left(z_{(l)}\,|\delta\right)} - \frac{\varphi'_2\left(z_{(m^{\circ\circ}-l+1)}\,|\delta\right)}{\bar{F}\left(z_{(m^{\circ\circ}-l+1)}\,|\delta\right)}\right\} = 0,$$

where

$$\varphi'_1\left(z_{(l)}\,|\delta\right) = \frac{\tan^{-1}(\delta z_{(l)})}{\left[\tan^{-1}(\delta)\right]^2(1+\delta^2)} - \frac{z_{(m^{\circ\circ}-l+1)}}{(1+\delta^2 z_{(m^{\circ\circ}-l+1)})\tan^{-1}(\delta)}, \tag{3.6}$$

and

$$\varphi'_2\left(z_{(m^{\circ\circ}-l+1)}\,|\delta\right) = \frac{\tan^{-1}(\delta z_{(m^{\circ\circ}-l+1)})}{\left[\tan^{-1}(\delta)\right]^2(1+\delta^2)} - \frac{z_{(m^{\circ\circ}-l+1)}}{\left(1+\delta^2 z^2_{(m^{\circ\circ}-l+1)}\right)\tan^{-1}(\delta)}. \tag{3.7}$$

The following function is minimized for obtaining the RADE $\ddddot{\delta}_3$ for the AUD.

$$\vartheta_2{}^{\bullet} = \frac{m^{\circ\circ}}{2} - 2\sum_{l=1}^{m^{\circ\circ}} F\left(z_{(l)}\,|\delta\right) - \frac{1}{m^{\circ\circ}}\sum_{l=1}^{m^{\circ\circ}} (2l-1)\log \bar{F}\left(z_{(m^{\circ\circ}+1-l)}\,|\delta\right). \tag{3.8}$$

The RTDE $\ddddot{\delta}_3$ of the AUD is determined by solving the numerically the following nonlinear equation rather than using (3.8):

$$-2\sum_{l=1}^{m^{\circ\circ}} \varphi'_1\left(z_{(l)}\,|\delta\right) + \frac{1}{m^{\circ\circ}}\sum_{l=1}^{m^{\circ\circ}} \frac{(2l-1)\varphi'_2\left(z_{(m^{\circ\circ}+1-l)}\,|\delta\right)}{\bar{F}\left(z_{(m^{\circ\circ}+1-l)}\,|\delta\right)} = 0,$$

where $\varphi_1'(.)$ and $\varphi_2'(.)$ are defined in (3.6) and (3.7).

The following function is used to provide the LADE $\ddddot{\delta}_4$ for δ of the AUD:

$$\vartheta_3{}^{\bullet} = \frac{-3m^{\circ\circ}}{2} + 2\sum_{l=1}^{m^{\circ\circ}} F\left(z_{(l)}\,|\delta\right) - \frac{1}{m^{\circ\circ}}\sum_{l=1}^{m^{\circ\circ}} (2l-1)\log F\left(z_{l)}\,|\delta\right).$$

To obtain the LADE $\ddddot{\delta}_4$ of the AUD, the following nonlinear equation may be solved:

$$\vartheta_3{}^{\bullet} = 2\sum_{l=1}^{m^{\circ\circ}} \varphi'_1\left(z_{(l)}\,|\delta\right) - \frac{1}{m^{\circ\circ}}\sum_{l=1}^{m^{\circ\circ}} \frac{(2l-1)\varphi'_1\left(z_{(l)}\,|\delta\right)}{F\left(z_{1)}\,|\delta\right)} = 0,$$

where $\varphi_1'(.)$ is defined in (3.6).

3.2. Cramér-von Mises estimators

To support the decision to use minimal distance estimators of the CM type, [44] offered empirical evidence showing the estimator's bias is less than that of other minimum distance estimators. Here, the RSS and SRS techniques are used to produce the CM estimate (CME) for the AUD parameter.

Let us assume that the ordered items $Z_{(1:m^{\circ\circ})}, Z_{(2:m^{\circ\circ})}, ..., Z_{(m^{\circ\circ}:m^{\circ\circ})}$, with sample size $m^{\circ\circ} = sv$, where s is set size and v is the cycle numbers, are the selected RSS from CDF (1.2). In order to obtain CME $\hat{\delta}_5$ of δ, the following function is minimized with regard to δ :

$$\psi = \frac{1}{12m^{\circ\circ}} + \sum_{k=1}^{m^{\circ\circ}} \left\{F\left(z_{(k:m^{\circ\circ})}\,|\delta\right) - \frac{2k-1}{2m^{\circ\circ}}\right\}^2. \tag{3.9}$$

Instead of using (3.9), CME can be derived by resolving the following nonlinear equation:

$$\sum_{k=1}^{m^{\circ\circ}} \left\{F\left(z_{(k:m^{\circ\circ})}\,|\delta\right) - \frac{2k-1}{2m^{\circ\circ}}\right\} \varphi_1\left(z_{(k:m^{\circ\circ})}\,|\delta\right) = 0,$$

where $\varphi_1(.)$ is defined in (3.2).

Currently, suppose that the ordered items $Z_{(1)}, Z_{(2)}, ..., Z_{(m^{\circ\circ})}$ are the seen SRS from the AUD with sample size $m^{\circ\circ}$. So, the following function is minimized to determine the CME $\ddddot{\delta}_5$ of δ:

$$\psi' = \frac{1}{12m^{\circ\circ}} + \sum_{l=1}^{m^{\circ\circ}} \left\{F\left(z_{(l)}\,|\delta\right) - \frac{2l-1}{2m^{\circ\circ}}\right\}^2. \tag{3.10}$$

Or equivalent to (3.10), the CME $\dddot{\delta}_5$ of δ is produced by minimizing the following function

$$\sum_{l=1}^{m^{\circ\circ}} \left\{ F\left(z_{(l)}\,|\delta\right) - \frac{2l-1}{2m^{\circ\circ}} \right\} \varphi_1'\left(z_{(l)}\,|\delta\right) \;\; = 0,$$

where $\varphi_1'(.)$ is defined in (3.6).

4. Maximum and minimum product of spacings estimators

The concept of differences in the values of the CDF at successive data points, according to Cheng and Amin [45], may be used to get the MPS estimate (MPSE) of the unknown parameter of the AUD. This approach is just as effective as ML estimators and consistent under a wider range of conditions.

Let $Z_{(1:m^{\circ\circ})}, Z_{(2:m^{\circ\circ})}, ..., Z_{(m^{\circ\circ}:m^{\circ\circ})}$ be ordered items of the RSS drawn from the AUD with sample size $m^{\circ\circ} = sv$, where s is set size and v is the cycle numbers. The uniform spacings may be defined as follows based on a random sample taken from the AUD.

$$\hbar_k(\delta) = F\left(z_{(k:m^{\circ\circ})}\,|\delta\right) - F\left(z_{(k-1:m^{\circ\circ})}\,|\delta\right), \quad k = 1, 2, ..., m,$$

where $F\left(z_{(0:m^{\circ\circ})}\,|\delta\right) = 0, \quad F\left(z_{(m^{\circ\circ}+1:m^{\circ\circ})}\,|\delta\right) = 1$, such that $\sum_{k=1}^{m^{\circ\circ}+1} \hbar_k(\delta) = 1$.

To get the MPSE $\hat{\delta}_6$ of δ, the geometric mean of the spacing should be maximized.

$$K(\delta) = \left\{ \prod_{k=1}^{m^{\circ\circ}+1} \hbar_k(\delta) \right\}^{\frac{1}{m^{\circ\circ}+1}},$$

or, alternatively, there is maximizing the function that follows:

$$H(\delta) = \frac{1}{m^{\circ\circ}+1} \sum_{k=1}^{m^{\circ\circ}+1} \ln\left[\hbar_k(\delta)\right].$$

The MPSE $\hat{\delta}_6$ of δ can also be obtained by numerically resolving the following nonlinear equations:

$$\frac{\partial H(\delta)}{\partial \delta} = \frac{1}{1+m^{\circ\circ}} \sum_{k=1}^{m^{\circ\circ}+1} \frac{1}{[\hbar(\delta)]} \left[\varphi_1\left(z_{(k:m^{\circ\circ})}\,|\delta\right) - \varphi_1\left(z_{(k-1:m^{\circ\circ})}\,|\delta\right)\right] = 0,$$

where $\varphi_1(z_{(k:m^{\circ\circ})}\,|\delta)$ is defined in (3.2) and $\varphi_1\left(z_{(k-1:m^{\circ\circ})}\,|\delta\right)$ has the same expression with $z_{(k-1:m^{\circ\circ})}$.
Similarly, the minimum spacing distance estimator of δ is created by minimizing the following function.

$$H^{\bullet}(\delta) = \sum_{k=1}^{m^{\circ\circ}+1} \Delta\left[\hbar_k(\delta), \frac{1}{m^{\circ\circ}+1}\right],$$

where $\Delta(u_1, u_2)$ is the suitable distance. According to Ref. [46], for $\Delta(u_1, u_2) = |u_1 - u_2|$, $\Delta(u_1, u_2) = \left|\log u_1 - \log u_2\right|$ are referred to the MSAD and MSALD, respectively. As a result, the MSAD estimate (MSADE) and MSALD estimate (MSALDE) of δ are provided by minimizing the following functions:

$$H^{\bullet}(\delta) = \sum_{k=1}^{m^{\circ\circ}+1} \left|\hbar_k(\delta) - \frac{1}{m^{\circ\circ}+1}\right|, \tag{4.1}$$

and

$$H^{\bullet}(\delta) = \sum_{k=1}^{m^{\circ\circ}+1} \left| \log(\hbar_k(\delta)) - \log\left(\frac{1}{m^{\circ\circ}+1}\right) \right|, \tag{4.2}$$

with respect to δ. Equivalently to (4.1) and (4.2), the MSADE $\hat{\delta}_7$ and MSALDE $\hat{\delta}_8$ are provided by solving the nonlinear equations

$$\frac{\partial H^{\bullet}(\delta)}{\partial \delta} = \sum_{k=1}^{m^{\circ\circ}+1} \frac{\hbar_k(\delta) - \frac{1}{m^{\circ\circ}+1}}{\left|\hbar_k(\delta) - \frac{1}{m^{\circ\circ}+1}\right|} \left[\varphi_1\left(z_{(k:m^{\circ\circ})} \,|\delta\right) - \varphi_1\left(z_{(k-1:m^{\circ\circ})} \,|\delta\right)\right] = 0,$$

and

$$\frac{\partial H^{\bullet}(\delta)}{\partial \delta} = \sum_{k=1}^{m^{\circ\circ}+1} \frac{\log(\hbar_k(\delta)) - \log\left(\frac{1}{m^{\circ\circ}+1}\right)}{\left|\log(\hbar_k(\delta)) - \log\left(\frac{1}{m^{\circ\circ}+1}\right)\right|} \left[\varphi_1\left(z_{(k:m^{\circ\circ})} \,|\delta\right) - \varphi_1\left(z_{(k-1:m^{\circ\circ})} \,|\delta\right)\right] = 0,$$

where $\varphi_1\left(z_{(k:m^{\circ\circ})} \,|\delta\right)$ and $\varphi_1\left(z_{(k-1:m^{\circ\circ})} \,|\delta\right)$ are defined above.

In addition to the above, the MPSE $\ddddot{\delta}_6$ of δ for the AUD under SRS is obtained. Let $Z_{(1)}, Z_{(2)}, ..., Z_{(m^{\circ\circ})}$ be SRS of size $m^{\circ\circ}$ from CDF (1.2), and the uniform spacings in this situation are

$$\hbar_l^{\bullet}(\delta) = F\left(z_{(l)} \,|\delta\right) - F\left(z_{(l-1)} \,|\delta\right), \quad l = 1, 2, ..., m^{\circ\circ},$$

where $F\left(z_{(0)} \,|\delta\right) = 0, \quad F\left(z_{(m^{\circ\circ}+1)} \,|\delta\right) = 1$, such as $\sum_{l=1}^{m^{\circ\circ}+1} \hbar_l^{\bullet}(\delta) = 1$.

The MPSE $\ddddot{\delta}_6$ of δ is provided by maximizing the following function:

$$K(\delta) = \frac{1}{1+m^{\circ\circ}} \sum_{l=1}^{m^{\circ\circ}+1} \ln\left[\hbar_l^{\bullet}(\delta)\right]. \tag{4.3}$$

Equivalent to (4.3), the MPSE $\ddddot{\delta}_6$ of δ is produced by solving the following nonlinear equation numerically:

$$\frac{\partial K(\delta)}{\partial \delta} = \frac{1}{1+m^{\circ\circ}} \sum_{l=1}^{m^{\circ\circ}+1} \frac{1}{\left[\hbar_l^{\bullet}(\delta)\right]} \left[\varphi'_1\left(z_{(l)} \,|\delta\right) - \varphi'_1\left(z_{(l-1)} \,|\delta\right)\right] = 0,$$

where $\varphi_1'(z_{(l)} \,|\delta)$ is defined in (3.6), and $\varphi_1'(z_{(l-1)} \,|\delta)$ has the same expression with $z_{(l-1)}$.

Furthermore, MSADE $\ddddot{\delta}_7$ and MSALDE $\ddddot{\delta}_8$ are obtained by solving numerically the following equations:

$$K^{\bullet}(\delta) = \sum_{k=1}^{m^{\circ\circ}+1} \left| \hbar_l^{\bullet}(\delta) - \frac{1}{m^{\circ\circ}+1} \right|, \tag{4.4}$$

and

$$K^{\bullet}(\delta) = \sum_{k=1}^{m^{\circ\circ}+1} \left| \log(\hbar_l^{\bullet}(\delta)) - \log\left(\frac{1}{m^{\circ\circ}+1}\right) \right|, \tag{4.5}$$

with respect to δ. Equivalently to (4.4) and (4.5), the MSADE $\ddddot{\delta}_7$ and MSALDE $\ddddot{\delta}_8$ are provided by solving the nonlinear equations

$$\frac{\partial K^{\bullet}(\delta)}{\partial \delta} = \sum_{l=1}^{m^{\circ\circ}+1} \frac{\hbar_l^{\bullet}(\delta) - \frac{1}{m^{\circ\circ}+1}}{\left|\hbar_l^{\bullet}(\delta) - \frac{1}{m^{\circ\circ}+1}\right|} \left[\varphi_1\left(z_{(l)} \,|\delta\right) - \varphi_1\left(z_{(l-1)} \,|\delta\right)\right] = 0,$$

and

$$\frac{\partial K^{\bullet}(\delta)}{\partial \delta}=\sum_{l=1}^{m^{\circ\circ}+1}\frac{\log(\hbar_l^{\bullet}(\delta))-\log\left(\frac{1}{m^{\circ\circ}+1}\right)}{\left|\log(\hbar_l^{\bullet}(\delta))-\log\left(\frac{1}{m^{\circ\circ}+1}\right)\right|}\left[\varphi_1\left(z_{(l)}\left|\delta\right.\right)-\varphi_1\left(z_{(l-1)}\left|\delta\right.\right)\right]=0,$$

where, $\varphi_1\left(z_{(l)}\left|\delta\right.\right)$ and $\varphi_1\left(z_{(l-1)}\left|\delta\right.\right)$ are defined above.

5. Other estimation methods

This section offers the LS estimate (LSE), WLS estimate (WLSE), and PS estimate (PSE) for the AUD parameter based on RSS and SRS methods.

5.1. Least and weighted least squares estimators

Let $Z_{(1:m^{\circ\circ})}, Z_{(2:m^{\circ\circ})}, ..., Z_{(m^{\circ\circ}:m^{\circ\circ})}$ be an observed ordered RSS with size $m^{\circ\circ}=sv$, from the AUD. The LSE $\hat{\delta}_9$ and WLSE $\hat{\delta}_{10}$ are derived by minimizing the following functions with regard to δ:

$$\gamma=\sum_{k=1}^{m^{\circ\circ}}\left[F\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)-\frac{k}{m^{\circ\circ}+1}\right]^2, \tag{5.1}$$

and

$$\gamma'=\sum_{k=1}^{m^{\circ\circ}}\frac{(m^{\circ\circ}+1)^2(m^{\circ\circ}+2)}{k(m^{\circ\circ}-k+1)}\left[F\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)-\frac{k}{m^{\circ\circ}+1}\right]^2. \tag{5.2}$$

These estimators $\hat{\delta}_9$ are $\hat{\delta}_{10}$, which are equivalent to (5.1) and (5.2), and can be obtained by solving the following equations numerically:

$$\sum_{k=1}^{m^{\circ\circ}}\left[F\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)-\frac{k}{m^{\circ\circ}+1}\right]\varphi_1\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)=0,$$

and

$$\sum_{k=1}^{m^{\circ\circ}}\frac{(m^{\circ\circ}+1)^2(m^{\circ\circ}+2)}{k(m^{\circ\circ}-k+1)}\left[F\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)-\frac{k}{m^{\circ\circ}+1}\right]\varphi_1\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)=0,$$

where $\varphi_1\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)$ is defined before.

Additionally, suppose that $Z_{(1)}, Z_{(2)}, ..., Z_{(m^{\circ\circ})}$ is an ordered SRS of size $m^{\circ\circ}$ taken from the AUD. The LSE and WLSE $\dddot{\delta}_9$, $\dddot{\delta}_{10}$ of δ are produced by solving numerically the following equations:

$$\sum_{l=1}^{m^{\circ\circ}}\left[F\left(z_{(l)}\left|\delta\right.\right)-\frac{l}{m^{\circ\circ}+1}\right]\varphi_1\left(z_{(l)}\left|\delta\right.\right)=0,$$

and

$$\sum_{l=1}^{m^{\circ\circ}}\frac{(m^{\circ\circ}+1)^2(m^{\circ\circ}+2)}{l(m^{\circ\circ}-l+1)}\left[F\left(z_{(l)}\left|\delta\right.\right)-\frac{l}{m^{\circ\circ}+1}\right]\varphi_1\left(z_{(l)}\left|\delta\right.\right)=0,$$

where $\varphi_1\left(z_{(k:m^{\circ\circ})}\left|\delta\right.\right)$ is defined before.

5.2. Percentiles estimators

One of the often employed methods for estimating the Weibull distribution's parameters is the percentile approach, which differs from other estimation techniques in terms of its ease of computation and effectiveness in parameter estimation [47]. Here, PSE $\hat{\delta}_{11}$ of δ of the AUD is provided using RSS and SRS methods.

Consider $Z_{(1:m^{\circ\circ})}, Z_{(2:m^{\circ\circ})}, ..., Z_{(m^{\circ\circ}:m^{\circ\circ})}$ as an observed ordered RSS, with size $m^{\circ\circ} = sv$, available from the AUD. From the PSE of the AUD's parameter one may get $\hat{\delta}_{11}$ by minimizing the following function and assuming that $p_k = \frac{k}{m^{\circ\circ}+1}$ is the estimate of $F\left(z_{(k:m^{\circ\circ})}\,|\delta\right)$

$$\Lambda = \sum_{k=1}^{m^{\circ\circ}} \left[z_{(k:m^{\circ\circ})} - \frac{1}{\delta} \tan\left(p_{(k:m^{\circ\circ})} \tan^{-1}(\delta) \right) \right],$$

with respect to δ.

In the case of the SRS method, let $Z_{(1)}, Z_{(2)}, ..., Z_{(m^{\circ\circ})}$ be an ordered SRS of size $m^{\circ\circ}$ drawn from AUD. The PSE $\dddot{\delta}_{11}$ of δ is obtained, by minimizing the following equation:

$$\Lambda_1 = \sum_{l=1}^{m^{\circ\circ}} \left[z_{(l)} - \frac{1}{\delta} \tan\left(p_{(l)} \tan^{-1}(\delta) \right) \right],$$

with respect to δ.

6. Simulation

This section focuses on the examination of various estimation methods presented in this paper. The goal is to assess the efficacy of these methods in estimating model parameters through the generation of random datasets derived from the proposed model. Subsequently, these datasets will be ranked, and the estimation methods will be employed to identify the most recommended one. The simulation will be conducted with the assumption of a flawless ranking, outlined as follows:

- To generate an RSS from the AUD with a fixed set size $s = 5$ and different cycle numbers $v = 3, 10, 24, 40, 60$, and 90, the corresponding sample sizes $m^{\circ\circ} = sv = 15, 50, 120, 200, 300$, and 450 are employed.
- Generate an SRS from the AUD with the specified sample sizes, $m^{\circ\circ}$ = 15, 50, 120, 200, 300, and 450.
- We have a set of estimates corresponding to each sample size, using the true parameter values (δ) of 0.15, 0.6, 1.0, 1.5, 2.0, and 2.5.
- To evaluate the effectiveness of the estimation methods, three measures are employed, which include the following:
  Average of absolute bias (bias), $|bias(\widehat{\boldsymbol{\delta}})| = \frac{1}{M}\sum_{i=1}^{M} |\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}|$, mean squared errors (MSE), $MSE = \frac{1}{M}\sum_{i=1}^{M} (\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta})^2$, mean absolute relative errors (MRE) $MRE = \frac{1}{M}\sum_{i=1}^{M} |\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}|/\boldsymbol{\delta}$.
- The measures outlined in the previous step serve as objective benchmarks for evaluating the accuracy and reliability of the estimated parameters. Utilizing these evaluation metrics enables a comprehensive assessment of the performance of the estimation techniques. This evaluation process provides valuable insights into the effectiveness and appropriateness of these techniques for the particular model under consideration.

- By repeating this process multiple times through numerous iterations, we can obtain a reliable and robust assessment of the estimation techniques. This repeated evaluation helps ensure that the performance results are consistent and representative, contributing to a more thorough understanding of the effectiveness of these techniques in estimating the model parameters.
- The results of the evaluation measures are presented in Tables 1–12, encompassing both SRS and RSS. These tables offer a comprehensive summary of the outcomes obtained. In these tables, the magnitude of each value signifies its relative effectiveness when compared to all the estimation approaches examined in the study. Lower-ranked values indicate stronger and more significant performance relative to the investigated estimation methods. These tables serve as a valuable reference for assessing the relative power and significance of the different estimation techniques.
- The ratio of the MSE for SRS to the MSE for RSS is provided in Table 13. This ratio helps to gauge the comparative performance of SRS and RSS in terms of MSE, offering insights into the efficiency of these sampling methods.
- For a thorough and detailed analysis of the estimates, we present both their partial and total ranks in Tables 14 and 15 for the SRS and RSS, respectively. These rank tables offer a more nuanced and comprehensive perspective on the performance and comparative effectiveness of each estimation approach, facilitating a deeper understanding of their relative strengths and weaknesses.

Upon careful analysis of the simulation results and the rankings presented in the tables, several conclusions can be drawn:

- It is noteworthy that for both SRS and RSS datasets, our model estimates exhibit the consistency property. This property implies that as the sample size increases, the estimates converge to the true parameter values.
- All three measures used exhibit a consistent trend: They decrease as the sample size increases. This pattern suggests that larger sample sizes lead to more accurate and precise parameter estimates.
- Based on our simulation results for both SRS and RSS datasets, it appears that the MPSE has the advantage in determining the quality of our estimates.
- From Table 13, it can be observed that the estimates obtained from the RSS datasets are more efficient compared to those obtained from the SRS datasets. This suggests that RSS is a more efficient sampling method in terms of producing estimates with lower MSE.

Table 1. Bias, MSE, and MRE values for ($\delta = 0.15$) under SRS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\dddot{\delta}$	0.5128[5]	0.5108[4]	0.5268[8]	0.4521[1]	0.5305[9]	0.4879[3]	0.4799[2]	0.5257[7]	0.5755[11]	0.5345[10]	0.5223[6]
	MSE	$\dddot{\delta}$	0.5886[4]	0.5957[5]	0.6356[9]	0.4704[1]	0.6343[8]	0.5022[2]	0.5088[3]	0.6309[7]	0.806[11]	0.7126[10]	0.6245[6]
	MRE	$\dddot{\delta}$	3.4184[5]	3.405[4]	3.5117[8]	3.0141[1]	3.5368[9]	3.2523[3]	3.1997[2]	3.5049[7]	3.8366[11]	3.563[10]	3.4823[6]
	$\sum$ *Ranks*		14[5]	13[4]	25[8]	3[1]	26[9]	8[3]	7[2]	21[7]	33[11]	30[10]	18[6]
50	bias	$\dddot{\delta}$	0.3349[4]	0.3408[6]	0.3462[8]	0.3171[1]	0.337[5]	0.329[3]	0.327[2]	0.3414[7]	0.3703[11]	0.361[10]	0.3535[9]
	MSE	$\dddot{\delta}$	0.1991[4]	0.2068[6]	0.2138[8]	0.1837[1]	0.205[5]	0.1905[2]	0.1917[3]	0.2082[7]	0.2532[10]	0.2582[11]	0.2295[9]
	MRE	$\dddot{\delta}$	2.2326[4]	2.2718[6]	2.3083[8]	2.1143[1]	2.2469[5]	2.1934[3]	2.18[2]	2.2762[7]	2.4686[11]	2.4064[10]	2.357[9]
	$\sum$ *Ranks*		12[4]	18[6]	24[8]	3[1]	15[5]	8[3]	7[2]	21[7]	32[11]	31[10]	27[9]
120	bias	$\dddot{\delta}$	0.2637[7]	0.2659[8]	0.2568[2]	0.2537[1]	0.2628[5]	0.2617[4]	0.2591[3]	0.2635[6]	0.2767[9]	0.2825[11]	0.279[10]
	MSE	$\dddot{\delta}$	0.1088[5]	0.1116[8]	0.1055[2]	0.1044[1]	0.1099[7]	0.1089[6]	0.1061[3]	0.1073[4]	0.1246[9]	0.1374[11]	0.1272[10]
	MRE	$\dddot{\delta}$	1.7582[7]	1.7728[8]	1.7119[2]	1.6911[1]	1.7523[5]	1.7446[4]	1.7271[3]	1.7564[6]	1.8445[9]	1.8832[11]	1.86[10]
	$\sum$ *Ranks*		19[7]	24[8]	6[2]	3[1]	17[6]	14[4]	9[3]	16[5]	27[9]	33[11]	30[10]
200	bias	$\dddot{\delta}$	0.2313[8]	0.2301[6]	0.2294[4]	0.2173[1]	0.2272[2]	0.2307[7]	0.2277[3]	0.2297[5]	0.2403[9]	0.2447[11]	0.2438[10]
	MSE	$\dddot{\delta}$	0.0796[8]	0.0781[5]	0.0788[6]	0.0722[1]	0.0761[2]	0.0791[7]	0.0764[3]	0.0769[4]	0.0874[9]	0.0958[11]	0.0907[10]
	MRE	$\dddot{\delta}$	1.5418[8]	1.534[6]	1.5293[4]	1.4486[1]	1.5149[2]	1.5379[7]	1.5181[3]	1.5314[5]	1.6019[9]	1.6315[11]	1.6256[10]
	$\sum$ *Ranks*		24[8]	17[6]	14[4.5]	3[1]	6[2]	21[7]	9[3]	14[4.5]	27[9]	33[11]	30[10]
300	bias	$\dddot{\delta}$	0.2075[4]	0.2098[6]	0.2108[8]	0.1925[1]	0.2059[3]	0.2084[5]	0.205[2]	0.2106[7]	0.2183[9]	0.223[11]	0.2207[10]
	MSE	$\dddot{\delta}$	0.0603[4]	0.062[7]	0.0629[8]	0.0549[1]	0.0602[3]	0.0614[5]	0.0593[2]	0.0619[6]	0.0686[9]	0.0763[11]	0.0701[10]
	MRE	$\dddot{\delta}$	1.3836[4]	1.3986[6]	1.4053[8]	1.283[1]	1.3724[3]	1.3896[5]	1.3666[2]	1.4037[7]	1.4553[9]	1.4864[11]	1.4712[10]
	$\sum$ *Ranks*		12[4]	19[6]	24[8]	3[1]	9[3]	15[5]	6[2]	20[7]	27[9]	33[11]	30[10]
450	bias	$\dddot{\delta}$	0.191[8]	0.1908[7]	0.1869[3.5]	0.1744[1]	0.1891[5]	0.1853[2]	0.1869[3.5]	0.1905[6]	0.1957[9]	0.2039[11]	0.1996[10]
	MSE	$\dddot{\delta}$	0.0488[6]	0.0492[8]	0.047[3.5]	0.0436[1]	0.0487[5]	0.0463[2]	0.047[3.5]	0.0489[7]	0.0524[9]	0.0611[11]	0.0548[10]
	MRE	$\dddot{\delta}$	1.2732[8]	1.2723[7]	1.2459[3]	1.1625[1]	1.261[5]	1.2351[2]	1.2463[4]	1.2698[6]	1.3047[9]	1.3591[11]	1.3308[10]
	$\sum$ *Ranks*		22[7.5]	22[7.5]	10[3]	3[1]	15[5]	6[2]	11[4]	19[6]	27[9]	33[11]	30[10]

Table 2. Bias, MSE, and MRE values for ($\delta = 0.15$) under RSS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\hat{\delta}$	0.3505[1]	0.395[5]	0.4091[7]	0.3525[2]	0.4098[8]	0.3829[4]	0.37[3]	0.3965[6]	0.4737[11]	0.4317[10]	0.4258[9]
	MSE	$\hat{\delta}$	0.2218[1]	0.3044[5]	0.3299[7]	0.2503[2]	0.3354[8]	0.2813[4]	0.2668[3]	0.3068[6]	0.4836[11]	0.4161[10]	0.383[9]
	MRE	$\hat{\delta}$	2.3369[1]	2.6331[5]	2.7272[7]	2.3501[2]	2.7317[8]	2.5528[4]	2.4666[3]	2.6436[6]	3.1579[11]	2.8777[10]	2.8388[9]
	$\sum$ *Ranks*		3[1]	15[5]	21[7]	6[2]	24[8]	12[4]	9[3]	18[6]	33[11]	30[10]	27[9]
50	bias	$\hat{\delta}$	0.2485[9]	0.2123[5]	0.2129[6]	0.1925[1]	0.213[7]	0.2075[2]	0.2086[3]	0.212[4]	0.2296[8]	0.2735[11]	0.2679[10]
	MSE	$\hat{\delta}$	0.094[9]	0.0642[5]	0.0652[7]	0.0544[1]	0.0648[6]	0.0606[2]	0.0624[3]	0.0636[4]	0.0773[8]	0.1338[11]	0.1145[10]
	MRE	$\hat{\delta}$	1.6565[9]	1.4152[5]	1.4193[6]	1.2837[1]	1.4197[7]	1.383[2]	1.3908[3]	1.4134[4]	1.5306[8]	1.823[11]	1.7858[10]
	$\sum$ *Ranks*		27[9]	15[5]	19[6]	3[1]	20[7]	6[2]	9[3]	12[4]	24[8]	33[11]	30[10]
120	bias	$\hat{\delta}$	0.1991[9]	0.1406[2]	0.1423[4]	0.1232[1]	0.1427[5]	0.1412[3]	0.143[6]	0.1462[7]	0.1486[8]	0.2087[11]	0.2026[10]
	MSE	$\hat{\delta}$	0.0542[9]	0.025[2]	0.0258[6]	0.0201[1]	0.0257[5]	0.0253[3]	0.0256[4]	0.0266[7]	0.0283[8]	0.0667[11]	0.057[10]
	MRE	$\hat{\delta}$	1.3271[9]	0.9374[2]	0.9488[4]	0.8213[1]	0.9511[5]	0.9411[3]	0.9531[6]	0.9747[7]	0.9906[8]	1.3916[11]	1.3509[10]
	$\sum$ *Ranks*		27[9]	6[2]	14[4]	3[1]	15[5]	9[3]	16[6]	21[7]	24[8]	33[11]	30[10]
200	bias	$\hat{\delta}$	0.1806[10]	0.1085[2]	0.1113[5]	0.0928[1]	0.1112[4]	0.1106[3]	0.1139[6]	0.1148[7]	0.1173[8]	0.1853[11]	0.1793[9]
	MSE	$\hat{\delta}$	0.0422[9]	0.0152[2]	0.0157[3.5]	0.0117[1]	0.0158[5]	0.0157[3.5]	0.0165[6]	0.0169[7]	0.0174[8]	0.05[11]	0.0429[10]
	MRE	$\hat{\delta}$	1.204[10]	0.7233[2]	0.7423[5]	0.6186[1]	0.7416[4]	0.7376[3]	0.7596[6]	0.7652[7]	0.7821[8]	1.2355[11]	1.1956[9]
	$\sum$ *Ranks*		29[10]	6[2]	13.5[5]	3[1]	13[4]	9.5[3]	18[6]	21[7]	24[8]	33[11]	28[9]
300	bias	$\hat{\delta}$	0.1596[9]	0.0902[4]	0.09[3]	0.0691[1]	0.0905[5]	0.0891[2]	0.0912[6]	0.0994[8]	0.0934[7]	0.1697[11]	0.1663[10]
	MSE	$\hat{\delta}$	0.0325[9]	0.0112[5]	0.011[3]	0.0073[1]	0.0111[4]	0.0109[2]	0.0114[6]	0.0142[8]	0.0118[7]	0.0405[11]	0.0354[10]
	MRE	$\hat{\delta}$	1.0638[9]	0.6017[4]	0.6001[3]	0.4603[1]	0.6035[5]	0.5937[2]	0.6078[6]	0.6626[8]	0.6227[7]	1.1314[11]	1.1087[10]
	$\sum$ *Ranks*		27[9]	13[4]	9[3]	3[1]	14[5]	6[2]	18[6]	24[8]	21[7]	33[11]	30[10]
450	bias	$\hat{\delta}$	0.1462[9]	0.0824[8]	0.0689[2]	0.0478[1]	0.0694[3]	0.0698[4]	0.0729[6]	0.0761[7]	0.0728[5]	0.1577[11]	0.1518[10]
	MSE	$\hat{\delta}$	0.0273[9]	0.0114[8]	0.0073[2]	0.0041[1]	0.0074[3.5]	0.0074[3.5]	0.0079[5.5]	0.0097[7]	0.0079[5.5]	0.0337[11]	0.0293[10]
	MRE	$\hat{\delta}$	0.9747[9]	0.5493[8]	0.4592[2]	0.3183[1]	0.4629[3]	0.4656[4]	0.486[6]	0.507[7]	0.4853[5]	1.0514[11]	1.0123[10]
	$\sum$ *Ranks*		27[9]	24[8]	6[2]	3[1]	9.5[3]	11.5[4]	17.5[6]	21[7]	15.5[5]	33[11]	30[10]

Table 3. Bias, MSE, and MRE values for ($\delta = 0.6$) under SRS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\dddot{\delta}$	0.6207{5}	0.6075{4}	0.6427{8}	0.5986{2}	0.6209{6}	0.5592{1}	0.6055{3}	0.6366{7}	0.7045{10}	0.7067{11}	0.6969{9}
	MSE	$\dddot{\delta}$	0.6293{5}	0.5692{3}	0.6515{7}	0.5246{2}	0.6296{6}	0.4705{1}	0.6066{4}	0.6556{8}	0.8866{10}	0.9346{11}	0.7847{9}
	MRE	$\dddot{\delta}$	1.0346{5}	1.0126{4}	1.0711{8}	0.9977{2}	1.0349{6}	0.932{1}	1.0091{3}	1.061{7}	1.1742{10}	1.1778{11}	1.1615{9}
	$\sum$ *Ranks*		15{5}	11{4}	23{8}	6{2}	18{6}	3{1}	10{3}	22{7}	30{10}	33{11}	27{9}
50	bias	$\dddot{\delta}$	0.4557{8}	0.4571{9}	0.4129{4}	0.4021{2}	0.4086{3}	0.3988{1}	0.4154{5}	0.4482{7}	0.4369{6}	0.4633{10}	0.4807{11}
	MSE	$\dddot{\delta}$	0.297{9}	0.2957{8}	0.246{5}	0.226{2}	0.2334{3}	0.2179{1}	0.241{4}	0.2817{7}	0.2767{6}	0.3094{10}	0.3237{11}
	MRE	$\dddot{\delta}$	0.7596{8}	0.7618{9}	0.6882{4}	0.6701{2}	0.6809{3}	0.6647{1}	0.6924{5}	0.7471{7}	0.7282{6}	0.7722{10}	0.8012{11}
	$\sum$ *Ranks*		25{8}	26{9}	13{4}	6{2}	9{3}	3{1}	14{5}	21{7}	18{6}	30{10}	33{11}
120	bias	$\dddot{\delta}$	0.3261{7}	0.3161{5}	0.2963{3}	0.2886{2}	0.3027{4}	0.2852{1}	0.3573{8}	0.3194{6}	0.3818{11}	0.3685{10}	0.3595{9}
	MSE	$\dddot{\delta}$	0.1668{7}	0.1613{5}	0.135{3}	0.1274{2}	0.138{4}	0.1264{1}	0.2051{9}	0.1637{6}	0.2333{11}	0.2082{10}	0.1993{8}
	MRE	$\dddot{\delta}$	0.5435{7}	0.5268{5}	0.4939{3}	0.4811{2}	0.5044{4}	0.4754{1}	0.5955{8}	0.5324{6}	0.6364{11}	0.6142{10}	0.5991{9}
	$\sum$ *Ranks*		21{7}	15{5}	9{3}	6{2}	12{4}	3{1}	25{8}	18{6}	33{11}	30{10}	26{9}
200	bias	$\dddot{\delta}$	0.3121{8}	0.2876{6}	0.2792{4}	0.2252{1}	0.2871{5}	0.2263{2}	0.264{3}	0.3165{9}	0.291{7}	0.3194{10}	0.3455{11}
	MSE	$\dddot{\delta}$	0.1814{9}	0.1597{7}	0.144{4}	0.0808{1}	0.1537{5}	0.0831{2}	0.1324{3}	0.1883{10}	0.1551{6}	0.1698{8}	0.2015{11}
	MRE	$\dddot{\delta}$	0.5202{8}	0.4793{6}	0.4653{4}	0.3753{1}	0.4785{5}	0.3771{2}	0.44{3}	0.5275{9}	0.485{7}	0.5324{10}	0.5758{11}
	$\sum$ *Ranks*		25{8}	19{6}	12{4}	3{1}	15{5}	6{2}	9{3}	28{9.5}	20{7}	28{9.5}	33{11}
300	bias	$\dddot{\delta}$	0.2609{6}	0.2871{10}	0.2688{8}	0.1889{1}	0.2707{9}	0.1897{2}	0.2442{5}	0.2357{3}	0.244{4}	0.2623{7}	0.2952{11}
	MSE	$\dddot{\delta}$	0.1497{7}	0.182{11}	0.1598{8}	0.059{1}	0.1632{9}	0.062{2}	0.1263{5}	0.1124{3.5}	0.1124{3.5}	0.1267{6}	0.1724{10}
	MRE	$\dddot{\delta}$	0.4348{6}	0.4786{10}	0.448{8}	0.3148{1}	0.4511{9}	0.3162{2}	0.4071{5}	0.3928{3}	0.4067{4}	0.4372{7}	0.492{11}
	$\sum$ *Ranks*		19{6}	31{10}	24{8}	3{1}	27{9}	6{2}	15{5}	9.5{3}	11.5{4}	20{7}	32{11}
450	bias	$\dddot{\delta}$	0.2359{8.5}	0.2146{3}	0.2196{4}	0.1484{1}	0.2215{5}	0.1498{2}	0.239{10}	0.2304{6}	0.2359{8.5}	0.2309{7}	0.2523{11}
	MSE	$\dddot{\delta}$	0.1406{9}	0.1139{4}	0.1222{5}	0.0368{1}	0.1232{6}	0.0375{2}	0.1471{11}	0.1298{8}	0.1262{7}	0.1117{3}	0.1454{10}
	MRE	$\dddot{\delta}$	0.3932{9}	0.3576{3}	0.3661{4}	0.2474{1}	0.3692{5}	0.2497{2}	0.3983{10}	0.384{6}	0.3931{8}	0.3848{7}	0.4205{11}
	$\sum$ *Ranks*		26.5{9}	10{3}	13{4}	3{1}	16{5}	6{2}	31{10}	20{7}	23.5{8}	17{6}	32{11}

Table 4. Bias, MSE, and MRE values for ($\delta = 0.6$) under RSS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\hat{\delta}$	0.4538{1}	0.472{3}	0.5065{7}	0.4879{5}	0.4887{6}	0.4678{2}	0.4746{4}	0.5331{8}	0.543{9}	0.5913{11}	0.5554{10}
	MSE	$\hat{\delta}$	0.2993{1}	0.316{3}	0.3685{7}	0.3255{5}	0.3619{6}	0.3087{2}	0.3177{4}	0.4032{8}	0.4637{10}	0.5488{11}	0.4368{9}
	MRE	$\hat{\delta}$	0.7563{1}	0.7867{3}	0.8442{7}	0.8132{5}	0.8145{6}	0.7797{2}	0.7909{4}	0.8885{8}	0.9049{9}	0.9854{11}	0.9257{10}
	$\sum$ *Ranks*		3{1}	9{3}	21{7}	15{5}	18{6}	6{2}	12{4}	24{8}	28{9}	33{11}	29{10}
50	bias	$\hat{\delta}$	0.3076{9}	0.2821{8}	0.201{1}	0.2118{4}	0.2073{3}	0.2034{2}	0.2238{6}	0.2503{7}	0.2234{5}	0.3601{10}	0.365{11}
	MSE	$\hat{\delta}$	0.1571{8}	0.1676{9}	0.0668{1}	0.0736{4}	0.072{3}	0.0685{2}	0.085{6}	0.128{7}	0.0804{5}	0.1958{10}	0.2026{11}
	MRE	$\hat{\delta}$	0.5127{9}	0.4701{8}	0.3351{1}	0.353{4}	0.3455{3}	0.3389{2}	0.373{6}	0.4171{7}	0.3723{5}	0.6002{10}	0.6083{11}
	$\sum$ *Ranks*		26{9}	25{8}	3{1}	12{4}	9{3}	6{2}	18{6}	21{7}	15{5}	30{10}	33{11}
120	bias	$\hat{\delta}$	0.2623{10}	0.146{8}	0.083{2}	0.0856{4}	0.0844{3}	0.0799{1}	0.1431{7}	0.1349{6}	0.1195{5}	0.2844{11}	0.2593{9}
	MSE	$\hat{\delta}$	0.1549{11}	0.0865{8}	0.0114{2.5}	0.0125{4}	0.0114{2.5}	0.0103{1}	0.0809{7}	0.0747{6}	0.0508{5}	0.1489{10}	0.1346{9}
	MRE	$\hat{\delta}$	0.4372{10}	0.2433{8}	0.1383{2}	0.1426{4}	0.1406{3}	0.1332{1}	0.2385{7}	0.2248{6}	0.1992{5}	0.4739{11}	0.4322{9}
	$\sum$ *Ranks*		31{10}	24{8}	6.5{2}	12{4}	8.5{3}	3{1}	21{7}	18{6}	15{5}	32{11}	27{9}
200	bias	$\hat{\delta}$	0.192{9}	0.0877{8}	0.0497{1}	0.0499{2}	0.051{4}	0.0502{3}	0.0738{7}	0.0695{6}	0.0688{5}	0.2446{11}	0.2316{10}
	MSE	$\hat{\delta}$	0.0998{9}	0.0529{8}	0.0039{1}	0.0041{3}	0.0049{4}	0.004{2}	0.0325{7}	0.0291{6}	0.0222{5}	0.1347{11}	0.1316{10}
	MRE	$\hat{\delta}$	0.3201{9}	0.1461{8}	0.0829{1}	0.0832{2}	0.0851{4}	0.0836{3}	0.123{7}	0.1159{6}	0.1146{5}	0.4077{11}	0.386{10}
	$\sum$ *Ranks*		27{9}	24{8}	3{1}	7{2}	12{4}	8{3}	21{7}	18{6}	15{5}	33{11}	30{10}
300	bias	$\hat{\delta}$	0.1497{9}	0.0474{7}	0.0327{3.5}	0.0323{1.5}	0.0323{1.5}	0.0327{3.5}	0.0511{8}	0.0426{5.5}	0.0426{5.5}	0.214{11}	0.1883{10}
	MSE	$\hat{\delta}$	0.0736{9}	0.0199{7}	0.0017{2.5}	0.0017{2.5}	0.0017{2.5}	0.0017{2.5}	0.0223{8}	0.0133{6}	0.0112{5}	0.1135{11}	0.108{10}
	MRE	$\hat{\delta}$	0.2496{9}	0.0791{7}	0.0545{3.5}	0.0539{1.5}	0.0539{1.5}	0.0545{3.5}	0.0852{8}	0.071{5.5}	0.071{5.5}	0.3567{11}	0.3139{10}
	$\sum$ *Ranks*		27{9}	21{7}	9.5{3.5}	5.5{1.5}	5.5{1.5}	9.5{3.5}	24{8}	17{6}	16{5}	33{11}	30{10}
450	bias	$\hat{\delta}$	0.1307{9}	0.0459{8}	0.0214{1}	0.0223{4}	0.022{3}	0.0219{2}	0.0336{6}	0.0359{7}	0.0289{5}	0.1672{11}	0.1428{10}
	MSE	$\hat{\delta}$	0.0693{9}	0.0304{8}	$7e-04${1}	$8e-04${3}	$8e-04${3}	$8e-04${3}	0.0136{6}	0.0193{7}	0.0082{5}	0.0865{11}	0.0754{10}
	MRE	$\hat{\delta}$	0.2179{9}	0.0765{8}	0.0356{1}	0.0371{4}	0.0367{3}	0.0365{2}	0.056{6}	0.0599{7}	0.0481{5}	0.2786{11}	0.238{10}
	$\sum$ *Ranks*		23{9}	20{8}	10{1}	18{7}	16{5}	14{3.5}	14{3.5}	17{6}	11{2}	29{11}	26{10}

Table 5. Bias, MSE, and MRE values for (δ = 1.0) under SRS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\dddot{\delta}$	0.7122[6]	0.6873[3]	0.7245[7]	0.7082[5]	0.6979[4]	0.6344[1]	0.6844[2]	0.8253[10]	0.8014[8]	0.8066[9]	0.8412[11]
	MSE	$\dddot{\delta}$	0.8474[6]	0.7629[3]	0.8495[7]	0.7981[4]	0.8277[5]	0.6614[1]	0.7295[2]	1.1297[9]	1.1745[10]	1.3548[11]	1.1033[8]
	MRE	$\dddot{\delta}$	0.7122[6]	0.6873[3]	0.7245[7]	0.7082[5]	0.6979[4]	0.6344[1]	0.6844[2]	0.8253[10]	0.8014[8]	0.8066[9]	0.8412[11]
	$\sum$ *Ranks*		18[6]	9[3]	21[7]	14[5]	13[4]	3[1]	6[2]	29[9.5]	26[8]	29[9.5]	30[11]
50	bias	$\dddot{\delta}$	0.4803[6]	0.506[9]	0.4083[3]	0.3938[2]	0.412[4]	0.3902[1]	0.5212[10]	0.484[7]	0.4515[5]	0.5029[8]	0.5381[11]
	MSE	$\dddot{\delta}$	0.4336[7]	0.4738[9]	0.2674[3]	0.2539[2]	0.2684[4]	0.2452[1]	0.5006[10]	0.4405[8]	0.3571[5]	0.4333[6]	0.5013[11]
	MRE	$\dddot{\delta}$	0.4803[6]	0.506[9]	0.4083[3]	0.3938[2]	0.412[4]	0.3902[1]	0.5212[10]	0.484[7]	0.4515[5]	0.5029[8]	0.5381[11]
	$\sum$ *Ranks*		19[6]	27[9]	9[3]	6[2]	12[4]	3[1]	30[10]	22[7.5]	15[5]	22[7.5]	33[11]
120	bias	$\dddot{\delta}$	0.3303[7]	0.356[8]	0.3065[4]	0.2456[1]	0.2842[3]	0.2541[2]	0.3263[6]	0.3208[5]	0.3763[10]	0.3653[9]	0.394[11]
	MSE	$\dddot{\delta}$	0.2801[8]	0.3012[9]	0.2087[4]	0.1004[1]	0.1845[3]	0.1067[2]	0.2702[6]	0.26[5]	0.3326[10]	0.2761[7]	0.3575[11]
	MRE	$\dddot{\delta}$	0.3303[7]	0.356[8]	0.3065[4]	0.2456[1]	0.2842[3]	0.2541[2]	0.3263[6]	0.3208[5]	0.3763[10]	0.3653[9]	0.394[11]
	$\sum$ *Ranks*		22[7]	25[8.5]	12[4]	3[1]	9[3]	6[2]	18[6]	15[5]	30[10]	25[8.5]	33[11]
200	bias	$\dddot{\delta}$	0.2903[8]	0.319[10]	0.2642[3]	0.1795[1]	0.2686[5]	0.1896[2]	0.2804[7]	0.2646[4]	0.2985[9]	0.2799[6]	0.3676[11]
	MSE	$\dddot{\delta}$	0.2652[9]	0.3161[10]	0.2133[4]	0.0533[1]	0.2146[5]	0.0587[2]	0.2464[7]	0.2208[6]	0.2651[8]	0.1644[3]	0.3641[11]
	MRE	$\dddot{\delta}$	0.2903[8]	0.319[10]	0.2642[3]	0.1795[1]	0.2686[5]	0.1896[2]	0.2804[7]	0.2646[4]	0.2985[9]	0.2799[6]	0.3676[11]
	$\sum$ *Ranks*		25[8]	30[10]	10[3]	3[1]	15[5.5]	6[2]	21[7]	14[4]	26[9]	15[5.5]	33[11]
300	bias	$\dddot{\delta}$	0.2353[6]	0.2451[9]	0.2423[8]	0.1477[1]	0.2386[7]	0.1489[2]	0.2484[10]	0.2341[5]	0.2254[3]	0.2319[4]	0.2941[11]
	MSE	$\dddot{\delta}$	0.2032[6]	0.2231[9]	0.2123[8]	0.0349[1]	0.2096[7]	0.0354[2]	0.2314[10]	0.1956[5]	0.175[4]	0.1323[3]	0.2854[11]
	MRE	$\dddot{\delta}$	0.2353[6]	0.2451[9]	0.2423[8]	0.1477[1]	0.2386[7]	0.1489[2]	0.2484[10]	0.2341[5]	0.2254[3]	0.2319[4]	0.2941[11]
	$\sum$ *Ranks*		18[6]	27[9]	24[8]	3[1]	21[7]	6[2]	30[10]	15[5]	10[3]	11[4]	33[11]
450	bias	$\dddot{\delta}$	0.2014[7]	0.2114[10]	0.1867[6]	0.1257[2]	0.1797[3]	0.1243[1]	0.211[9]	0.2015[8]	0.1802[4]	0.1845[5]	0.2343[11]
	MSE	$\dddot{\delta}$	0.1805[8]	0.2013[10]	0.1525[6]	0.0253[2]	0.1369[5]	0.0241[1]	0.1993[9]	0.176[7]	0.1145[4]	0.0973[3]	0.215[11]
	MRE	$\dddot{\delta}$	0.2014[7]	0.2114[10]	0.1867[6]	0.1257[2]	0.1797[3]	0.1243[1]	0.211[9]	0.2015[8]	0.1802[4]	0.1845[5]	0.2343[11]
	$\sum$ *Ranks*		22[7]	30[10]	18[6]	6[2]	11[3]	3[1]	27[9]	23[8]	12[4]	13[5]	33[11]

Table 6. Bias, MSE, and MRE values for (δ = 1.0) under RSS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\hat{\delta}$	0.5275[6]	0.5141[2]	0.5221[3]	0.5273[5]	0.5268[4]	0.4992[1]	0.5335[7]	0.614[9]	0.5662[8]	0.6926[11]	0.6648[10]
	MSE	$\hat{\delta}$	0.5039[7]	0.4254[2]	0.4337[5]	0.4281[3]	0.4289[4]	0.3892[1]	0.4383[6]	0.6202[9]	0.5341[8]	1.147[11]	0.6721[10]
	MRE	$\hat{\delta}$	0.5275[6]	0.5141[2]	0.5221[3]	0.5273[5]	0.5268[4]	0.4992[1]	0.5335[7]	0.614[9]	0.5662[8]	0.6926[11]	0.6648[10]
	$\sum$ *Ranks*		19[6]	6[2]	11[3]	13[5]	12[4]	3[1]	20[7]	27[9]	24[8]	33[11]	30[10]
50	bias	$\hat{\delta}$	0.3107[9]	0.235[7]	0.1668[2]	0.1707[3]	0.1729[4]	0.1579[1]	0.2146[6]	0.238[8]	0.1894[5]	0.3776[10]	0.416[11]
	MSE	$\hat{\delta}$	0.2806[10]	0.1836[7]	0.0433[2]	0.0468[3]	0.0486[4]	0.0388[1]	0.1309[6]	0.1993[8]	0.0699[5]	0.2681[9]	0.3842[11]
	MRE	$\hat{\delta}$	0.3107[9]	0.235[7]	0.1668[2]	0.1707[3]	0.1729[4]	0.1579[1]	0.2146[6]	0.238[8]	0.1894[5]	0.3776[10]	0.416[11]
	$\sum$ *Ranks*		28[9]	21[7]	6[2]	9[3]	12[4]	3[1]	18[6]	24[8]	15[5]	29[10]	33[11]
120	bias	$\hat{\delta}$	0.2243[9]	0.0964[7]	0.0706[3]	0.0681[1]	0.0713[4]	0.0699[2]	0.0799[6]	0.114[8]	0.0776[5]	0.251[11]	0.2387[10]
	MSE	$\hat{\delta}$	0.2005[10]	0.061[7]	0.0077[2]	0.0074[1]	0.008[4]	0.0078[3]	0.0229[6]	0.098[8]	0.0142[5]	0.1692[9]	0.2032[11]
	MRE	$\hat{\delta}$	0.2243[9]	0.0964[7]	0.0706[3]	0.0681[1]	0.0713[4]	0.0699[2]	0.0799[6]	0.114[8]	0.0776[5]	0.251[11]	0.2387[10]
	$\sum$ *Ranks*		28[9]	21[7]	8[3]	3[1]	12[4]	7[2]	18[6]	24[8]	15[5]	31[10.5]	31[10.5]
200	bias	$\hat{\delta}$	0.1698[11]	0.0626[7]	0.0417[1]	0.0426[4]	0.0421[3]	0.0418[2]	0.0448[6]	0.0739[8]	0.0441[5]	0.1689[9]	0.1696[10]
	MSE	$\hat{\delta}$	0.1455[11]	0.0457[7]	0.0027[1]	0.0029[3.5]	0.0028[2]	0.0029[3.5]	0.0032[6]	0.0701[8]	0.003[5]	0.0785[9]	0.1166[10]
	MRE	$\hat{\delta}$	0.1698[11]	0.0626[7]	0.0417[1]	0.0426[4]	0.0421[3]	0.0418[2]	0.0448[6]	0.0739[8]	0.0441[5]	0.1689[9]	0.1696[10]
	$\sum$ *Ranks*		33[11]	21[7]	3[1]	11.5[4]	8[3]	7.5[2]	18[6]	24[8]	15[5]	27[9]	30[10]
300	bias	$\hat{\delta}$	0.1399[9]	0.0433[8]	0.0279[1]	0.0289[4]	0.0284[2]	0.0287[3]	0.0293[5]	0.0406[7]	0.0294[6]	0.1418[11]	0.1411[10]
	MSE	$\hat{\delta}$	0.118[11]	0.031[8]	0.0012[1]	0.0013[4]	0.0013[4]	0.0013[4]	0.0013[4]	0.0283[7]	0.0013[4]	0.0708[9]	0.1043[10]
	MRE	$\hat{\delta}$	0.1399[9]	0.0433[8]	0.0279[1]	0.0289[4]	0.0284[2]	0.0287[3]	0.0293[5]	0.0406[7]	0.0294[6]	0.1418[11]	0.1411[10]
	$\sum$ *Ranks*		29[9]	24[8]	3[1]	12[4]	8[2]	10[3]	14[5]	21[7]	16[6]	31[11]	30[10]
450	bias	$\hat{\delta}$	0.1036[9]	0.0221[7]	0.0192[3]	0.019[2]	0.0187[1]	0.0195[4]	0.0196[5]	0.0261[8]	0.0211[6]	0.113[10]	0.1484[11]
	MSE	$\hat{\delta}$	0.0744[10]	0.0081[7]	$6e-04$[3.5]	$6e-04$[3.5]	$5e-04$[1]	$6e-04$[3.5]	$6e-04$[3.5]	0.0159[8]	$7e-04$[6]	0.0457[9]	0.1471[11]
	MRE	$\hat{\delta}$	0.1036[9]	0.0221[7]	0.0192[3]	0.019[2]	0.0187[1]	0.0195[4]	0.0196[5]	0.0261[8]	0.0211[6]	0.113[10]	0.1484[11]
	$\sum$ *Ranks*		22[8]	15[4]	14.5[3]	12.5[2]	8[1]	16.5[5]	18.5[7]	18[6]	23[9.5]	23[9.5]	27[11]

Table 7. Bias, MSE, and MRE values for ($\delta = 1.5$) under SRS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RTADE	WLSE	LTADE	MSADE	MSALDE
15	bias	$\dddot{\delta}$	0.8259[6]	0.7838[3]	0.8036[5]	0.7894[4]	0.8341[7]	0.7285[1]	0.7378[2]	0.9492[9]	0.8732[8]	1.0216[11]	0.9697[10]
	MSE	$\dddot{\delta}$	1.2727[7]	1.0399[3]	1.0808[4]	1.0933[5]	1.202[6]	0.8775[1]	0.9547[2]	1.6659[10]	1.4583[8]	2.1721[11]	1.6264[9]
	MRE	$\dddot{\delta}$	0.5506[6]	0.5225[3]	0.5358[5]	0.5263[4]	0.5561[7]	0.4857[1]	0.4919[2]	0.6328[9]	0.5821[8]	0.681[11]	0.6465[10]
	$\sum$ *Ranks*		19[6]	9[3]	14[5]	13[4]	20[7]	3[1]	6[2]	28[9]	24[8]	33[11]	29[10]
50	bias	$\dddot{\delta}$	0.5183[7]	0.586[10]	0.4[2]	0.4047[3]	0.4139[4]	0.3985[1]	0.5567[9]	0.5348[8]	0.489[5]	0.5173[6]	0.6337[11]
	MSE	$\dddot{\delta}$	0.6396[7]	0.8036[10]	0.2791[2]	0.2805[3]	0.288[4]	0.2613[1]	0.7203[9]	0.6572[8]	0.4544[5]	0.5237[6]	0.8491[11]
	MRE	$\dddot{\delta}$	0.3455[7]	0.3907[10]	0.2666[2]	0.2698[3]	0.276[4]	0.2656[1]	0.3711[9]	0.3565[8]	0.326[5]	0.3449[6]	0.4225[11]
	$\sum$ *Ranks*		21[7]	30[10]	6[2]	9[3]	12[4]	3[1]	27[9]	24[8]	15[5]	18[6]	33[11]
120	bias	$\dddot{\delta}$	0.3961[10]	0.355[5]	0.2711[3]	0.2496[1]	0.2806[4]	0.2575[2]	0.393[9]	0.3723[7]	0.3904[8]	0.3606[6]	0.4223[11]
	MSE	$\dddot{\delta}$	0.5109[10]	0.4097[6]	0.1418[3]	0.1005[1]	0.1596[4]	0.1041[2]	0.5078[9]	0.4587[7]	0.4643[8]	0.2852[5]	0.5133[11]
	MRE	$\dddot{\delta}$	0.2641[10]	0.2367[5]	0.1807[3]	0.1664[1]	0.1871[4]	0.1717[2]	0.262[9]	0.2482[7]	0.2603[8]	0.2404[6]	0.2815[11]
	$\sum$ *Ranks*		30[10]	16[5]	9[3]	3[1]	12[4]	6[2]	27[9]	21[7]	24[8]	17[6]	33[11]
200	bias	$\dddot{\delta}$	0.2728[6]	0.2857[7]	0.2125[4]	0.1934[1]	0.2094[3]	0.1975[2]	0.2908[8]	0.3021[9]	0.3034[10]	0.2703[5]	0.3321[11]
	MSE	$\dddot{\delta}$	0.2776[6]	0.3128[7]	0.1143[4]	0.0596[1]	0.1093[3]	0.0614[2]	0.3489[8]	0.3503[9]	0.3585[10]	0.1747[5]	0.4016[11]
	MRE	$\dddot{\delta}$	0.1819[6]	0.1905[7]	0.1417[4]	0.1289[1]	0.1396[3]	0.1316[2]	0.1939[8]	0.2014[9]	0.2023[10]	0.1802[5]	0.2214[11]
	$\sum$ *Ranks*		18[6]	21[7]	12[4]	3[1]	9[3]	6[2]	24[8]	27[9]	30[10]	15[5]	33[11]
300	bias	$\dddot{\delta}$	0.2244[5]	0.2549[8]	0.1945[4]	0.1567[1]	0.1871[3]	0.1656[2]	0.2356[6]	0.2598[10]	0.2597[9]	0.2369[7]	0.2841[11]
	MSE	$\dddot{\delta}$	0.2361[6]	0.3395[10]	0.1524[4]	0.0393[1]	0.1275[3]	0.0433[2]	0.257[7]	0.3273[9]	0.3211[8]	0.1709[5]	0.3588[11]
	MRE	$\dddot{\delta}$	0.1496[5]	0.1699[8]	0.1297[4]	0.1045[1]	0.1248[3]	0.1104[2]	0.1571[6]	0.1732[10]	0.1731[9]	0.1579[7]	0.1894[11]
	$\sum$ *Ranks*		16[5]	26[8.5]	12[4]	3[1]	9[3]	6[2]	19[6.5]	29[10]	26[8.5]	19[6.5]	33[11]
450	bias	$\dddot{\delta}$	0.1999[8]	0.1892[7]	0.1614[3]	0.1295[2]	0.1625[4]	0.1291[1]	0.2165[10]	0.1861[6]	0.2033[9]	0.1765[5]	0.2949[11]
	MSE	$\dddot{\delta}$	0.2334[9]	0.222[7]	0.13[5]	0.0268[2]	0.1241[4]	0.0263[1]	0.2636[10]	0.2028[6]	0.2307[8]	0.0745[3]	0.4434[11]
	MRE	$\dddot{\delta}$	0.1333[8]	0.1262[7]	0.1076[3]	0.0864[2]	0.1083[4]	0.086[1]	0.1443[10]	0.124[6]	0.1355[9]	0.1177[5]	0.1966[11]
	$\sum$ *Ranks*		25[8]	21[7]	11[3]	6[2]	12[4]	3[1]	30[10]	18[6]	26[9]	13[5]	33[11]

Table 8. Bias, MSE, and MRE values for ($\delta = 1.5$) under RSS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\hat{\delta}$	0.5613[5]	0.5416[3]	0.5415[2]	0.5732[7]	0.5575[4]	0.512[1]	0.5678[6]	0.7597[10]	0.6059[8]	0.7398[9]	0.7922[11]
	MSE	$\hat{\delta}$	0.7281[7]	0.5044[2]	0.5163[3]	0.5459[6]	0.5328[4]	0.4343[1]	0.5351[5]	1.1729[11]	0.7324[8]	0.9953[9]	1.083[10]
	MRE	$\hat{\delta}$	0.3742[5]	0.3611[3]	0.361[2]	0.3821[7]	0.3717[4]	0.3413[1]	0.3785[6]	0.5065[10]	0.404[8]	0.4932[9]	0.5281[11]
	$\sum$ *Ranks*		17[5.5]	8[3]	7[2]	20[7]	12[4]	3[1]	17[5.5]	31[10]	24[8]	27[9]	32[11]
50	bias	$\hat{\delta}$	0.3325[9]	0.1833[6]	0.1701[2]	0.1707[3]	0.1671[1]	0.1759[4]	0.1762[5]	0.2899[8]	0.1844[7]	0.3829[10]	0.4402[11]
	MSE	$\hat{\delta}$	0.4045[10]	0.1073[7]	0.0447[1]	0.0466[3]	0.0463[2]	0.0489[4]	0.0492[5]	0.4029[9]	0.0559[6]	0.3246[8]	0.5328[11]
	MRE	$\hat{\delta}$	0.2216[9]	0.1222[6]	0.1134[2]	0.1138[3]	0.1114[1]	0.1172[4]	0.1175[5]	0.1933[8]	0.123[7]	0.2552[10]	0.2935[11]
	$\sum$ *Ranks*		28[9.5]	19[6]	5[2]	9[3]	4[1]	12[4]	15[5]	25[8]	20[7]	28[9.5]	33[11]
120	bias	$\hat{\delta}$	0.1905[9]	0.085[7]	0.0726[2]	0.0724[1]	0.0746[4]	0.0735[3]	0.0763[5]	0.1106[8]	0.0801[6]	0.2246[10]	0.256[11]
	MSE	$\hat{\delta}$	0.1833[10]	0.0383[7]	0.0084[2.5]	0.0083[1]	0.0086[4]	0.0084[2.5]	0.0093[5]	0.1197[8]	0.01[6]	0.1394[9]	0.3088[11]
	MRE	$\hat{\delta}$	0.127[9]	0.0567[7]	0.0484[2]	0.0483[1]	0.0498[4]	0.049[3]	0.0509[5]	0.0737[8]	0.0534[6]	0.1497[10]	0.1707[11]
	$\sum$ *Ranks*		28[9]	21[7]	6.5[2]	3[1]	12[4]	8.5[3]	15[5]	24[8]	18[6]	29[10]	33[11]
200	bias	$\hat{\delta}$	0.1872[10]	0.0448[2]	0.0431[1]	0.0457[3]	0.0462[4]	0.0472[6]	0.0469[5]	0.0762[8]	0.0488[7]	0.1621[9]	0.2017[11]
	MSE	$\hat{\delta}$	0.2301[10]	0.0031[2]	0.003[1]	0.0034[4]	0.0033[3]	0.0035[5.5]	0.0035[5.5]	0.0961[9]	0.0038[7]	0.0727[8]	0.2481[11]
	MRE	$\hat{\delta}$	0.1248[10]	0.0299[2]	0.0287[1]	0.0304[3]	0.0308[4]	0.0314[6]	0.0312[5]	0.0508[8]	0.0325[7]	0.1081[9]	0.1345[11]
	$\sum$ *Ranks*		30[10]	6[2]	3[1]	10[3]	11[4]	17.5[6]	15.5[5]	25[8]	21[7]	26[9]	33[11]
300	bias	$\hat{\delta}$	0.1573[10]	0.0282[1]	0.0305[4]	0.0307[5]	0.0295[2]	0.0304[3]	0.031[6.5]	0.0368[8]	0.031[6.5]	0.135[9]	0.1622[11]
	MSE	$\hat{\delta}$	0.2056[11]	0.0013[1]	0.0015[5.5]	0.0015[5.5]	0.0014[2.5]	0.0014[2.5]	0.0015[5.5]	0.0276[8]	0.0015[5.5]	0.0507[9]	0.183[10]
	MRE	$\hat{\delta}$	0.1049[10]	0.0188[1]	0.0203[4]	0.0205[5]	0.0197[2]	0.0202[3]	0.0207[6.5]	0.0246[8]	0.0207[6.5]	0.09[9]	0.1081[11]
	$\sum$ *Ranks*		31[10]	3[1]	13.5[4]	15.5[5]	6.5[2]	8.5[3]	18.5[6.5]	24[8]	18.5[6.5]	27[9]	32[11]
450	bias	$\hat{\delta}$	0.1396[11]	0.0196[2]	0.0207[6]	0.0195[1]	0.0204[3.5]	0.0204[3.5]	0.0214[7]	0.034[8]	0.0205[5]	0.1083[9]	0.1232[10]
	MSE	$\hat{\delta}$	0.199[11]	$6e-04$[1.5]	$7e-04$[5]	$6e-04$[1.5]	$7e-04$[5]	$7e-04$[5]	$7e-04$[5]	0.0449[8]	$7e-04$[5]	0.0486[9]	0.1356[10]
	MRE	$\hat{\delta}$	0.0931[11]	0.0131[2]	0.0138[6]	0.013[1]	0.0136[3.5]	0.0136[3.5]	0.0143[7]	0.0227[8]	0.0137[5]	0.0722[9]	0.0821[10]
	$\sum$ *Ranks*		26[11]	9.5[2]	21[8]	7.5[1]	16[3.5]	16[3.5]	23[9.5]	17[5]	19[6]	20[7]	23[9.5]

Table 9. Bias, MSE, and MRE values for ($\delta = 2.0$) under SRS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\hat{\delta}$	0.8662[3]	0.89[4]	0.936[5]	0.9702[7]	0.9394[6]	0.8391[1]	0.8576[2]	1.139[11]	1.0388[8]	1.0907[10]	1.0534[9]
	MSE	$\dddot{\delta}$	1.4064[3]	1.4249[4]	1.6365[5]	1.7863[7]	1.6449[6]	1.2631[1]	1.3114[2]	2.5605[10]	2.2348[9]	2.9563[11]	2.0211[8]
	MRE	$\dddot{\delta}$	0.4331[3]	0.445[4]	0.468[5]	0.4851[7]	0.4697[6]	0.4196[1]	0.4288[2]	0.5695[11]	0.5194[8]	0.5453[10]	0.5267[9]
	$\sum$ *Ranks*		9[3]	12[4]	15[5]	21[7]	18[6]	3[1]	6[2]	32[11]	25[8]	31[10]	26[9]
50	bias	$\dddot{\delta}$	0.5893[7]	0.6143[9]	0.4517[2]	0.4631[4]	0.4561[3]	0.4439[1]	0.5144[6]	0.6613[10]	0.4828[5]	0.6069[8]	0.7456[11]
	MSE	$\dddot{\delta}$	0.8981[8]	0.9587[9]	0.3418[2]	0.3854[4]	0.3458[3]	0.3249[1]	0.5695[6]	1.1788[10]	0.4004[5]	0.7547[7]	1.3034[11]
	MRE	$\dddot{\delta}$	0.2946[7]	0.3071[9]	0.2259[2]	0.2316[4]	0.2281[3]	0.2219[1]	0.2572[6]	0.3307[10]	0.2414[5]	0.3035[8]	0.3728[11]
	$\sum$ *Ranks*		22[7]	27[9]	6[2]	12[4]	9[3]	3[1]	18[6]	30[10]	15[5]	23[8]	33[11]
120	bias	$\dddot{\delta}$	0.4366[8]	0.4407[9]	0.2887[1]	0.291[2]	0.2921[4]	0.292[3]	0.38[6]	0.4637[10]	0.3547[5]	0.3908[7]	0.4989[11]
	MSE	$\dddot{\delta}$	0.7198[8]	0.7406[9]	0.1352[1]	0.1371[2]	0.1377[4]	0.1372[3]	0.4774[7]	0.8288[11]	0.33[5]	0.362[6]	0.8257[10]
	MRE	$\dddot{\delta}$	0.2183[8]	0.2204[9]	0.1444[1]	0.1455[2]	0.146[3.5]	0.146[3.5]	0.19[6]	0.2318[10]	0.1773[5]	0.1954[7]	0.2494[11]
	$\sum$ *Ranks*		24[8]	27[9]	3[1]	6[2]	11.5[4]	9.5[3]	19[6]	31[10]	15[5]	20[7]	32[11]
200	bias	$\dddot{\delta}$	0.318[8]	0.3186[9]	0.2241[3]	0.2219[1]	0.2224[2]	0.2255[4]	0.3266[10]	0.3167[7]	0.3049[6]	0.3037[5]	0.3701[11]
	MSE	$\dddot{\delta}$	0.4703[8]	0.4776[9]	0.0866[4]	0.0783[1]	0.0803[2]	0.0811[3]	0.4872[10]	0.4586[7]	0.361[6]	0.2278[5]	0.5503[11]
	MRE	$\dddot{\delta}$	0.159[8]	0.1593[9]	0.112[3]	0.111[1]	0.1112[2]	0.1128[4]	0.1633[10]	0.1584[7]	0.1525[6]	0.1518[5]	0.185[11]
	$\sum$ *Ranks*		24[8]	27[9]	10[3]	3[1]	6[2]	11[4]	30[10]	21[7]	18[6]	15[5]	33[11]
300	bias	$\dddot{\delta}$	0.258[6]	0.261[7]	0.1862[3]	0.1821[1]	0.1857[2]	0.1869[4]	0.2793[8]	0.2918[11]	0.2893[10]	0.2431[5]	0.2841[9]
	MSE	$\dddot{\delta}$	0.3621[7]	0.3677[8]	0.0643[3]	0.0516[1]	0.0651[4]	0.0548[2]	0.4523[10]	0.4936[11]	0.4307[9]	0.1549[5]	0.3537[6]
	MRE	$\dddot{\delta}$	0.129[6]	0.1305[7]	0.0931[3]	0.091[1]	0.0929[2]	0.0935[4]	0.1396[8]	0.1459[11]	0.1447[10]	0.1216[5]	0.142[9]
	$\sum$ *Ranks*		19[6]	22[7]	9[3]	3[1]	8[2]	10[4]	26[9]	33[11]	29[10]	15[5]	24[8]
450	bias	$\dddot{\delta}$	0.212[7]	0.2088[6]	0.1503[3]	0.1492[1]	0.1583[4]	0.1497[2]	0.2339[9]	0.2292[8]	0.2624[11]	0.1916[5]	0.2373[10]
	MSE	$\dddot{\delta}$	0.2839[7]	0.2713[6]	0.0526[3]	0.0349[1]	0.0706[4]	0.035[2]	0.3701[10]	0.3634[9]	0.459[11]	0.0959[5]	0.3072[8]
	MRE	$\dddot{\delta}$	0.106[7]	0.1044[6]	0.0752[3]	0.0746[1]	0.0791[4]	0.0748[2]	0.1169[9]	0.1146[8]	0.1312[11]	0.0958[5]	0.1186[10]
	$\sum$ *Ranks*		21[7]	18[6]	9[3]	3[1]	12[4]	6[2]	28[9.5]	25[8]	33[11]	15[5]	28[9.5]

Table 10. Bias, MSE, and MRE values for ($\delta = 2.0$) under RSS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\hat{\delta}$	0.6322[4]	0.6102[2]	0.6497[5]	0.6604[7]	0.6505[6]	0.5827[1]	0.6233[3]	0.8329[10]	0.7071[8]	0.8762[11]	0.8269[9]
	MSE	$\hat{\delta}$	0.9731[8]	0.6199[2]	0.781[4]	0.8186[6]	0.792[5]	0.5512[1]	0.6432[3]	1.5585[11]	0.9725[7]	1.5475[10]	1.2327[9]
	MRE	$\hat{\delta}$	0.3161[4]	0.3051[2]	0.3248[5]	0.3302[7]	0.3253[6]	0.2913[1]	0.3116[3]	0.4165[10]	0.3536[8]	0.4381[11]	0.4135[9]
	$\sum$ *Ranks*		16[5]	6[2]	14[4]	20[7]	17[6]	3[1]	9[3]	31[10]	23[8]	32[11]	27[9]
50	bias	$\hat{\delta}$	0.3897[9]	0.193[4]	0.1927[2]	0.1894[1]	0.1929[3]	0.2032[5]	0.2036[6]	0.289[8]	0.2086[7]	0.3972[11]	0.397[10]
	MSE	$\hat{\delta}$	0.6241[11]	0.0645[4]	0.0593[2]	0.0566[1]	0.0596[3]	0.0651[5]	0.066[6]	0.4495[9]	0.0692[7]	0.3584[8]	0.4557[10]
	MRE	$\hat{\delta}$	0.1948[9]	0.0965[3.5]	0.0964[2]	0.0947[1]	0.0965[3.5]	0.1016[5]	0.1018[6]	0.1445[8]	0.1043[7]	0.1986[11]	0.1985[10]
	$\sum$ *Ranks*		29[9]	11.5[4]	6[2]	3[1]	9.5[3]	15[5]	18[6]	25[8]	21[7]	30[10.5]	30[10.5]
120	bias	$\hat{\delta}$	0.2299[9]	0.0806[1]	0.0822[2]	0.0823[3]	0.0832[4]	0.0907[7]	0.0868[5]	0.1273[8]	0.0889[6]	0.2521[10]	0.2691[11]
	MSE	$\hat{\delta}$	0.2926[10]	0.0103[1]	0.0106[2]	0.0107[3]	0.0108[4]	0.0128[7]	0.0117[5]	0.1932[8]	0.0123[6]	0.2008[9]	0.3613[11]
	MRE	$\hat{\delta}$	0.1149[9]	0.0403[1]	0.0411[2]	0.0412[3]	0.0416[4]	0.0454[7]	0.0434[5]	0.0636[8]	0.0445[6]	0.1261[10]	0.1346[11]
	$\sum$ *Ranks*		28[9]	3[1]	6[2]	9[3]	12[4]	21[7]	15[5]	24[8]	18[6]	29[10]	33[11]
200	bias	$\hat{\delta}$	0.1779[9]	0.0504[3]	0.0495[1]	0.0502[2]	0.0508[4]	0.0547[7]	0.0527[5]	0.0742[8]	0.0536[6]	0.1814[10]	0.2188[11]
	MSE	$\hat{\delta}$	0.224[10]	0.004[3]	0.0038[1]	0.004[3]	0.004[3]	0.0047[7]	0.0043[5]	0.103[9]	0.0046[6]	0.0966[8]	0.3339[11]
	MRE	$\hat{\delta}$	0.0889[9]	0.0252[3]	0.0247[1]	0.0251[2]	0.0254[4]	0.0273[7]	0.0264[5]	0.0371[8]	0.0268[6]	0.0907[10]	0.1094[11]
	$\sum$ *Ranks*		28[9.5]	9[3]	3[1]	7[2]	11[4]	21[7]	15[5]	25[8]	18[6]	28[9.5]	33[11]
300	bias	$\hat{\delta}$	0.1527[9]	0.0336[2.5]	0.0336[2.5]	0.0334[1]	0.0339[4]	0.037[7]	0.0353[5]	0.0595[8]	0.0356[6]	0.1543[10]	0.1965[11]
	MSE	$\hat{\delta}$	0.2146[10]	0.0018[2.5]	0.0018[2.5]	0.0018[2.5]	0.0018[2.5]	0.0022[7]	0.0019[5]	0.1098[9]	0.002[6]	0.0868[8]	0.3452[11]
	MRE	$\hat{\delta}$	0.0764[9]	0.0168[2.5]	0.0168[2.5]	0.0167[1]	0.0169[4]	0.0185[7]	0.0176[5]	0.0297[8]	0.0178[6]	0.0771[10]	0.0983[11]
	$\sum$ *Ranks*		28[9.5]	7.5[2.5]	7.5[2.5]	4.5[1]	10.5[4]	21[7]	15[5]	25[8]	18[6]	28[9.5]	33[11]
450	bias	$\hat{\delta}$	0.1452[10]	0.0224[2]	0.023[4]	0.0222[1]	0.0227[3]	0.025[7]	0.0239[5]	0.0359[8]	0.0243[6]	0.1194[9]	0.1483[11]
	MSE	$\hat{\delta}$	0.2516[11]	$8e-04$[2.5]	$8e-04$[2.5]	$8e-04$[2.5]	$8e-04$[2.5]	0.001[7]	$9e-04$[5.5]	0.0574[9]	$9e-04$[5.5]	0.047[8]	0.2342[10]
	MRE	$\hat{\delta}$	0.0726[10]	0.0112[2]	0.0115[4]	0.0111[1]	0.0114[3]	0.0125[7]	0.0119[5]	0.018[8]	0.0121[6]	0.0597[9]	0.0742[11]
	$\sum$ *Ranks*		25[10]	11.5[2]	15.5[5]	9.5[1]	13.5[3]	15[4]	20.5[8]	19[6]	22.5[9]	20[7]	26[11]

Table 11. Bias, MSE, and MRE values for (δ = 2.5) under SRS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\dddot{\delta}$	0.9599[2]	0.9855[3]	1.1014[5]	1.1374[6]	1.1475[7]	0.9352[1]	1.0073[4]	1.3039[11]	1.157[8]	1.1786[9]	1.1824[10]
	MSE	$\dddot{\delta}$	1.7187[3]	1.6923[2]	2.3321[5]	2.4383[6]	2.5806[8]	1.5886[1]	1.8703[4]	3.5516[11]	2.8096[9]	2.8769[10]	2.4776[7]
	MRE	$\dddot{\delta}$	0.384[2]	0.3942[3]	0.4406[5]	0.4549[6]	0.459[7]	0.3741[1]	0.4029[4]	0.5216[11]	0.4628[8]	0.4714[9]	0.473[10]
	$\sum$ *Ranks*		7[2]	8[3]	15[5]	18[6]	22[7]	3[1]	12[4]	33[11]	25[8]	28[10]	27[9]
50	bias	$\dddot{\delta}$	0.5636[7]	0.5986[8]	0.5265[2]	0.53[3]	0.5391[5]	0.5024[1]	0.5325[4]	0.7652[11]	0.5466[6]	0.6536[9]	0.7347[10]
	MSE	$\dddot{\delta}$	0.7067[7]	0.8341[8]	0.4737[2]	0.5305[6]	0.503[4]	0.435[1]	0.482[3]	1.7478[11]	0.5132[5]	0.9315[9]	1.3096[10]
	MRE	$\dddot{\delta}$	0.2254[7]	0.2395[8]	0.2106[2]	0.212[3]	0.2156[5]	0.201[1]	0.213[4]	0.3061[11]	0.2187[6]	0.2614[9]	0.2939[10]
	$\sum$ *Ranks*		21[7]	24[8]	6[2]	12[4]	14[5]	3[1]	11[3]	33[11]	17[6]	27[9]	30[10]
120	bias	$\dddot{\delta}$	0.4462[7]	0.4787[9]	0.334[1]	0.3388[2]	0.3415[3]	0.3453[4]	0.3614[6]	0.5359[11]	0.3552[5]	0.457[8]	0.5352[10]
	MSE	$\dddot{\delta}$	0.7607[8]	0.877[9]	0.1793[1]	0.1808[2]	0.1832[3]	0.189[4]	0.3259[6]	1.2275[11]	0.2236[5]	0.4848[7]	1.0708[10]
	MRE	$\dddot{\delta}$	0.1785[7]	0.1915[9]	0.1336[1]	0.1355[2]	0.1366[3]	0.1381[4]	0.1446[6]	0.2144[11]	0.1421[5]	0.1828[8]	0.2141[10]
	$\sum$ *Ranks*		22[7]	27[9]	3[1]	6[2]	9[3]	12[4]	18[6]	33[11]	15[5]	23[8]	30[10]
200	bias	$\dddot{\delta}$	0.4358[10]	0.4021[8]	0.248[1]	0.2526[3]	0.2509[2]	0.2633[4]	0.3284[6]	0.4068[9]	0.2919[5]	0.3362[7]	0.4777[11]
	MSE	$\dddot{\delta}$	0.9925[10]	0.8236[9]	0.0967[1]	0.1001[2]	0.1006[3]	0.1128[4]	0.4511[7]	0.8184[8]	0.2007[5]	0.2805[6]	1.0014[11]
	MRE	$\dddot{\delta}$	0.1743[10]	0.1609[8]	0.0992[1]	0.1011[3]	0.1004[2]	0.1053[4]	0.1314[6]	0.1627[9]	0.1167[5]	0.1345[7]	0.1911[11]
	$\sum$ *Ranks*		30[10]	25[8]	3[1]	8[3]	7[2]	12[4]	19[6]	26[9]	15[5]	20[7]	33[11]
300	bias	$\dddot{\delta}$	0.3835[11]	0.3358[8]	0.206[2]	0.2007[1]	0.2086[3]	0.2142[4]	0.2656[5]	0.3661[10]	0.2788[7]	0.2723[6]	0.3589[9]
	MSE	$\dddot{\delta}$	0.9468[11]	0.7071[8]	0.0672[2]	0.0627[1]	0.0681[3]	0.0749[4]	0.3498[6]	0.8202[10]	0.363[7]	0.1516[5]	0.7172[9]
	MRE	$\dddot{\delta}$	0.1534[11]	0.1343[8]	0.0824[2]	0.0803[1]	0.0835[3]	0.0857[4]	0.1062[5]	0.1464[10]	0.1115[7]	0.1089[6]	0.1435[9]
	$\sum$ *Ranks*		33[11]	24[8]	6[2]	3[1]	9[3]	12[4]	16[5]	30[10]	21[7]	17[6]	27[9]
450	bias	$\dddot{\delta}$	0.2793[9]	0.2534[7]	0.1634[2]	0.1628[1]	0.1714[3]	0.182[4]	0.2577[8]	0.295[11]	0.243[6]	0.2212[5]	0.291[10]
	MSE	$\dddot{\delta}$	0.5946[10]	0.4777[8]	0.0423[2]	0.0405[1]	0.0468[3]	0.0521[4]	0.4753[7]	0.6681[11]	0.3999[6]	0.1162[5]	0.5314[9]
	MRE	$\dddot{\delta}$	0.1117[9]	0.1013[7]	0.0653[2]	0.0651[1]	0.0686[3]	0.0728[4]	0.1031[8]	0.118[11]	0.0972[6]	0.0885[5]	0.1164[10]
	$\sum$ *Ranks*		28[9]	22[7]	6[2]	3[1]	9[3]	12[4]	23[8]	33[11]	18[6]	15[5]	29[10]

Table 12. Bias, MSE, and MRE values for (δ = 2.5) under RSS.

$m^{\circ\circ}$	Measure	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
15	bias	$\hat{\delta}$	0.6278[1]	0.6804[2]	0.7479[6]	0.7486[7]	0.7441[5]	0.6932[3]	0.7387[4]	0.9368[10]	0.8054[8]	0.9378[11]	0.9052[9]
	MSE	$\hat{\delta}$	0.8588[3]	0.7567[1]	1.1269[7]	1.108[6]	1.1045[5]	0.7758[2]	0.9299[4]	2.1615[11]	1.3529[8]	1.6869[10]	1.432[9]
	MRE	$\hat{\delta}$	0.2511[1]	0.2721[2]	0.2991[6]	0.2994[7]	0.2976[5]	0.2773[3]	0.2955[4]	0.3747[10]	0.3222[8]	0.3751[11]	0.3621[9]
	$\sum$ *Ranks*		5[1.5]	5[1.5]	19[6]	20[7]	15[5]	8[3]	12[4]	31[10]	24[8]	32[11]	27[9]
50	bias	$\hat{\delta}$	0.4159[10]	0.2217[2]	0.2222[3]	0.2189[1]	0.2275[5]	0.2386[6]	0.2239[4]	0.3112[8]	0.2453[7]	0.4475[11]	0.3909[9]
	MSE	$\hat{\delta}$	0.7329[11]	0.0773[2]	0.0774[3]	0.077[1]	0.0823[5]	0.0898[6]	0.0815[4]	0.5289[10]	0.0992[7]	0.4726[9]	0.3113[8]
	MRE	$\hat{\delta}$	0.1664[10]	0.0887[2]	0.0889[3]	0.0876[1]	0.091[5]	0.0955[6]	0.0896[4]	0.1245[8]	0.0981[7]	0.179[11]	0.1564[9]
	$\sum$ *Ranks*		31[10.5]	6[2]	9[3]	3[1]	15[5]	18[6]	12[4]	26[8.5]	21[7]	31[10.5]	26[8.5]
120	bias	$\hat{\delta}$	0.3167[11]	0.0899[1]	0.0967[3]	0.097[4]	0.0963[2]	0.1102[7]	0.0999[5]	0.156[8]	0.1032[6]	0.2892[10]	0.244[9]
	MSE	$\hat{\delta}$	0.6865[11]	0.0126[1]	0.0149[4]	0.0147[3]	0.0145[2]	0.0185[7]	0.0158[5]	0.3438[10]	0.0169[6]	0.2536[9]	0.1786[8]
	MRE	$\hat{\delta}$	0.1267[11]	0.036[1]	0.0387[3]	0.0388[4]	0.0385[2]	0.0441[7]	0.0399[5]	0.0624[8]	0.0413[6]	0.1157[10]	0.0976[9]
	$\sum$ *Ranks*		33[11]	3[1]	10[3]	11[4]	6[2]	21[7]	15[5]	26[8.5]	18[6]	29[10]	26[8.5]
200	bias	$\hat{\delta}$	0.2146[9]	0.056[2]	0.0592[4]	0.0554[1]	0.0586[3]	0.0649[7]	0.0611[5]	0.0712[8]	0.0625[6]	0.2248[10]	0.2259[11]
	MSE	$\hat{\delta}$	0.362[11]	0.0049[1]	0.0055[4]	0.005[2]	0.0053[3]	0.0064[7]	0.0057[5]	0.0783[8]	0.0062[6]	0.1747[9]	0.3609[10]
	MRE	$\hat{\delta}$	0.0859[9]	0.0224[2]	0.0237[4]	0.0221[1]	0.0234[3]	0.026[7]	0.0244[5]	0.0285[8]	0.025[6]	0.0899[10]	0.0903[11]
	$\sum$ *Ranks*		29[9.5]	5[2]	12[4]	4[1]	9[3]	21[7]	15[5]	24[8]	18[6]	29[9.5]	32[11]
300	bias	$\hat{\delta}$	0.1586[9]	0.0373[1.5]	0.0385[3]	0.0373[1.5]	0.0403[4]	0.0445[7]	0.042[5.5]	0.0619[8]	0.042[5.5]	0.1637[10]	0.1866[11]
	MSE	$\hat{\delta}$	0.2146[10]	0.0022[1.5]	0.0023[3]	0.0022[1.5]	0.0026[4]	0.0031[7]	0.0028[6]	0.1238[9]	0.0027[5]	0.1014[8]	0.2935[11]
	MRE	$\hat{\delta}$	0.0635[9]	0.0149[1.5]	0.0154[3]	0.0149[1.5]	0.0161[4]	0.0178[7]	0.0168[5.5]	0.0247[8]	0.0168[5.5]	0.0655[10]	0.0746[11]
	$\sum$ *Ranks*		28[9.5]	4.5[1.5]	9[3]	4.5[1.5]	12[4]	21[7]	17[6]	25[8]	16[5]	28[9.5]	33[11]
450	bias	$\hat{\delta}$	0.1285[9]	0.0248[1.5]	0.0256[3]	0.0248[1.5]	0.0257[4]	0.0301[7]	0.0276[5]	0.0441[8]	0.028[6]	0.1326[10]	0.1563[11]
	MSE	$\hat{\delta}$	0.1537[10]	0.001[2.5]	0.001[2.5]	0.001[2.5]	0.001[2.5]	0.0014[7]	0.0012[5]	0.0994[9]	0.0013[6]	0.07[8]	0.2914[11]
	MRE	$\hat{\delta}$	0.0514[9]	0.0099[1.5]	0.0102[3]	0.0099[1.5]	0.0103[4]	0.0121[7]	0.011[5]	0.0177[8]	0.0112[6]	0.053[10]	0.0625[11]
	$\sum$ *Ranks*		28[9.5]	5.5[1.5]	8.5[3]	5.5[1.5]	10.5[4]	21[7]	15[5]	25[8]	18[6]	28[9.5]	33[11]

Table 13. Numerical values for MSE of SRS divided by MSE of RSS for all estimates.

$m^{\circ\circ}$	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
						$\delta = 0.15$						
15	$\hat{\delta}$	2.65374	1.95696	1.92664	1.87934	1.89117	1.78528	1.90705	2.05639	1.66667	1.71257	1.63055
50	$\hat{\delta}$	2.11809	3.22118	3.27914	3.37684	3.16358	3.14356	3.07212	3.27358	3.27555	1.92975	2.00437
120	$\hat{\delta}$	2.00738	4.46400	4.08915	5.19403	4.27626	4.30435	4.14453	4.03383	4.40283	2.05997	2.23158
200	$\hat{\delta}$	1.88626	5.13816	5.01911	6.17094	4.81646	5.03822	4.63030	4.55030	5.02299	1.91600	2.11422
300	$\hat{\delta}$	1.85538	5.53571	5.71818	7.52055	5.42342	5.63303	5.20175	4.35915	5.81356	1.88395	1.98023
450	$\hat{\delta}$	1.78755	4.31579	6.43836	10.63415	6.58108	6.25676	5.94937	5.04124	6.63291	1.81306	1.87031
						$\delta = 0.6$						
15	$\hat{\delta}$	2.10257	1.80127	1.76798	1.61167	1.73971	1.52413	1.90935	1.62599	1.91201	1.70299	1.79647
50	$\hat{\delta}$	1.89052	1.76432	3.68263	3.07065	3.24167	3.18102	2.83529	2.20078	3.44154	1.58018	1.59773
120	$\hat{\delta}$	1.07682	1.86474	11.84211	10.19200	12.10526	12.27184	2.53523	2.19143	4.59252	1.39825	1.48068
200	$\hat{\delta}$	1.81764	3.01890	36.92308	19.70732	31.36735	20.77500	4.07385	6.47079	6.98649	1.26058	1.53116
300	$\hat{\delta}$	2.03397	9.14573	94.00000	34.70588	96.00000	36.47059	5.66368	8.45113	10.03571	1.11630	1.59630
450	$\hat{\delta}$	2.02886	3.74671	174.57143	46.00000	154.00000	46.87500	10.81618	6.72539	15.39024	1.29133	1.92838
						$\delta = 1.0$						
15	$\hat{\delta}$	1.68168	1.79337	1.95873	1.86428	1.92982	1.69938	1.66439	1.82151	2.19903	1.18117	1.64157
50	$\hat{\delta}$	1.54526	2.58061	6.17552	5.42521	5.52263	6.31959	3.82429	2.21024	5.10873	1.61619	1.30479
120	$\hat{\delta}$	1.39701	4.93770	27.10390	13.56757	23.06250	13.67949	11.79913	2.65306	23.42254	1.63180	1.75935
200	$\hat{\delta}$	1.82268	6.91685	79.00000	18.37931	76.64286	20.24138	77.00000	3.14979	88.36667	2.09427	3.12264
300	$\hat{\delta}$	1.72203	7.19677	176.91667	26.84615	161.23077	27.23077	178.00000	6.91166	134.61538	1.86864	2.73634
450	$\hat{\delta}$	2.42608	24.85185	254.16667	42.16667	273.80000	40.16667	332.16667	11.06918	163.57143	2.12910	1.46159
						$\delta = 1.5$						
15	$\hat{\delta}$	1.74797	2.06166	2.09336	2.00275	2.25601	2.02049	1.78415	1.42033	1.99113	2.18236	1.50175
50	$\hat{\delta}$	1.58121	7.48928	6.24385	6.01931	6.22030	5.34356	14.64024	1.63117	8.12880	1.61337	1.59366
120	$\hat{\delta}$	2.78723	10.69713	16.88095	12.10843	18.55814	12.39286	54.60215	3.83208	46.43000	2.04591	1.66224
200	$\hat{\delta}$	1.20643	100.90323	38.10000	17.52941	33.12121	17.54286	99.68571	3.64516	94.34211	2.40303	1.61870
300	$\hat{\delta}$	1.14835	261.15385	101.60000	26.20000	91.07143	30.92857	171.33333	11.85870	214.06667	3.37081	1.96066
450	$\hat{\delta}$	1.17286	370.00000	185.71429	44.66667	177.28571	37.57143	376.57143	4.51670	329.57143	1.53292	3.26991
						$\delta = 2.0$						
15	$\hat{\delta}$	1.44528	2.29860	2.09539	2.18214	2.07689	2.29155	2.03887	1.64293	2.29799	1.91037	1.63957
50	$\hat{\delta}$	1.43903	14.86357	5.76391	6.80919	5.80201	4.99078	8.62879	2.62247	5.78613	2.10575	2.86022
120	$\hat{\delta}$	2.46001	71.90291	12.75472	12.81308	12.75000	10.71875	40.80342	4.28986	26.82927	1.80279	2.28536
200	$\hat{\delta}$	2.09955	119.40000	22.78947	19.57500	20.07500	17.25532	113.30233	4.45243	78.47826	2.35818	1.64810
300	$\hat{\delta}$	1.68733	204.27778	35.72222	28.66667	36.16667	24.90909	238.05263	4.49545	215.35000	1.78456	1.02462
450	$\hat{\delta}$	1.12838	339.12500	65.75000	43.62500	88.25000	35.00000	411.22222	6.33101	510.00000	2.04043	1.31170
						$\delta = 2.5$						
15	$\hat{\delta}$	2.00128	2.23642	2.06948	2.20063	2.33644	2.04769	2.01129	1.64312	2.07672	1.70544	1.73017
50	$\hat{\delta}$	0.96425	10.79043	6.12016	6.88961	6.11179	4.84410	5.91411	3.30459	5.17339	1.97101	4.20687
120	$\hat{\delta}$	1.10808	69.60317	12.03356	12.29932	12.63448	10.21622	20.62658	3.57039	13.23077	1.91167	5.99552
200	$\hat{\delta}$	2.74171	168.08163	17.58182	20.02000	18.98113	17.62500	79.14035	10.45211	32.37097	1.60561	2.77473
300	$\hat{\delta}$	4.41193	321.40909	29.21739	28.50000	26.19231	24.16129	124.92857	6.62520	134.44444	1.49507	2.44361
450	$\hat{\delta}$	3.86858	477.70000	42.30000	40.50000	46.80000	37.21429	396.08333	6.72133	307.61538	1.66000	1.82361

Table 14. Partial and total rankings of all AUD estimate techniques by SRS for different values of δ.

Parameter	$m^{\circ\circ}$	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
$\delta = 0.15$	15	5.0	4.0	8.0	1.0	9.0	3.0	2.0	7.0	11.0	10.0	6.0
	50	4.0	6.0	8.0	1.0	5.0	3.0	2.0	7.0	11.0	10.0	9.0
	120	7.0	8.0	2.0	1.0	6.0	4.0	3.0	5.0	9.0	11.0	10.0
	200	8.0	6.0	4.5	1.0	2.0	7.0	3.0	4.5	9.0	11.0	10.0
	300	4.0	6.0	8.0	1.0	3.0	5.0	2.0	7.0	9.0	11.0	10.0
	450	7.5	7.5	3.0	1.0	5.0	2.0	4.0	6.0	9.0	11.0	10.0
$\delta = 0.6$	15	5.0	4.0	8.0	2.0	6.0	1.0	3.0	7.0	10.0	11.0	9.0
	50	8.0	9.0	4.0	2.0	3.0	1.0	5.0	7.0	6.0	10.0	11.0
	120	7.0	5.0	3.0	2.0	4.0	1.0	8.0	6.0	11.0	10.0	9.0
	200	8.0	6.0	4.0	1.0	5.0	2.0	3.0	9.5	7.0	9.5	11.0
	300	6.0	10.0	8.0	1.0	9.0	2.0	5.0	3.0	4.0	7.0	11.0
	450	9.0	3.0	4.0	1.0	5.0	2.0	10.0	7.0	8.0	6.0	11.0
$\delta = 1.0$	15	6.0	3.0	7.0	5.0	4.0	1.0	2.0	9.5	8.0	9.5	11.0
	50	6.0	9.0	3.0	2.0	4.0	1.0	10.0	7.5	5.0	7.5	11.0
	120	7.0	8.5	4.0	1.0	3.0	2.0	6.0	5.0	10.0	8.5	11.0
	200	8.0	10.0	3.0	1.0	5.5	2.0	7.0	4.0	9.0	5.5	11.0
	300	6.0	9.0	8.0	1.0	7.0	2.0	10.0	5.0	3.0	4.0	11.0
	450	7.0	10.0	6.0	2.0	3.0	1.0	9.0	8.0	4.0	5.0	11.0
$\delta = 1.5$	15	6.0	3.0	5.0	4.0	7.0	1.0	2.0	9.0	8.0	11.0	10.0
	50	7.0	10.0	2.0	3.0	4.0	1.0	9.0	8.0	5.0	6.0	11.0
	120	10.0	5.0	3.0	1.0	4.0	2.0	9.0	7.0	8.0	6.0	11.0
	200	6.0	7.0	4.0	1.0	3.0	2.0	8.0	9.0	10.0	5.0	11.0
	300	5.0	8.5	4.0	1.0	3.0	2.0	6.5	10.0	8.5	6.5	11.0
	450	8.0	7.0	3.0	2.0	4.0	1.0	10.0	6.0	9.0	5.0	11.0
$\delta = 2.0$	15	3.0	4.0	5.0	7.0	6.0	1.0	2.0	11.0	8.0	10.0	9.0
	50	7.0	9.0	2.0	4.0	3.0	1.0	6.0	10.0	5.0	8.0	11.0
	120	8.0	9.0	1.0	2.0	4.0	3.0	6.0	10.0	5.0	7.0	11.0
	200	8.0	9.0	3.0	1.0	2.0	4.0	10.0	7.0	6.0	5.0	11.0
	300	6.0	7.0	3.0	1.0	2.0	4.0	9.0	11.0	10.0	5.0	8.0
	450	7.0	6.0	3.0	1.0	4.0	2.0	9.5	8.0	11.0	5.0	9.5
$\delta = 2.5$	15	2.0	3.0	5.0	6.0	7.0	1.0	4.0	11.0	8.0	10.0	9.0
	50	7.0	8.0	2.0	4.0	5.0	1.0	3.0	11.0	6.0	9.0	10.0
	120	7.0	9.0	1.0	2.0	3.0	4.0	6.0	11.0	5.0	8.0	10.0
	200	10.0	8.0	1.0	3.0	2.0	4.0	6.0	9.0	5.0	7.0	11.0
	300	11.0	8.0	2.0	1.0	3.0	4.0	5.0	10.0	7.0	6.0	9.0
	450	9.0	7.0	2.0	1.0	3.0	4.0	8.0	11.0	6.0	5.0	10.0
$\sum$ Ranks		245.5	251.5	146.5	72.0	157.5	84.0	213.0	284.0	273.5	282.0	366.5
Overall Rank		6	7	3	1	4	2	5	10	8	9	11

Table 15. Partial and total rankings of all AUD estimate techniques by RSS for different values of δ.

Parameter	$m^{\circ\circ}$	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
δ = 0.15	15	1.0	5.0	7.0	2.0	8.0	4.0	3.0	6.0	11.0	10.0	9.0
	50	9.0	5.0	6.0	1.0	7.0	2.0	3.0	4.0	8.0	11.0	10.0
	120	9.0	2.0	4.0	1.0	5.0	3.0	6.0	7.0	8.0	11.0	10.0
	200	10.0	2.0	5.0	1.0	4.0	3.0	6.0	7.0	8.0	11.0	9.0
	300	9.0	4.0	3.0	1.0	5.0	2.0	6.0	8.0	7.0	11.0	10.0
	450	9.0	8.0	2.0	1.0	3.0	4.0	6.0	7.0	5.0	11.0	10.0
δ = 0.6	15	1.0	3.0	7.0	5.0	6.0	2.0	4.0	8.0	9.0	11.0	10.0
	50	9.0	8.0	1.0	4.0	3.0	2.0	6.0	7.0	5.0	10.0	11.0
	120	10.0	8.0	2.0	4.0	3.0	1.0	7.0	6.0	5.0	11.0	9.0
	200	9.0	8.0	1.0	2.0	4.0	3.0	7.0	6.0	5.0	11.0	10.0
	300	9.0	7.0	3.5	1.5	1.5	3.5	8.0	6.0	5.0	11.0	10.0
	450	9.0	8.0	1.0	7.0	5.0	3.5	3.5	6.0	2.0	11.0	10.0
δ = 1.0	15	6.0	2.0	3.0	5.0	4.0	1.0	7.0	9.0	8.0	11.0	10.0
	50	9.0	7.0	2.0	3.0	4.0	1.0	6.0	8.0	5.0	10.0	11.0
	120	9.0	7.0	3.0	1.0	4.0	2.0	6.0	8.0	5.0	10.5	10.5
	200	11.0	7.0	1.0	4.0	3.0	2.0	6.0	8.0	5.0	9.0	10.0
	300	9.0	8.0	1.0	4.0	2.0	3.0	5.0	7.0	6.0	11.0	10.0
	450	8.0	4.0	3.0	2.0	1.0	5.0	7.0	6.0	9.5	9.5	11.0
δ = 1.5	15	5.5	3.0	2.0	7.0	4.0	1.0	5.5	10.0	8.0	9.0	11.0
	50	9.5	6.0	2.0	3.0	1.0	4.0	5.0	8.0	7.0	9.5	11.0
	120	9.0	7.0	2.0	1.0	4.0	3.0	5.0	8.0	6.0	10.0	11.0
	200	10.0	2.0	1.0	3.0	4.0	6.0	5.0	8.0	7.0	9.0	11.0
	300	10.0	1.0	4.0	5.0	2.0	3.0	6.5	8.0	6.5	9.0	11.0
	450	11.0	2.0	8.0	1.0	3.5	3.5	9.5	5.0	6.0	7.0	9.5
δ = 2.0	15	5.0	2.0	4.0	7.0	6.0	1.0	3.0	10.0	8.0	11.0	9.0
	50	9.0	4.0	2.0	1.0	3.0	5.0	6.0	8.0	7.0	10.5	10.5
	120	9.0	1.0	2.0	3.0	4.0	7.0	5.0	8.0	6.0	10.0	11.0
	200	9.5	3.0	1.0	2.0	4.0	7.0	5.0	8.0	6.0	9.5	11.0
	300	9.5	2.5	2.5	1.0	4.0	7.0	5.0	8.0	6.0	9.5	11.0
	450	10.0	2.0	5.0	1.0	3.0	4.0	8.0	6.0	9.0	7.0	11.0
δ = 2.5	15	1.5	1.5	6.0	7.0	5.0	3.0	4.0	10.0	8.0	11.0	9.0
	50	10.5	2.0	3.0	1.0	5.0	6.0	4.0	8.5	7.0	10.5	8.5
	120	11.0	1.0	3.0	4.0	2.0	7.0	5.0	8.5	6.0	10.0	8.5
	200	9.5	2.0	4.0	1.0	3.0	7.0	5.0	8.0	6.0	9.5	11.0
	300	9.5	1.5	3.0	1.5	4.0	7.0	6.0	8.0	5.0	9.5	11.0
	450	9.5	1.5	3.0	1.5	4.0	7.0	5.0	8.0	6.0	9.5	11.0
$\sum$ Ranks		304.5	148.0	113.0	100.5	138.0	135.5	200.0	270.0	237.0	362.0	367.5
Overall Rank		9	5	2	1	4	3	6	8	7	10	11

7. Application

To highlight the practical utility of the proposed estimation methods, a real dataset was meticulously selected and is comprehensively elucidated in this section. The objective was to illustrate the practical applications of these proposed estimates by conducting an in-depth analysis of the real-world dataset. This analysis serves as a demonstration of how these estimation techniques can be applied to real-world data, showcasing their effectiveness and relevance in practical research and decision-making contexts.

The used real dataset presents the firm's risk management cost-effectiveness and it was studied by Abd El-Bar et al. [48]. Its values are: 0.0432, 0.1271, 0.793, 0.0407, 0.0076, 0.037, 0.18, 0.1129, 0.09, 0.0535, 0.0783, 0.0093, 0.0851, 0.1753, 0.0036, 0.1597, 0.002, 0.1357, 0.0215, 0.0065, 0.079, 0.0329, 0.0458, 0.1192, 0.0431, 0.1245, 0.0255, 0.1396, 0.0122, 0.15, 0.14, 0.0529, 0.2222, 0.0315, 0.0389, 0.0297, 0.0608, 0.1833, 0.0279, 0.0694, 0.15, 0.0818, 0.2912, 0.1261, 0.0931, 0.0216, 0.0525, 0.1938, 0.0433, 0.0351, 0.0629, 0.0125, 0.0571, 0.0094, 0.0885, 0.0411, 0.004, 0.0582, 0.2172, 0.0434, 0.0509, 0.65, 0.0913, 0.1, 0.0375, 0.2886, 0.0206, 0.0028, 0.0407, 0.0849, 0.0612, 0.1333, 0.9755.

Table 16 provides a comprehensive summary of the descriptive analyses performed on the dataset under investigation. Figure 2 displays various graphical representations, including histograms, kernel density plots, violin plots, box plots, total time on test (TTT) plots, and quantile-quantile (Q-Q) plots. These visualizations and descriptive statistics collectively offer insights into the characteristics and distributions of the data, enhancing our understanding of the dataset's key features and patterns. The dataset was subjected to a Kolmogorov-Smirnov (KS) test to assess its compatibility with a specific model. The MLE was utilized to obtain the parameter estimates. The K-S distance (KSD) was computed to be 0.0812933, and the p-value (KSP) was found to be 0.720254. Based on these results, it is apparent that the AUD is a suitable candidate for fitting the firm's real dataset. To visually demonstrate this suitability, Figure 3 presents various graphical representations, including the probability-probability (P-P) plot, estimated CDF, estimated survival function (SF), and a histogram with the estimated PDF. These visualizations collectively suggest that the AUD is a suitable choice for modeling and fitting the firm's real dataset, as they align well with the distributional characteristics of the model.

Based on the theoretical findings discussed earlier, the dataset underwent an examination using two sampling techniques, SRS and RSS. Tables 17 and 18 present the SRS and RSS estimates, respectively, derived from the AUD. These estimates are provided for different sample sizes over five cycles, employing various estimation techniques. The process of generating the RSS and SRS observations was facilitated using the R-package. These tables collectively display the results of the estimation techniques applied to the dataset, allowing for a comprehensive comparison of the sampling methods and estimation procedures used. To demonstrate the superiority of RSS over SRS to various estimation methods, we conducted an evaluation using several goodness-of-fit statistics for the model. These statistics encompassed the Anderson-Darling test statistics (ADTS), Cramér-von Mises test statistics (CMTS), and KS test statistics (KSTS), along with their KSP. These tests and their p-values were utilized to assess how well the data conforms to the model, and their results can provide insights into the effectiveness of RSS compared to SRS in capturing the underlying distribution of the dataset. Estimates that outperformed their counterparts typically displayed larger p-values (greater than 5%) and lower goodness-of-fit values. Table 19 provides a comparative analysis between the SRS and RSS designs in terms of their goodness-of-fit values and KSPs. This comparison helps in assessing the relative effectiveness of SRS and RSS in fitting the dataset to the model, with a focus on identifying which design and estimation techniques yield better goodness-of-fit results. The fitting of the model to the dataset can be observed in Figures 4 and 5. Notably, the RSS design demonstrates superior performance compared to the SRS design in terms of efficiency. This is evident from the smaller goodness-of-fit values and the correspondingly larger KSPs. This superiority is consistently observed across all estimates, even when the same number of measurement units is considered. These findings underscore the advantages of RSS over SRS in terms of fitting the dataset to the model and obtaining more efficient estimates.

Table 16. Descriptive analyses of the firm's real dataset.

	$m^{\circ\circ}$	Mean	Median	Skewness	Kurtosis	Range	Minimum	Maximum	Sum
data	73	0.109733	0.0608	3.71542	17.9579	0.9735	0.002	0.9755	8.0105

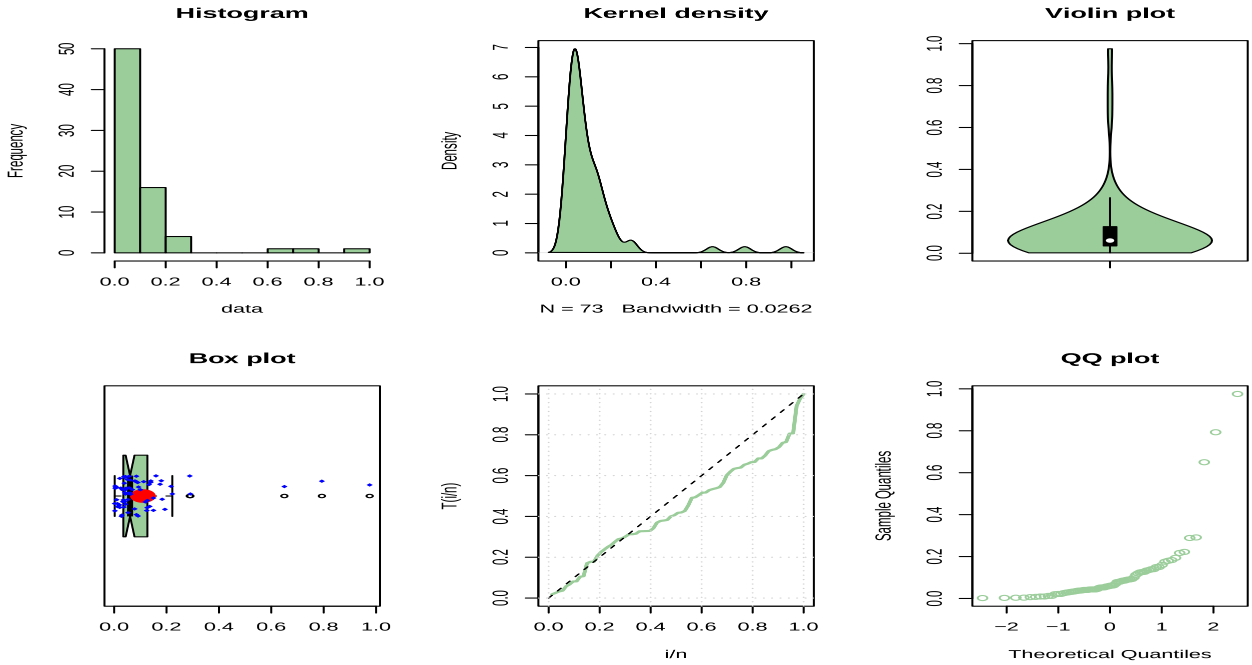

Figure 2. Some plots for the firm's real dataset.

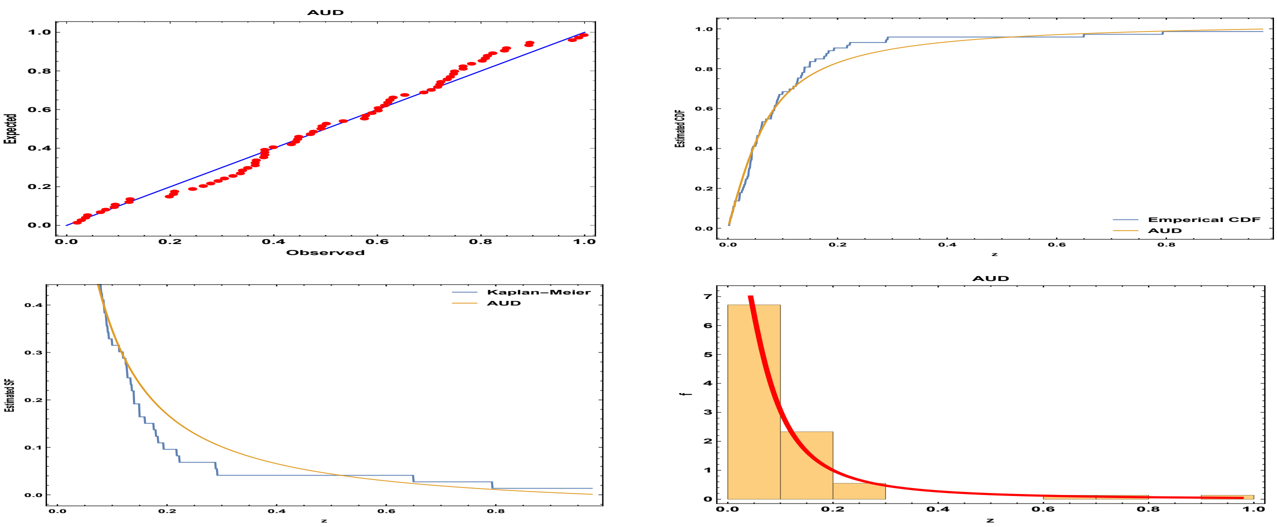

Figure 3. P-P plot, estimated CDF, estimated SF, and histogram with the estimated PDF for the AUD.

Table 17. Values of AUD estimate for various estimating techniques using the SRS dataset.

$m^{\circ\circ}$	Estimate	MLE	ADE	CME	MPSE	LSE	PSE	RADE	WLSE	LADE	MSADE	MSALDE
20	$\dddot{\delta}$	14.2997	14.4682	14.3253	14.2344	14.2934	7.22796	13.5736	14.591	15.5268	23.1179	14.2344
35	$\dddot{\delta}$	18.2715	18.3864	18.1745	18.2414	18.1516	51.3402	20.4428	19.3072	16.4006	15.145	17.8123
50	$\dddot{\delta}$	14.7089	14.8449	14.6641	14.6879	14.6491	16.5	15.777	15.078	13.8907	13.9479	15.4683
65	$\dddot{\delta}$	16.1061	16.1716	16.031	16.0896	16.0229	16.2323	17.0406	16.2854	15.298	14.9776	15.7568

Table 18. Values of AUD estimate for various estimating techniques using the RSS dataset with set size $s = 5$.

$m^{\circ\circ}$	Estimate	MLE	ADE	CME	MPSE	LSE	PCE	RADE	WLSE	LADE	MSADE	MSALDE
20	$\hat{\delta}$	14.2503	14.2662	13.8488	13.6955	13.7732	7.56834	13.3422	14.3021	15.4774	13.5672	17.3995
35	$\hat{\delta}$	12.5203	12.5202	12.3418	12.2565	12.3128	12.4091	13.0535	12.8861	11.94	14.5198	17.4009
50	$\hat{\delta}$	17.1027	17.0984	16.9476	16.9634	16.9351	15.6024	17.708	17.2244	16.471	18.3951	14.3959
65	$\hat{\delta}$	14.4861	14.4806	14.3786	14.4454	14.3728	14.4125	15.1468	14.4975	13.8117	13.7231	6.63895

Table 19. The estimates, KSTS, ADTS, CMTS, and KSP in the SRS and RSS designs for the dataset at $m^{\circ\circ} = 50$.

Method	design	$\hat{\delta}$	ADTS	CMTS	KSTS	KSP
MLE	SRS	14.7089	0.761824	0.114911	0.117979	0.489566
	RSS	17.1027	0.387748	0.0472457	0.0751288	0.940393
ADE	SRS	14.8449	0.760685	0.115343	0.115999	0.511594
	RSS	17.0984	0.387747	0.0472331	0.0751671	0.940158
CME	SRS	14.6641	0.762704	0.114883	0.118639	0.482331
	RSS	16.9476	0.388744	0.047017	0.0765033	0.931604
MPSE	SRS	14.6879	0.762205	0.114891	0.118288	0.48617
	RSS	16.9634	0.388545	0.0470194	0.0763622	0.932538
LSE	SRS	14.6491	0.763056	0.114886	0.118861	0.479908
	RSS	16.9351	0.388917	0.0470186	0.0766156	0.930856
PSE	SRS	16.5	0.91112	0.157392	0.117831	0.491197
	RSS	15.6024	0.494113	0.0658479	0.0894908	0.818117
RADE	SRS	15.777	0.810594	0.131257	0.105954	0.6285
	RSS	17.708	0.403364	0.052323	0.0830061	0.881087
WLSE	SRS	15.078	0.76395	0.117256	0.112676	0.549461
	RSS	17.2244	0.388432	0.0477398	0.0750625	0.940799
LADE	SRS	13.8907	0.81995	0.123882	0.13058	0.361317
	RSS	16.471	0.405507	0.0492597	0.0808795	0.899158
MSADE	SRS	13.9479	0.812856	0.12257	0.12966	0.369911
	RSS	18.3951	0.455783	0.0655095	0.0940681	0.768166
MSALDE	SRS	15.4683	0.783456	0.123611	0.107303	0.612458
	RSS	14.3959	0.762225	0.120147	0.106991	0.616163

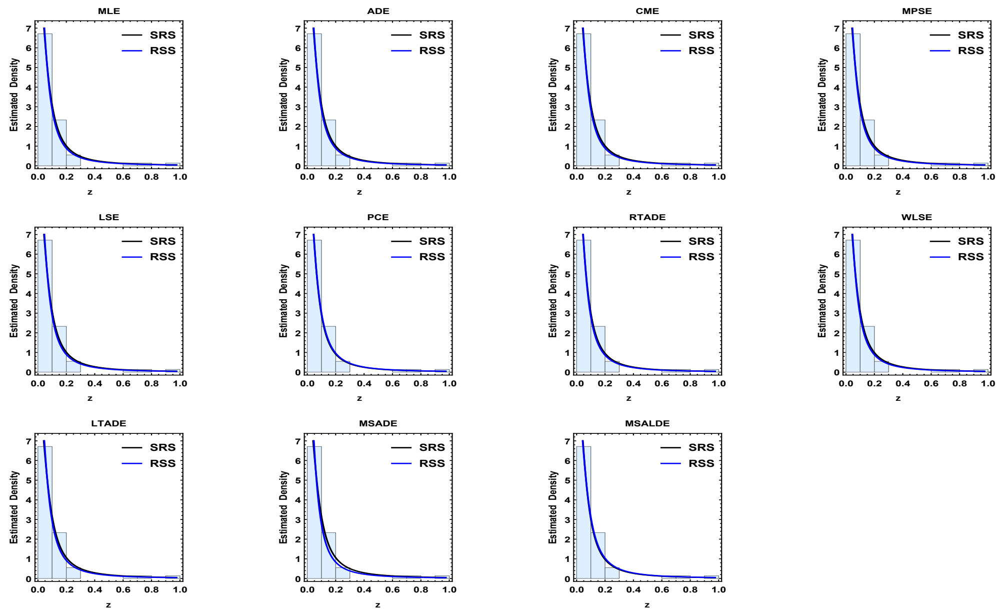

Figure 4. Plots of the estimated PDFs of the AUD with histogram for the two sampling methods at $m^{\circ\circ} = 50$.

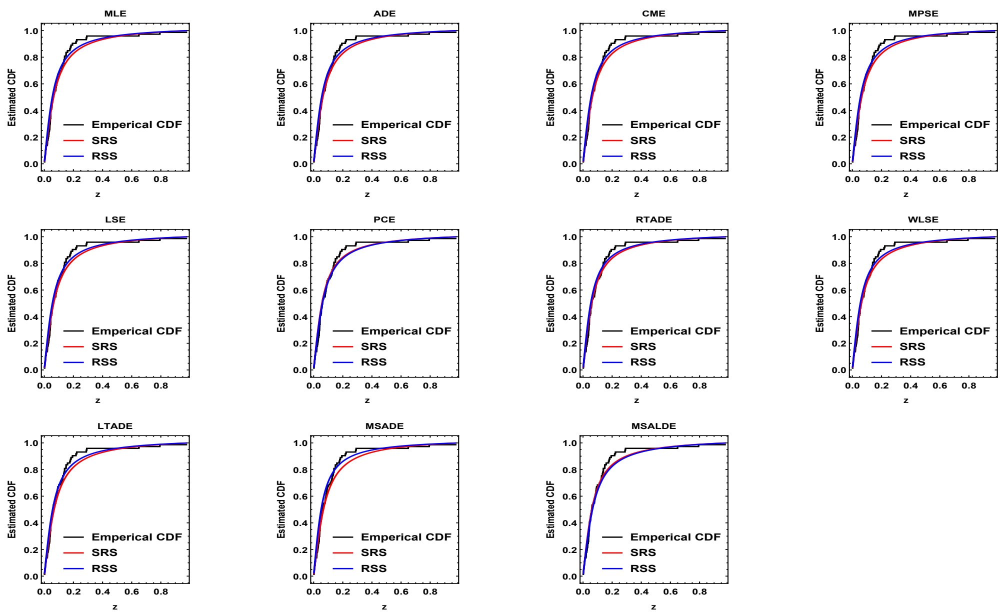

Figure 5. Plots of the estimated CDFs of the AUD for the two sampling methods at $m^{\circ\circ} = 50$.

8. Conclusions

A brand-new bounded distribution called the arctan uniform distribution may be used to simulate several bounded real-world datasets that already exist. When accurately measuring the observation is difficult or expensive, RSS is a valuable strategy. In the present work, the parameter estimator of the arctan uniform distribution is regarded using RSS and SRS approaches. The PS, WLS, AD, ML, MSALD, CM, LS, MPS, RAD, LAD, and MSAD are a few of the well-known conventional estimating techniques that are used. A Monte Carlo simulation based on some accuracy measures is used to assess the effectiveness of the generated estimates. Based on the results of our simulations for both the SRS and RSS datasets, it appears that the MPS approach is preferred in evaluating the quality of suggested estimates compared to the others. A similar pattern of decline with larger sample sizes is seen in all criteria measures. This trend indicates that parameter estimates are more accurate and trustworthy with higher sample numbers. Estimates derived from the RSS datasets are more trustworthy than those derived from the SRS datasets. This suggests that RSS is a sampling strategy that produces estimates with a lower mean squared error than other sampling methods. Real data findings provide more evidence that the RSS design is superior to the SRS approach.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RG23048).

Conflict of interest

The authors declare no conflict of interest.

References

1. C. W. Topp, F. C. Leone, A family of j-shaped frequency functions, *J. Amer. Stat. Assoc.*, **50** (1995), 209–219.
2. P. Kumaraswamy, A generalized probability density function for double-bounded random processes, *J. Hydro.*, **46** (1980), 79–88. http://dx.doi.org/10.1016/0022-1694(80)90036-0
3. A. F. B. Menezes, J. Mazucheli, S. Dey, The unit-logistic distribution: Different methods of estimation, *Pes. Oper.*, **9** (2018), 555–578. https://doi.org/10.1590/0101-7438.2018.038.03.0555
4. J. Mazucheli, A. F. B. Menezes, S. Dey, The unit-Birnbaum-Saunders distribution with applications, *Chil. J. Stat.*, **9** (2018), 47–57.
5. J. Mazucheli, A. F. Menezes, S. Dey, Unit-Gompertz distribution with applications, *Statistica*, **79** (2019), 25–43. https://doi.org/10.6092/issn.1973-2201/8497
6. M. A. Almuqrin, A. M. Gemeay, M. M. Abd El-Raouf, M. Kilai, R. Aldallal, E. Hussam, A flexible extension of reduced Kies distribution: Properties, inference, and applications in biology, *Complexity*, **2022** (2022), 6078567. https://doi.org/10.1155/2022/6078567
7. J. Mazucheli, A. F. B. Menezes, L. B. Fernandes, R. P. De Oliveira, M. E. Ghitany, The unit-Weibull distribution as an alternative to the Kumaraswamy distribution for the modeling of quantiles conditional on covariates, *J. Appl. Stat.*, **47** (2020), 954–974. https://doi.org/10.1080/02664763.2019.1657813

8. M. C. Korkmaz, The unit generalized half normal distribution: A new bounded distribution with inference and application, *U.P.B. Sci. Bull. Series A*, **82** (2020), 954–974.

9. E. Altun, M. El-Morshedy, M. S. Eliwa, A new regression model for bounded response variable: An alternative to the beta and unit-Lindley regression models, *Plos one*, **16** (2021), e0245627. https://doi.org/10.1371/journal.pone.0245627

10. M. C. Korkmaz, C. Chesneau, On the unit Burr-XII distribution with the quantile regression modeling and applications, *Comput. Appl. Math.*, **40** (2021). https://doi.org/10.1007/s40314-021-01418-5

11. E. Yıldırım, E. S. Ilıkkan, A. M. Gemeay, N. Makumi, M. E. Bakr, O. S. Balogun, Power unit Burr-XII distribution: Statistical inference with applications, *AIP Adv.*, **13** (2023), 105107. https://doi.org/10.1063/5.0171077

12. R. A. R. Bantan, F. Jamal, C. Chesneau, M. Elgarhy, Theory and applications of the unit gamma/Gompertz distribution, *Mathematics*, **9** (2021). https://doi.org/10.3390/math9161850

13. A. Krishna, R. Maya, C. Chesneau, M. R. Irshad, The unit Teissier distribution and its applications, *Math. Comput. Appl.*, **27** (2022), 12. https://doi.org/10.3390/mca27010012

14. S. Nasiru, C. Chesneau, A. G. Abubakari, I. D. Angbing, Generalized unit half-logistic geometric distribution: Properties and regression with applications to insurance, *Analytics*, **2** (2023), 438–462. https://doi.org/10.3390/analytics2020025

15. O. Kharazmi, M. Alizadeh, J. E. Contreras-Reyes, H. Haghbin, Arctan-based family of distributions: Properties, survival regression, Bayesian analysis and applications, *Axioms*, **11** (2022), 399. https://doi.org/10.3390/axioms11080399

16. G. A. McIntyre, A method for unbiased selective sampling, using ranked sets, *Aust. J. Agri. Res.*, **3** (1952), 385–390.

17. H. M. Aljohani, E. M. Almetwally, A. S. Alghamdi, E. H. Hafez, Ranked set sampling with application of modified Kies exponential distribution, *Alexandria Eng. J.*, **60** (2021), 4041–4046. https://doi.org/10.1016/j.aej.2021.02.043

18. L. K. Halls, T. R. Dell, Trial of ranked-set sampling for forage yields, *Forest Sci.*, **12** (1966), 22–26.

19. M. H. Sabry, E. M. Almetwally, Estimation of the exponential Pareto distributions parameters under ranked and double ranked set sampling designs, *Pak. J. Stat. Oper. Res.*, **17** (2021), 169–184. https://doi.org/10.18187/PJSOR.v17i1.3448

20. R. Alharbi, M. S. Mustafa, A. Al-Mutairi, M. Hussein, M. Yusuf, A. Elshenawy, S. G. Nassr, Enhancing mean estimators in median ranked set sampling with dual auxiliary information, *Heliyon*, **9** (2023), E21427. https://doi.org/10.1016/j.heliyon.2023.e21427

21. J. M. Cobby, M. S. Ridout, P. J. Bassett, R. V. Large, An investigation into the use of ranked set sampling on grass and grass-clover swards, *Grass Forage Sci.*, **40** (1985), 257–263. https://doi.org/10.1111/j.1365-2494.1985.tb01753.x

22. G. P. Patil, A. K. Sinha, C. Taille, Relative precision of ranked set sampling: A comparison with the regression estimator, *Environmetrics*, **4** (1993), 399–412. https://doi.org/10.1111/j.1365-2494.1985.tb01753.x

23. H. Muttlak, W. Al-Sabah, Statistical quality control based on ranked set sampling, *J. Appl. Stat.*, **30** (2003), 1055–1078. https://doi.org/10.1080/0266476032000076173

24. N. Alotaibi, A. S. Al-Moisheer, I. Elbatal, M. Shrahili, M. Elgarhy, E. M. Almetwally, Half logistic inverted Nadarajah-Haghighi distribution under ranked set sampling with applications, *Mathematics*, **11** (2023), 1693. https://doi.org/10.3390/math11071693

25. F. H. Riad, M. A. Sabry, E. M. Almetwally, R. Aldallal, R. Alharbi, M. M. Hossain, On extended neoteric ranked set sampling plan: Likelihood function derivation and parameter estimation, *Complexity*, **2022** (2022), 1697481. https://doi.org/10.1155/2022/1697481

26. L. Stokes, Parametric ranked set sampling, *Ann. Instit. Statist. Math.*, **47** (1995), 465–482. https://doi.org/10.1007/BF00773396

27. A. B. Shaibu, H. A. Muttlak, Estimating the parameters of the normal, exponential and gamma distributions using median and extreme ranked set samples, *Statistica*, **64** (2004), 75–98. https://doi.org/10.6092/issn.1973-2201/25

28. A. Adatia, Estimation of parameters of the half-logistic distribution using generalized ranked set sampling, *Comput. Stat. Data Analy.*, **33** (2000), 1–13. https://doi.org/10.1016/S0167-9473(99)00035-3

29. O. M. Yousef, S. A. Al-Subh, Estimation of Gumbel parameters under ranked set sampling, *J. Modern Appl. Stat. Methods*, **13** (2014), 24. https://doi.org/10.56801/10.56801/v13.i.741

30. M. Esemen, S. Gurler, Parameter estimation of generalized Rayleigh distribution based on ranked set sample, *J. Statist. Comput. Simul.*, **88** (2018), 615–628. https://doi.org/10.1080/00949655.2017.1398256

31. W. Chen, R. Yang, D. Yao, C. Long, Pareto parameters estimation using moving extremes ranked set sampling, *Statist. Papers*, **62** (2021), 1195–1211. https://doi.org/10.1007/s00362-019-01132-9

32. W. Qian, W. Chen, X. He, Parameter estimation for the Pareto distribution based on ranked set sampling, *Statist. Papers*, **62** (2021), 395–417. https://doi.org/10.1007/s00362-019-01102-1

33. A. I. Al-Omari, S. Benchiha, I. M. Almanjahie, Efficient estimation of two-parameter xgamma distribution parameters using ranked set sampling design, *Mathematics*, **10** (2022), 3170. https://doi.org/10.3390/math10173170

34. A. I. Samuh, M. H. Al-Omari, N. Koyuncu, Estimation of the parameters of the new Weibull-Pareto distribution using ranked set sampling, *Statistica*, **80** (2020), 103–123. https://doi.org/10.6092/issn.1973-2201/9368

35. R. Bantan, M. Elsehetry, A. S. Hassan, M. Elgarhy, D. Sharma, C. Chesneau, et al., A two-parameter model: properties and estimation under ranked sampling, *Mathematics*, **9** (2013), 1214. https://doi.org/10.3390/math9111214

36. A. I. Al-Omari, S. Benchiha, I. M. Almanjahie, Efficient estimation of the generalized quasi-Lindley distribution parameters under ranked set sampling and applications, *Math. Prob. Eng.*, **2021** (2021), 1214. https://doi.org/10.1155/2021/9982397

37. H. F. Nagy, A. I. Al-Omari, A. S. Hassan, G. A. Alomani, Improved estimation of the inverted kumaraswamy distribution parameters based on ranked set sampling with an application to real data, *Mathematics*, **10** (2022), 4102. https://doi.org/10.3390/math10214102

38. A. S. Hassan, N. Alsadat, M. Elgarhy, C. Chesneau, H. F. Nagy, Analysis of $R = P[Y < X < Z]$ using ranked set sampling for a generalized inverse exponential model, *Axioms*, **12** (2023), 302. https://doi.org/10.3390/axioms12030302

39. N. Alsadat, A. S. Hassan, M. Elgarhy, C. Chesneau, R. E. Mohamed, An efficient stress-strength reliability estimate of the unit Gompertz distribution using ranked set sampling, *Symmetry*, **15** (2023), 1121. https://doi.org/10.3390/sym15051121

40. R. Yang, W. Chen, Y. Dong, Log-extended exponential-geometric parameters estimation using simple random sampling and moving extremes ranked set sampling, *Commun. Stat. Simul. Comput.*, **52** (2023), 267–277. https://doi.org/10.1080/03610918.2020.1853167

41. A. S. Hassan, N. Alsadat, M. Elgarhy, C. Chesneau, R. E. Mohamed, Different classical estimation methods using ranked set sampling and data analysis for the inverse power Cauchy distribution, *J. Radiation Res. Appl. Sci.*, **16** (2023), 100685. https://doi.org/10.1016/j.jrras.2023.100685

42. N. Alsadat, A. S. Hassan, A. M. Gemeay, C. Chesneau, M. Elgarhy, Different estimation methods for the generalized unit half-logistic geometric distribution: Using ranked set sampling, *J. Radiation Res. Appl. Sci.*, **13** (2023), 085230. https://doi.org/10.1063/5.0169140

43. T. W. Anderson, D. A. Darling, Asymptotic theory of certain goodness of fit criteria based on stochastic processes, *The Ann. Math. Stat.*, **13** (1952), 193–212. https://doi.org/10.1214/aoms/1177729437

44. P. D. M. Macdonald, Comments and queries comment on an estimation procedure for mixtures of distributions by Choi and Bulgren, *J. Royal Statist. Society Ser. B Statist. Methodol.*, **33** (1971), 326–329. https://doi.org/10.1111/j.2517-6161.1971.tb00884.x

45. R. C. H. Cheng, N. A. K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, *J. Royal Statist. Society Ser. B Statist. Methodol.*, **45** (1983), 394–403. https://doi.org/10.1111/j.2517-6161.1983.tb01268.x

46. H. Torabi, A general method for estimating and hypotheses testing using spacings, *J. Statist. Theory Appl.*, **8** (2008), 163–168.

47. E. Castillo, A. S. Hadi, A method for estimating parameters and quantiles of distributions of continuous random variables, *Comput. Stat. Data Anal.*, **20** (1995), 421–439. https://doi.org/10.1016/0167-9473(94)00049-O

48. A. Abd El-Bar, H. S. Bakouch, S. Chowdhury, A new trigonometric distribution with bounded support and an application, *Rev. Unión Mat. Arge.*, **62** (2021), 459–473. https://doi.org/10.33044/revuma.1872