Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 10289–10303.
DOI: 10.3934/math.2024503
Received: 20 December 2023
Revised: 26 February 2024
Accepted: 04 March 2024
Published: 14 March 2024

Research article

Symmetry reductions and conservation laws of a modified-mixed KdV equation: exploring new interaction solutions

Nauman Raza[1,2,*], **Maria Luz Gandarias**[3] **and Ghada Ali Basendwah**[4]

[1] Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore 54590, Pakistan
[2] Department of Mathematics, Near East University, TRNC, Mersin 10, Nicosia 99138, Turkey
[3] Department of Mathematics, University of Cadiz, Puerto Real, Cádiz 11510, Spain
[4] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: nauman.math@pu.edu.pk.

Abstract: This article represented the investigation of the modified mixed Korteweg-de Vries equation using different versatile approaches. First, the Lie point symmetry approach was used to determine all possible symmetry generators. With the help of these generators, we reduced the dimension of the proposed equation which leads to the ordinary differential equation. Second, we employed the unified Riccati equation expansion technique to construct the abundance of soliton dynamics. A group of kink solitons and other solitons related to hyperbolic functions were among these solutions. To give the physical meaning of these theoretical results, we plotted these solutions in 3D, contour, and 2D graphs using suitable physical parameters. The comprehend outcomes were reported, which can be useful and beneficial in the future investigation of the studied equation. The results showed that applied techniques are very useful to study the other nonlinear physical problems in nonlinear sciences.

Keywords: symmetry reductions; invariant solution; unified Riccati equation expansion method; exact solution; modified mixed KdV equation
Mathematics Subject Classification: 35B32, 35C08

1. Introduction

Over the previous few decades, researchers have shown considerable interest in the extraction of solutions of nonlinear systems such as traveling wave solutions, including solitary waves, periodic waves, kink and anti-kink solutions, solitons, and some integrability of some intriguing nonlinear partial differential equations (PDEs) [1–5], for instance, the nonlinear Schrodinger equation [6], Burgers equation [7], modified equal width [8], modified equal width-Burgers equation [9], the KdV model [10], combined KdV-mKdV equation [11], etc. Solitons are solitary waves that possess elastic scattering properties. These waves emerge as a result of the intricate equilibrium between dispersion, and nonlinearity. The origination of solitons can be attributed to the pioneering formulation of the KdV equation [12, 13]. The KdV equation is extensively employed for the simulation of the propagation of shallow, thin, and elongated water waves. It provides an explanation for a multitude of physical processes, encompassing acoustic, hydromagnetic, and ion-acoustic waves. The standardized form of the KdV model is expressed as follows:

$$\Omega_t + 6\Omega\Omega_x + \Omega_{xxx} = 0. \tag{1.1}$$

This equation has a number of essential applications in a variety of scientific domains. Various extensions of KdV equations have been extracted in recent decades. In [14], Wang and Kara have introduced KdV and modified KdV equations. They have used the (2+1)-dimensional mKdV equations to derive conservation laws by performing Lie symmetry analysis. In [15], the authors studied the bifurcation and established some exact solutions for (2+1)-dimensional KdV equation. Later, Wazwaz [16] developed new (3+1)-dimensional KdV and mKdV equations. The author demonstrated their integrability using Painlevé analysis in [16] and established several multiple-solution solutions for these equations. Moreover, the combined KdV-mKdV model has been investigated in [17] and derive the integrability, stability analysis and soliton structures. Now, we have considered another extension of the KdV equations called the the modified-mixed KdV (mmKdV) model, read as:

$$\Omega_t + (\alpha\sqrt{\Omega} + \beta\Omega)\Omega_x + \delta\Omega_{xxx} = 0, \tag{1.2}$$

where $\Omega = \Omega(x, t)$. The mmKdV equation characterizes an electron distribution with a flat top, wherein the nonlinearity is more pronounced when the width is smaller and the velocity is higher [18–20].

In [20], the following assumption has been proposed

$$\Omega = w^2, \quad w = w(x, t). \tag{1.3}$$

By substituting (1.3) into (1.2), the following equation was obtained

$$ww_t + (\alpha w^2 + \beta w^3)w_x + \alpha(ww_{xxx} + 3w_x w_{xx}) = 0. \tag{1.4}$$

The polynomial function, and rational sinh-cosh methods were used to derive some results.

The Lie symmetry method applied to PDEs, is a powerful approach to obtain reductions and invariant solutions [21].

Recently, Tian et al. [22] successfully proposed an effective and direct approach to study the symmetry-preserving discretization for a class of generalized higher order equations, and proposed an open problem about symmetries and the multipliers of conservation law.

In this paper we consider Eq (1.4) from the viewpoint of symmetry reductions in PDEs. We obtain the Lie point symmetries admitted by (1.4) for arbitrary constants. We derived conservation laws for Eq (1.4).

Taking into account the relationship between symmetries and conservation laws by using the invariance of two conservation laws under translations, we derive two first integrals. By combining these first integrals, we obtain a triple reduction to a first-order autonomous equation.

2. Lie point symmetries

A Lie point symmetry [23–25] for the Eq (1.4) pertains to a set of point transformations that maintain the equations invariance. These transformations can be expressed in infinitesimal form:

$$\begin{aligned}\tilde{t} =&t + \varepsilon\tau(t, x, w) + O(\varepsilon^2),\\ \tilde{x} =&x + \varepsilon\xi^1(t, x, w) + O(\varepsilon^2),\\ \tilde{w} =&w + \varepsilon\eta(t, x, w) + O(\varepsilon^2),\end{aligned}$$

where ε symbolizes the group parameter, and the related vector field is described as follows:

$$\mathrm{X} = \tau(t, x, w)\partial_t + \xi(t, x, w)\partial_x + \eta(t, x, w)\partial_w. \tag{2.1}$$

The transformation group will exhibit point symmetry if

$$\overset{(3)}{\mathrm{Pr}}\,\mathrm{X}\left(ww_t + (\alpha w^2 + \beta w^3)w_x + \alpha(ww_{xxx} + 3w_xw_{xx})\right)|_{\mathcal{E}} = 0. \tag{2.2}$$

The expression "$\mathrm{Pr}^{(3)}\,\mathrm{X}$" is the third prolongation of the vector field given by (2.1), while $\mathcal{E}$ refers to the solution space of Eq (1.4). The Eq (2.2), which determines the outcome, separates based on the derivatives of w. This division results in a linear system that is over-determined, involving the infinitesimals $\tau(t, x, w)$, $\xi(t, x, w)$, and $\eta(t, x, w)$. This system is known as the determining system. By solving the determining system, we are able to derive the subsequent outcomes:

A single-parameter Lie group on the variables (t, x, w) associated with Eq (1.4) is considered as a point symmetry. This Lie group is characterized by a vector field that preserves the solution space of the equation.

The incorporation of each identified symmetry enables a reduction in the number of independent variables within Eq (1.4). Specifically, this reduction leads to the transformation of PDEs into ordinary differential equations (ODEs). Moreover, these ODEs may possess additional symmetries that facilitate a further reduction in the equation's order. The solutions of these reduced ODEs correspond to invariant solutions denoted as $u(t, x, w)$ in relation to Eq (1.4).

Theorem 2.1. *(i) The point symmetries for Eq* (1.4)*, with $\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$, are generated by:*

$$\mathrm{X}_1 =\partial_t, \tag{2.3}$$

$$\mathrm{X}_2 =\partial_x. \tag{2.4}$$

(ii) For some particular parameters α and β, there are additional generators given below. For $\alpha = 0$,

$$\mathrm{X}_3 =t\partial_x + \frac{1}{2\beta w}\partial_w, \tag{2.5}$$

$$\mathrm{X}_4 =3t\partial_t + x\partial_x - w\partial_w. \tag{2.6}$$

(iii) For $\beta = 0$,

$$\mathrm{X}_5 =3t\partial_t + x\partial_x - 2w\partial_w. \tag{2.7}$$

(i) Their commutator is given for $\alpha = 0$ by

$$[\mathrm{X}_1, \mathrm{X}_3] = X_2, \tag{2.8}$$
$$[\mathrm{X}_1, \mathrm{X}_4] = 3X_1, \tag{2.9}$$
$$[\mathrm{X}_2, \mathrm{X}_4] = X_2, \tag{2.10}$$
$$[\mathrm{X}_3, \mathrm{X}_4] = -2X_3. \tag{2.11}$$

(ii) Their commutator is given for $\beta = 0$ by

$$[\mathrm{X}_1, \mathrm{X}_5] = 3X_1, \tag{2.12}$$
$$[\mathrm{X}_2, \mathrm{X}_5] = X_2. \tag{2.13}$$

The corresponding optimal systems are:
(i) For $\alpha = 0$,

$$X_1 + cX_2, \quad X_4.$$

(ii) For $\beta = 0$,

$$X_1 + cX_2, \quad X_5.$$

2.1. Conservation laws

The conservation law for the PDE denoted by [26–30]: $G(t, x, w, w_t, w_x, w_{xx}, \ldots) = 0$, can be expressed as $D_t T + D_x \Phi = 0$. Here, T represents the conserved density and Φ is the flux vector, both functions of t, x, w. The conserved current is denoted as (T, Φ).

Each conservation law for the PDE $G = 0$ is linked to a corresponding multiplier. An injective relationship exists between nontrivial conserved currents $(T, \Phi)|\mathcal{E}$ modulo trivial ones and nonzero multipliers $Q|\mathcal{E}$. This relationship is characterized by $QG = D_t T + D_x \Phi$ holding as an identity, where Q is a function of t, x, w, and derivatives of w, ensuring that $Q|\mathcal{E}$ is non-singular. Various explicit methods can be employed to obtain a conserved current for each solution Q.

We will focus on examining low-order multipliers of the form $Q(t, x, w, w_x, w_{xx})$. The determining equations can be decomposed into an over-determined linear system of equations. Solving this system for Q is straightforward, subject to the conditions $\alpha \neq 0, \beta \neq 0, \delta \neq 0$.

Proposition 2.1. *All multipliers [31] admitted by the mmKdV equation (1.4), with $\alpha \neq 0, \beta \neq 0, \delta \neq 0$, are given by:*

$$Q_1 = 1, \tag{2.14}$$
$$Q_2 = \frac{w^2}{2}, \tag{2.15}$$
$$Q_3 = ww_{xx} + w_x^2 + \tfrac{\beta}{4\delta}w^4 + \tfrac{\alpha}{3\delta}w^3. \tag{2.16}$$

These multipliers yield all the nontrivial conservation laws, summarized as below.

Theorem 2.2. *The conservation laws for the mmKdV equation (1.4), with $\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$ are given by:*

$$T_1 = \frac{w^2}{2},$$

$$
\begin{aligned}
X_1 =&\delta(ww_{xx} + w_x^2) + \frac{\beta}{4}w^4 + \frac{\alpha}{3}w^3, && (2.17)\\
T_2 =&\frac{w^4}{8},\\
X_2 =&\frac{\delta}{2}w^3 w_{xx} + \frac{\beta}{12}w^6 + \frac{\alpha}{10}w^5, && (2.18)\\
T_3 =&-\frac{1}{2}w^2 w_x^2 + \frac{\beta}{24\delta}w^6 + \frac{\alpha}{15\delta}w^5,\\
X_3 =&\frac{\delta}{2}w^2 w_{xx}^2 + (\delta w w_x^2 + \frac{\beta}{4}w^5 + \frac{\alpha}{3}w^4)w_{xx}\\
&+ w^2 w_t w_x + \frac{\delta}{2}w_x^4 + (\frac{\beta}{4}w^4 + \frac{\alpha}{3}w^3)w_x^2 + \frac{\beta^2}{32\delta}w^8 + \frac{\beta\alpha}{12\delta}w^7 + \frac{\alpha^2}{18\delta}w^6. && (2.19)
\end{aligned}
$$

2.2. Traveling wave reduction, scaling reduction, and first integrals from symmetry reduction

Symmetry reduction [32] is frequently employed, with one prevalent application being the simplification to ODEs.

A traveling wave has the form

$$
w(t, x) = U(\xi) \quad \xi = x - ct, \tag{2.20}
$$

where c is the velocity of the traveling wave.

Inserting Eq (2.20) into the Eq (1.2) results in a following ODE:

$$
\delta(UU''' + 3U'U'') + (\beta U^3 + \alpha U^2 - cU)U' = 0. \tag{2.21}
$$

In [33], a comprehensive multi-reduction approach was presented, demonstrating that symmetry-based conservation laws give rise to first integrals. This method directly utilizes the inherent symmetry to identify all first integrals. As a result of this reduction, two distinct first integrals are obtained. The functionally independent first integrals of the ODE given by (2.21) are subsequently derived from the corresponding symmetry-invariant multipliers. $Q_1 = 1,\ Q_2 = \frac{w^2}{2}$ are given by

$$
\begin{aligned}
\Psi_1 &= \frac{c}{2}U^2 - \delta UU'' - \delta(U')^2 - \frac{\beta}{4}U^4 - \frac{\alpha}{3}U^3 = C_1,\\
\Psi_2 &= \frac{c}{4}U^8 - \frac{\delta}{2}U^3U'' - \frac{\beta}{12}U^6 - \frac{\alpha}{3}U^5 = C_2 = C_2,
\end{aligned} \tag{2.22}
$$

and eliminating U'' yields an autonomous nonlinear first-order ODE

$$
(U')^2 + \frac{\beta}{12\delta}U^4 + \frac{2\alpha}{15\delta}U^3 - \frac{c}{4\delta}U^2 + \frac{C_1}{\delta} + \frac{2C_2}{\delta U^2} = 0, \tag{2.23}
$$

with C_1, C_2 arbitrary constants.

We point out that, for special values of the constants, the general solution of Eq (2.23) can be written in terms of the Jacobi elliptic functions [34].

Now, we consider the scaling symmetry

$$
X_4 = 3t\partial_t + x\partial_x - w\partial_w,
$$

which gives similarity solutions

$$w(t,x) = \frac{U(z)}{x}, \quad z = \frac{x^3}{t}, \tag{2.24}$$

where z and U are the scaling invariants.

The similarity reduced ODE for this solution is a third-order nonlinear ODE

$$\delta z^3(27U^2U''' + 81UU'U'') + (3\beta z U^4 + z(24\delta - z)U^2)U' - \beta U^5 - 12\delta U^3 = 0. \tag{2.25}$$

In [35–37], it is observed that when considering ODEs resulting from a symmetry reduction under the scaling of the Eq (1.2), the conservation laws of the equation that are invariant under scalings reduce to a first integral of the ODE. Additionally, the work in [33] introduced a general multi-reduction method, demonstrating that all first integrals arising from conservation laws can be directly obtained using the symmetry. This reduction process results in two first integrals. The derived first integral of the ODE (2.25) is obtained from the corresponding symmetry-invariant multiplier. $Q_3 = \beta t u^2 - x$ is given by

$$\Psi_1 = -27\delta U(\beta U^2 - z)U'' + 27\delta z(U')^2 - \frac{U^2}{2z^2}((\beta U^2 - z)^2 + 12\delta(\beta U^2 - 2z)) = C_1, \tag{2.26}$$

which is a nonlinear second-order ODE.

3. Exact solution through unified Riccati equation expansion method

Here, we will gain the exact solution of Eq (2.21) by using the unified Riccati equation approach [38]. According to this, the proposed method has the following finite form of the analytic solution:

$$U(\tau) = B_0 + \sum_{k=1}^{n} B_k G^k(\tau), \tag{3.1}$$

where $B_k, (k = 1, 2, 3, ..., n)$ are arbitrary parameters such that $B_n \neq 0$ and n is a positive integer. Further, the function $G(\tau)$ must satisfy the following equation:

$$G'(\tau) = p_0 + p_1 G(\tau) + p_2 G^2(\tau). \tag{3.2}$$

On solving Eq (3.2) and taking constant of integration as zero, yields the following solution:

(1) If $\Delta > 0$,

$$G_1(\tau) = -\frac{p_1}{2p_2} - \frac{\sqrt{\Delta}}{2l_2}\tanh\left(\frac{\sqrt{\Delta}\tau}{2}\right), \tag{3.3}$$

$$G_2(\tau) = -\frac{p_1}{2p_2} - \frac{\sqrt{\Delta}}{2l_2}\coth\left(\frac{\sqrt{\Delta}\tau}{2}\right). \tag{3.4}$$

(2) If $\Delta < 0$,

$$G_3(\tau) = -\frac{p_1}{2p_2} - \frac{\sqrt{\Delta}}{2l_2}\tanh\left(\frac{\sqrt{\Delta}\tau}{2}\right), \tag{3.5}$$

$$G_4(\tau) = -\frac{p_1}{2p_2} - \frac{\sqrt{\Delta}}{2l_2}\coth\left(\frac{\sqrt{\Delta}\tau}{2}\right). \tag{3.6}$$

(3) If $\Delta = 0$,

$$G_5(\tau) = -\frac{p_1}{2p_2} - \frac{1}{l_2\tau + c_1}, \tag{3.7}$$

where $\Delta = p_1^2 - 4p_0p_2$.

3.1. Solution for Eq (2.21)

Apply the homogeneous balance principle to calculate the balancing number n, which will be used to define the degree of the analytic solution. Balancing the dispersive and highest nonlinear terms, we get $n = 1$. Therefore, the method of solution takes the linear form as

$$U(\tau) = B_0 + B_1 G(\tau). \tag{3.8}$$

By inserting Eqs (3.8) and (3.2) into Eq (2.21), collecting terms having identical powers of $G^k(\tau)$, and equating the coefficients to zero leads to the derivation of the subsequent outcomes.

Family 1.

$$B_0 = \frac{p_1B_1}{2p_2}, \qquad \alpha = 0, \qquad p_0 = \frac{2\delta p_1^2 + c}{8\delta p_2}, \qquad \beta = -\frac{12\delta p_2^2}{B_1^2}. \tag{3.9}$$

Family 2.

$$\begin{aligned} \alpha &= \frac{15cp_2}{2B_0p_2 - B_1p_1}, \qquad p_0 = -\frac{B_0(B_0p_2 - B_1p_1)}{B_1^2}, \\ \beta &= -\frac{3cp_2^2}{(2B_0p_2 - B_1p_1)^2}, \qquad \delta = \frac{cB_1^2}{(2B_0p_2 - B_1p_1)^2}. \end{aligned} \tag{3.10}$$

Use Family 1:

Case 1: If $\Delta > 0$, then

$$U_{1,1}(x,t) = B_1\left(-\frac{\sqrt{\Delta}\tanh\left(\frac{1}{2}\sqrt{\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + \frac{B_1p_1}{2p_2}, \tag{3.11}$$

$$U_{1,2}(x,t) = B_1\left(-\frac{\sqrt{\Delta}\coth\left(\frac{1}{2}\sqrt{\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + \frac{B_1p_1}{2p_2}. \tag{3.12}$$

Case 2: If $\Delta < 0$, then

$$U_{1,3}(x,t) = B_1\left(-\frac{\sqrt{-\Delta}\tan\left(\frac{1}{2}\sqrt{-\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + \frac{B_1p_1}{2p_2}, \tag{3.13}$$

$$U_{1,4}(x,t) = B_1\left(-\frac{\sqrt{-\Delta}\cot\left(\frac{1}{2}\sqrt{-\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + \frac{B_1p_1}{2p_2}. \tag{3.14}$$

Case 3: If $\Delta = 0$, then

$$U_{1,5}(x,t) = B_1\left(-\frac{1}{p_2(x-ct) + c_1} - \frac{p_1}{2p_2}\right) + \frac{B_1p_1}{2p_2}. \tag{3.15}$$

Use Family 2:

Case 1: If $\Delta > 0$, then

$$U_{2,1}(x,t) = B_1\left(-\frac{\sqrt{\Delta}\tanh\left(\frac{1}{2}\sqrt{\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + B_0, \tag{3.16}$$

$$U_{2,2}(x,t) = B_1\left(-\frac{\sqrt{\Delta}\coth\left(\frac{1}{2}\sqrt{\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + B_0. \tag{3.17}$$

Case 2: If $\Delta < 0$, then

$$U_{2,3}(x,t) = B_1\left(-\frac{\sqrt{-\Delta}\tan\left(\frac{1}{2}\sqrt{-\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + B_0, \tag{3.18}$$

$$U_{2,4}(x,t) = B_1\left(-\frac{\sqrt{-\Delta}\cot\left(\frac{1}{2}\sqrt{-\Delta}(x-ct)\right)}{2p_2} - \frac{p_1}{2p_2}\right) + B_0. \tag{3.19}$$

Case 3: If $\Delta = 0$, then

$$U_{2,5}(x,t) = B_1\left(-\frac{1}{p_2(x-ct)+c_1} - \frac{p_1}{2p_2}\right) + B_0. \tag{3.20}$$

3.2. Solution for Eq (2.26)

By applying the homogeneous balance principle to Eq (2.26), we get $n = 1$ and Eq (3.1) yields to the following:

$$U(\tau) = B_0 + B_1 G(\tau). \tag{3.21}$$

On substituting Eqs (3.2) and (3.21) into Eq (2.26) with $C_1 = 0$, collecting all terms having identical powers of $G^k(\tau)$ and setting each coefficient to zero, and solving the algebraic equations by Maple, we obtain the following result:

$$\delta = \frac{z}{24}, \quad p_0 = -\frac{2\beta a_0^2}{a_1 z^2\sqrt{-\frac{2\beta}{9z}}}, \quad p_1 = \frac{2\sqrt{-\frac{2\beta}{9z}}a_0}{z}, \quad p_2 = \frac{\sqrt{-\frac{2\beta}{9z}}a_1}{z}. \tag{3.22}$$

As $\Delta = p_1^2 - 4p_0p_1 = 0$ for arbitrary values of p_0, p_1, and p_2, we have

$$U(x,t) = -\frac{3B_1 z}{\sqrt{-\frac{2\beta}{z}}B_1(x-ct) + 3c_1 z}, \tag{3.23}$$

where $z = \frac{x^3}{t}$ and c_1 is constant of integration.

4. Conclusions

In this study, we have successfully obtained the ODE by utilizing all available Lie symmetry generators. Through the utilization of these symmetry generators, the proposed equation has been reduced to an ODE via symmetry reductions. The unified Riccati equation expansion method has been effectively employed to obtain the ODE, as well as new interaction solutions such as kink solitons and other solitons associated with hyperbolic functions. These solutions have been retrieved with great success. Additionally, in order to demonstrate the visual representation of several wave patterns with different system features and to validate the accuracy of our findings as shown in Figures 1–7. The outcomes of this endeavor will serve as a source of inspiration and motivation for future discussions in the realm of nonlinear physical sciences. Through the analysis of the computations, we are able to ascertain the significant value of this method in terms of its ability to locate precise wave solutions in a more comprehensive manner. In subsequent endeavors, we can expand upon the methods provided to incorporate various other nonlinear models. The resulting solutions present themselves as innovative, captivating, and potentially instrumental in enhancing our comprehension of energy transfer and diffusion processes within mathematical models of diverse fields that are pertinent, wherein nonlinear challenging matters are encountered.

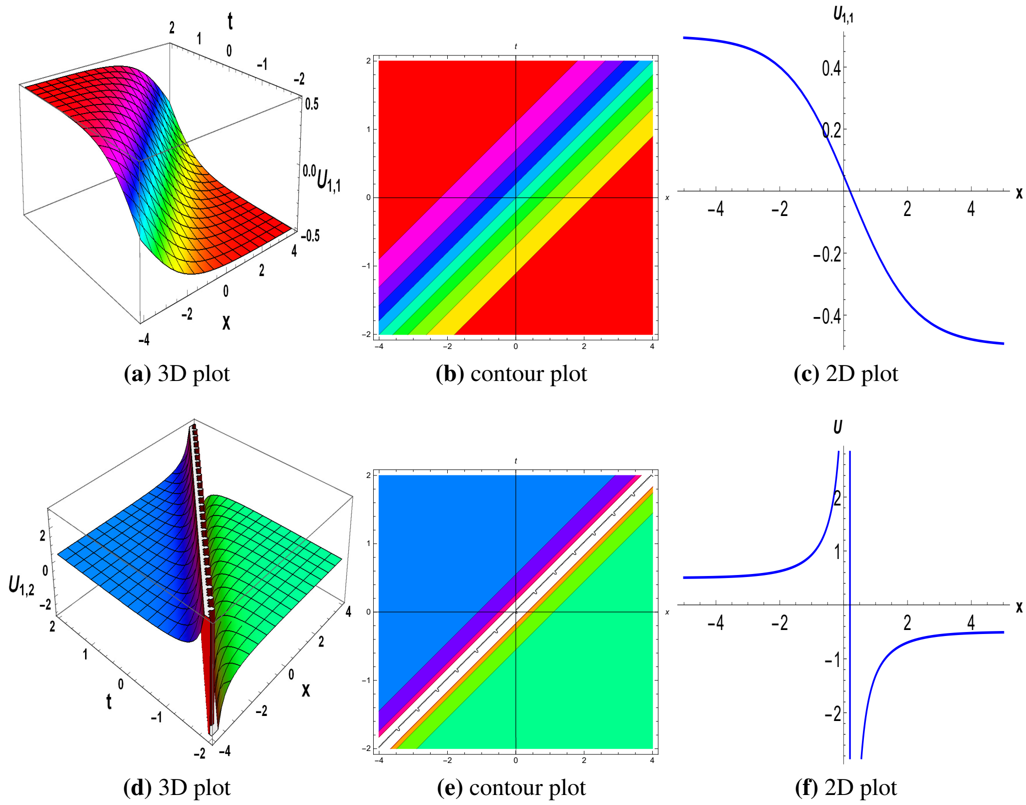
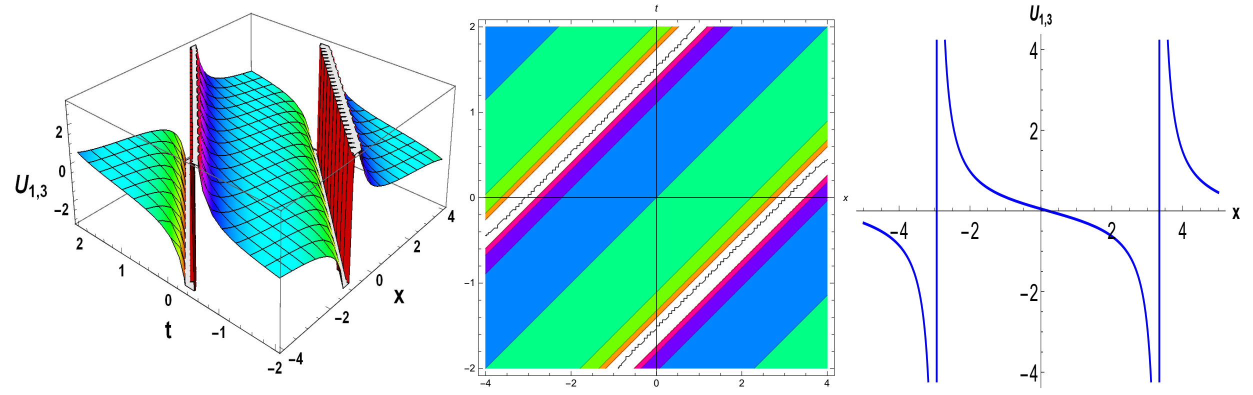
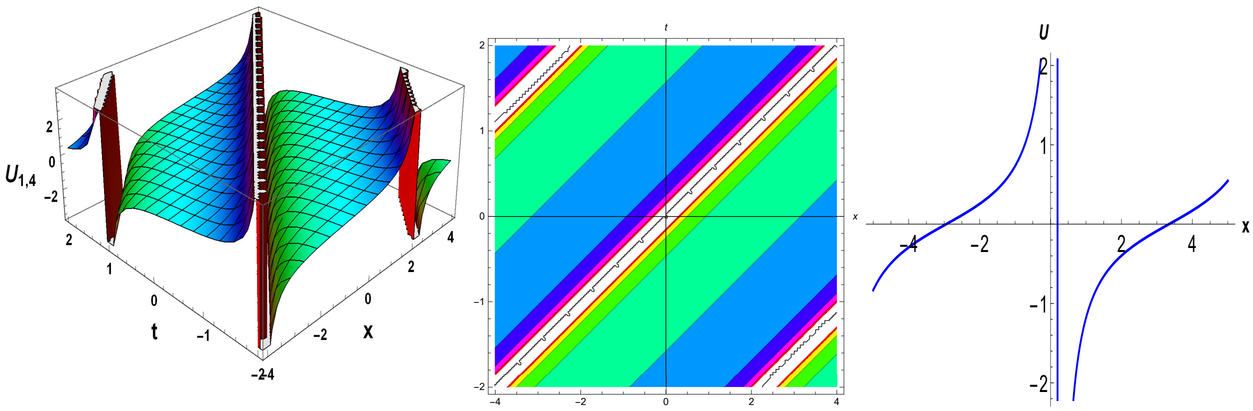


Figure 1. Graphical representation of Eqs (3.11) and (3.12) using suitable parameters $B_1 = 3, c = 2, \Delta = 1, p_1 = 2$ and $p_2 = 3$.

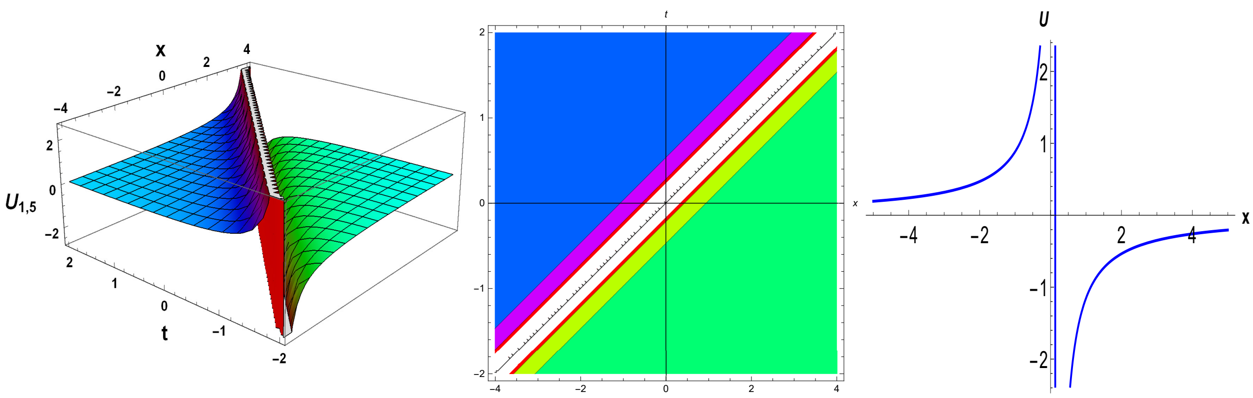


(a) 3D plot **(b)** contour plot **(c)** 2D plot



(d) 3D plot **(e)** contour plot **(f)** 2D plot

Figure 2. Graphical representation of Eqs (3.13) and (3.14) using suitable parameters $B_1 = 3, c = 2, \Delta = -1, p_1 = 2$ and $p_2 = 3$.



(a) 3D plot **(b)** contour plot **(c)** 2D plot

Figure 3. Graphical representation of Eq (3.15) using suitable parameters $B_1 = 3, c = 2, \Delta = 0, p_1 = 2$ and $p_2 = 3$.

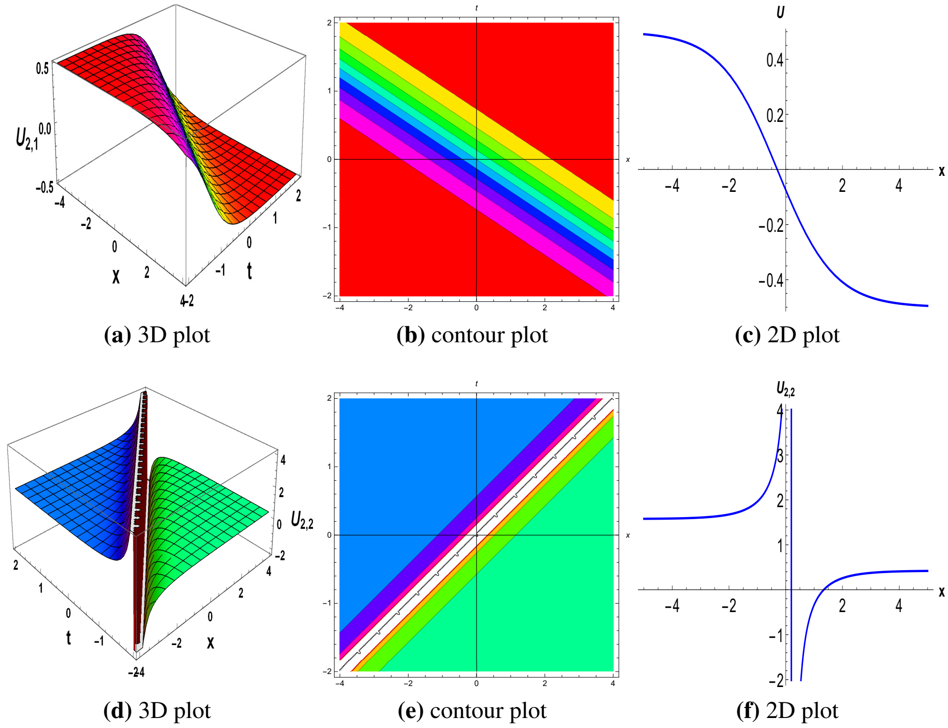


Figure 4. Graphical representation of Eqs (3.16) and (3.17) using suitable parameters $B_1 = 3, c = 2, \Delta = 1,\ p_1 = 2$ and $p_2 = 3$.

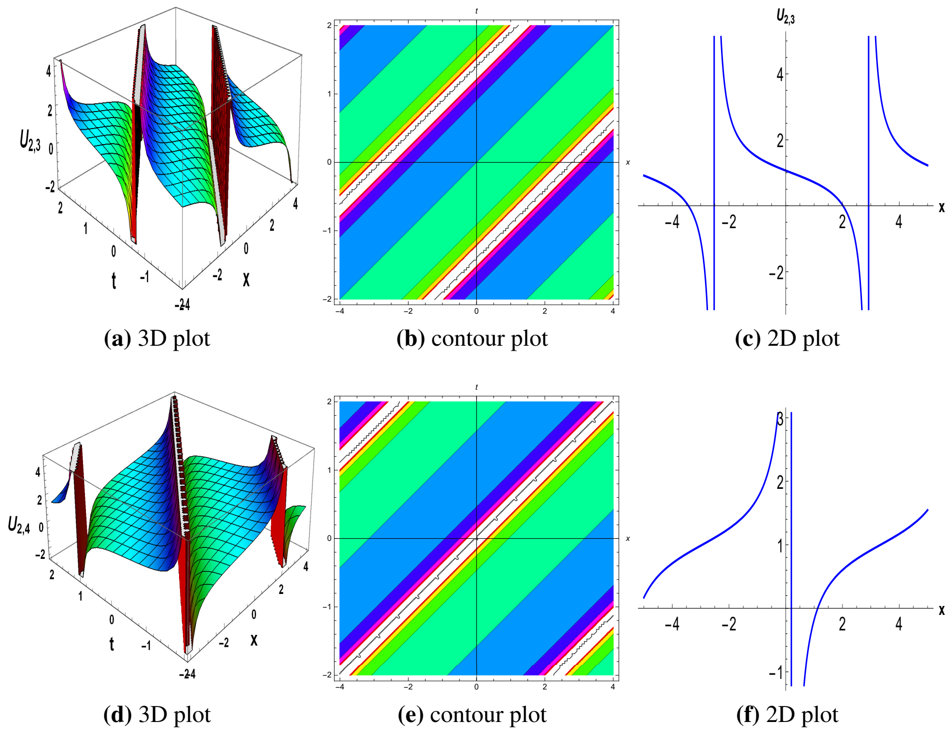
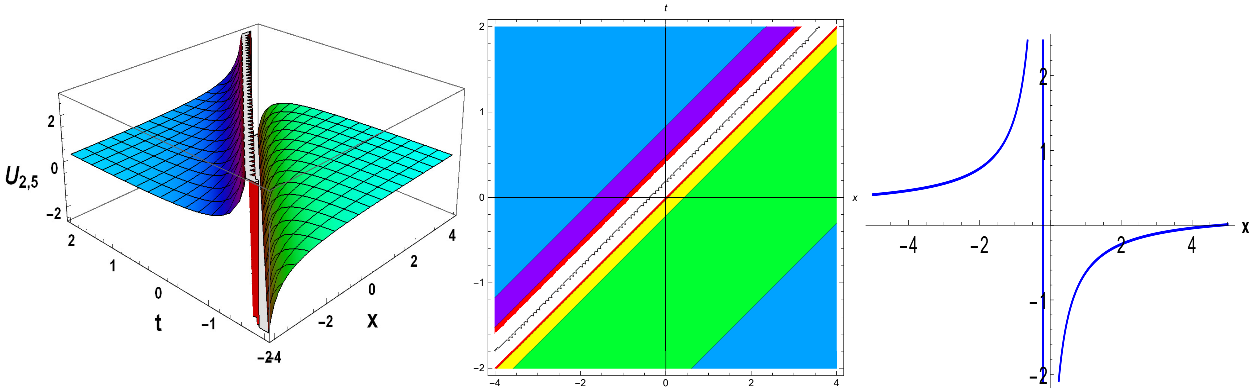
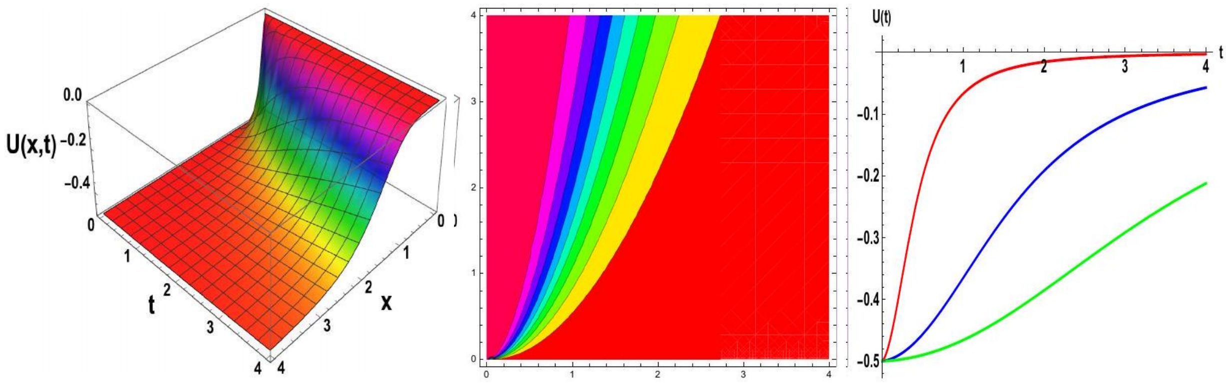


Figure 5. Graphical representation of Eqs (3.18) and (3.19) using suitable parameters $B_1 = 3, c = 2, \Delta = -1, p_1 = 2$ and $p_2 = 3$.



(a) 3D plot **(b)** contour plot **(c)** 2D plot

Figure 6. Graphical representation of Eq (3.20) using suitable parameters $B_1 = 3, c = 2, \Delta = 0, p_1 = 2$ and $p_2 = 3$.



(a) 3D plot **(b)** contour plot **(c)** 2D plot

Figure 7. Graphical representation of Eq (3.23) using suitable parameters $B_1 = 1, c = -1.2, \beta = -0.5$ and $c_1 = 2$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Prof. Maria Luz Gandarias is the Guest Editor of special issue "Lie Symmetry Analysis and Conservation Laws for Nonlinear Differential Equations and Applications" for AIMS Mathematics. Prof. Maria Luz Gandarias was not involved in the editorial review and the decision to publish this article.

All authors declare no conflicts of interest in this paper.

References

1. Z. Lin, J. Wang, W. Wei, Multipoint BVPs for generalized impulsive fractional differential equations, *Appl. Math. Comput.*, **258** (2015), 608–616. https://doi.org/10.1016/j.amc.2014.12.092
2. Z. Lin, W. Wei, J. Wang, Existence and stability results for impulsive integro-differential equations, *Facta Univ.*, **29** (2014), 119–130.
3. N. Raza, M. H. Rafiq, M. Kaplan, S. Kumar, Y. M. Chu, The unified method for abundant soliton solution of local time fractional nonlinear evolution equations, *Results Phys.*, **22** (2021), 103979. https://doi.org/10.1016/j.rinp.2021.103979
4. M. H. Rafiq, A. Jhangeer, N. Raza, The analysis of solitonic, supernonlinear, periodic, quasiperiodic, bifurcation and chaotic patterns of perturbed Gerdjikov-Ivanov model with full nonlinearity, *Commun. Nonlinear Sci. Numer. Simul.*, **116** (2023), 106818. https://doi.org/10.1016/j.cnsns.2022.106818
5. N. Raza, A. Javid, Optical dark and singular solitons to the Biswas-Milovic equation in nonlinear optics with spatio-temporal dispersion, *Optik*, **158** (2018), 1049–1057. https://doi.org/10.1016/j.ijleo.2017.12.186
6. S. Lou, Searching for higher dimensional integrable models from lower ones via Painleve analysis, *Phys. Rev. Lett.*, **80** (1998), 5027. https://doi.org/10.1103/PhysRevLett.80.5027
7. M. B. Hossen, H. O. Roshid, M. Z. Ali, Dynamical structures of exact soliton solutions to Burgers' equation via the bilinear approach, *Partial Differ. Equ. Appl. Math.*, **3** (2021), 100035. https://doi.org/10.1016/j.padiff.2021.100035
8. A. Zafar, M. Raheel, M. Mirzazadeh, M. Eslami, Different soliton solutions to the modified equal-width wave equation with Beta-time fractional derivative via two different methods, *Rev. Mex. Fis.*, **68** (2022), 1–14. https://doi.org/10.31349/revmexfis.68.010701
9. C. M. Khalique, I. Simbanefayi, Exact solutions and conservation laws for the modified equal width-Burgers equation, *Open Phys.*, **16** (2018), 795–800. https://doi.org/10.1515/phys-2018-0099
10. R. K. Bhowmik, M. F. Al-Asad, M. R. Karim, Soliton solution of Korteweg-de Vries equation, *Int. J. Appl. Math. Stat.*, **4** (2019), 45–48.
11. S. Malik, S. Kumar, A. Das, A (2+1)-dimensional combined KdV-mKdV equation: integrability, stability analysis and soliton solutions, *Nonlinear Dyn.*, **1** (2022), 2689–2701. https://doi.org/10.1007/s11071-021-07075-x
12. S. Naowarat, S. Saifullah, S. Ahmad, M. De la Sen, Periodic, singular and dark solitons of a generalized geophysical KdV equation by using the tanh-coth method, *Symmetry*, **15** (2023), 135. https://doi.org/10.3390/sym15010135
13. G. Wang, T. Xu, Symmetry properties and explicit solutions of the nonlinear time fractional KdV equation, *Bound. Value Probl.*, **2013** (2013), 232. https://doi.org/10.1186/1687-2770-2013-232
14. G. Wang, A. H. Kara, A (2+1)-dimensional KdV equation and mKdV equation: symmetries, group invariant solutions and conservation laws, *Phys. Lett. A*, **383** (2019), 728–731. https://doi.org/10.1016/j.physleta.2018.11.040

15. A. A. Elmandouha, A. G. Ibrahim, Bifurcation and travelling wave solutions for a (2+1)-dimensional KdV equation, *J. Taibah Univ. Sci.*, **14** (2020), 139–147. https://doi.org/10.1080/16583655.2019.1709271

16. A. M. Wazwaz, New Painlevé-integrable (2+1)-and (3+1)-dimensional KdV and mKdV equations, *Rom. J. Phys.*, **65** (2020), 108.

17. S. Malik, S. Kumar, A. Das, A (2+1)-dimensional combined KdV-mKdV equation: integrability, stability analysis and soliton solutions, *Nonlinear Dyn.*, **107** (2022), 2689–2701. https://doi.org/10.1007/s11071-021-07075-x

18. S. G. Tagare, A. Chakrabarti, Solution of a generalized Korteweg-de Vries equation, *Phys. Fluids*, **17** (1974), 1331–1332. https://doi.org/10.1063/1.1694886

19. G. C. Das, S. G. Tagare, J. Sarma, Quasipotential analysis for ion-acoustic solitary waves and double layers in plasmas, *Planet. Space Sci.*, **46** (1998), 417–424. https://doi.org/10.1016/S0032-0633(97)00142-6

20. S. Sain, A. Ghose-Choudhury, S. Garai, Solitary wave solutions for the KdV-type equations in plasma: a new approach with the Kudryashov function, *Eur. Phys. J. Plus*, **136** (2021), 226. https://doi.org/10.1140/epjp/s13360-021-01217-1

21. N. Raza, F. Salman, A. R. Butt, M. L. Gandarias, Lie symmetry analysis, soliton solutions and qualitative analysis concerning to the generalized q-deformed Sinh-Gordon equation, *Commun. Nonlinear Sci. Numer. Simul.*, **116** (2023), 106824. https://doi.org/10.1016/j.cnsns.2022.106824

22. S. F. Tian, M. J. Xu, T. T. Zhang, A symmetry-preserving difference scheme and analytical solutions of a generalized higher-order beam equation, *P. Roy. Soc. A*, **477** (2021), 20210455. https://doi.org/10.1098/rspa.2021.0455

23. P. J. Olver, *Applications of Lie groups to differential equations*, New York: Springer, 1993. https://doi.org/10.1007/978-1-4684-0274-2

24. G. W. Bluman, A. F. Cheviakov, S. C. Anco, *Applications of symmetry methods to partial differential equations*, New York: Springer, 2010. https://doi.org/10.1007/978-0-387-68028-6

25. S. C. Anco, Generalization of Noether's theorem in modern form to non-variational partial differential equations, In: R. Melnik, R. Makarov, J. Belair, *Recent progress and modern challenges in applied mathematics, modeling and computational science*, Fields Institute Communications, **79** (2017), 119–182. https://doi.org/10.1007/978-1-4939-6969-2_5

26. S. C. Anco, G. Bluman, Direct construction of conservation laws from field equations, *Phys. Rev. Lett.*, **78** (1997), 2869. https://doi.org/10.1103/PhysRevLett.78.2869

27. S. C. Anco, G. Bluman, Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications, *Eur. J. Appl. Math.*, **13** (2002), 545–566. https://doi.org/10.1017/S095679250100465X

28. S. C. Anco, G. Bluman, Direct construction method for conservation laws of partial differential equations Part II: General treatment, *Eur. J. Appl. Math.*, **13** (2002), 567–585. https://doi.org/10.1017/S0956792501004661

29. T. Wolf, A comparison of four approaches to the calculation of conservation laws, *Eur. J. Appl. Math.*, **13** (2002), 129–152. https://doi.org/10.1017/S0956792501004715

30. W. X. Ma, Conservation laws by symmetries and adjoint symmetries, *Discrete Cont. Dyn. Syst.-Ser. S*, **11** (2018), 707–721. https://doi.org/10.3934/dcdss.2018044
31. M. L. Gandarias, M. Rosa, E. Recio, S. Anco, Conservation laws and symmetries of a generalized Kawahara equation, *AIP Conf. Proc.*, **1836** (2017), 020072. https://doi.org/10.1063/1.4982012
32. G. Bluman, S. Anco, *Symmetry and integration methods for differential equations*, Springer Science & Business Media, 2008.
33. S. C. Anco, M. L. Gandarias, Symmetry multi-reduction method for partial differential equations with conservation laws, *Commun. Nonlinear Sci. Numer. Simul.*, **91** (2020), 105349. https://doi.org/10.1016/j.cnsns.2020.105349
34. N. A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, *Chaos Soliton. Fract.*, **24** (2005), 1217–1231. https://doi.org/10.1016/j.chaos.2004.09.109
35. A. Sjöberg, Double reduction of PDEs from the association of symmetries with conservation laws with applications, *Appl. Math. Comput.*, **184** (2007), 608–616. https://doi.org/10.1016/j.amc.2006.06.059
36. A. Sjöberg, On double reductions from symmetries and conservation laws, *Nonlinear Anal.: Real World Appl.*, **10** (2009), 3472–3477. https://doi.org/10.1016/j.nonrwa.2008.09.029
37. A. H. Bokhari, A. Y. Al-Dweik, F. D. Zaman, A. H. Kara, F. M. Mahomed, Generalization of the double reduction theory, *Nonlinear Anal.: Real World Appl.*, **11** (2010), 3763–3769. https://doi.org/10.1016/j.nonrwa.2010.02.006
38. Sirendaoreji, Unified Riccati equation expansion method and its application to two new classes of Benjamin-Bona-Mahony equations, *Nonlinear Dyn.*, **89** (2017), 333–344. https://doi.org/10.1007/s11071-017-3457-6

AIMS Press