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Research article

On Hermite-Hadamard type inequalities for co-ordinated convex function via conformable fractional integrals

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Abstract: In this study, some new Hermite-Hadamard type inequalities for co-ordinated convex functions were obtained with the help of conformable fractional integrals. We have presented some remarks to give the relation between our results and earlier obtained results. Moreover, an identity for partial differentiable functions has been established. By using this equality and concept of co-ordinated convexity, we have proven a trapezoid type inequality for conformable fractional integrals.

Keywords: convex functions; co-ordinated convex mapping; Hermite-Hadamard inequality; conformable fractional integral; trapezoid type inequalities
Mathematics Subject Classification: 26A33, 26A51, 26B25, 26D10, 26D15

1. Introduction

The Hermite-Hadamard inequality stands as a cornerstone in the realm of convex functions, boasting a geometric interpretation and having broad applicability. Countless mathematicians have dedicated their endeavors to extending, refining, and providing counterparts for this inequality across various classes of functions, often involving convex mappings. The inequalities originally formulated by C. Hermite and J. Hadamard for convex functions hold significant importance in the existing literature. (see, e.g., [1], [2, p. 137]). The Hermite-Hadamard inequality is stated as follows:

If $\mathcal{F} : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\eta_1, \eta_2 \in I$ with $\eta_1 < \eta_2$ then,

$$\mathcal{F}\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \mathcal{F}(\delta) d\delta \leq \frac{\mathcal{F}(\eta_1) + \mathcal{F}(\eta_2)}{2}. \tag{1.1}$$

The Hermite-Hadamard inequality has attracted the attention of many mathematicians since the day it was proved. Especially in recent years, many generalizations and extensions of this inequality have been created. The definition of convexity is used a lot when creating new versions of the Hermite-Hadamard inequality. To define convexity on coordinates let us first consider a bidimensional interval $\Delta := [\eta_1, \eta_2] \times [\upsilon_1, \upsilon_2]$ in $\mathbb{R}^2$.

Definition 1.1. *[3] A function $\mathcal{F} : \Delta \to \mathbb{R}$ is called a co-ordinated convex on Δ, if it satifies the inequality*

$$\begin{aligned} & \mathcal{F}(\mu t + (1-t)\tau, s\rho + (1-s)\sigma) \\ \leq \; & ts\mathcal{F}(\mu, \rho) + t(1-s)\mathcal{F}(\mu, \sigma) + s(1-t)\mathcal{F}(\tau, \rho) + (1-t)(1-s)\mathcal{F}(\tau, \sigma) \end{aligned} \tag{1.2}$$

for all $(\mu, \rho), (\tau, \sigma) \in \Delta$ and $t, s \in [0, 1]$.

In [3], Dragomir proved the Hermite-Hadamard inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$. For several results concerning the Hermite-Hadamard type inequality for co-ordinated convex functions. Some papers devoted Hermite-Hadamard inequalities for co-ordinated convex functions [4–6]. Alomari and Darus presented some inequalities for s-convex function on co-ordinates. [7]. Vivas et al. proved some Hermite-Hadamard inequalities for co-ordinated convex interval valued functions [8].

Fractional analysis is a current field of study with various uses in fields such as physics, engineering and biology. Fractional integral operators are also very important for mathematics because the generalization of many integral inequalities has been introduced to the literature thanks to the fractional integral operators For more information please refeer to the books [9–11]. Multiple fractional operators have been defined so far, i.e. Caputo, Riemann-Liouville, Hadamard, and Katugampola to name a few. The concept of conformable fractional integrals was given by Khalil et al. in 2014 [12]. The conformable fractional integral operator was used throughout this study. For all this, please see [13–18].

Definition 1.2. *[9] Let $\mathcal{F} \in L_1[\eta_1, \eta_2]$. The Riemann-Liouville fractional integrals $I^{\beta}_{\eta_1^+}\mathcal{F}$ and $I^{\beta}_{\eta_2^-}\mathcal{F}$ of order $\beta > 0$ are given by*

$$I^{\alpha}_{\eta_1^+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_{\eta_1}^{x} (x - \delta)^{\beta-1} \mathcal{F}(\delta) d\delta, \quad x > \eta_1, \tag{1.3}$$

$$I^{\alpha}_{\eta_2^-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\eta_2} (\delta - x)^{\beta-1} \mathcal{F}(\delta) d\delta, \quad x < \eta_2. \tag{1.4}$$

The Riemann-Liouville fractional integrals will be provided for order $\beta > 0$. The Riemann Liouville integrals will be equal to the classical Riemann integral for the condition $\beta = 1$.

Definition 1.3. *[9] Let $\mathcal{F} \in L_1(\Delta)$ .Riemann-Liouville fractional integrals $I^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+,}\mathcal{F}$, $I^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-,}\mathcal{F}$, $I^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}$ and $I^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}$ of orders $\alpha,\beta > 0$ with $\eta_1, \upsilon_1 \geq 0$ are defined by*

$$I^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+,}\mathcal{F}(\delta,\xi) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\upsilon_1}^{\xi}(\delta-t)^{\alpha-1}(\xi-s)^{\beta-1}\mathcal{F}(t,s)dsdt, \quad \delta > \eta_1, \xi > \upsilon_1,$$

$$I^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-,}\mathcal{F}(\delta,\xi) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\xi}^{\upsilon_2}(\delta-t)^{\alpha-1}(s-\xi)^{\beta-1}\mathcal{F}(t,s)dsdt, \quad \delta > \eta_1, \xi < \upsilon_2,$$

$$I^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\delta,\xi) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\delta}^{\eta_2}\int_{\upsilon_1}^{\xi}(t-\delta)^{\alpha-1}(\xi-s)^{\beta-1}\mathcal{F}(t,s)dsdt, \quad \delta < \eta_2, \xi > \upsilon_1,$$

and

$$I^{\alpha,\beta}_{\eta_2-,\upsilon_2^-}\mathcal{F}(\delta,\xi) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\delta}^{\eta_2}\int_{\xi}^{\upsilon_2}(t-\delta)^{\alpha-1}(s-\xi)^{\beta-1}\mathcal{F}(t,s)dsdt, \quad \delta < \eta_2, \xi < \upsilon_2,$$

respectively. Here, Γ is the gamma function.

Definition 1.4. *[19] For $\mathcal{F} \in L_1[\eta_1,\eta_2]$, the conformable fractional integral operators ${}^{\beta}I^{\alpha}_{\eta_1^+}\mathcal{F}$ and ${}^{\beta}I^{\alpha}_{\eta_2^-}\mathcal{F}$ of orders $\beta > 0$ and $\alpha \in (0,1]$ are given by*

$${}^{\beta}J^{\alpha}_{\eta_1^+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\eta_1}^{x}\left(\frac{(x-\eta_1)^{\alpha}-(t-\eta_1)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(t)}{(t-\eta_1)^{1-\alpha}}dt, \quad x > \eta_1, \tag{1.5}$$

and

$${}^{\beta}J^{\alpha}_{\eta_2^-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\eta_2}\left(\frac{(\eta_2-x)^{\alpha}-(\eta_2-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(t)}{(\eta_2-t)^{1-\alpha}}dt, \quad x < \eta_2, \tag{1.6}$$

respectively.

Remark 1.1. *If we consider that $\alpha = 1$ in Definition 1.4, then the fractional integrals (1.5) and (1.6) reduce to the Riemann-Liouville fractional integrals (1.3) and (1.4), respectively.*

Definition 1.5. *[20] Let $\mathcal{F} \in L_1([\eta_1,\eta_2]\times[\upsilon_1,\upsilon_2])$, $\gamma_1, \gamma_2 \in (0,1]$, $\alpha > 0$ and $\beta > 0$. The conformable fractional integrals of orders α, β of $\mathcal{F}(\delta,\xi)$ are defined by*

$$\begin{aligned}\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}\right)(\delta,\xi) &= \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\upsilon_1}^{\xi}\left(\frac{(\delta-\eta_1)^{\gamma_1}-(t-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\right.\\ &\quad\left.\times\left(\frac{(\xi-\upsilon_1)^{\gamma_2}-(s-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(t,s)}{(t-\eta_1)^{1-\gamma_1}(s-\upsilon_1)^{1-\gamma_2}}dsdt\right],\end{aligned} \tag{1.7}$$

$$\begin{aligned}\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2-,\upsilon_1^+}\mathcal{F}\right)(\delta,\xi) &= \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\delta}^{\eta_2}\int_{\upsilon_1}^{\xi}\left(\frac{(\eta_2-\delta)^{\gamma_1}-(\eta_2-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\right.\\ &\quad\left.\times\left(\frac{(\xi-\upsilon_1)^{\gamma_2}-(s-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(t,s)}{(\eta_2-t)^{1-\gamma_1}(s-\upsilon_1)^{1-\gamma_2}}dsdt\right],\end{aligned} \tag{1.8}$$

$$\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}\right)(\delta,\xi) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\xi}^{\upsilon_2}\left(\frac{(\delta-\eta_1)^{\gamma_1}-(t-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times\left(\frac{(\upsilon_2-\xi)^{\gamma_2}-(\upsilon_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(t,s)}{(t-\eta_1)^{1-\gamma_1}(\upsilon_2-s)^{1-\gamma_2}}dsdt\right], \tag{1.9}$$

and

$$\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}\right)(\delta,\xi) = \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\delta}^{\eta_2}\int_{\xi}^{\upsilon_2}\left(\frac{(\eta_2-\delta)^{\gamma_1}-(\eta_2-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times\left(\frac{(\upsilon_2-\xi)^{\gamma_2}-(\upsilon_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(t,s)}{(\eta_2-t)^{1-\gamma_1}(\upsilon_2-s)^{1-\gamma_2}}dsdt\right]. \tag{1.10}$$

Remark 1.2. *If we consider that $\gamma_1 = \gamma_2 = 1$ in Definition 1.5, then Definition 1.5 reduces to Definition 1.3.*

By Definition 1.5, we can write the following conformable fractional integrals:

Definition 1.6. *Let $\mathcal{F} \in L_1([\eta_1,\eta_2]\times[\upsilon_1,\upsilon_2])$, $\gamma_1,\ \gamma_2 \in (0,1]$, $\alpha > 0$ and $\beta > 0$. In this case, the following equations can be written:*

$$\left({}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}\right)\left(\delta,\frac{\upsilon_1+\upsilon_2}{2}\right) = \frac{1}{\Gamma(\alpha)}\int_{\eta_1}^{\delta}\left(\frac{(\delta-\eta_1)^{\gamma_1}-(t-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(t,\frac{\upsilon_1+\upsilon_2}{2})}{(t-\eta_1)^{1-\gamma_1}}dt,\ \delta>\eta_1, \tag{1.11}$$

$$\left({}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}\right)\left(\delta,\frac{\upsilon_1+\upsilon_2}{2}\right) = \frac{1}{\Gamma(\alpha)}\int_{\delta}^{\eta_2}\left(\frac{(\eta_2-\delta)^{\gamma_1}-(\eta_2-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(t,\frac{\upsilon_1+\upsilon_2}{2})}{(\eta_2-t)^{1-\gamma_1}}dt,\ \delta<\eta_2, \tag{1.12}$$

$$\left({}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}\right)\left(\frac{\eta_1+\eta_2}{2},\xi\right) = \frac{1}{\Gamma(\beta)}\int_{\upsilon_1}^{\xi}\left(\frac{(\xi-\upsilon_1)^{\gamma_2}-(s-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\frac{\eta_1+\eta_2}{2},s)}{(s-\upsilon_1)^{1-\gamma_2}}ds,\ \xi>\upsilon_1, \tag{1.13}$$

and

$$\left({}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}\right)\left(\frac{\eta_1+\eta_2}{2},\xi\right) = \frac{1}{\Gamma(\beta)}\int_{\xi}^{\upsilon_2}\left(\frac{(\upsilon_2-\xi)^{\gamma_2}-(\upsilon_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\frac{\eta_1+\eta_2}{2},s)}{(\upsilon_2-s)^{1-\gamma_2}}ds,\ \xi<\upsilon_2. \tag{1.14}$$

Theorem 1.1. *[21] Assume that $\mathcal{F} : [\eta_1,\eta_2] \to \mathbb{R}$ is a convex function. Then, for $\beta > 0$ and $\alpha \in (0,1]$, the following inequalities for the fractional conformable integrals hold:*

$$\mathcal{F}\left(\frac{\eta_1+\eta_2}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(\eta_2-\eta_1)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2) + {}^{\beta}J^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1)\right] \le \frac{\mathcal{F}(\eta_1)+\mathcal{F}(\eta_2)}{2}. \tag{1.15}$$

For some results connected with fractional integral inequalities, see [22–25].

The purpose of this article is to establish the Hermite-Hadamard-type inequality for co-ordinated convex mappings by using the conformable fractional integral operators.

2. New versions of Hermite-Hadamard inequalities

In this part, we obtain new versions of the Hermite-Hadamard inequality for co-ordinated convex functions involving conformable fractional integrals.

Theorem 2.1. *Let $\mathcal{F} : \Delta \to \mathbb{R}$ be co-ordinated convex on Δ and $\mathcal{F} \in L_1(\Delta)$. Then, we have the following Hermite-Hadamard inequality for conformable fractional integrals*

$$\begin{aligned} & \mathcal{F}\left(\frac{\eta_1+\eta_2}{2}, \frac{\upsilon_1+\upsilon_2}{2}\right) \qquad (2.1) \\ \leq\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2) + {}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\right. \\ & \left. + {}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^+,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2) + {}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)\right] \\ \leq\ & \frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}. \end{aligned}$$

Proof. For $t,\ s \in [0,1]$, we can write

$$\begin{aligned} & \mathcal{F}\left(\frac{\eta_1+\eta_2}{2}, \frac{\upsilon_1+\upsilon_2}{2}\right) \\ =\ & \mathcal{F}\left(\frac{1}{4}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2) + \frac{1}{4}(t\eta_1+(1-t)\eta_2+(1-s)\upsilon_1+s\upsilon_2)\right. \\ & \left. +\frac{1}{4}(1-t)\eta_1+t\eta_2, s\upsilon_1+(1-s)\upsilon_2) + \frac{1}{4}(1-t)\eta_1+t\eta_2+(1-s)\upsilon_1+s\upsilon_2\right). \end{aligned}$$

With the help of the co-ordinated convexity of $\mathcal{F}$, we have

$$\begin{aligned} & \mathcal{F}\left(\frac{\eta_1+\eta_2}{2}, \frac{\upsilon_1+\upsilon_2}{2}\right) \qquad (2.2) \\ \leq\ & \frac{1}{4}(\mathcal{F}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2) + \mathcal{F}(t\eta_1+(1-t)\eta_2+(1-s)\upsilon_1+s\upsilon_2) \\ & +\mathcal{F}(1-t)\eta_1+t\eta_2, s\upsilon_1+(1-s)\upsilon_2) + \mathcal{F}(1-t)\eta_1+t\eta_2+(1-s)\upsilon_1+s\upsilon_2) \\ \leq\ & \frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}. \end{aligned}$$

If we multiply the inequality (2.2) by $\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}$ and integrate the resulting inequality on $[0,1]\times[0,1]$, we have

$$\begin{aligned} & \mathcal{F}\left(\frac{\eta_1+\eta_2}{2}, \frac{\upsilon_1+\upsilon_2}{2}\right) \qquad (2.3) \\ & \times\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}dsdt \\ \leq\ & \frac{1}{4}\left[\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\right. \end{aligned}$$

$$\times\left(\mathcal{F}\left(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2\right)+\mathcal{F}\left(t\eta_1+(1-t)\eta_2,(1-s)\upsilon_1+s\upsilon_2\right)\right)dsdt$$
$$+\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}$$
$$\times\left(\mathcal{F}\left((1-t)\eta_1+t\eta_2, s\upsilon_1+(1-s)\upsilon_2\right)+\mathcal{F}\left((1-t)\eta_1+t\eta_2,(1-s)\upsilon_1+s\upsilon_2\right)\right)dsdt]$$
$$\leq \frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}$$
$$\times\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}dsdt.$$

By applying the change of variables technique, we get

$$\begin{aligned}
&\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1} \qquad (2.4)\\
&\times\mathcal{F}\left(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2\right)dsdt\\
=\ &\frac{1}{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{1-\left(\frac{\delta-\eta_1}{\eta_2-\eta_1}\right)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{\delta-\eta_1}{\eta_2-\eta_1}\right)^{\gamma_1-1}\\
&\times\left(\frac{1-\left(\frac{\xi-\upsilon_1}{\upsilon_2-\upsilon_1}\right)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\left(\frac{\xi-\upsilon_1}{\upsilon_2-\upsilon_1}\right)^{\gamma_2-1}d\xi d\delta\\
=\ &\left(\frac{1}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\left(\frac{1}{\upsilon_2-\upsilon_1}\right)^{\gamma_2\beta}\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\\
&\times\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{d\xi}{(\xi-\upsilon_1)^{1-\gamma_2}}\frac{d\delta}{(\delta-\eta_1)^{1-\gamma_1}}\\
=\ &\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}\right)(\eta_2,\upsilon_2).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1} \qquad (2.5)\\
&\times\mathcal{F}\left(t\eta_1+(1-t)\eta_2,(1-s)\upsilon_1+s\upsilon_2\right)dsdt\\
=\ &\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}\right)(\eta_2,\upsilon_1),
\end{aligned}$$

$$\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1} \qquad (2.6)$$

$$
\begin{aligned}
&\times\mathcal{F}((1-t)\eta_1+t\eta_2, s\upsilon_1+(1-s)\upsilon_2)\,dsdt\\
=\;&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}\right)(\eta_1,\upsilon_2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1} \qquad (2.7)\\
&\times\mathcal{F}((1-t)\eta_1+t\eta_2,(1-s)\upsilon_1+s\upsilon_2)\,dsdt\\
=\;&\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-\upsilon_2^-}\mathcal{F}\right)(\eta_1,\upsilon_1).
\end{aligned}
$$

On the other side, we have

$$
\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}dsdt=\frac{1}{\gamma_1^\alpha\gamma_2^\beta\alpha\beta}. \qquad (2.8)
$$

If we substitute the Eqs (2.4)–(2.8) in (2.3), then we get

$$
\begin{aligned}
&\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right)\frac{1}{\gamma_1^\alpha\gamma_2^\beta\alpha\beta} \qquad (2.9)\\
\le\;&\frac{1}{4}\left[\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}\right)(\eta_2,\upsilon_2)\right.\\
&+\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}\right)(\eta_2,\upsilon_1)\\
&+\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}\right)(\eta_1,\upsilon_2)\\
&\left.+\frac{\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}\right)(\eta_1,\upsilon_1)\right]\\
\le\;&\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}\frac{1}{\gamma_1^\alpha\gamma_2^\beta\alpha\beta},
\end{aligned}
$$

which concludes the proof. □

Remark 2.1. *In Theorem 2.1, if we choose $\gamma_1 = 1$ and $\gamma_2 = 1$, then we have the following inequalities for Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right) \qquad (2.10)\\
\le\;&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_2-\eta_1)^\alpha(\upsilon_2-\upsilon_1)^\beta}\left[I^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+I^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)\right.\\
&\left.+I^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+I^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\right]\\
\le\;&\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4},
\end{aligned}
$$

which is proved by Sarikaya in [23, Theorem 3].

Remark 2.2. *In Theorem 2.1, if we choose $\gamma_1 = 1,\ \gamma_2 = 1,\ \alpha = 1$ and $\beta = 1$, then we have the following inequalities*

$$\begin{aligned} &\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right) \\ \leq\ & \frac{1}{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\mathcal{F}(t,s)dsdt \\ \leq\ & \frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}, \end{aligned} \tag{2.11}$$

which is given by Alomari and Darus [26, Theorem 1.1].

Theorem 2.2. *Let $\mathcal{F} : \Delta \to \mathbb{R}$ be co-ordinated convex on Δ and $\mathcal{F} \in L_1(\Delta)$. Then the following Hermite-Hadamard inequality for conformable fractional integrals holds:*

$$\begin{aligned} &\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right) \\ \leq\ & \frac{\Gamma(\alpha+1)\gamma_1^{\alpha}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}\left(\eta_2,\frac{\upsilon_1+\upsilon_2}{2}\right)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}\left(\eta_1,\frac{\upsilon_1+\upsilon_2}{2}\right)\right] \\ &+\frac{\Gamma(\beta+1)\gamma_2^{\beta}}{4(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_2\right)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_1\right)\right] \\ \leq\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\Big[{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1) \\ &+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\Big] \\ \leq\ & \frac{\Gamma(\alpha+1)\gamma_1^{\alpha}}{8(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_1)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_2)\right] \\ &+\frac{\Gamma(\beta+1)\gamma_2^{\beta}}{8(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)\right] \\ \leq\ & \frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4} \end{aligned} \tag{2.12}$$

for $\gamma_1,\gamma_2 \in (0,1]$, $\alpha > 0, \beta > 0$.

Proof. Since $\mathcal{F} : \Delta \to \mathbb{R}$ is a co-ordinated convex function, then the function $h_\delta : [\upsilon_1,\upsilon_2] \to \mathbb{R}$, $h_\delta(\xi) = \mathcal{F}(\delta,\xi)$ is convex on $[\upsilon_1,\upsilon_2]$ for all $\delta \in [\eta_1,\eta_2]$. Then, by applying (1.15), we can write

$$h_\delta\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{\Gamma(\beta+1)\gamma_2^{\beta}}{2(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[{}^{\beta}J^{\gamma_2}_{\upsilon_1^+}h_\delta(\upsilon_2)+{}^{\beta}J^{\gamma_2}_{\upsilon_2^-}h_\delta(\upsilon_1)\right] \leq \frac{h_\delta(\upsilon_2)+h_\delta(\upsilon_1)}{2},\ \delta \in [\eta_1,\eta_2]. \tag{2.13}$$

That is,

$$\begin{aligned} \mathcal{F}\left(\delta,\frac{\upsilon_1+\upsilon_2}{2}\right) \leq\ & \frac{\beta\gamma_2^{\beta}}{2(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\xi-\upsilon_1)^{1-\gamma_2}}d\xi\right. \\ &\left.+\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\upsilon_2-\xi)^{1-\gamma_2}}d\xi\right] \leq \frac{\mathcal{F}(\delta,\upsilon_1)+\mathcal{F}(\delta,\upsilon_2)}{2} \end{aligned} \tag{2.14}$$

for all $\delta \in [\eta_1, \eta_2]$. Then multiplying both sides of (2.14) by

$$\frac{\alpha\gamma_1^{\alpha}}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{1}{(\delta-\eta_1)^{1-\gamma_1}}$$

and integrating with respect to δ over $[\eta_1, \eta_2]$, we have

$$\begin{aligned}
&\frac{\alpha\gamma_1^{\alpha}}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}\left(\delta,\frac{\upsilon_1+\upsilon_2}{2}\right)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta \qquad (2.15)\\
\leq\;& \frac{\alpha\beta\gamma_1^{\alpha}\gamma_2^{\beta}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\\
&\times\left[\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\delta-\eta_1)^{1-\gamma_1}(\xi-\upsilon_1)^{1-\gamma_2}}d\xi d\delta\right.\\
&\left.+\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\delta-\eta_1)^{1-\gamma_1}(\upsilon_2-\xi)^{1-\gamma_2}}d\xi d\delta\right]\\
\leq\;& \frac{\alpha\gamma_1^{\alpha}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_1)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right.\\
&\left.+\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_2)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right].
\end{aligned}$$

Similarly, let us multiply both sides of (2.14) by

$$\frac{\alpha\gamma_1^{\alpha}}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{1}{(\eta_2-\delta)^{1-\gamma_1}}$$

and integrate with respect to δ on the interval $[\eta_1, \eta_2]$; then, we have

$$\begin{aligned}
&\frac{\alpha\gamma_1^{\alpha}}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}\left(\delta,\frac{\upsilon_1+\upsilon_2}{2}\right)}{(\eta_2-\eta_1)^{1-\gamma_1}}d\delta \qquad (2.16)\\
\leq\;& \frac{\alpha\beta\gamma_1^{\alpha}\gamma_2^{\beta}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\\
&\times\left[\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\eta_2-\delta)^{1-\gamma_1}(\xi-\upsilon_1)^{1-\gamma_2}}d\xi d\delta\right.\\
&\left.+\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\eta_2-\delta)^{1-\gamma_1}(\upsilon_2-\xi)^{1-\gamma_2}}d\xi d\delta\right]
\end{aligned}$$

$$\leq \frac{\alpha\gamma_1^\alpha}{4(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_1)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right.$$
$$\left.+\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_2)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right].$$

In the same way, since $\mathcal{F}:\Delta\to\mathbb{R}$ is a co-ordinated convex function, the function $g_\xi:[\eta_1,\eta_1]\to\mathbb{R}$, $g_\xi(\delta)=\mathcal{F}(\delta,\xi)$ is convex on $[\eta_1,\eta_1]$ for all $\xi\in[\upsilon_{1,}\upsilon_2]$. Then, by applying (1.15), we can write,

$$g_\xi\left(\frac{\eta_1+\eta_2}{2}\right)\leq\frac{\Gamma(\alpha+1)\gamma_1^\alpha}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\alpha}J_{\eta_1^+}^{\gamma_1}g_\xi(\eta_2)+{}^{\alpha}J_{\eta_2^-}^{\gamma_1}g_\xi(\eta_1)\right]\leq\frac{g_\xi(\eta_1)+g_\xi(\eta_2)}{2}. \tag{2.17}$$

That is,

$$\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\xi\right)\leq\frac{\alpha\gamma_1^\alpha}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\xi)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta\right. \tag{2.18}$$
$$\left.+\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\xi)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right]\leq\frac{\mathcal{F}(\eta_1,\xi)+\mathcal{F}(\eta_2,\xi)}{2}$$

for all $\xi\in[\upsilon_{1,}\upsilon_2]$. Then multiplying both sides of (2.18) by

$$\frac{\beta\gamma_2^\beta}{2(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{1}{(\xi-\upsilon_1)^{1-\gamma_2}}$$

and integrating with respect to ξ over $[\upsilon_{1,}\upsilon_2]$, we have,

$$\frac{\beta\gamma_2^\beta}{2(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\xi\right)}{(\xi-\upsilon_1)^{1-\gamma_2}}d\xi \tag{2.19}$$
$$\leq\frac{\alpha\beta\gamma_1^\alpha\gamma_2^\beta}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}$$
$$\times\left[\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\delta-\eta_1)^{1-\gamma_1}(\xi-\upsilon_1)^{1-\gamma_2}}d\xi d\delta\right.$$
$$\left.+\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\eta_2-\delta)^{1-\gamma_1}(\xi-\upsilon_1)^{1-\gamma_2}}d\xi d\delta\right]$$
$$\leq\frac{\beta\gamma_2^\beta}{4(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\eta_1,\xi)}{(\xi-\upsilon_1)^{1-\gamma_2}}d\xi\right.$$
$$\left.+\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\eta_2,\xi)}{(\xi-\upsilon_1)^{1-\gamma_2}}d\xi\right].$$

Similarly, let us multiply both sides of (2.18) by

$$\frac{\beta\gamma_2^{\beta}}{2\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{1}{(\upsilon_2-\xi)^{1-\gamma_2}}$$

and integrate with respect to ξ in the interval $[\upsilon_{1,}\upsilon_2]$; then, we have

$$\begin{aligned}
&\frac{\beta\gamma_2^{\beta}}{2\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\xi\right)}{(\upsilon_2-\xi)^{1-\gamma_2}}d\xi \qquad (2.20)\\
\leq\;& \frac{\alpha\beta\gamma_1^{\alpha}\gamma_2^{\beta}}{4\left(\upsilon_2-\upsilon_1\right)^{\gamma_1\alpha}\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\\
&\times\left[\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\delta-\eta_1)^{1-\gamma_1}(\upsilon_2-\xi)^{1-\gamma_2}}d\xi d\delta\right.\\
&\left.+\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\delta,\xi)}{(\eta_2-\delta)^{1-\gamma_1}(\upsilon_2-\xi)^{1-\gamma_2}}d\xi d\delta\right]\\
\leq\;& \frac{\beta\gamma_2^{\beta}}{4\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\left[\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}\left(\eta_1,\xi\right)}{(\upsilon_2-\xi)^{1-\gamma_2}}d\xi\right.\\
&\left.+\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}\left(\eta_2,\xi\right)}{(\upsilon_2-\xi)^{1-\gamma_2}}d\xi\right].
\end{aligned}$$

If we add (2.15), (2.16), (2.19) and (2.20) and divide by 2, we get,

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\gamma_1^{\alpha}}{4\left(\eta_2-\eta_1\right)^{\gamma_1\alpha}}\left[{}^{\gamma_1}J_{\eta_1^+}^{\alpha}\mathcal{F}\left(\eta_2,\frac{\upsilon_1+\upsilon_2}{2}\right)+{}^{\gamma_1}J_{\eta_2^-}^{\alpha}\mathcal{F}\left(\eta_1,\frac{\upsilon_1+\upsilon_2}{2}\right)\right] \qquad (2.21)\\
&+\frac{\Gamma(\beta+1)\gamma_2^{\beta}}{2\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\left[{}^{\gamma_2}J_{\upsilon_1^+}^{\beta}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_2\right)+{}^{\gamma_2}J_{\upsilon_2^-}^{\beta}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_1\right)\right]\\
\leq\;& \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{4\left(\eta_2-\eta_1\right)^{\gamma_1\alpha}\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\left[{}^{\gamma_1\gamma_2}J_{\eta_1^+,\upsilon_1^+}^{\alpha,\beta}\mathcal{F}\left(\eta_2,\upsilon_2\right)+{}^{\gamma_1\gamma_2}J_{\eta_1^+,\upsilon_2^-}^{\alpha,\beta}\mathcal{F}\left(\eta_2,\upsilon_1\right)\right.\\
&\left.+{}^{\gamma_1\gamma_2}J_{\eta_2^-,\upsilon_1{}^+}^{\alpha,\beta}\mathcal{F}\left(\eta_1,\upsilon_2\right)+{}^{\gamma_1\gamma_2}J_{\eta_2^-,\upsilon_2^-}^{\alpha,\beta}\mathcal{F}\left(\eta_1,\upsilon_1\right)\right]\\
\leq\;& \frac{\Gamma(\alpha+1)\gamma_1^{\alpha}}{8\left(\eta_2-\eta_1\right)^{\gamma_1\alpha}}\left[{}^{\gamma_1}J_{\eta_1^+}^{\alpha}\mathcal{F}\left(\eta_2,\upsilon_1\right)+{}^{\gamma_1}J_{\eta_1^+}^{\alpha}\mathcal{F}\left(\eta_2,\upsilon_2\right)+{}^{\gamma_1}J_{\eta_2^-}^{\alpha}\mathcal{F}\left(\eta_1,\upsilon_1\right)+{}^{\gamma_1}J_{\eta_2^-}^{\alpha}\mathcal{F}\left(\eta_1,\upsilon_2\right)\right]\\
&+\frac{\Gamma(\beta+1)\gamma_2^{\beta}}{8\left(\eta_2-\eta_1\right)^{\gamma_2\beta}}\left[{}^{\gamma_2}J_{\upsilon_1^+}^{\beta}\mathcal{F}\left(\eta_1,\upsilon_2\right)+{}^{\gamma_2}J_{\upsilon_1^+}^{\beta}\mathcal{F}\left(\eta_2,d\right)+{}^{\gamma_2}J_{\upsilon_2^-}^{\beta}\mathcal{F}\left(\eta_1,\upsilon_1\right)+{}^{\gamma_2}J_{\upsilon_2^-}^{\beta}\mathcal{F}\left(\eta_2,\upsilon_1\right)\right],
\end{aligned}$$

which give the second and the third inequalities in (2.12).

Now, let us write $\delta=\frac{\eta_1+\eta_2}{2}$ on the left side of the inequality (2.14); we have

$$\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right) \qquad (2.22)$$

$$\leq \frac{\beta\gamma_2^{\beta}}{2(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[\int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\frac{\eta_1+\eta_2}{2},\xi)}{(\xi-\upsilon_1)^{1-\gamma_2}}d\xi + \int_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\frac{\eta_1+\eta_2}{2},\xi)}{(\upsilon_2-\xi)^{1-\gamma_2}}d\xi\right],$$

and then, incorporating $\xi=\frac{\upsilon_1+\upsilon_2}{2}$ on the left side of the inequality (2.18), we have

$$\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{\alpha\gamma_1^{\alpha}}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}\left(\delta,\frac{\upsilon_1+\upsilon_2}{2}\right)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta + \int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}\left(\delta,\frac{\upsilon_1+\upsilon_2}{2}\right)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right]. \tag{2.23}$$

If we add the inequalities (2.22) and (2.23) and divide by 2, then we get the following inequality

$$\begin{aligned}&\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right)\\ \leq\ & \frac{\Gamma(\alpha+1)\gamma_1^{\alpha}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}\left(\eta_2,\frac{\upsilon_1+\upsilon_2}{2}\right)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}\left(\eta_1,\frac{\upsilon_1+\upsilon_2}{2}\right)\right.\\ &+\left.\frac{\Gamma(\beta+1)\gamma_2^{\beta}}{4(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[{}^{\gamma_1}J^{\alpha}_{\upsilon_1^+}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_2\right)+{}^{\gamma_1}J^{\alpha}_{\upsilon_2^-}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_1\right)\right]\right].\end{aligned} \tag{2.24}$$

The inequality in (2.24) is the first inequality of (2.12).

Finally, assuming that $\xi=\upsilon_1$ and $\xi=\upsilon_2$ on the right-hand side of (2.18), we have

$$\frac{\alpha\gamma_1^{\alpha}}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_1)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta + \int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_1)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right] \leq \frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_1)}{2} \tag{2.25}$$

and

$$\frac{\alpha\gamma_1^{\alpha}}{2(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[\int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\delta-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_2)}{(\delta-\eta_1)^{1-\gamma_1}}d\delta + \int_{\eta_1}^{\eta_2}\left(\frac{(\eta_2-\eta_1)^{\gamma_1}-(\eta_2-\delta)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\frac{\mathcal{F}(\delta,\upsilon_2)}{(\eta_2-\delta)^{1-\gamma_1}}d\delta\right] \leq \frac{\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_2)}{2}, \tag{2.26}$$

respectively. Likewise, assuming that $\delta=\eta_1$ and $\delta=\eta_2$ on the right-hand side of (2.14), we have

$$\frac{\beta\gamma_2^{\beta}}{2\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\left[\int\limits_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\eta_1,\xi)}{(\xi-\upsilon_1)^{1-\gamma_2}}d\xi\right.$$
$$\left.+\int\limits_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\eta_1,\xi)}{(\upsilon_2-\xi)^{1-\gamma_2}}d\xi\right]\le\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)}{2} \tag{2.27}$$

and

$$\frac{\beta\gamma_2^{\beta}}{2\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\left[\int\limits_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\xi-\upsilon_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\eta_2,\xi)}{(\xi-\upsilon_1)^{1-\gamma_2}}d\xi\right.$$
$$\left.+\int\limits_{\upsilon_1}^{\upsilon_2}\left(\frac{(\upsilon_2-\upsilon_1)^{\gamma_2}-(\upsilon_2-\xi)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\mathcal{F}(\eta_2,\xi)}{(\upsilon_2-\xi)^{1-\gamma_2}}d\xi\right]\le\frac{\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{2}, \tag{2.28}$$

respectively. If we add the inequalites (2.25)–(2.28) and divide by 4, then we get the following inequality

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\gamma_1^{\alpha}}{8\left(\eta_2-\eta_1\right)^{\gamma_1\alpha}}\left[{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}\left(\eta_2,\upsilon_1\right)+{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}\left(\eta_2,\upsilon_2\right)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}\left(\eta_1,\upsilon_1\right)+{}^{\gamma_1}J^{\alpha}_{\eta_2-}\mathcal{F}\left(\eta_1,\upsilon_2\right)\right]\\
&+\frac{2^{\gamma_1\beta-3}\Gamma(\beta+1)\gamma_2^{\beta}}{8\left(\upsilon_2-\upsilon_1\right)^{\gamma_2\beta}}\left[{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}\left(\eta_1,\upsilon_2\right)+{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}\left(\eta_2,\upsilon_2\right)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}\left(\eta_1,\upsilon_1\right)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}\left(\eta_2,\upsilon_1\right)\right]\\
\le\;&\frac{\mathcal{F}\left(\eta_1,\upsilon_1\right)+\mathcal{F}\left(\eta_1,\upsilon_2\right)+\mathcal{F}\left(\eta_2,\upsilon_1\right)+\mathcal{F}\left(\eta_2,\upsilon_2\right)}{4},
\end{aligned}$$

which gives the last inequality in (2.12). This completes the proof. □

Remark 2.3. *In Theorem 2.2, if we choose $\gamma_1=1$ and $\gamma_2=1$, then we have the following Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right)\\
\le\;&\frac{\Gamma(\alpha+1)}{4\left(\eta_2-\eta_1\right)^{\alpha}}\left[I^{\alpha}_{\eta_1^+}\mathcal{F}\left(\eta_2,\frac{\upsilon_1+\upsilon_2}{2}\right)+I^{\alpha}_{\eta_2^-}\mathcal{F}\left(\eta_1,\frac{\upsilon_1+\upsilon_2}{2}\right)\right]\\
&+\frac{\Gamma(\beta+1)}{4\left(\upsilon_2-\upsilon_1\right)^{\beta}}\left[I^{\beta}_{\upsilon_1^+}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_2\right)+I^{\beta}_{\upsilon_2^-}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\upsilon_1\right)\right]\\
\le\;&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left(\eta_2-\eta_1\right)^{\alpha}\left(\upsilon_2-\upsilon_1\right)^{\beta}}\left[I^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}\left(\eta_2,\upsilon_2\right)+I^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}\left(\eta_2,\upsilon_1\right)\right.\\
&\left.+I^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}\left(\eta_1,\upsilon_2\right)+I^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}\left(\eta_1,\upsilon_1\right)\right]\\
\le\;&\frac{\Gamma(\alpha+1)}{8\left(\eta_2-\eta_1\right)^{\alpha}}\left[I^{\alpha}_{\eta_1^+}\mathcal{F}\left(\eta_2,\upsilon_1\right)+I^{\alpha}_{\eta_1^+}\mathcal{F}\left(\eta_2,\upsilon_2\right)+I^{\alpha}_{\eta_2^-}\mathcal{F}\left(\eta_1,\upsilon_1\right)+I^{\alpha}_{\eta_2^-}\mathcal{F}\left(\eta_1,\upsilon_2\right)\right]\\
&+\frac{\Gamma(\beta+1)}{8\left(\upsilon_2-\upsilon_1\right)^{\beta}}\left[I^{\beta}_{\upsilon_1^+}\mathcal{F}\left(\eta_1,\upsilon_2\right)+I^{\beta}_{\upsilon_1^+}\mathcal{F}\left(\eta_2,\upsilon_2\right)+I^{\beta}_{\upsilon_2^-}\mathcal{F}\left(\eta_1,\upsilon_1\right)+I^{\beta}_{\upsilon_2^-}\mathcal{F}\left(\eta_2,\upsilon_1\right)\right]\\
\le\;&\frac{\mathcal{F}\left(\eta_1,\upsilon_1\right)+\mathcal{F}\left(\eta_1,\upsilon_2\right)+\mathcal{F}\left(\eta_2,\upsilon_1\right)+\mathcal{F}\left(\eta_2,\upsilon_2\right)}{4}
\end{aligned} \tag{2.29}$$

which is proved by Sarikaya in [23, Theorem 4].

Remark 2.4. *In Theorem 2.2, if we choose $\gamma_1 = 1$, $\gamma_2 = 1$, $\alpha = 1$ and $\beta = 1$,we have the following Hermite-Hadamard inequalities*

$$\begin{aligned}
&\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},\frac{\upsilon_1+\upsilon_2}{2}\right) &(2.30)\\
\le\ & \frac{1}{2(\eta_2-\eta_1)}\int_{\eta_1}^{\eta_2}\mathcal{F}\left(t,\frac{\upsilon_1+\upsilon_2}{2}\right)dt+\frac{1}{2(\upsilon_2-\upsilon_1)}\int_{\upsilon_1}^{\upsilon_2}\mathcal{F}\left(\frac{\eta_1+\eta_2}{2},s\right)ds\\
\le\ & \frac{1}{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\mathcal{F}(t,s)dsdt\\
\le\ & \frac{1}{4(\eta_2-\eta_1)}\left[\int_{\eta_1}^{\eta_2}\mathcal{F}(t,\upsilon_1)dt+\int_{\eta_1}^{\eta_2}\mathcal{F}(t,\upsilon_2)dt\right]\\
&+\frac{1}{4(\upsilon_2-\upsilon_1)}\left[\int_{\upsilon_1}^{\upsilon_2}\mathcal{F}(\eta_1,s)ds+\int_{\upsilon_1}^{\upsilon_2}\mathcal{F}(\eta_2,s)ds\right]\\
\le\ & \frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4},
\end{aligned}$$

which is proved by Dragomir in [3, Theorem 1].

3. Trapezoid-type inequalities

In this section, we prove a trapezoid type inequality by using conformable fractional integrals. First, we need the following lemma.

Lemma 3.1. *Let $\mathcal{F}:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping. If $\partial^2\mathcal{F}/\partial t\partial s\in L_1(\Delta)$, then the following equality holds:*

$$\begin{aligned}
&\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}} &(3.1)\\
&\times\left[{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\right]-A\\
=\ &\frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)\gamma_1^{\alpha}\gamma_2^{\beta}}{4}\int_0^1\int_0^1\left[\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}-\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\\
&\times\left[\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}-\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]\frac{\partial^2\mathcal{F}}{\partial t\partial s}(t\eta_1+(1-t)\eta_2,s\upsilon_1+(1-s)\upsilon_2)dsdt,
\end{aligned}$$

where

$$\begin{aligned}
A\ =\ &\frac{\Gamma(\beta+1)\gamma_2^{\beta}}{4(\upsilon_2-\upsilon_1)^{\gamma_2\beta}}\left[{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)\right] &(3.2)\\
&+\frac{\Gamma(\alpha+1)\gamma_1^{\alpha}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_1)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_2)\right].
\end{aligned}$$

Proof. By integration by parts , we get

$$\begin{aligned}
I_1 &= \int_0^1\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\mathcal{F}}{\partial t\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dsdt \\
&= \int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left\{\frac{1}{\eta_2-\eta_1}\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\frac{\partial\mathcal{F}}{\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)\Big|_0^1\right. \\
&\left.+\frac{\alpha}{\eta_2-\eta_1}\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\frac{\partial\mathcal{F}}{\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)\,dt\right\}ds \\
&= \int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\left\{-\frac{1}{\gamma_1^{\alpha}(\eta_2-\eta_1)}\frac{\partial\mathcal{F}}{\partial s}(\eta_1, s\upsilon_1+(1-s)\upsilon_2)\right. \qquad (3.3)\\
&\left.+\frac{\alpha}{\eta_2-\eta_1}\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\frac{\partial\mathcal{F}}{\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dt\right\}ds \\
&= -\frac{1}{\gamma_1^{\alpha}(\eta_2-\eta_1)}\int_0^1\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial\mathcal{F}}{\partial s}(\eta_1, s\upsilon_1+(1-s)\upsilon_2)ds \\
&+\frac{\alpha}{\eta_2-\eta_1}\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left[\int_0^1\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right. \\
&\left.\times\frac{\partial\mathcal{F}}{\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)ds\right]dt \\
&= \frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\mathcal{F}(\eta_1,\upsilon_1) \\
&-\frac{\beta}{\gamma_1^{\alpha}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\mathcal{F}(\eta_1, s\upsilon_1+(1-s)\upsilon_2)ds \\
&-\frac{\alpha}{\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\mathcal{F}(t\eta_1+(1-t)\eta_2,\upsilon_1)dt \\
&+\frac{\alpha\beta}{(\upsilon_2-\upsilon_1)(\eta_2-\eta_1)}\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1} \\
&\times\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\mathcal{F}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dsdt.
\end{aligned}$$

Similarly, by integration by parts, it follows that

$$\begin{aligned}
I_2 &= \int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\mathcal{F}}{\partial t\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dsdt\\
&= -\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\mathcal{F}(\eta_1,\upsilon_2)\\
&\quad+\frac{\beta}{\gamma_1^{\alpha}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta-1}s^{\gamma_2-1}\mathcal{F}(\eta_1, s\upsilon_1+(1-s)\upsilon_2)ds\\
&\quad+\frac{\alpha}{\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\mathcal{F}(t\eta_1+(1-t)\eta_2,\upsilon_2)dt\\
&\quad-\frac{\alpha\beta}{(\upsilon_2-\upsilon_1)(\eta_2-\eta_1)}\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta-1}s^{\gamma_2-1}\\
&\quad\times\mathcal{F}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dsdt, \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_0^1\int_0^1\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\mathcal{F}}{\partial t\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dsdt\\
&= -\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\mathcal{F}(\eta_2,\upsilon_1)\\
&\quad+\frac{\beta}{\gamma_1^{\alpha}(\eta_2-\eta_1(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\mathcal{F}(\eta_2, s\upsilon_1+(1-s)\upsilon_2)ds\\
&\quad+\frac{\alpha}{\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}t^{\gamma_1-1}\mathcal{F}(t\eta_1+(1-t)\eta_2,\upsilon_1)dt\\
&\quad-\frac{\alpha\beta}{(\upsilon_2-\upsilon_1)(\eta_2-\eta_1)}\int_0^1\int_0^1\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}t^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\\
&\quad\times(1-s)^{\gamma_2-1}\mathcal{F}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dsdt, \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_0^1\int_0^1\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^{\alpha}\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta}\frac{\partial^2\mathcal{F}}{\partial t\partial s}(t\eta_1+(1-t)\eta_2, s\upsilon_1+(1-s)\upsilon_2)dsdt\\
&= \frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\mathcal{F}(\eta_2,\upsilon_2)
\end{aligned}$$

$$-\frac{\beta}{\gamma_1^{\alpha}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta-1}s^{\gamma_2-1}\mathcal{F}(\eta_2,s\upsilon_1+(1-s)\upsilon_2)ds$$
$$-\frac{\alpha}{\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_0^1\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}t^{\gamma_1-1}\mathcal{F}(t\eta_1+(1-t)\eta_2,\upsilon_2)dt$$
$$+\frac{\alpha\beta}{(\upsilon_2-\upsilon_1)(\eta_2-\eta_1)}\int_0^1\int_0^1\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}t^{\gamma_1-1}\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta-1}s^{\gamma_2-1}$$
$$\times\mathcal{F}(t\eta_1+(1-t)\eta_2,s\upsilon_1+(1-s)\upsilon_2)dsdt. \tag{3.6}$$

By using the inequalities comprising (3.3)–(3.6) and applying the change of variables technique to $\delta=t\eta_1+(1-t)\eta_2$ and $\xi=s\upsilon_1+(1-s)\upsilon_2$ for $(t,s)\in[0,1]$, we can write

$$\begin{aligned}
&I-I_2-I_3+I_4 \qquad (3.7)\\
=\;&\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\\
&-\frac{\Gamma(\beta+1)}{\gamma_1^{\alpha}(\eta_2-\eta_1)^{\gamma_1\alpha+1}(\upsilon_2-\upsilon_1)}\left[{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_2}J^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)+{}^{\gamma_2}J^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)\right]\\
&-\frac{\Gamma(\alpha+1)}{\gamma_2^{\beta}(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)^{\gamma_2\beta+1}}\left[{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1}J^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_1)+{}^{\gamma_1}J^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_2)\right]\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\eta_2-\eta_1)^{\gamma_1\alpha+1}(\upsilon_2-\upsilon_1)^{\gamma_2\beta+1}}\\
&\times\left[{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\right].
\end{aligned}$$

Multiplying the both sides of (3.7) by $\frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)\gamma_1^{\alpha}\gamma_2^{\beta}}{4}$, we obtain the required result (3.1). □

Now, we can present the following trapezoid-type inequality.

Theorem 3.1. *Let $\mathcal{F}:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[\eta_1,\eta_2]\times[\upsilon_1,\upsilon_2]$ in $\mathbb{R}^2$ with $0\le\eta_1\le\eta_2, 0\le\upsilon_1\le\upsilon_2, \gamma_1,\gamma_2\neq 0$, and $\alpha,\beta\in(0,1]$. If $\left|\partial^2\mathcal{F}/\partial t\partial s\right|$ is a convex function on the Δ, then the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}} \qquad (3.8)\\
&\times\left[{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\right]-A\Bigg|\\
\le\;&\frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}{4\gamma_1\gamma_2}\left[2B\left(\frac{1}{\gamma_1},\alpha+1,\left(\frac{1}{2}\right)^{\gamma_1}\right)-B\left(\frac{1}{\gamma_1},\alpha+1\right)\right]\\
&\times\left[2\mathfrak{B}\left(\frac{1}{\gamma_2},\beta+1,\left(\frac{1}{2}\right)^{\gamma_2}\right)-\mathrm{B}\left(\frac{1}{\gamma_2},\beta+1\right)\right]\\
&\times\left[\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_2)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_2)\right|\right],
\end{aligned}$$

where A is defined as in (3.2), and B and $\mathfrak{B}$ are the beta function and the incomplete beta function, respectively, defined by

$$\begin{aligned} B(\mu,\nu) &= \int_0^1 \zeta^{\mu-1}(1-\zeta)^{\nu-1}d\zeta, \\ \mathfrak{B}(\mu,\nu,r) &= \int_0^r \zeta^{\mu-1}(1-\zeta)^{\nu-1}d\zeta. \end{aligned}$$

Proof. From Lemma 3.1, we have

$$\begin{aligned} &\Bigg|\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^\alpha\gamma_2^\beta}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}} \qquad (3.9)\\ &\times\Big[{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\Big]-A\Bigg| \\ \le\ & \frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)\gamma_1^\alpha\gamma_2^\beta}{4}\int_0^1\int_0^1\left|\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha-\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^\alpha\right|\left|\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta-\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^\beta\right| \\ &\times\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(t\eta_1+(1-t)\eta_2,s\upsilon_1+(1-s)\upsilon_2)\right|dsdt. \end{aligned}$$

Since $\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}\right|$ is a co-ordinated convex function on Δ, then one has

$$\begin{aligned} &\Bigg|\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^\alpha\gamma_2^\beta}{4(\eta_2-\eta_1)^{\gamma_1\alpha}(\upsilon_2-\upsilon_1)^{\gamma_2\beta}} \qquad (3.10)\\ &\times\Big[{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+{}^{\gamma_1\gamma_2}J^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\Big]-A\Bigg| \\ \le\ & \frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)\gamma_1^\alpha\gamma_2^\beta}{4}\int_0^1\int_0^1\left|\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha-\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^\alpha\right|\left|\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta-\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^\beta\right| \\ &\times\left[ts\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_1)\right|+s(1-t)\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_1)\right|+t(1-s)\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_2)\right|+(1-s)(1-t)\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_2)\right|\right]dsdt \\ =\ & \frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)\gamma_1^\alpha\gamma_2^\beta}{4}\left(\int_0^1\int_0^1 ts\left|\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha-\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^\alpha\right|\left|\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta-\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^\beta\right|dsdt\right) \\ &\times\left[\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_2)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_2)\right|\right]. \end{aligned}$$

Here, we have

$$\begin{aligned} &\int_0^1 t\left|\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha-\left(\frac{1-t^{\gamma_1}}{\gamma_1}\right)^\alpha\right|dt \\ =\ & \frac{1}{\gamma_1^\alpha}\left[\int_0^{\frac{1}{2}} t\left[(1-t^{\gamma_1})^\alpha-(1-(1-t)^{\gamma_1})^\alpha\right]dt+\int_{\frac{1}{2}}^1 t\left[(1-(1-t)^{\gamma_1})^\alpha-(1-t^{\gamma_1})^\alpha\right]dt\right] \end{aligned}$$

$$= \frac{1}{\gamma_1^{\alpha}}\left[\int_0^{\frac{1}{2}} t\left[(1-t^{\gamma_1})^{\alpha}-(1-(1-t)^{\gamma_1})^{\alpha}\right]dt+\int_0^{\frac{1}{2}}(1-t)\left[(1-t^{\gamma_1})^{\alpha}-(1-(1-t)^{\gamma_1})^{\alpha}\right]dt\right]$$

$$= \frac{1}{\gamma_1^{\alpha}}\int_0^{\frac{1}{2}}\left[(1-t^{\gamma_1})^{\alpha}-(1-(1-t)^{\gamma_1})^{\alpha}\right]dt$$

$$= \frac{1}{\gamma_1^{\alpha+1}}\left[2\mathfrak{B}\left(\frac{1}{\gamma_1},\alpha+1,\left(\frac{1}{2}\right)^{\gamma_1}\right)-B\left(\frac{1}{\gamma_1},\alpha+1\right)\right]$$

and similarly

$$\int_0^1 s\left|\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}-\left(\frac{1-s^{\gamma_2}}{\gamma_2}\right)^{\beta}\right|ds$$
$$= \frac{1}{\gamma_2^{\beta+1}}\left[2\mathfrak{B}\left(\frac{1}{\gamma_2},\beta+1,\left(\frac{1}{2}\right)^{\gamma_2}\right)-B\left(\frac{1}{\gamma_2},\beta+1\right)\right].$$

This completes the proof. □

Corollary 3.1. *In Theorem 3.1, if we choose $\gamma_1 = 1$ and $\gamma_2 = 1$, then we have the following trapezoid type inequalities for Riemann-Liouville fractional integrals*

$$\left|\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_2-\eta_1)^{\alpha}(\upsilon_2-\upsilon_1)^{\beta}}\right.$$
$$\left.\times\left[I^{\alpha,\beta}_{\eta_1^+,\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+I^{\alpha,\beta}_{\eta_1^+,\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)+I^{\alpha,\beta}_{\eta_2^-,\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+I^{\alpha,\beta}_{\eta_2^-,\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)\right]-B\right|$$
$$\leq \frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}{4(\alpha+1)(\beta+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(1-\frac{1}{2^{\beta}}\right)$$
$$\times\left[\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_2)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_2)\right|\right],$$

where

$$B = \frac{\Gamma(\beta+1)}{4(\upsilon_2-\upsilon_1)^{\beta}}\left[I^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_1,\upsilon_2)+I^{\beta}_{\upsilon_1^+}\mathcal{F}(\eta_2,\upsilon_2)+I^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_1,\upsilon_1)+I^{\beta}_{\upsilon_2^-}\mathcal{F}(\eta_2,\upsilon_1)\right]$$
$$+\frac{\Gamma(\alpha+1)}{4(\eta_2-\eta_1)^{\alpha}}\left[I^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_1)+I^{\alpha}_{\eta_1^+}\mathcal{F}(\eta_2,\upsilon_2)+I^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_1)+I^{\alpha}_{\eta_2^-}\mathcal{F}(\eta_1,\upsilon_2)\right].$$

Remark 3.1. *In Theorem 3.1, if we choose $\gamma_1 = 1$, $\gamma_2 = 1$, $\alpha = 1$ and $\beta = 1$,we have the following trapezoid type inequality*

$$\left|\frac{\mathcal{F}(\eta_1,\upsilon_1)+\mathcal{F}(\eta_1,\upsilon_2)+\mathcal{F}(\eta_2,\upsilon_1)+\mathcal{F}(\eta_2,\upsilon_2)}{4}+\frac{1}{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}\int_{\eta_1}^{\eta_2}\int_{\upsilon_1}^{\upsilon_2}\mathcal{F}(t,s)dsdt\right.$$

$$-\frac{1}{2(\eta_2-\eta_1)}\int\limits_{\eta_1}^{\eta_2}\left[\mathcal{F}(t,\upsilon_1)+\mathcal{F}(t,\upsilon_2)\right]dt-\frac{1}{2(\upsilon_2-\upsilon_1)}\int\limits_{\upsilon_1}^{\upsilon_2}\left[\mathcal{F}(\eta_1,s)+\mathcal{F}(\eta_2,s)\right]ds\Bigg|$$
$$\leq \frac{(\eta_2-\eta_1)(\upsilon_2-\upsilon_1)}{64}\left[\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_1)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_1,\upsilon_2)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial t\partial s}(\eta_2,\upsilon_2)\right|\right],$$

which is proved by Sarikaya et al. in [27, Theorem 2].

4. Conclusions

In this study, new Hermite-Hadamard type inequalities for coordinated convex functions were obtained through the use of conformable fractional integrals. Some remarks have been presented to show the relationship between our results and earlier obtained results. Furthermore, an identity has been established for partially differentiable functions. By using this equality and the concept of coordinated convexity, a trapezoid type inequality for conformable fractional integrals has been proved. This study demonstrates how conformable fractional integrals can be used in Hermite-Hadamard type inequalities for coordinated convex functions. It also introduces a new identity for partially differentiable functions. These results indicate that such inequalities and identities can be applied to a wide range of studies. For researchers, the findings of this study can provide a basis for further studies in this field.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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