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Research article

Generalized approximation spaces generation from $\mathbb{I}_j$-neighborhoods and ideals with application to Chikungunya disease

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Abstract: Rough set theory is an advanced uncertainty tool that is capable of processing sophisticated real-world data satisfactorily. Rough approximation operators are used to determine the confirmed and possible data that can be obtained by using subsets. Numerous rough approximation models, inspired by neighborhood systems, have been proposed in earlier studies for satisfying axioms of Pawlak approximation spaces (P-approximation spaces) and improving the accuracy measures. This work provides a formulation a novel type of generalized approximation spaces (G-approximation spaces) based on new neighborhood systems inspired by $\mathbb{I}_j$-neighborhoods and ideal structures. The originated G-approximation spaces are offered to fulfill the axiomatic requirements of P-approximation spaces and give more information based on the data subsets under study. That is, they are real simulations of the P-approximation spaces and provide more accurate decisions than the previous models. Several examples are provided to compare the suggested G-approximation spaces with existing ones. To illustrate the application potentiality and efficiency of the provided approach, a numerical example for Chikungunya disease is presented. Ultimately, we conclude our study with a summary and direction for further research.

Keywords: $\mathbb{I}_j$-neighborhoods; $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods; approximation operators; rough set; topology; Chikungunya disease
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1. Introduction

In 1982, Pawlak [48, 49] established the concept of rough set theory as a crucial foundation for the analysis of information systems that involve ambiguity/imperfect data. This theory starts from an equivalence relation R over the universe U, which serves to define the granule (or block) bases. These bases are known as equivalent classes and it is said that the elements lie in the same class or category provided that they are associated with each other by R. To decide the nature of the information obtained from a rough set of data, the concepts of lower approximation and upper approximation which are central in the rough set theory have been introduced. The ordered pair consisting of lower and upper approximations known as P-approximation space. For more illustrative descriptions of the obtained information in terms of structure and completeness of knowledge, researchers have put forward the concepts of a boundary region and an accuracy measure.

The condition of equivalence relation that is imposed for P-approximation spaces is a strict condition that limits the applications of rough sets. Therefore, Yao [55] relaxed this condition to an arbitrary relation and defined the after and before neighborhoods as novel granule bases to analyze information systems. This model (called G-approximation space) overcame the drawbacks of P-approximation space and helped decision-makers to deal with a wide range of practical problems. Afterward, many researchers and scholars introduced new sorts of neighborhood systems [1, 5, 6, 59]. By using operators like union, intersection, subset and superset, various scholars have proposed another category of rough neighborhoods. Among them, Mareay [43] applied the equality relation between Yao's neighborhoods to present new rough set paradigms. Further discussion about these paradigms has been conducted in [21, 22]. Dai et al. [24] displayed three types of rough models by using the concept of maximal right neighborhoods inspired by a similarity relation. Then, Al-shami [9] provided seven rough set paradigms from the perspective of different kinds of maximal neighborhoods they generated from arbitrary binary relations. He exploited these models to rank suspected individuals of COVID-19. Fresh techniques to create G-approximation spaces utilizing maximal neighborhoods have been suggested by Azzam and Al-shami [23]. In the previous studies, various types of G-approximation spaces have been introduced with the aim to develop characterizations of boundary regions and accuracy measures such as $\mathbb{I}_j$-neighborhoods [14], containment rough neighborhoods [8] and subset rough neighborhoods [13]. Abu-Donia [2, 3] exhibited approximation operators using finite set relations instead of one relation, which offers more advantages for G-approximation spaces and can handle some real-life issues. To cope with some complicated problems and widen the scope of applications, several authors have hybridized the rough set with fuzzy set and soft set theories [18–20, 27, 37, 42, 44–46, 52].

Given the similarity between interior and closure operators and lower and upper rough approximation operators, scholars have studied G-approximation spaces from the perspective of topology [41, 51, 53, 54, 60]. Moreover, some generalizations of topology such as supra topology [12], infra topology [16], minimal structure [7, 26], generalized open sets [4, 10, 11] and bitopology [50] have been used to describe G-approximation spaces. In 2013, Kandil et al. [39] introduced the abstract principle of the so-called ideal $\mathcal{K}$ with rough neighborhoods to provide the ideal G-approximation spaces (I-G-approximation spaces) as new rough set paradigms, which maximize the accuracy. This combination has been approved by several authors, who made use of it to reduce the upper approximation and increase the lower approximation. The aforementioned neighborhood systems have been reformulated within the frameworks of I-G-approximation spaces [30, 32, 40, 47], topological structures [17, 28, 31, 58] and graph theory [29]. It was combined ideals and diverse types of maximal neighborhoods to construct some approximation spaces that have desirable properties to cope with some practical issues by some authors [15, 33–35].

Al-shami et al. [14] presented the concept of E_j-neighborhoods (studied here under the name of

$\mathbb{I}_j$-neighborhoods) using the intersection operator between Yao's neighborhoods. Then, Hosny et al. [36] analyzed the rough set models induced from $\mathbb{I}_j$-neighborhoods via I-G-approximation spaces and topologies. Here, we replace the nonempty intersection of $\mathbb{I}_j$-neighborhoods by the belong relation to ideal with the aim to establish new types of neighborhood systems, namely, $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods. We demonstrate in this context the advantages of the proposed models as tools to enlarge lower (shrink upper) approximation and reduce ambiguity regions, which results in a more accurate decision. Also, we illustrate via our experimental results on Chikungunya's information system that the proposed approach outperforms state-of-the-art methods [14,36] in terms of improving approximation operators and increasing accuracy measures. Finally, it should be noted that the current methodology is valid for any arbitrary binary relation, which means that the strict condition of an equivalence relation for P-approximation spaces can be eliminated.

The layout of this article is as follows. Section 2 describes the basic concepts of rough neighborhoods, topologies, and ideals. Then, Section 3 is devoted to presenting new rough neighborhood systems, i.e., $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods, and exploring their basic properties. In Section 4, we employ $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods to present some rough set paradigms and elucidate their merits compared to the state-of-the-art methods given in [14, 36]. These models are discussed from the perspective of topology in Section 5. In Section 6 we give a medical example on the subject of Chikungunya's information system to illustrate how the current models effectively assist to reduce the amount of uncertain information and increase the decision-making accuracy. Finally, in Section 7, we conclude our paper with a summary of the paper's contributions and a direction for further research.

2. Preliminaries

This section is dedicated to recalling the preliminaries and fundamentals that are necessary for the readers to be aware of the manuscript content.

It is well known that a (binary) relation R on a nonempty set U is a subset of $U \times U$. For $s, t \in U$ we write sRt if $(s, t) \in R$.

Definition 2.1. *(see [25]) A relation R on U is said to be:*

1) reflexive if $sRs, \forall s \in U$.
2) symmetric if $sRt \Leftrightarrow tRs$.
3) transitive if sRt whenever tRp and sRp.
4) preorder (or quasi-order) if it is reflexive and transitive.
5) equivalence if it is reflexive, symmetric and transitive.
6) serial if for each $s \in U$ there exists $t \in U$ such that sRt.

Henceforth, we consider U as a nonempty finite set and R as an arbitrary relation unless we state otherwise.

Definition 2.2. *The following ω-neighborhoods of an element $s \in U$ inspired by a relation R are defined as follows:*

1) after neighborhood of s, denoted by $\omega_a(s)$ is given by $\omega_a(s) = \{t \in U : (s, t) \in R\}$ [55].
2) before neighborhood of s, denoted by $\omega_b(s)$ is given by $\omega_b(s) = \{t \in U : (t, s) \in R\}$ [55].

3) minimal-after neighborhood of s, denoted by $\omega_{\langle a\rangle}(s)$, is the intersection of all after neighborhoods containing s [5,6].
4) minimal-before neighborhood of s, denoted by $\omega_{\langle b\rangle}(s)$, is the intersection of all before neighborhoods containing s [5,6].

Remark 2.3. *Some authors studied "after neighborhoods" and "before neighborhoods" under the names "right neighborhoods" and "left neighborhoods", respectively.*

Definition 2.4. *[14] The following $\mathbb{I}$-neighborhoods of an element $s \in U$ inspired by a relation R are defined as follows:*

1) $\mathbb{I}_a(s) = \{t \in U : \omega_a(t) \cap \omega_a(s) \neq \phi\}$.
2) $\mathbb{I}_b(s) = \{t \in U : \omega_b(t) \cap \omega_b(s) \neq \phi\}$.
3) $\mathbb{I}_i(s) = \mathbb{I}_a(s) \cap \mathbb{I}_b(s)$.
4) $\mathbb{I}_u(s) = \mathbb{I}_a(s) \cup \mathbb{I}_b(s)$.
5) $\mathbb{I}_{\langle a\rangle}(s) = \{t \in U : \omega_{\langle a\rangle}(t) \cap \omega_{\langle a\rangle}(s) \neq \phi\}$.
6) $\mathbb{I}_{\langle b\rangle}(s) = \{t \in U : \omega_{\langle b\rangle}(t) \cap \omega_{\langle b\rangle}(s) \neq \phi\}$.
7) $\mathbb{I}_{\langle i\rangle}(s) = \mathbb{I}_{\langle a\rangle}(s) \cap \mathbb{I}_{\langle b\rangle}(s)$.
8) $\mathbb{I}_{\langle u\rangle}(s) = \mathbb{I}_{\langle a\rangle}(s) \cup \mathbb{I}_{\langle b\rangle}(s)$.

In [14], $\mathbb{I}$-neighborhoods were studied under the name "E-neighborhoods".

Definition 2.5. *[22,43] The following ρ-neighborhoods of an element $s \in U$ inspired by a relation R are defined as follows:*

1) $\rho_a(s) = \{t \in U : \omega_a(t) = \omega_a(s)\}$.
2) $\rho_b(s) = \{t \in U : \omega_b(t) = \omega_b(s)\}$.
3) $\rho_i(s) = \rho_a(s) \cap \rho_b(s)$.
4) $\rho_u(s) = \rho_a(s) \cup \rho_b(s)$.
5) $\rho_{\langle a\rangle}(s) = \{t \in U : \omega_{\langle a\rangle}(t) = \omega_{\langle a\rangle}(s)\}$.
6) $\rho_{\langle b\rangle}(s) = \{t \in U : \omega_{\langle b\rangle}(t) = \omega_{\langle b\rangle}(s)\}$.
7) $\rho_{\langle i\rangle}(s) = \rho_{\langle a\rangle}(s) \cap \rho_{\langle b\rangle}(s)$.
8) $\rho_{\langle u\rangle}(s) = \rho_{\langle a\rangle}(s) \cup \rho_{\langle b\rangle}(s)$.

For simplicity, the set $\{a, b, \langle a\rangle, \langle b\rangle, i, u, \langle i\rangle, \langle u\rangle\}$ will be denoted by $\mho$.

Definition 2.6. *[56,57] For ω-neighborhoods and for each $j \in \mho$, the approximation operators (lower and upper), boundary region, and measures of accuracy and roughness of a nonempty subset F of U are respectively given by*

$R_{\star}^{\omega_j}(F) = \{s \in U : \omega_j(s) \subseteq F\}$.
$R^{\star\omega_j}(F) = \{s \in U : \omega_j(s) \cap F \neq \phi\}$.
$BND_R^{\star\omega_j}(F) = R^{\star\omega_j}(F) - R_{\star}^{\omega_j}(F)$.
$ACC_R^{\star\omega_j}(F) = \frac{|R_{\star}^{\omega_j}(F) \cap F|}{|R^{\star\omega_j}(F) \cup F|}$.
$Rough_R^{\star\omega_j}(F) = 1 - ACC_R^{\star\omega_j}(F)$.

Definition 2.7. *[14] For $\mathbb{I}$-neighborhoods and for each $j \in \mho$, the approximation operators (lower and upper), boundary region, and measures of accuracy and roughness of a nonempty subset F of U are respectively given by*

$R_{\bigstar}^{\mathbb{I}_j}(F) = \{s \in U : \mathbb{I}_j(s) \subseteq F\}$.
$R^{\bigstar\mathbb{I}_j}(F) = \{s \in U : \mathbb{I}_j(s) \cap F \neq \phi\}$.
$BND_R^{\bigstar\mathbb{I}_j}(F) = R^{\bigstar\mathbb{I}_j}(F) - R_{\bigstar}^{\mathbb{I}_j}(F)$.
$ACC_R^{\bigstar\mathbb{I}_j}(F) = \frac{|R_{\bigstar}^{\mathbb{I}_j}(F) \cap F|}{|R^{\bigstar\mathbb{I}_j}(F) \cup F|}$.
$Rough_R^{\bigstar\mathbb{I}_j}(F) = 1 - ACC_R^{\bigstar\mathbb{I}_j}(F)$.

Definition 2.8. *[38] An ideal $\mathcal{K}$ on any nonempty set U is a nonempty collection of subsets of U that is closed under finite unions and subsets, i.e., it satisfies the following conditions:*

1) $F \in \mathcal{K}$ and $H \in \mathcal{J} \Rightarrow F \cup H \in \mathcal{K}$,
2) $F \in \mathcal{K}$ and $H \subseteq F \Rightarrow H \in \mathcal{K}$.

Definition 2.9. *[36] Let R and $\mathcal{K}$ respectively denote the binary relation and ideal on a nonempty set U. The approximation operators (lower and upper), boundary region, accuracy and roughness of a nonempty subset L of U derived from R and $\mathcal{K}$ through the use of $\mathbb{I}$-neighborhoods respectively given by*

$L_{\bigstar}^{\mathbb{I}_j}(L) = \{s \in U : \mathbb{I}_j(s) \cap L^c \in \mathcal{K}\}$.
$U^{\bigstar\mathbb{I}_j}(L) = \{s \in U : \mathbb{I}_j(s) \cap L \notin \mathcal{K}\}$.
$\triangle_R^{\bigstar\mathbb{I}_j}(L) = U^{\bigstar\mathbb{I}_j}(L) - L_{\bigstar}^{\mathbb{I}_j}(L)$.
$\mathcal{M}_R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = \frac{|L_{\bigstar}^{\mathbb{I}_j}(L) \cap L|}{|U^{\bigstar\mathbb{I}_j}(L) \cup L|}$.
$\mathcal{R}_R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = 1 - \mathcal{M}_R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L)$.

Definition 2.10. *[22] For ρ-neighborhoods and for each $j \in \mho$, the approximation operators (lower and upper), boundary region, and measures of accuracy and roughness of a nonempty subset F of U are respectively given by*

$R_{\bigstar}^{\rho_j}(F) = \{s \in U : \rho_j(s) \subseteq F\}$.
$R^{\bigstar\rho_j}(F) = \{s \in U : \rho_j(s) \cap F \neq \phi\}$.
$BND_R^{\bigstar\rho_j}(F) = R^{\bigstar\rho_j}(F) - R_{\bigstar}^{\rho_j}(F)$.
$ACC_R^{\bigstar\rho_j}(F) = \frac{|R_{\bigstar}^{\rho_j}(F) \cap F|}{|R^{\bigstar\rho_j}(F) \cup F|}$.
$Rough_R^{\bigstar\rho_j}(F) = 1 - ACC_R^{\bigstar\rho_j}(F)$.

Theorem 2.11. *[14] Let $\mathcal{K}$ be an ideal on U. Then, $\forall j \in \mho$, the following holds:*

1) $\tau^{\omega_j} = \{A \subseteq U : \forall s \in A, \omega_j(s) \subseteq A\}$ is a topology on U.
2) $\tau^{\rho_j} = \{A \subseteq U : \forall s \in A, \rho_j(s) \subseteq A\}$ is a topology on U.
3) $\tau^{\mathbb{I}_j} = \{A \subseteq U : \forall s \in A, \mathbb{I}_j(s) \subseteq A\}$ is a topology on U.

Definition 2.12. *[14] Let $\tau^{\mathbb{I}_j}$ be a topology on U that is described by the above theorem $\forall j \in \mho$ and $A \subseteq U$. Then, the interior and closure operators of A in $(U, \tau^{\mathbb{I}_j})$, denoted by $\underline{\tau^{\mathbb{I}_j}}(A)$ and $\overline{\tau^{\mathbb{I}_j}}(A)$, are called the $\tau^{\mathbb{I}_j}$-lower approximation and $\tau^{\mathbb{I}_j}$-upper approximation, respectively.*

Definition 2.13. *[14] The $\tau^{\mathbb{I}_j}$-boundary and $\tau^{\mathbb{I}_j}$-accuracy induced by a topological space $(U, \tau^{\mathbb{I}_j})$ are respectively given by $BND^{\tau^{\mathbb{I}_j}}(A) = \overline{\tau^{\mathbb{I}_j}}(A) - \underline{\tau^{\mathbb{I}_j}}(A)$ and $ACC^{\tau^{\mathbb{I}_j}}(A) = \frac{|\underline{\tau^{\mathbb{I}_j}}(A)|}{|\overline{\tau^{\mathbb{I}_j}}(A)|}$.*

3. Novel types of rough set neighborhoods based on $\mathbb{I}$-neighborhoods and ideals

In this section, we put forward a new type of neighborhood system generated by $\mathbb{I}$-neighborhoods and an ideal structure. We scrutinize the main features of these neighborhoods and elucidate the relationships between them as well as their relationships with the aforementioned types.

Definition 3.1. *Let R be a relation on U and $\mathcal{K}$ an ideal on U. Then, the $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods of $s \in U$ are defined as follows:*

1) $\mathbb{I}_a^{\mathcal{K}}(s) = \{t \in U : \omega_a(t) \cap \omega_a(s) \notin \mathcal{K}\}$.
2) $\mathbb{I}_b^{\mathcal{K}}(s) = \{t \in U : \omega_b(t) \cap \omega_b(s) \notin \mathcal{K}\}$.
3) $\mathbb{I}_i^{\mathcal{K}}(s) = \mathbb{I}_a^{\mathcal{K}}(s) \cap \mathbb{I}_b^{\mathcal{K}}(s)$.
4) $\mathbb{I}_u^{\mathcal{K}}(s) = \mathbb{I}_a^{\mathcal{K}}(s) \cup \mathbb{I}_b^{\mathcal{K}}(s)$.
5) $\mathbb{I}_{\langle a \rangle}^{\mathcal{K}}(s) = \{t \in U : \omega_{\langle a \rangle}(t) \cap \omega_{\langle a \rangle}(s) \notin \mathcal{K}\}$.
6) $\mathbb{I}_{\langle b \rangle}^{\mathcal{K}}(s) = \{t \in U : \omega_{\langle b \rangle}(t) \cap \omega_{\langle b \rangle}(s) \notin \mathcal{K}\}$.
7) $\mathbb{I}_{\langle i \rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle a \rangle}^{\mathcal{K}}(s) \cap \mathbb{I}_{\langle b \rangle}^{\mathcal{K}}(s)$.
8) $\mathbb{I}_{\langle u \rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle a \rangle}^{\mathcal{K}}(s) \cup \mathbb{I}_{\langle b \rangle}^{\mathcal{K}}(s)$.

It should be noted that if $\mathcal{K} = \phi$ in Theorem 3.2, then Definition 3.1 is equivalent to the previous one in Definition 2.4 [14]. So, the current work is considered as a real extension of the work in [14].

Theorem 3.2. *Let $(U, R, \mathcal{K})$ be an I-G approximation space. Then, $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \mathbb{I}_j(s), \forall j \in \mho$.*

Proof: We prove the case for $j = a$ and the other cases similarly. Let $t \in \mathbb{I}_a^{\mathcal{K}}(s)$. Then, $\omega_a(t) \cap \omega_a(s) \notin \mathcal{K}$. Thus, $\omega_a(t) \cap \omega_a(s) \neq \phi$. So, $t \in \mathbb{I}_a(s)$. Hence, $\mathbb{I}_a^{\mathcal{K}}(s) \subseteq \mathbb{I}_a(s)$.

The converse of Theorem 3.2 is not true in general as shown in the following example.

Example 3.3. *Let $U = \{p, q, s, t\}$, $R = \{(p, p), (t, t), (p, s), (p, t), (t, q), (q, t)\}$ and $\mathcal{K} = \{\phi, \{t\}\}$. In Table 1, we compute ω_j-neighborhoods, ρ_j-neighborhoods, $\mathbb{I}_j$-neighborhoods and $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods.*

Table 1. ω_j-neighborhoods, ρ_j-neighborhoods, $\mathbb{I}_j$-neighborhoods and $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods.

	p	q	s	t
ω_a	$\{p,s,t\}$	$\{t\}$	$\{\phi\}$	$\{q,t\}$
ω_b	$\{p\}$	$\{t\}$	$\{p\}$	$\{p,q,t\}$
ω_i	$\{p\}$	$\{t\}$	$\{\phi\}$	$\{q,t\}$
ω_u	$\{p,s,t\}$	$\{t\}$	$\{p\}$	$\{p,q,t\}$
$\omega_{\langle a\rangle}$	$\{p,s,t\}$	$\{q,t\}$	$\{p,s,t\}$	$\{t\}$
$\omega_{\langle b\rangle}$	$\{p\}$	$\{p,q,t\}$	$\{\phi\}$	$\{t\}$
$\omega_{\langle i\rangle}$	$\{p\}$	$\{q,t\}$	$\{\phi\}$	$\{t\}$
$\omega_{\langle u\rangle}$	$\{p,s,t\}$	$\{p,q,t\}$	$\{p,s,t\}$	$\{t\}$
ρ_a	$\{p\}$	$\{q\}$	$\{s\}$	$\{t\}$
ρ_b	$\{p,s\}$	$\{q\}$	$\{p,s\}$	$\{t\}$
ρ_i	$\{p\}$	$\{q\}$	$\{s\}$	$\{t\}$
ρ_u	$\{p,s\}$	$\{q\}$	$\{p,s\}$	$\{t\}$
$\rho_{\langle a\rangle}$	$\{p,s\}$	$\{q\}$	$\{p,s\}$	$\{t\}$
$\rho_{\langle b\rangle}$	$\{p\}$	$\{q\}$	$\{s\}$	$\{t\}$
$\rho_{\langle i\rangle}$	$\{p\}$	$\{q\}$	$\{s\}$	$\{t\}$
$\rho_{\langle u\rangle}$	$\{p,s\}$	$\{q\}$	$\{p,s\}$	$\{t\}$
$\mathbb{I}_a$	$\{p,q,t\}$	$\{p,q,t\}$	$\{\phi\}$	$\{p,q,t\}$
$\mathbb{I}_b$	$\{p,s,t\}$	$\{q,t\}$	$\{p,s,t\}$	U
$\mathbb{I}_i$	$\{p,t\}$	$\{q,t\}$	$\{\phi\}$	$\{p,q,t\}$
$\mathbb{I}_u$	U	$\{p,q,t\}$	$\{p,s,t\}$	U
$\mathbb{I}_{\langle a\rangle}$	U	U	U	U
$\mathbb{I}_{\langle b\rangle}$	$\{p,q\}$	$\{p,q,t\}$	$\{\phi\}$	$\{q,t\}$
$\mathbb{I}_{\langle i\rangle}$	$\{p,q\}$	$\{p,q,t\}$	$\{\phi\}$	$\{q,t\}$
$\mathbb{I}_{\langle u\rangle}$	U	U	U	U
$\mathbb{I}_a^{\mathcal{K}}$	$\{p\}$	$\{\phi\}$	$\{\phi\}$	$\{t\}$
$\mathbb{I}_b^{\mathcal{K}}$	$\{p,s,t\}$	$\{\phi\}$	$\{p,s,t\}$	$\{p,s,t\}$
$\mathbb{I}_i^{\mathcal{K}}$	$\{p\}$	$\{\phi\}$	$\{\phi\}$	$\{t\}$
$\mathbb{I}_u^{\mathcal{K}}$	$\{p,s,t\}$	$\{\phi\}$	$\{p,s,t\}$	$\{p,s,t\}$
$\mathbb{I}_{\langle a\rangle}^{\mathcal{K}}$	$\{p,s\}$	$\{q\}$	$\{p,s\}$	$\{\phi\}$
$\mathbb{I}_{\langle b\rangle}^{\mathcal{K}}$	$\{p,q\}$	$\{p,q\}$	$\{\phi\}$	$\{\phi\}$
$\mathbb{I}_{\langle i\rangle}^{\mathcal{K}}$	$\{p\}$	$\{q\}$	$\{\phi\}$	$\{\phi\}$
$\mathbb{I}_{\langle u\rangle}^{\mathcal{K}}$	$\{p,q,s\}$	$\{p,q\}$	$\{p,s\}$	$\{\phi\}$

Theorem 3.4. *Let $(U,R,\mathcal{K})$ be an I-G approximation space and $s\in U$. Then, the following holds:*

1) $\mathbb{I}_i^{\mathcal{K}}(s)\subseteq\mathbb{I}_a^{\mathcal{K}}(s)\cap\mathbb{I}_b^{\mathcal{K}}(s)\subseteq\mathbb{I}_a^{\mathcal{K}}(s)\cup\mathbb{I}_b^{\mathcal{K}}(s)\subseteq\mathbb{I}_u^{\mathcal{K}}(s)$.
2) $\mathbb{I}_{\langle i\rangle}^{\mathcal{K}}(s)\subseteq\mathbb{I}_{\langle a\rangle}^{\mathcal{K}}(s)\cap\mathbb{I}_{\langle b\rangle}^{\mathcal{K}}(s)\subseteq\mathbb{I}_{\langle a\rangle}^{\mathcal{K}}(s)\cup\mathbb{I}_{\langle b\rangle}^{\mathcal{K}}(s)\subseteq\mathbb{I}_{\langle u\rangle}^{\mathcal{K}}(s)$.
3) $t\in\mathbb{I}_j^{\mathcal{K}}(s)\Leftrightarrow s\in\mathbb{I}_j^{\mathcal{K}}(t),\forall j\in\mho$.
4) if R is reflexive, then $\mathbb{I}_{\langle j\rangle}^{\mathcal{K}}(s)\subseteq\mathbb{I}_j^{\mathcal{K}}(s)$ and $\rho_j(s)\cup\omega_j(s)\subseteq\mathbb{I}_j^{\mathcal{K}}(s),\forall j\in\mho$.

5) If R is symmetric, then $\mathbb{I}_i^{\mathcal{K}}(s) = \mathbb{I}_a^{\mathcal{K}}(s) = \mathbb{I}_b^{\mathcal{K}}(s) = \mathbb{I}_u^{\mathcal{K}}(s)$ and $\mathbb{I}_{\langle i\rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle a\rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle b\rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle u\rangle}^{\mathcal{K}}(s)$.
6) If R is transitive, then $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \mathbb{I}_{\langle j\rangle}^{\mathcal{K}}(s), \forall j \in \{a, b, i, u\}$.
7) If R is serial, then $\rho_j(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(s), \forall j \in \mho$.
8) If R is symmetric and transitive, then $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \omega_j(s)$ and $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(t)$ (if $s \in \mathbb{I}_j^{\mathcal{K}}(t), \forall j \in \mho$.
9) If R is an equivalence relation, then all instances of $\mathbb{I}_j^{\mathcal{K}}(s)$ are identical $\forall j \in \mho$.

Proof:

(3) $p \in \mathbb{I}_j^{\mathcal{K}}(s) \Leftrightarrow \omega_j(p) \cap \omega_j(s) \notin \mathcal{K} \Leftrightarrow s \in \mathbb{I}_j^{\mathcal{K}}(p), \forall j \in \{a, b, \langle a\rangle, \langle b\rangle\}$. Hence, $p \in \mathbb{I}_j^{\mathcal{K}}(s) \Leftrightarrow s \in \mathbb{I}_j^{\mathcal{K}}(p), \forall j \in \{i, u, \langle i\rangle, \langle u\rangle\}$.

(4) First, $\cap_{p\in\omega_a(s)}\omega_a(s) \subseteq \omega_a(p)$ and $\cap_{p\in\omega_b(s)}\omega_a(s) \subseteq \omega_b(p)$ for R is reflexive. Therefore, $\mathbb{I}_{\langle a\rangle}^{\mathcal{K}}(s) \subseteq \mathbb{I}_a^{\mathcal{K}}(s)$ and $\mathbb{I}_{\langle b\rangle}^{\mathcal{K}}(s) \subseteq \mathbb{I}_b^{\mathcal{K}}(s)$. So, $\mathbb{I}_{\langle i\rangle}^{\mathcal{K}}(s) \subseteq \mathbb{I}_i^{\mathcal{K}}(s)$ and $\mathbb{I}_{\langle u\rangle}^{\mathcal{K}}(s) \subseteq \mathbb{I}_u^{\mathcal{K}}(s)$.
Second, Let $p \in \omega_j(s)$. Since R is reflexive, then $p \in \omega_j(p)$. Consequently, $p \in \omega_j(p) \cap \omega_j(s)$. Hence, $\omega_j(p) \cap \omega_j(s) \neq \phi$. So, $\omega_j(p) \cap \omega_j(s) \notin \mathcal{K}$. Then, $p \in \mathbb{I}_j^{\mathcal{K}}(s)$ and consequently, $\omega_j(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(s)$. Let $p \in P_j(s)$. Then, $\omega_j(p) = \omega_j(s) \neq \phi$ for R is reflexive. So, $\omega_j(p) \cap \omega_j(s) \neq \phi$. Hence, $\omega_j(p) \cap \omega_j(s) \notin \mathcal{K}$. Then, $p \in \mathbb{I}_j^{\mathcal{K}}(s)$ and consequently, $P_j(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(s)$.

(5) $\omega_a(s) = \omega_b(s), \forall s \in U$ for R is symmetric. Consequently, $\omega_a(s) \cap \omega_a(p) \notin \mathcal{K} \Leftrightarrow \omega_b(s) \cap \omega_b(p) \notin \mathcal{K}$. So, $\mathbb{I}_a^{\mathcal{K}}(s) = \mathbb{I}_b^{\mathcal{K}}(s)$. Thus, $\mathbb{I}_a^{\mathcal{K}}(s) = \mathbb{I}_b^{\mathcal{K}}(s) = \mathbb{I}_i^{\mathcal{K}}(s) = \mathbb{I}_u^{\mathcal{K}}(s)$. Similarly, $\mathbb{I}_{\langle a\rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle b\rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle i\rangle}^{\mathcal{K}}(s) = \mathbb{I}_{\langle u\rangle}^{\mathcal{K}}(s)$.

(6) It follows from the fact that $\omega_a(s) \subseteq \omega_{\langle a\rangle}(p)$ for R is transitive. Consequently, $\mathbb{I}_a^{\mathcal{K}}(s) \subseteq \mathbb{I}_{\langle a\rangle}^{\mathcal{K}}(s)$. Similarly, $\mathbb{I}_b^{\mathcal{K}}(s) \subseteq \mathbb{I}_{\langle b\rangle}^{\mathcal{K}}(s)$. Thus, $\mathbb{I}_i^{\mathcal{K}}(s) \subseteq \mathbb{I}_{\langle i\rangle}^{\mathcal{K}}(s)$ and $\mathbb{I}_u^{\mathcal{K}}(s) \subseteq \mathbb{I}_{\langle u\rangle}^{\mathcal{K}}(s)$.

(7) Follows by (3) and (4).

(8) First, let $p \in \mathbb{I}_{\langle i\rangle}^{\mathcal{K}}(s)$. Then, $\omega_a(p) \cap \omega_a(s) \notin \mathcal{K}$. So, $\omega_a(p) \cap \omega_a(s) \neq \phi$. Thus, there exists $q \in \omega_a(p) \cap \omega_a(s)$. So, pRq and xRq. Since R is symmetric and transitive, xRp. Therefore, $p \in \omega_a(s)$ So, $\mathbb{I}_a^{\mathcal{K}}(s) \subseteq \omega_a(s)$.
Second, since R is symmetric, $\mathbb{I}_a^{\mathcal{K}}(s) = \mathbb{I}_b^{\mathcal{K}}(s) = \mathbb{I}_i^{\mathcal{K}}(s) = \mathbb{I}_u^{\mathcal{K}}(s)$. Let $s \in \mathbb{I}_a^{\mathcal{K}}(t)$. Then, $\omega_a(s) \cap \omega_a(t) \notin \mathcal{K}$. Thus, $\omega_a(s) \cap \omega_a(t) \neq \phi$. So, there exists $a \in U$ such that xRa and yRa. Let $p \in \mathbb{I}_b^{\mathcal{K}}(s)$. Then, $\omega_b(p) \cap \omega_b(s) \notin \mathcal{K}$. So, there exists $b \in U$ such that bRp and bRx. Consequently, aRp and aRy since R is symmetric and transitive. Therefore, $a \in \omega_b(p) \cap \omega_b(t)$. So, $\omega_b(p) \cap \omega_b(s) \neq \phi$. Thus, $p \in \mathbb{I}_b^{\mathcal{K}}(t)$.

(9) First, by (4) and (8), $\mathbb{I}_j^{\mathcal{K}}(s) = \omega_j(s)$ and $P_j(s) \subseteq \mathbb{I}_j^{\mathcal{K}}(s)$. It remains to prove that $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq P_j(s)$. Since R is symmetric, $\mathbb{I}_a^{\mathcal{K}}(s) = \mathbb{I}_b^{\mathcal{K}}(s) = \mathbb{I}_i^{\mathcal{K}}(s) = \mathbb{I}_u^{\mathcal{K}}(s)$ by (5). Let $p \in \mathbb{I}_b^{\mathcal{K}}(s)$. Then, $\omega_b(p) \cap \omega_b(s) \notin \mathcal{K}$. Thus, $\omega_b(p) \cap \omega_b(s) \neq \phi$. Since R is an equivalence relation, $\omega_b(p) = \omega_b(s)$. Therefore, $p \in P_b(s)$. Consequently, $\mathbb{I}_b^{\mathcal{K}}(s) \subseteq P_b(s)$. Thus, the result is obtained.
Second, let $x \in \mathbb{I}_b^{\mathcal{K}}(t)$. Then, $\omega_b(s) \cap \omega_b(t) \notin \mathcal{K}$. Thus, $\omega_b(s) \cap \omega_b(t) \neq \phi$. Since R is an equivalence relation, $\omega_b(s) = \omega_b(t) \neq \phi$. Therefore, $\mathbb{I}_j^{\mathcal{K}}(s) = \mathbb{I}_j^{\mathcal{K}}(t)$. Since, R is reflexive, $s \in \omega_b(s)$. Hence, $s \in \mathbb{I}_j^{\mathcal{K}}(t)$.

4. Some new rough set models based on $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods

This part is devoted to setting up new rough set models that are defined by using $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods. Here, we research their main characterizations and compare them. We also demonstrate, with the help of illustrative examples, how the proposed models improve the approximation operators and increase the accuracy measures of subsets relative to those presented in [14, 36].

Definition 4.1. *Let R and $\mathcal{K}$ respectively denote the binary relation and ideal on a nonempty set U. The improved operators (lower and upper), boundary region, accuracy and roughness of a nonempty subset L of U derived from R and $\mathcal{K}$ are respectively given by*

$R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) = \{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap L^c \in \mathcal{K}\}.$

$R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = \{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap L \notin \mathcal{K}\}.$

$BND_R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) - R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L).$

$ACC_R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = \frac{|R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L)\cap L|}{|R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L)\cup L|}.$

$Rough_R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = 1 - ACC_R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L).$

In Table 2, we offer a comparison of the approximation operators, boundary region, and accuracy measure results for a set L as based on Definition 4.1 in the cases of $j \in \{a, b, i, u\}$.

Proposition 4.2. *Consider $L, D \subseteq U$ and let $\mathcal{K}$ be an ideal and R be a binary relation on U. Then the following holds:*

1) if $L^c \in \mathcal{K}$, then $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) = U$.
2) if $\mathcal{K} = P(U)$, then $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) = U$.
3) $\phi \subseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(\phi)$.
4) $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(U) = U$.
5) $L \subseteq D \Rightarrow R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \subseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$.
6) $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L \cup D) \supseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \cup R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$.
7) $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L \cap D) = R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \cap R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$.
8) $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) = (R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L^c))^c$.

Proof:

(4)

$$\begin{aligned} R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(U) &= \{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap U^c \in \mathcal{K}\}. \\ &= U. \end{aligned}$$

(5) Let $s \in R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L)$. Then, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L^c \in \mathcal{K}$. Since $D^c \subseteq L^c$ and $\mathcal{K}$ is an ideal, it follows that $\mathbb{I}_j^{\mathcal{K}}(s) \cap D^c \in \mathcal{K}$. Therefore, $s \in R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L)$. Hence, $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \subseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$.

Table 2. Comparison of the boundary region and accuracy measure results for a set L, as obtained by using Definition 4.1 with $j \in \{a, b, i, u\}$.

L	The present Definition 4.1 at $j = a$				The present Definition 4.1 at $j = b$				The present Definition 4.1 at $j = i$				The present Definition 4.1 at $j = u$			
	$R_{\star}^{\mathbb{I}_a^{\mathcal{K}}}(L)$	$R^{\star\mathbb{I}_a^{\mathcal{K}}}(L)$	$BND^{\star\mathbb{I}_a^{\mathcal{K}}}(L)$	$ACC^{\star\mathbb{I}_a^{\mathcal{K}}}(L)$	$R_{\star}^{\mathbb{I}_b^{\mathcal{K}}}(L)$	$R^{\star\mathbb{I}_b^{\mathcal{K}}}(L)$	$BND^{\star\mathbb{I}_b^{\mathcal{K}}}(L)$	$ACC^{\star\mathbb{I}_b^{\mathcal{K}}}(L)$	$R_{\star}^{\mathbb{I}_i^{\mathcal{K}}}(L)$	$R^{\star\mathbb{I}_i^{\mathcal{K}}}(L)$	$BND^{\star\mathbb{I}_i^{\mathcal{K}}}(L)$	$ACC^{\star\mathbb{I}_i^{\mathcal{K}}}(L)$	$R_{\star}^{\mathbb{I}_u^{\mathcal{K}}}(L)$	$R^{\star\mathbb{I}_u^{\mathcal{K}}}(L)$	$BND^{\star\mathbb{I}_u^{\mathcal{K}}}(L)$	$ACC^{\star\mathbb{I}_u^{\mathcal{K}}}(L)$
ϕ	$\{q, s, t\}$	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	$\{q, s, t\}$	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
U	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	ϕ	1	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	ϕ	1
$\{p\}$	U	$\{p\}$	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	0	U	$\{p\}$	ϕ	1	ϕ	$\{p, s, t\}$	$\{p, s, t\}$	0
$\{q\}$	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	ϕ	ϕ	1	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	ϕ	ϕ	1
$\{s\}$	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	0	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	0
$\{t\}$	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	ϕ	ϕ	0	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	ϕ	ϕ	0
$\{p, q\}$	U	$\{p\}$	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4	U	$\{p\}$	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4
$\{p, s\}$	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	ϕ	2/3	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	ϕ	2/3
$\{p, t\}$	U	$\{p\}$	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	0	U	$\{p\}$	$\{q\}$	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	0
$\{q, s\}$	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4
$\{q, t\}$	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	ϕ	ϕ	1/2	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	ϕ	ϕ	1/2
$\{s, t\}$	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	0	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	0
$\{p, q, s\}$	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	$\{p, s, t\}$	3/4	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	ϕ	3/4
$\{p, q, t\}$	U	$\{p\}$	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4	U	$\{p\}$	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4
$\{p, s, t\}$	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	ϕ	1	U	$\{p\}$	ϕ	1	U	$\{p, s, t\}$	ϕ	1
$\{q, s, t\}$	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4	$\{q, s, t\}$	ϕ	ϕ	1	$\{q\}$	$\{p, s, t\}$	$\{p, s, t\}$	1/4

(6) It follows immediately by part (5).

(7) $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L \cap D) \subseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \cap R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$ by part (2). Let $s \in R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \cap R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$. Then, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L^c \in \mathcal{K}$ and $\mathbb{I}_j^{\mathcal{K}}(s) \cap D^c \in \mathcal{K}$. It follows that $(\mathbb{I}_j^{\mathcal{K}}(s) \cap (L^c \cup D^c)) \in \mathcal{K}$. So, $(\mathbb{I}_j^{\mathcal{K}}(s) \cap (L \cap D)^c) \in \mathcal{K}$. Therefore, $s \in R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L \cap D)$. Thus, $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \cap R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D) \subseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L \cap D)$. Hence, $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L \cap D) = R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) \cap R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$.

(8)

$$\begin{aligned}(R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L^c))^c &= (\{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap L^c \notin \mathcal{K}\})^c.\\ &= \{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap L^c \in \mathcal{K}\}.\\ &= R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L).\end{aligned}$$

Proposition 4.3. *Consider $L, D \subseteq U$ and let $\mathcal{K}$ be an ideal and R be a binary relation on U. Then the following holds:*

1) if $L \in \mathcal{K}$, then $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = \phi$.
2) if $\mathcal{K} = P(U)$, then $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = \phi$.
3) $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(U) \supseteq U$.
4) $\phi = R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(\phi)$.
5) $L \subseteq D \Rightarrow R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \subseteq R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(D)$.
6) $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L \cap D) \subseteq R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \cap R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(D)$.
7) $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L \cup D) = R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \cup R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(D)$.
8) $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) = (R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L^c))^c$.

Proof:

(4)

$$\begin{aligned}R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(\phi) &= \{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap \phi \notin \mathcal{K}\}.\\ &= \phi.\end{aligned}$$

(5) Let $s \in R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L)$. Then, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L \notin \mathcal{K}$. Since $L \subseteq D$ and $\mathcal{K}$ is an ideal, it follows that $\mathbb{I}_j^{\mathcal{K}}(s) \cap D \notin \mathcal{K}$. Therefore, $s \in R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(D)$. Hence, $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \subseteq R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(D)$.

(6) It follows immediately by part (5).

(7) $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L \cup D) \supseteq R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \cup R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$ by part (2). Let $x \in R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L \cup D)$. Then, $\mathbb{I}_j^{\mathcal{K}}(s) \cap (L \cup D) \notin \mathcal{K}$. It follows that $((\mathbb{I}_j^{\mathcal{K}}(s) \cap L) \cup (\mathbb{I}_j^{\mathcal{K}}(s) \cap D)) \notin \mathcal{K}$. Therefore, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L \notin \mathcal{K}$ or $\mathbb{I}_j^{\mathcal{K}}(s) \cap D \notin \mathcal{K}$, which means that $s \in R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L)$ or $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$. Then, $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \cup R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$. Thus, $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \cup R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D) \supseteq R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L \cup D)$. Hence, $R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L \cup D) = R^{\bigstar\mathbb{I}_j^{\mathcal{K}}}(L) \cup R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(D)$.

(8)

$$\begin{aligned}(R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L^c))^c &= (\{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap L \in \mathcal{K}\})^c. \\ &= \{s \in U : \mathbb{I}_j^{\mathcal{K}}(s) \cap L \notin \mathcal{K}\}. \\ &= R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(L).\end{aligned}$$

Remark 4.4. *By Example 3.3, we elucidate that*

1) the converse of part (5) in Propositions 4.2 and 4.3 is generally incorrect.
2) the inclusion relations of parts (3) and (6) in Propositions 4.2 and 4.3 are generally proper.

Theorem 4.5. *Let $(U, R, \mathcal{K})$ be an I-G approximation space such that $L \subseteq U$. Then, $\forall j \in \mho$, the following holds:*

1) $R_{\bigstar}^{\mathbb{I}_j}(L) \subseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L)$.
2) $R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(L) \subseteq R^{\bigstar \mathbb{I}_j}(L)$.
3) $BND_R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(L) \subseteq BND_R^{\bigstar \mathbb{I}_j}(L)$.
4) $ACC_R^{\bigstar \mathbb{I}_j}(L) \leq ACC_R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(L)$.

Proof:

(1) Let $s \in R_{\bigstar}^{\mathbb{I}_j}(L)$. Then, $\mathbb{I}_j(s) \subseteq L$. Since $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \mathbb{I}_j(s)$ (by Theorem 3.2), then $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq L$. Thus, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L^c \in \mathcal{K}$. Hence, $s \in R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L)$. Therefore, $R_{\bigstar}^{\mathbb{I}_j}(L) \subseteq R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L)$.

(2) Let $s \in R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(L)$. Then, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L \notin \mathcal{K}$. So, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L \neq \phi$. Consequently, $\mathbb{I}_j(s) \cap L \neq \phi$ given that $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq \mathbb{I}_j(s)$ (by Theorem 3.2). Thus, $s \in R^{\bigstar \mathbb{I}_j}(L)$. Therefore, $R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(L) \subseteq R^{\bigstar \mathbb{I}_j}(L)$.

(3)–(4) It follow by (1) and (2).

Remark 4.6. *Consider Example 3.3 and the following:*

1) Table 3 shows that the inclusion relation and less than relation of parts in Theorem 4.5 are generally proper.
2) Table 4 shows that the approximations in [22, 55] (see 2.6 and 2.10) and the proposed approximation operators in Definition 4.1 are incomparable.

Table 3. Comparison of the results obtained based on the proposed Definition 4.1 and Definition 2.7 [14] for $j = a$.

L	The proposed Definition 4.1 at $j = a$				Definition [14] 2.7 at $j = a$			
	$R_{\bigstar}^{\mathbb{I}_a^{\mathcal{K}}}(L)$	$R^{\bigstar^{\mathbb{I}_a^{\mathcal{K}}}}(L)$	$BND^{\bigstar^{\mathbb{I}_a^{\mathcal{K}}}}(L)$	$ACC^{\bigstar^{\mathbb{I}_a^{\mathcal{K}}}}(L)$	$R_{\bigstar}^{\mathbb{I}_a}(L)$	$R^{\bigstar^{\mathbb{I}_a}}(L)$	$BND_R^{\bigstar^{\mathbb{I}_a}}(L)$	$ACC_R^{\bigstar^{\mathbb{I}_a}}(L)$
ϕ	$\{q,s,t\}$	ϕ	ϕ	0	$\{s\}$	ϕ	ϕ	0
U	U	$\{p\}$	ϕ	1	U	$\{p,q,t\}$	ϕ	1
$\{p\}$	U	$\{p\}$	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	0
$\{q\}$	$\{q,s,t\}$	ϕ	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	0
$\{s\}$	$\{q,s,t\}$	ϕ	ϕ	1	$\{s\}$	ϕ	ϕ	1
$\{t\}$	$\{q,s,t\}$	ϕ	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	0
$\{p,q\}$	U	$\{p\}$	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	0
$\{p,s\}$	U	$\{p\}$	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	1/4
$\{p,t\}$	U	$\{p\}$	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	0
$\{q,s\}$	$\{q,s,t\}$	ϕ	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	1/4
$\{q,t\}$	$\{q,s,t\}$	ϕ	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	0
$\{s,t\}$	$\{q,s,t\}$	ϕ	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	1/4
$\{p,q,s\}$	U	$\{p\}$	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	1/4
$\{p,q,t\}$	U	$\{p\}$	ϕ	1	U	$\{p,q,t\}$	ϕ	1
$\{p,s,t\}$	U	$\{p\}$	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	1/4
$\{q,s,t\}$	$\{q,s,t\}$	ϕ	ϕ	1	$\{s\}$	$\{p,q,t\}$	$\{p,q,t\}$	1/4

Table 4. Comparison of the results obtained by using the proposed Definition 4.1, Definition [55] 2.6, and Definition 2.10 [22] for $j = b$.

L	The present Definition 4.1 at $j = b$				Definition [55] 2.6 at $j = b$				Definition [22] 2.10 at $j = b$			
	$R_{\star}^{\mathbb{I}_b^{\mathcal{K}}}(L)$	$R^{\star\mathbb{I}_b^{\mathcal{K}}}(L)$	$BND^{\star\mathbb{I}_b^{\mathcal{K}}}(L)$	$ACC^{\star\mathbb{I}_b^{\mathcal{K}}}(L)$	$R_{\star}^{\omega_b}(L)$	$R^{\star\omega_b}(L)$	$BND_R^{\star\omega_b}(L)$	$ACC_R^{\star\omega_b}(L)$	$R_{\star}^{\rho_b}(L)$	$R^{\star\rho_b}(L)$	$BND_R^{\star\rho_b}(L)$	$ACC_R^{\star\rho_b}(L)$
ϕ	ϕ	ϕ	ϕ	0	$\{q\}$	ϕ	ϕ	0	$\{t\}$	ϕ	ϕ	0
U	U	$\{p,s,t\}$	ϕ	1	U	U	ϕ	1	U	$\{p,q,s\}$	ϕ	1
$\{p\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{p,q,s\}$	$\{p,s,t\}$	$\{t\}$	1/3	$\{t\}$	$\{p,q\}$	$\{p,q\}$	0
$\{q\}$	$\{q\}$	ϕ	ϕ	1	$\{q\}$	$\{t\}$	$\{t\}$	1/2	$\{q,t\}$	$\{q\}$	$\{q\}$	1
$\{s\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{q\}$	ϕ	ϕ	0	$\{t\}$	$\{p,s\}$	$\{p,s\}$	0
$\{t\}$	$\{q\}$	ϕ	ϕ	0	$\{q\}$	$\{q,t\}$	$\{t\}$	0	$\{t\}$	ϕ	ϕ	1
$\{p,q\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	1/4	U	$\{p,s,t\}$	ϕ	1/2	$\{q,t\}$	$\{p,q,s\}$	$\{p,s\}$	1/3
$\{p,s\}$	U	$\{p,s,t\}$	ϕ	2/3	$\{p,q,s\}$	$\{p,s,t\}$	$\{t\}$	2/3	$\{p,s,t\}$	$\{p,s\}$	ϕ	1
$\{p,t\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{p,q,s\}$	U	$\{t\}$	1/4	$\{p,s,t\}$	$\{p,s\}$	ϕ	2/3
$\{q,s\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	1/4	$\{q\}$	$\{t\}$	$\{t\}$	1/3	$\{q,t\}$	$\{p,q,s\}$	$\{p,s\}$	1/3
$\{q,t\}$	$\{q\}$	ϕ	ϕ	1/2	$\{q\}$	$\{q,t\}$	$\{t\}$	1/2	$\{q,t\}$	$\{q\}$	ϕ	1
$\{s,t\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{q\}$	$\{q,t\}$	$\{t\}$	0	$\{t\}$	$\{p,s\}$	$\{p,s\}$	1/3
$\{p,q,s\}$	U	$\{p,s,t\}$	ϕ	3/4	U	$\{p,s,t\}$	ϕ	3/4	U	$\{p,q,s\}$	ϕ	1
$\{p,q,t\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	1/4	U	U	ϕ	3/4	$\{q,t\}$	$\{p,q\}$	$\{p\}$	2/3
$\{p,s,t\}$	U	$\{p,s,t\}$	ϕ	1	$\{p,q,s\}$	U	$\{t\}$	1/2	$\{p,s\}$	$\{p,s\}$	ϕ	2/3
$\{q,s,t\}$	$\{q\}$	$\{p,s,t\}$	$\{p,s,t\}$	1/4	$\{q\}$	$\{q,t\}$	$\{t\}$	1/3	$\{q,t\}$	$\{p,q\}$	ϕ	1/2

5. Generalized topology based on $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods

One of the well-known approaches to initiating new rough set models is the use of a topology. Here, we provide a method to generate topological spaces from $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods. Then, we exploit these topologies to introduce new approximation operators and accuracy measures. We make comparisons between them and explore their main properties.

Theorem 5.1. *Let $(U, R, \mathcal{K})$ be an I-G approximation space. Then, $\forall j \in \mho$, the collection $\tau^{\mathbb{I}_j^{\mathcal{K}}} = \{L \subseteq U : \forall s \in L, \mathbb{I}_j^{\mathcal{K}}(s) \cap L^c \in \mathcal{K}\}$ is a topology on U.*

Proof:

(1) Clearly U and ϕ belong to $\tau^{\mathbb{I}_j^{\mathcal{K}}}$.

(2) Let $L_i \in \tau^{\mathbb{I}_j^{\mathcal{K}}}$ $(\forall i \in I)$ and $a \in \cup_{i \in I} L_i$. Then,
$\exists\ i_0 \in I$ such that $a \in L_{i_0}$
$\Rightarrow \mathbb{I}_j^{\mathcal{K}}(a) \cap L_{i_0}^c \in \mathcal{I}$
$\Rightarrow \mathbb{I}_j^{\mathcal{K}}(a) \cap (\cup_{i \in I} L_i)^c \in \mathcal{I}$
$\Rightarrow \cup_{i \in I} L_i \in \tau^{\mathbb{I}_j^{\mathcal{K}}}$.

(3) Let $L, D \in \tau^{\mathbb{I}_j^{\mathcal{K}}}$, and $a \in L \cap D$.
$\Rightarrow \mathbb{I}_j^{\mathcal{K}}(a) \cap L^c \in \mathcal{I}$ *and* $\mathbb{I}_j^{\mathcal{K}}(a) \cap D^c \in \mathcal{I}$
$\Rightarrow (\mathbb{I}_j^{\mathcal{K}}(a) \cap L^c) \cup (\mathbb{I}_j^{\mathcal{K}}(a) \cap D^c) \in \mathcal{I}$
$\Rightarrow \mathbb{I}_j^{\mathcal{K}}(a) \cap (L^c \cup D^c) \in \mathcal{I}$
$\Rightarrow (\mathbb{I}_j^{\mathcal{K}}(a) \cap (L \cap D)^c \in \mathcal{I}$
$\Rightarrow L \cap D \in \tau^{\mathbb{I}_j^{\mathcal{K}}}$.

From (1)–(3), we obtain $\tau^{\mathbb{I}_j^{\mathcal{K}}}$ as a topology on U.

The proposed topologies, mentioned above, are finer than the previous ones [14] as is shown in the following result.

Theorem 5.2. *Let $(U, R, \mathcal{K})$ be an I-G approximation space. Then, $\tau^{\mathbb{I}_j} \subseteq \tau^{\mathbb{I}_j^{\mathcal{K}}}$ $\forall j \in \mho$.*

Proof: Let $L \in \tau^{\mathbb{I}_j}$. Then, $\mathbb{I}_j(s) \subseteq L$ $\forall s \in L$ and consequently $\mathbb{I}_j^{\mathcal{K}}(s) \subseteq L$ $\forall s \in L$ by Theorem 3.2. Thus, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L^c = \phi \in \mathcal{K}$ $\forall s \in L$. Therefore, $L \in \tau^{\mathbb{I}_j^{\mathcal{K}}}$. Hence, $\tau^{\mathbb{I}_j} \subseteq \tau^{\mathbb{I}_j^{\mathcal{K}}}$.

Remark 5.3. *It should be noted that the following holds:*

1) If $\mathcal{K} = \phi$ in Theorem 5.2, then the proposed generated topologies are equivalent to those in Theorem 2.11 [14]. So, the current work is considered as a proper extension of the work in [14].
2) In Example 3.3, we obtain for $j = a$ the following: $P(U) = \tau^{\mathbb{I}_a^{\mathcal{K}}} \subsetneq \tau^{\mathbb{I}_a} = \{U, \phi, \{s\}, \{p, q, t\}\}$. This means that $\tau^{\mathbb{I}_j^{\mathcal{K}}} \subsetneq \tau^{\mathbb{I}_j}$.

Proposition 5.4. *Let $(U, R, \mathcal{K})$ be an I-G approximation space. Then the following holds:*

1) $\tau^{\mathbb{I}_u^{\mathcal{K}}} \subseteq \tau^{\mathbb{I}_a^{\mathcal{K}}}$ and $\tau^{\mathbb{I}_u^{\mathcal{K}}} \subseteq \tau^{\mathbb{I}_b^{\mathcal{K}}}$.
2) $\tau^{\mathbb{I}_a^{\mathcal{K}}} \subseteq \tau^{\mathbb{I}_i^{\mathcal{K}}}$ and $\tau^{\mathbb{I}_b^{\mathcal{K}}} \subseteq \tau^{\mathbb{I}_i^{\mathcal{K}}}$.

3) $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle u\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle a\rangle}}$ and $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle u\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle b\rangle}}$.
4) $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle a\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle i\rangle}}$ and $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle b\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle i\rangle}}$.
5) If R is reflexive, then $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle j\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{j}}$ and $\tau^{\rho^{\mathcal{K}}_{j}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{j}}, \forall j \in \mho$.
6) If R is serial, then $\tau^{\rho^{\mathcal{K}}_{j}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{j}}, \forall j \in \{a, b, i, u\}$.
7) If R is symmetric, then $\tau^{\mathbb{I}^{\mathcal{K}}_{a}} = \tau^{\mathbb{I}^{\mathcal{K}}_{b}} = \tau^{\mathbb{I}^{\mathcal{K}}_{i}} = \tau^{\mathbb{I}^{\mathcal{K}}_{u}}$ and $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle a\rangle}} = \tau^{\mathbb{I}^{\mathcal{K}}_{\langle b\rangle}} = \tau^{\mathbb{I}^{\mathcal{K}}_{\langle i\rangle}} = \tau^{\mathbb{I}^{\mathcal{K}}_{\langle u\rangle}}$.
8) If R is transitive, then $\tau^{\mathbb{I}^{\mathcal{K}}_{j}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle j\rangle}}, \forall j \in \{a, b, i, u\}$.
9) If R is an equivalence relation, then $\forall j \in \mho$ all $\tau^{\mathbb{I}^{\mathcal{K}}_{j}}$ are identical.

Proof:

(1) Let $L \in \tau^{\mathbb{I}^{\mathcal{K}}_{u}}$. Then, $\mathbb{I}^{\mathcal{K}}_{u}(p) \cap L^c \in \mathcal{K}\ \forall p \in L$. Thus, $(\mathbb{I}^{\mathcal{K}}_{a}(p) \cup \mathbb{I}^{\mathcal{K}}_{b}(p)) \cap L^c \in \mathcal{K}\ \forall p \in L$. Hence, $\mathbb{I}^{\mathcal{K}}_{a}(p) \cap A^c \in \mathcal{K}\ \forall p \in L$ and $\mathbb{I}^{\mathcal{K}}_{b}(p) \cap L^c \in \mathcal{K}\ \forall p \in A$. Therefore, $L \in \tau^{\mathbb{I}^{\mathcal{K}}_{a}}$ and $L \in \tau^{\mathbb{I}^{\mathcal{K}}_{b}}$. Hence, $\tau^{\mathbb{I}^{\mathcal{K}}_{u}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{a}}$ and $\tau^{\mathbb{I}^{\mathcal{K}}_{u}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{b}}$. Similarly, we can prove 3).

(2) Let $L \in \tau^{\mathbb{I}^{\mathcal{K}}_{a}}$. Then, $\mathbb{I}^{\mathcal{K}}_{a}(p) \cap L^c \in \mathcal{K}\ \forall p \in L$. Thus, $(\mathbb{I}^{\mathcal{K}}_{a}(p) \cap \mathbb{I}^{\mathcal{K}}_{b}(p)) \cap L^c \in \mathcal{K}\ \forall p \in L$. Hence, $\mathbb{I}^{\mathcal{K}}_{i}(p) \cap L^c \in \mathcal{K}\ \forall p \in L$. Therefore, $L \in \tau^{\mathbb{I}^{\mathcal{K}}_{i}}$. Hence, $\tau^{\mathbb{I}^{\mathcal{K}}_{a}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{i}}$. Similarly, we can prove 4).

(5)–(9) It follows directly by Theorem 3.4.

Corollary 5.5. *Let $(U, R, \mathcal{K})$ be an I-G approximation space. Then, $\forall j \in \mho$ we have the following properties:*

1) $\tau^{\mathbb{I}^{\mathcal{K}}_{u}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{a}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{i}}$.
2) $\tau^{\mathbb{I}^{\mathcal{K}}_{u}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{b}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{i}}$.
3) $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle u\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle a\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle i\rangle}}$.
4) $\tau^{\mathbb{I}^{\mathcal{K}}_{\langle u\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle b\rangle}} \subseteq \tau^{\mathbb{I}^{\mathcal{K}}_{\langle i\rangle}}$.

Remark 5.6. *Example 3.3 shows that the inclusion in Proposition 5.4 and Corollary 5.5 cannot be replaced by an equality relation.*

Definition 5.7. *Let L be a subset of a topological space $(U, \tau^{\mathbb{I}^{\mathcal{K}}_{j}})$ $\forall j \in \mho$. The interior operator of L, denoted by $\underline{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L) = L^o$ and the closure operator of L, denoted by $\overline{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L) = \overline{L}$, are called the $\tau^{\mathbb{I}^{\mathcal{K}}_{j}}$-lower approximation and $\tau^{\mathbb{I}^{\mathcal{K}}_{j}}$-upper approximation, respectively.*

Definition 5.8. *The $\tau^{\mathbb{I}^{\mathcal{K}}_{j}}$-boundary and $\tau^{\mathbb{I}^{\mathcal{K}}_{j}}$-accuracy induced by a topological space $(U, \tau^{\mathbb{I}^{\mathcal{K}}_{j}})$ are respectively given by $BND^{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L) = \overline{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L) - \underline{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L)$ and $ACC^{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L) = \frac{|\underline{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L)|}{|\overline{\tau^{\mathbb{I}^{\mathcal{K}}_{j}}}(L)|}$.*

Table 5 presents the comparison of the boundary region and accuracy measure results for a set L as based on Definition 5.7 for $j \in \{a, b, i, u\}$.

Table 5. Comparison of the boundary region and accuracy measure results for a set L, as obtained by using the proposed Definition 5.7 for $j \in \{a, b, i, u\}$.

L	The present Definition 5.7 at $j = r$				The present Definition 5.7 at $j = l$				The present Definition 5.7 at $j = i$				The present Definition 5.7 at $j = u$			
	$\underline{\tau^{\mathbb{I}^K}_a}(L)$	$\overline{\tau^{\mathbb{I}^K}_a}(L)$	$BND^{\tau^{\mathbb{I}^K}_a}(L)$	$ACC^{\tau^{\mathbb{I}^K}_a}(L)$	$\underline{\tau^{\mathbb{I}^K}_b}(L)$	$\overline{\tau^{\mathbb{I}^K}_b}(L)$	$BND^{\tau^{\mathbb{I}^K}_b}(L)$	$ACC^{\tau^{\mathbb{I}^K}_b}(L)$	$\underline{\tau^{\mathbb{I}^K}_i}(L)$	$\overline{\tau^{\mathbb{I}^K}_i}(L)$	$BND^{\tau^{\mathbb{I}^K}_i}(L)$	$ACC^{\tau^{\mathbb{I}^K}_i}(L)$	$\underline{\tau^{\mathbb{I}^K}_u}(L)$	$\overline{\tau^{\mathbb{I}^K}_u}(L)$	$BND^{\tau^{\mathbb{I}^K}_u}(L)$	$ACC^{\tau^{\mathbb{I}^K}_u}(L)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
U	U	U	ϕ	1	U	U	ϕ	1	U	U	ϕ	1	U	U	ϕ	1
$\{p\}$	$\{p\}$	$\{p\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{p\}$	$\{p\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0
$\{q\}$	$\{q\}$	$\{q\}$	ϕ	1	$\{q\}$	$\{q\}$	ϕ	1	$\{q\}$	$\{q\}$	ϕ	1	$\{q\}$	$\{q\}$	ϕ	1
$\{s\}$	$\{s\}$	$\{s\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{s\}$	$\{s\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0
$\{t\}$	$\{t\}$	$\{t\}$	ϕ	1	ϕ	$\{t\}$	$\{t\}$	0	$\{t\}$	$\{t\}$	ϕ	1	ϕ	$\{t\}$	$\{t\}$	0
$\{p,q\}$	$\{p,q\}$	$\{p,q\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4	$\{p,q\}$	$\{p,q\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4
$\{p,s\}$	$\{p,s\}$	$\{p,s\}$	ϕ	1	$\{p,s\}$	$\{p,s,t\}$	$\{t\}$	2/3	$\{p,s\}$	$\{p,s\}$	ϕ	1	$\{p,s\}$	$\{p,s,t\}$	$\{t\}$	2/3
$\{p,t\}$	$\{p,t\}$	$\{p,t\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{p,t\}$	$\{p,t\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0
$\{q,s\}$	$\{q,s\}$	$\{q,s\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4	$\{q,s\}$	$\{q,s\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4
$\{q,t\}$	$\{q,t\}$	$\{q,t\}$	ϕ	1	$\{q\}$	$\{q,t\}$	$\{t\}$	1/2	$\{q,t\}$	$\{q,t\}$	ϕ	1	$\{q\}$	$\{q,t\}$	$\{t\}$	1/2
$\{s,t\}$	$\{s,t\}$	$\{s,t\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0	$\{s,t\}$	$\{s,t\}$	ϕ	1	ϕ	$\{p,s,t\}$	$\{p,s,t\}$	0
$\{p,q,s\}$	$\{p,q,s\}$	$\{p,q,s\}$	ϕ	1	$\{p,q,s\}$	U	$\{t\}$	3/4	$\{p,q,s\}$	$\{p,q,s\}$	ϕ	1	$\{p,q,s\}$	U	$\{t\}$	3/4
$\{p,q,t\}$	$\{p,q,t\}$	$\{p,q,t\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4	$\{p,q,t\}$	$\{p,q,t\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4
$\{p,s,t\}$	$\{p,s,t\}$	$\{p,s,t\}$	ϕ	1	$\{p,s,t\}$	$\{p,s,t\}$	ϕ	1	$\{p,s,t\}$	$\{p,s,t\}$	ϕ	1	$\{p,s,t\}$	$\{p,s,t\}$	ϕ	1
$\{q,s,t\}$	$\{q,s,t\}$	$\{q,s,t\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4	$\{q,s,t\}$	$\{q,s,t\}$	ϕ	1	$\{q\}$	U	$\{p,s,t\}$	1/4

By the next result, we confirm that the accuracy calculated by employing the approach of Section 4 is higher than that of this section.

Theorem 5.9. *Let $(U, R, \mathcal{K})$ be an I-G approximation space. Then, $ACC^{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L) \leq ACC_R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L)\ \forall j \in \mho$.*

Proof: First, let $s \in \underline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)$. Then, $s \in R_\bigstar^{\mathbb{I}_j^{\mathcal{K}}}(L)$. Thus, $\underline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L) \subseteq R_\bigstar^{\mathbb{I}_j^{\mathcal{K}}}(L)$. So, $|\underline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)| \leq |R_\bigstar^{\mathbb{I}_j^{\mathcal{K}}}(L)|$. Second, let $s \in R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L) \cup L$. Then, $s \in R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L)$ or $s \in L$. Thus, $\mathbb{I}_j^{\mathcal{K}}(s) \cap L \notin \mathcal{K}$ or $s \in \overline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)$. So, $\frac{1}{|\overline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)|} \leq \frac{1}{|R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L) \cup L|}$. Consequently, $ACC^{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(A) = \frac{|\underline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)|}{|\overline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)|} \leq \frac{|R_\bigstar^{\mathbb{I}_j^{\mathcal{K}}}(L) \cap L|}{|R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L) \cup L|} = ACC_R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}$.

Theorem 5.10. *Let L be a subset of an I-G approximation space $(U, R, \mathcal{K})$. Then, $\forall j \in \mho$, we have the following:*

1) $\underline{\tau^{\mathbb{I}_j}}(L) \subseteq \underline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)$.
2) $\overline{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L) \subseteq \overline{\tau^{\mathbb{I}_j}}(L)$.
3) $BND^{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L) \subseteq BND^{\tau^{\mathbb{I}_j}}(L)$.
4) $ACC^{\tau^{\mathbb{I}_j}}(L) \leq ACC^{\tau^{\mathbb{I}_j^{\mathcal{K}}}}(L)$.

Proof: It follows directly from Theorem 5.2.

Remark 5.11. *It should be noted that the following holds:*

1) The inclusion relations of parts in Theorem 5.10 are generally proper as demonstrated in Example 3.3 and Table 6.
2) If $\mathcal{K} = \phi$ in Theorem 5.10, then the approximations in Definition 2.13 [14] and the proposed approximations in Definition 5.7 are equivalent.

Table 6. Comparison of the boundary region and accuracy measure results for a set L based on the proposed Definition 5.7 and Definition 2.13 [14] for $j = a$.

L	The present Definition 5.7 at $j = a$				The previous Definition 2.13 at $j = a$			
	$\underline{\tau^{\mathbb{I}_a^{\mathcal{K}}}}(L)$	$\overline{\tau^{\mathbb{I}_a^{\mathcal{K}}}}(L)$	$BND^{\tau^{\mathbb{I}_a^{\mathcal{K}}}}(L)$	$ACC^{\tau^{\mathbb{I}_a^{\mathcal{K}}}}(L)$	$\underline{\tau^{\mathbb{I}_b}}(L)$	$\overline{\tau^{\mathbb{I}_b}}(L)$	$BND^{\tau^{\mathbb{I}_b}}(L)$	$ACC^{\tau^{\mathbb{I}_b}}(L)$
ϕ	ϕ	ϕ	ϕ	0	ϕ	ϕ	ϕ	0
U	U	U	ϕ	1	U	U	ϕ	1
$\{p\}$	$\{p\}$	$\{p\}$	ϕ	1	ϕ	$\{p, q, t\}$	$\{p, q, t\}$	0
$\{q\}$	$\{q\}$	$\{q\}$	ϕ	1	ϕ	$\{p, q, t\}$	$\{p, q, t\}$	0
$\{s\}$	$\{s\}$	$\{s\}$	ϕ	1	$\{s\}$	$\{s\}$	ϕ	1
$\{t\}$	$\{t\}$	$\{t\}$	ϕ	1	ϕ	$\{p, q, t\}$	$\{p, q, t\}$	0
$\{p, q\}$	$\{p, q\}$	$\{p, q\}$	ϕ	1	ϕ	$\{p, q, t\}$	$\{p, q, t\}$	0
$\{p, s\}$	$\{p, s\}$	$\{p, s\}$	ϕ	1	$\{s\}$	U	$\{p, q, t\}$	1/4
$\{p, t\}$	$\{p, t\}$	$\{p, t\}$	ϕ	1	ϕ	$\{p, q, t\}$	$\{p, q, t\}$	0
$\{q, s\}$	$\{q, s\}$	$\{q, s\}$	ϕ	1	$\{s\}$	U	$\{p, q, t\}$	1/4
$\{q, t\}$	$\{q, t\}$	$\{q, t\}$	ϕ	1	ϕ	$\{p, q, t\}$	$\{p, q, t\}$	0
$\{s, t\}$	$\{s, t\}$	$\{s, t\}$	ϕ	1	$\{s\}$	U	$\{p, q, t\}$	1/4
$\{p, q, s\}$	$\{p, q, s\}$	$\{p, q, s\}$	ϕ	1	$\{s\}$	U	$\{p, q, t\}$	1/4
$\{p, q, t\}$	$\{p, q, t\}$	$\{p, q, t\}$	ϕ	1	$\{p, q, t\}$	$\{p, q, t\}$	ϕ	1
$\{p, s, t\}$	$\{p, s, t\}$	$\{p, s, t\}$	ϕ	1	$\{s\}$	U	$\{p, q, t\}$	1/4
$\{q, s, t\}$	$\{q, s, t\}$	$\{q, s, t\}$	ϕ	1	$\{s\}$	U	$\{p, q, t\}$	1/4

6. Medical example: Chikungunya disease

In this section, we look at the performance of the method proposed here and the previous one introduced in [14,36] by applying them to the information system of Chikungunya disease. This illness is an infection caused by the Chikungunya virus and it spreads to humans through the bite of an infected mosquito. The symptoms of infection usually start within 3 days to a week after an infected mosquito bite. There are several symptoms of infection, and the most common ones are joint pain and fever; alternatively, joint swelling, headache, and rashes are other symptoms that are different for each individual. Up to now, there is no medicine to treat or vaccine to prevent Chikungunya. However, there are some procedures that may relieve some symptoms, such as the administration of fluids, rest, and over-the-counter pain medications. People with medical conditions such as diabetes, high blood pressure, or heart disease are at risk for more severe disease as well as older adults and newborns infected around the time of birth. In general, most patients feel better within a week; however, the seriousness of this disease stems from the fact that joint pain can be severe and disabling and may persist for months. This disease presents a challenge for medical care professionals in many countries around the world.

In what follows, we are going to utilize the proposed approaches to analyze the data of some patients given in Table 7, which will help the decision-makers to make an accurate decision for the patients under consideration. In Table 7, we display the set of patients $U = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ in rows, while we put the symptoms (or attributes) of Chikungunya disease in the columns as follows:

E_1 is a fever, E_2 is joint pain, E_3 is joint swelling, E_4 is a headache, and E_5 is a rashes, where E_1, E_2, E_3, E_4 take two values: 'T' and 'F' which respectively denote the occurrence or non-occurrence of a symptom. Whereas the attribute E_5 takes three values: first-degree (1-d), second-degree (2-d), and third-degree (3-d). In the last column, we set the decision of disease D as having two values "infected" or "uninfected".

Table 7. Information system for Chikungunya disease.

U	E_1	E_2	E_3	E_4	E_5	Chikungunya disease
s_1	T	F	F	F	(1-d)	uninfected
s_2	T	T	F	F	(3-d)	infected
s_3	F	F	T	T	(1-d)	uninfected
s_4	T	T	T	F	(3-d)	infected
s_5	F	F	T	T	(1-d)	uninfected
s_6	F	T	F	T	(2-d)	uninfected
s_7	T	F	T	T	(2-d)	infected

Now, we compute the values of the variable descriptions of the patient symptoms provided in Table 7. This procedure is completed, as shown in Table 8, by finding the similarity degrees between the patients' symptoms which are computed as follows:

$$\varphi(s_i, s_j) = \frac{\sum_{k=1}^{n}(E_k(s_i) = E_k(s_j))}{n}, \tag{6.1}$$

where $\varphi(s_i, s_j)$ denotes the similarity degree between two patients s_i, s_j and n denotes the number of patients' symptoms.

Table 8. Degrees of similarity for the symptoms of seven patients.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_1	1	0.6	0.4	0.4	0.4	0.2	0.4
s_2	0.6	1	0	0.8	0	0.4	0.2
s_3	0.4	0	1	0.2	1	0.4	0.6
s_4	0.4	0.8	0.2	1	0.2	0.2	0.4
s_5	0.4	0	1	0.2	1	0.4	0.6
s_6	0.2	0.4	0.4	0.2	0.4	1	0.4
s_7	0.4	0.2	0.6	0.4	0.6	0.4	1

Then, let us consider the patients that are associated with each other according to their existing symptoms by the relation $s_i\mathcal{R}s_j \iff \varphi(s_i, s_j) \geq 0.6$, where $\varphi(s_i, s_j)$ is calculated by using Eq (6.1). It is worth noting that the proposed relations are provided by the experts in charge of the system. This means that it may change according to the estimation of the experts.

To initiate the G-approximation spaces, we chose to build the basic neighborhood system ω_j. It is obvious that there exist two types of ω_j-neighborhoods because of the symmetry of the proposed relation $\mathcal{R}$. We remark that $(s_2, s_4), (s_4, s_7) \in \mathcal{R}$ but $(s_2, s_7) \notin \mathcal{R}$, so $\mathcal{R}$ is not a transitive relation. Hence, the P-approximation space fails to describe this type of information system.

By the proposed relation and the given similarities we obtain that $\mathcal{R} = \{(s_1, s_1), (s_2, s_2), (s_3, s_3), (s_4, s_4), (s_5, s_5), (s_6, s_6), (s_7, s_7), (s_1, s_2), (s_2, s_1), (s_2, s_4), (s_3, s_7), (s_4, s_2), (s_5, s_7), (s_7, s_3), (s_7, s_5)\}$.

Without loss of generality, let the ideal be $\mathcal{K} = \{\phi, \{s_2\}, \{s_7\}, \{s_2, s_7\}\}$.

In Table 9, we suffice by calculating the ω_j-neighborhoods, $\mathbb{I}_j$-neighborhoods and $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods for $j = a$.

Table 9. ω_a-neighborhoods, $\mathbb{I}_a$-neighborhoods, and $\mathbb{I}_a^{\mathcal{K}}$-neighborhoods.

	ω_a-neighborhood	$\mathbb{I}_a$-neighborhood	$\mathbb{I}_a^{\mathcal{K}}$-neighborhood
s_1	$\{s_1, s_2\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_2\}$
s_2	$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$
s_3	$\{s_3, s_7\}$	$\{s_3, s_5, s_7\}$	$\{s_3, s_7\}$
s_4	$\{s_2, s_4\}$	$\{s_1, s_2, s_4\}$	$\{s_2, s_4\}$
s_5	$\{s_5, s_7\}$	$\{s_3, s_5, s_7\}$	$\{s_5, s_7\}$
s_6	$\{s_6\}$	$\{s_6\}$	$\{s_6\}$
s_7	$\{s_3, s_5, s_7\}$	$\{s_3, s_5, s_7\}$	$\{s_3, s_5, s_7\}$

For the uninfected set with Chikungunya, i.e., $F = \{s_1, s_3, s_5, s_6\}$ and the infected set with Chikungunya, i.e., $L = \{s_2, s_4, s_7\}$, we compute their approximation operators (lower and upper), boundary regions, and measures of accuracy by employing the methodologies in [14, 36] and our methodology given in the previous section.

(i) For patients without infection with Chikungunya $F = \{s_1, s_3, s_5, s_6\}$, the settings are as follows:

1) Al-shami et al.'s technique [14]:
   - lower approximation: $R_{\bigstar}^{\mathbb{I}_j}(F) = \{s_6\}$;
   - upper approximation: $R^{\bigstar^{\mathbb{I}_j}}(F) = U$;
   - region of boundary: $BND_R^{\bigstar^{\mathbb{I}_j}}(F) = U \setminus \{s_6\}$
   - measure of accuracy: $ACC_R^{\bigstar^{\mathbb{I}_j}}(F) = \frac{1}{7}$
2) Hosny et al.'s technique [36]:
   - lower approximation: $L_{\bigstar}^{\mathbb{I}_j}(F) = \{s_3, s_5, s_6, s_7\}$;
   - upper approximation: $U^{\bigstar^{\mathbb{I}_j}}(F) = U$;
   - region of boundary: $\Delta_R^{\bigstar^{\mathbb{I}_j}}(F) = U \setminus \{s_3, s_5, s_6, s_7\}$;
   - measure of accuracy: $\mathcal{M}_R^{\bigstar^{\mathbb{I}_j}}(F) = \frac{3}{7}$.
3) Our technique:
   - lower approximation: $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(F) = \{s_1, s_3, s_5, s_6, s_7\}$;
   - upper approximation: $R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(F) = U \setminus \{s_4\}$;
   - region of boundary: $BND_R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(F) = \{s_2\}$;
   - measure of accuracy: $ACC_R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(F) = \frac{2}{3}$.

(ii) For patients with infection with Chikungunya $L = \{s_2, s_4, s_7\}$, the settings are as follows:

1) Al-shami et al.'s technique [14]:

- lower approximation: $R_{\bigstar}^{\mathbb{I}_j}(L) = \phi$;
- upper approximation: $R^{\bigstar^{\mathbb{I}_j}}(L) = U \setminus \{s_6\}$;
- region of boundary: $BND_R^{\bigstar^{\mathbb{I}_j}}(L) = U \setminus \{s_6\}$;
- measure of accuracy: $ACC_R^{\bigstar^{\mathbb{I}_j}}(L) = 0$.

2) Hosny et al.'s technique [36]:
- lower approximation: $L_{\bigstar}^{\mathbb{I}_j}(F) = \phi$;
- upper approximation: $U^{\bigstar^{\mathbb{I}_j}}(F) = U \setminus \{s_3, s_5, s_6, s_7\}$;
- region of boundary: $\Delta_R^{\bigstar^{\mathbb{I}_j}}(F) = U \setminus \{s_3, s_5, s_6, s_7\}$;
- measure of accuracy: $\mathcal{M}_R^{\bigstar^{\mathbb{I}_j}}(F) = 0$.

3) Our technique:
- lower approximation: $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(L) = \{s_4\}$;
- upper approximation: $R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L) = \{s_2, s_4\}$;
- region of boundary: $BND_R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L) = \{s_2\}$;
- measure of accuracy: $ACC_R^{\bigstar^{\mathbb{I}_j^{\mathcal{K}}}}(L) = \frac{1}{3}$.

According to the aforementioned calculations, it can be seen that the boundary regions of uninfected set with Chikungunya and infected set with Chikungunya, as obtained by using the approach in [14, 36] are $\mathcal{U} \setminus \{s_6\}$ and $U \setminus \{s_3, s_5, s_6, s_7\}$, respectively which means that we are unable, in this situation, to determine whether these people are infected with Chikungunya. Thus, the area of vagueness/uncertainty enlarges, and hence the decision-making accuracy decreases. On the other hand, the boundary region of these two subsets according to the current method is $\{s_2\}$. This means that we successfully reduced the area of vagueness/uncertainty for both subsets. This directly leads to maximizing the measure of accuracy and increasing the confidence in the decision.

Finally, we provide Algorithm 1 to show how the approximation operator, boundary region, and measure of accuracy are determined via our approach.

Algorithm 1: The algorithm for computing approximation operators and accuracy measure using $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods.

Input : The group of patients $U = \{s_1, s_2, ..., s_n\}$ under study, the common symptoms of Chikungunya disease $E = \{E_1, E_2, ..., E_m\}$

Output: Lower approximation, upper approximation, boundary region and accuracy measure of a subset F of patients.

Build the information system of Chikungunya by inputing the values of symptoms for each patient;
Construct similarity matrix $(\varphi(s_i, s_j))_{n\times n}$ using Eq (6.1);
Ask experts of the system to determine the relation between patients by considering the values of the similarity matrix;
Provide an ideal structure;
for $j \in \mho$ **do**
    Calculate the basic granule ω_j-neighborhoods;
    Find $\mathbb{I}_j$-neighborhoods;
    Find $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods;
end
for $F \subseteq U$ **do**
    Calculate lower approximation of F : $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(F)$;
    Calculate upper approximation of F : $R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F)$;
end
if $R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(F) = \phi$ **then**
    Set accuracy measure $ACC_R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F) = 0$;
    Set boundary region $BND_R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F) = R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F)$
else
    Find boundary region by applying $BND_R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F) = R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F) - R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(F)$;
    Find accuracy measure by applying $ACC_R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F) = \frac{|R_{\bigstar}^{\mathbb{I}_j^{\mathcal{K}}}(F)\cap F|}{|R^{\bigstar \mathbb{I}_j^{\mathcal{K}}}(F)\cup F|}$
end

7. Conclusions

The theory of rough sets is a robust approach to remedying uncertainty problems in a variety of situations by classifying their input into three essential areas. The key concepts of this theory are lower and upper approximations and accuracy measures. One of the existing techniques to improve the output of these concepts is the use of neighborhood systems, so many studies have been conducted in this line of research. Considering this, we chose this research as the focus of this manuscript.

First, we have established new types of rough set neighborhoods inspired by $\mathbb{I}_j$-neighborhoods and ideal structures $\mathcal{K}$, namely, $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods. We have demonstrated their main characterizations and derived some relationships under specific types of relations. We have also constructed a new rough set models using $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods as granular of computations. We have explored their basic features and elucidated their role in maximizing the measure of accuracy as compared to the existing models in [14,36]. Among the G-approximation spaces the best approximation operators and accuracy measures were obtained in the cases of i and $\langle i \rangle$, whereas less desirable results were produced as a result of using $\mathbb{I}_u^{\mathcal{K}}$-neighborhoods and $\mathbb{I}_{\langle i \rangle}^{\mathcal{K}}$-neighborhoods. After that, we applied $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods to build some topological spaces, which were then used to introduce G-approximation spaces. Ultimately, we have examined the performance of the proposed approach by analyzing the information system of Chikungunya disease; the numerical simulations proved the superiority of the proposed approach in terms of ability to reduce boundary regions and maximize accuracy.

Our future plan will be to combine the ideal structure with some types of the aforementioned neighborhood systems such as rough containment and subset neighborhoods to produce novel types of G-approximation spaces. The merits of such rough neighborhoods as a tool to increase the lower approximation and minimize the upper approximation constitute a strong motivation for further research. Description of real-life issues by making use of these G-approximation spaces and their topological models will be another pioneering topic for further research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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