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Research article

Dimension formulas of the highest weight exceptional Lie algebra-modules

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Abstract: Given a complex simple exceptional Lie algebra $\mathfrak{g}$, let $\mathscr{V}_\lambda$ be an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ. We provide the precise dimension formula of $\dim \mathscr{V}_\lambda$.

Keywords: Dynkin index; Weyl dimension formula; exceptional Lie algebra; highest weight module; symbolic computation
Mathematics Subject Classification: 17-04, 17B10, 17B25

1. Introduction

In his famous work on the classification of semisimple Lie subalgebras of semisimple Lie algebras, E. B. Dynkin defined an index to describe the different embeddings of subalgebras [2,3]. This index is now commonly referred to as the Dynkin index [9,10]. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, and θ be the highest root for the positive roots of $\mathfrak{g}$. We choose a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)_\mathfrak{g}$ on $\mathfrak{g}$ such that $(\theta,\theta)_\mathfrak{g} = 2$. Let $\mathfrak{s}$ and $\mathfrak{g}$ be two simple Lie algebras, with $(\cdot,\cdot)_\mathfrak{s}$ and $(\cdot,\cdot)_\mathfrak{g}$ being the corresponding normalized bilinear forms on $\mathfrak{s}$ and $\mathfrak{g}$, respectively. Now, let

$$\phi : \mathfrak{s} \longrightarrow \mathfrak{g}$$

be a Lie algebra homomorphism. Then, one can define a new bilinear form on $\mathfrak{s}$ as follows:

$$\begin{aligned} \langle\cdot,\cdot\rangle : \ \mathfrak{s}\times\mathfrak{s} &\longrightarrow \qquad \mathbb{C} \\ (x,y) &\longmapsto (\phi(x),\phi(y))_\mathfrak{g}. \end{aligned}$$

The bilinear form $\langle\cdot,\cdot\rangle$ is proportional to $(\cdot,\cdot)_\mathfrak{s}$, and the ratio is referred to as the **index** of ϕ:

$$\langle x,y\rangle = (\phi(x),\phi(y))_\mathfrak{g} = \mathrm{ind}(\mathfrak{s}\xrightarrow{\phi}\mathfrak{g})\cdot(x,y)_\mathfrak{s}, \ \ \forall x,y\in\mathfrak{s}.$$

If $\mathfrak{s}$ is a simple Lie subalgebra of $\mathfrak{g}$, then there is a canonical embedding $\phi: \mathfrak{s} \hookrightarrow \mathfrak{g}$. The **Dynkin index** of $\mathfrak{s}$ in $\mathfrak{g}$ is defined as

$$\operatorname{ind}(\mathfrak{s} \hookrightarrow \mathfrak{g}) := \frac{(x, x)_{\mathfrak{g}}}{(x, x)_{\mathfrak{s}}}, \quad x \in \mathfrak{s}.$$

Let $\pi: \mathfrak{g} \longrightarrow \mathfrak{sl}(V)$ be a representation of $\mathfrak{g}$, then the **Dynkin index of the representation** π is defined as

$$\operatorname{ind}_D(\mathfrak{g}, V) = \operatorname{ind}_D(\mathfrak{g}, \pi) := \operatorname{ind}(\mathfrak{g} \xrightarrow{\pi} \mathfrak{sl}(V)).$$

For a finite-dimensional irreducible representation (π, V_λ) of $\mathfrak{g}$ with highest weight λ, Dynkin derived the following equation to calculate $\operatorname{ind}_D(\mathfrak{g}, V_\lambda)$ (cf. [2, 3])

$$\operatorname{ind}_D(\mathfrak{g}, V_\lambda) = \frac{\dim V_\lambda}{\dim \mathfrak{g}} (\lambda, \lambda + 2\rho)_{\mathfrak{g}}, \tag{1.1}$$

where ρ is the half sum of the positive roots of $\mathfrak{g}$. The Dynkin index is not only used in representation theory (cf. [6–10]), but also used in mathematical physics (e.g., the Wess-Zumino-Witten model in conformal field theory [1]).

While studying the Dynkin index for exceptional Lie algebras (cf. [6, 7]), we found that it is necessary to derive precise dimension formulas of the highest weight finite-dimensional $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$-modules. Given a complex semisimple Lie algebra $\mathfrak{g}$, let $\mathscr{V}_\lambda$ be an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ. Then the Weyl dimension formula is

$$\dim \mathscr{V}_\lambda = \frac{\prod_{\alpha \in \Delta^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Delta^+} (\rho, \alpha)}, \tag{1.2}$$

where Δ^+ is the set of positive roots (cf. [5, Theorem 5.84]). Theoretically, the Weyl dimension formula can be used to get the dimension of a finite-dimensional highest weight module. However, as the number of positive roots increases, the computation becomes increasingly difficult. It is nearly impossible to calculate the dimension formula of a highest weight $\mathfrak{e}_8$-module by hand.

In this article, we use the computer algebra systems *SageMath*[*] and *MATHEMATICA*[†] to perform the symbolic computations. To calculate $\dim \mathscr{V}_\lambda$, we need to know the explicit set of positive roots Δ^+. We obtain the lists of positive roots Δ^+ using *SageMath*, and then we calculate the dimension formulas using the '**For**' loop in *MATHEMATICA*. We write *MATHEMATICA* programs to perform these calculations. Then, we obtain the precise dimension formulas of the irreducible finite-dimensional highest weight $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$-modules. The formulas are presented in Theorems 2.1, 3.1, 4.1, 5.1, and 6.1, respectively. With the value $(\lambda, \lambda+2\rho)_{\mathfrak{g}}$, we can compute the Dynkin index of the representation $\mathscr{V}_\lambda$ using formula (1.1). For example, if $\mathfrak{g} = \mathfrak{g}_2$ and $\mathscr{V}_\lambda$ is a finite-dimensional $\mathfrak{g}_2$-module of highest weight $\lambda = (a_1, a_2)$, then

$$(\lambda, \lambda + 2\rho)_{\mathfrak{g}_2} = \frac{2}{3}(a_1^2 + 3a_2^2 + 3a_1 a_2 + 5a_1 + 9a_2)$$

and the Dynkin index of $\mathscr{V}_\lambda$ is (cf. [6]):

$$\begin{aligned}\operatorname{ind}_D(\mathfrak{g}_2, \mathscr{V}_\lambda) = & \frac{1}{2520}(a_1 + 1)(a_2 + 1)(a_1 + a_2 + 2)(a_1 + 2a_2 + 3)(a_1 + 3a_2 + 4)(2a_1 + 3a_2 + 5) \\ & (a_1^2 + 3a_2^2 + 3a_1 a_2 + 5a_1 + 9a_2).\end{aligned}$$

For the Dynkin indices of $\mathfrak{f}_4$-, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$-modules, see [6].

[*] SageMath licensed under the GPL, Version 10.1, released on 20 August 2023.

[†] Wolfram Research, Version 13.3, license purchased by the Civil Aviation University of China.

2. Dimension formula of the highest weight $\mathfrak{g}_2$-module

Consider the $\mathcal{G}_2$ root system

$$\Delta_{\mathcal{G}_2} = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\} \cup \{\pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\},$$

and its corresponding complex simple exceptional Lie algebra $\mathfrak{g}_2$. The root system is located in $\mathbb{R}^3$ (cf. [5, page 692]), which is shown in Figure 1.

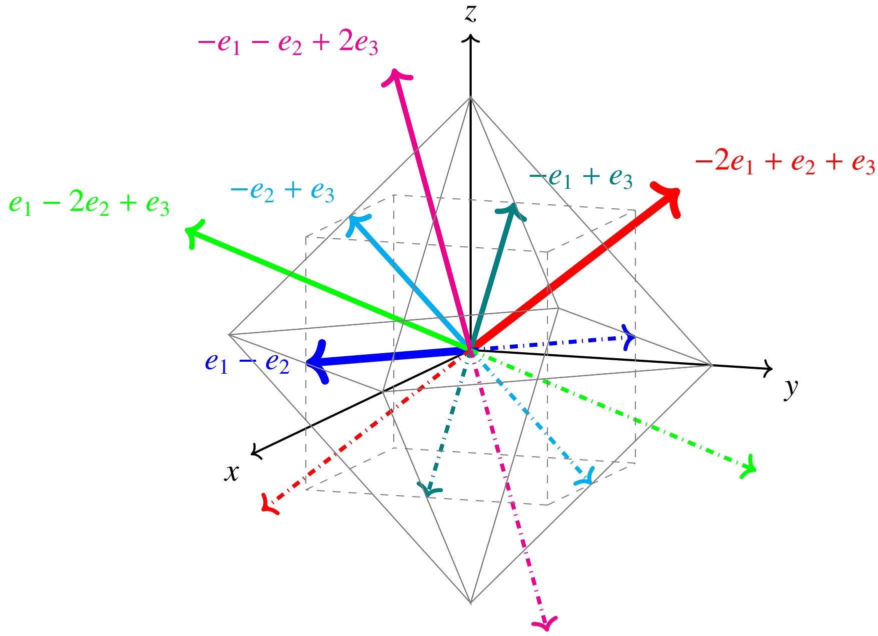

Figure 1. $\mathcal{G}_2$ root system in $\mathbb{R}^3$.

Let $\{w_1, w_2\}$ be the fundamental weights of the $\mathcal{G}_2$ root system, and $\mathscr{V}_\lambda$ an irreducible finite-dimensional $\mathfrak{g}_2$-module with the highest weight $\lambda = a_1w_1 + a_2w_2$. A detailed construction of the representation space $\mathscr{V}_\lambda$ is given in [4]. We calculate the dimension of $\mathscr{V}_\lambda$ in the following theorem.

Theorem 2.1. *Let $\mathscr{V}_\lambda$ be an irreducible finite-dimensional $\mathfrak{g}_2$-module with the highest weight $\lambda = a_1w_1 + a_2w_2$. The dimension of $\mathscr{V}_\lambda$ is equal to*

$$\frac{1}{120}(1 + a_1)(1 + a_2)(2 + a_1 + a_2)(3 + a_1 + 2a_2)(4 + a_1 + 3a_2)(5 + 2a_1 + 3a_2).$$

Proof. There are six positive roots in the $\mathcal{G}_2$ root system (cf. [5, page 692]):

$$\Delta^+_{\mathcal{G}_2} = \{e_1, e_2, e_1 + e_2, 2e_1 + e_2, 3e_1 + e_2, 3e_1 + 2e_2\}.$$

The half sum of the positive roots is equal to

$$\rho_{\mathcal{G}_2} = \frac{1}{2} \sum_{\alpha \in \Delta^+_{\mathcal{G}_2}} \alpha = -e_1 - 2e_2 + 3e_3.$$

First, we calculate the denominator of the Weyl dimension formula (1.2):

$$\prod_{\alpha\in\Delta^+_{\mathcal{G}_2}} (\rho_{\mathcal{G}_2}, \alpha) = 1 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 9 = 3240.$$

Then, we calculate the numerator part. There are two fundamental weights of the $\mathcal{G}_2$ root system (cf. [5, page 692])

$$w_1 = e_1 - e_3,\ w_2 = 2e_1 - e_2 - e_3.$$

For the $\mathfrak{g}_2$-module $\mathscr{V}_\lambda$ with the highest weight $\lambda = a_1 w_1 + a_2 w_2$, we calculate the numerator of the Weyl dimension formula (1.2):

$$\begin{aligned}&\prod_{\alpha\in\Delta^+_{\mathcal{G}_2}} (\lambda + \rho_{\mathcal{G}_2}, \alpha)\\ &= (a_1 + 1) \cdot (3a_2 + 3) \cdot (a_1 + 3a_2 + 4) \cdot (2a_1 + 3a_2 + 5) \cdot (3a_1 + 3a_2 + 6) \cdot (3a_1 + 6a_2 + 9).\end{aligned}$$

By combining both the numerator part and the denominator part, we obtain the dimension formula. □

Remark 2.1. *We can use the computer algebra system SageMath to list all the positive roots and the fundamental weights. The codes are:*

```
sage: RSgtwo=RootSystem(['G',2]).ambient_space()
sage: RSgtwo.fundamental_weights()
sage: RSgtwo.positive_roots()
```

The complete lists of the fundamental weights and positive roots are provided in Appendix A.1. We then employ MATHEMATICA for the symbolic computation of the Weyl dimension formula, with the corresponding codes attached in Appendix B.1. The MATHEMATICA output is consistent with our theorem.

3. Dimension formula of the highest weight $\mathfrak{f}_4$-module

Consider the $\mathcal{F}_4$ root system and its corresponding complex simple exceptional Lie algebra $\mathfrak{f}_4$. The root system is located in $\mathbb{R}^4$ (cf. [5, page 691]):

$$\Delta_{\mathcal{F}_4} = \{\pm e_j \pm e_k : j < k\} \cup \{\pm e_j\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

Let $\{w_1, w_2, w_3, w_4\}$ be the fundamental weights of the $\mathcal{F}_4$ root system, and $\mathscr{V}_\lambda$ an irreducible finite-dimensional $\mathfrak{f}_4$-module with the highest weight $\lambda = \sum_{j=1}^{4} a_j w_j$. We calculate the dimension of $\mathscr{V}_\lambda$ in the following theorem.

Theorem 3.1. *Let $\mathscr{V}_\lambda$ be an irreducible finite-dimensional $\mathfrak{f}_4$-module with the highest weight $\lambda = \sum_{j=1}^{4} a_j w_j$. The dimension of $\mathscr{V}_\lambda$ is equal to*

$$\frac{1}{24141680640000}(1+a_1)(1+a_2)(2+a_1+a_2)(1+a_3)(2+a_2+a_3)(3+a_1+a_2+a_3)(3+2a_2+a_3)(4+a_1+2a_2+a_3)$$
$$(5 + 2a_1 + 2a_2 + a_3)(1 + a_4)(2 + a_3 + a_4)(3 + a_2 + a_3 + a_4)(4 + a_1 + a_2 + a_3 + a_4)(4 + 2a_2 + a_3 + a_4)$$

$(5+a_1+2a_2+a_3+a_4)(6+2a_1+2a_2+a_3+a_4)(5+2a_2+2a_3+a_4)(6+a_1+2a_2+2a_3+a_4)(7+2a_1+2a_2+2a_3+a_4)$
$(7+a_1+3a_2+2a_3+a_4)(8+2a_1+3a_2+2a_3+a_4)(9+2a_1+4a_2+2a_3+a_4)(10+2a_1+4a_2+3a_3+a_4)$
$(11+2a_1+4a_2+3a_3+2a_4)$.

Proof. There are 24 positive roots in the $\mathcal{F}_4$ root system (cf. [5, page 691]):

$$\Delta^+_{\mathcal{F}_4} = \{e_j \pm e_k \ : \ j<k\} \cup \{e_j\} \cup \left\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\}$$
$$= \left\{\begin{matrix} e_1,\ e_2,\ e_3,\ e_4,\ e_1+e_2,\ e_1+e_3,\ e_1+e_4,\ e_2+e_3,\ e_2+e_4, \\ e_3+e_4,\ e_1-e_2,\ e_1-e_3,\ e_1-e_4,\ e_2-e_3,\ e_2-e_4,\ e_3-e_4, \\ \frac{1}{2}(e_1+e_2+e_3+e_4),\ \frac{1}{2}(e_1+e_2+e_3-e_4),\ \frac{1}{2}(e_1+e_2-e_3+e_4), \\ \frac{1}{2}(e_1+e_2-e_3-e_4),\ \frac{1}{2}(e_1-e_2+e_3+e_4),\ \frac{1}{2}(e_1-e_2+e_3-e_4), \\ \frac{1}{2}(e_1-e_2-e_3+e_4),\ \frac{1}{2}(e_1-e_2-e_3-e_4) \end{matrix}\right\}.$$

The half sum of the positive roots is equal to

$$\rho_{\mathcal{F}_4} = \frac{1}{2}\sum_{\alpha\in\Delta^+_{\mathcal{F}_4}} \alpha = \frac{11}{2}e_1 + \frac{5}{2}e_2 + \frac{3}{2}e_3 + \frac{1}{2}e_4.$$

First, we calculate the denominator of the Weyl dimension formula (1.2):

$$\prod_{\alpha\in\Delta^+_{\mathcal{F}_4}} (\rho_{\mathcal{F}_4}, \alpha)$$
$$= \frac{11}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot 8\cdot 7\cdot 6\cdot 4\cdot 3\cdot 2\cdot 3\cdot 4\cdot 5\cdot 1\cdot 2\cdot 1\cdot 5\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot 3\cdot\frac{5}{2}\cdot 2\cdot 1\cdot\frac{1}{2}$$
$$= 5893965000.$$

Then, we calculate the numerator part. There are four fundamental weights of the $\mathcal{F}_4$ root system (cf. [5, page 691]):

$$w_1 = e_1+e_2,\ w_2 = 2e_1+e_2+e_3,\ w_3 = \frac{3}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4,\ w_4 = e_1.$$

For the $\mathfrak{f}_4$-module $\mathscr{V}_\lambda$ with the highest weight $\lambda = a_1w_1+a_2w_2+a_3w_3+a_4w_4$, we calculate the numerator of the Weyl dimension formula (1.2):

$$\prod_{\alpha\in\Delta^+_{\mathcal{F}_4}} (\lambda+\rho_{\mathcal{F}_4}, \alpha) = \left(\frac{11}{2}+a_1+2a_2+\frac{3}{2}a_3+a_4\right)\cdot\left(\frac{5}{2}+a_1+a_2+\frac{1}{2}a_3\right)\cdot\left(\frac{3}{2}+a_2+\frac{1}{2}a_3\right)\cdot\left(\frac{1}{2}+\frac{1}{2}a_3\right)$$
$$\cdot(8+2a_1+3a_2+2a_3+a_4)\cdot(7+a_1+3a_2+2a_3+a_4)\cdot(6+a_1+2a_2+2a_3+a_4)$$
$$\cdot(4+a_1+2a_2+a_3)\cdot(3+a_1+a_2+a_3)\cdot(2+a_2+a_3)\cdot(3+a_2+a_3+a_4)\cdot(4+a_1+a_2+a_3+a_4)$$
$$\cdot(5+a_1+2a_2+a_3+a_4)\cdot(1+a_1)\cdot(2+a_1+a_2)\cdot(1+a_2)\cdot\left(5+a_1+2a_2+\frac{3}{2}a_3+\frac{1}{2}a_4\right)$$
$$\cdot\left(\frac{9}{2}+a_1+2a_2+a_3+\frac{1}{2}a_4\right)\cdot\left(\frac{7}{2}+a_1+a_2+a_3+\frac{1}{2}a_4\right)\cdot\left(3+a_1+a_2+\frac{1}{2}a_3+\frac{1}{2}a_4\right)$$
$$\cdot\left(\frac{5}{2}+a_2+a_3+\frac{1}{2}a_4\right)\cdot\left(2+a_2+\frac{1}{2}a_3+\frac{1}{2}a_4\right)\cdot\left(1+\frac{1}{2}a_3+\frac{1}{2}a_4\right)\cdot\left(\frac{1}{2}+\frac{1}{2}a_4\right).$$

By combining both the numerator part and the denominator part, we obtain the dimension formula. □

Remark 3.1. *We can use the computer algebra system SageMath to list all the positive roots and the fundamental weights. The codes are:*

```
sage: RSffour = RootSystem(['F',4 ]).ambient_space()
sage: RSffour.fundamental_weights()
sage: RSffour.positive_roots()
```

The complete lists of the fundamental weights and positive roots are provided in Appendix A.2. We then employ MATHEMATICA for the symbolic computation of the Weyl dimension formula, with the corresponding codes attached in Appendix B.2. The MATHEMATICA output is consistent with our theorem.

4. Dimension formula of the highest weight $\mathfrak{e}_6$-module

Consider the $\mathcal{E}_6$ root system and its corresponding complex simple exceptional Lie algebra $\mathfrak{e}_6$. The root system is located in the subspace of $\mathbb{R}^8$

$$V_{\mathcal{E}_6} = \{v \in \mathbb{R}^8 : \langle v, e_6 - e_7\rangle = \langle v, e_7 + e_8\rangle = 0\},$$

and the root system is (cf. [5, page 687])

$$\Delta_{\mathcal{E}_6} = \{\pm e_j \pm e_k : j < k \leq 5\} \cup \left\{ \frac{1}{2}\sum_{i=1}^{8} (-1)^{n(i)} e_i \in V_{\mathcal{E}_6} : \sum_{i=1}^{8} n(i) \text{ even} \right\}.$$

There are 36 positive roots in $\Delta_{\mathcal{E}_6}$, and we can use the computer algebra system *SageMath* to list all the positive roots and the fundamental weights. The codes are:

```
sage: RSesix = RootSystem(['E',6]).ambient_space()
sage: RSesix.fundamental_weights()
sage: RSesix.positive_roots()
```

The complete lists of the fundamental weights and positive roots are provided in Appendix A.3. For an $\mathfrak{e}_6$-module $\mathscr{V}_\lambda$, we can calculate $\dim \mathscr{V}_\lambda$ by the Weyl dimension formula (1.2). We employ *MATHEMATICA* for the symbolic computation, with the corresponding codes attached in Appendix B.3. We summarize the *MATHEMATICA* output in the following theorem.

Theorem 4.1. *Let $\{w_j : j = 1, 2, 3, 4, 5, 6\}$ be the fundamental weights of the $\mathcal{E}_6$ root system, and $\mathscr{V}_\lambda$ the irreducible finite-dimensional $\mathfrak{e}_6$-module with the highest weight $\lambda = \sum_{j=1}^{6} a_j w_j$. The dimension of $\mathscr{V}_\lambda$ is equal to*

$$\frac{1}{2336142152171520000}(1 + a_1)(1 + a_2)(1 + a_3)(2 + a_1 + a_3)(1 + a_4)(2 + a_2 + a_4)(2 + a_3 + a_4)(3 + a_1 + a_3 + a_4)$$
$$(3+a_2+a_3+a_4)(4+a_1+a_2+a_3+a_4)(1+a_5)(2+a_4+a_5)(3+a_2+a_4+a_5)(3+a_3+a_4+a_5)(4+a_1+a_3+a_4+a_5)$$
$$(4 + a_2 + a_3 + a_4 + a_5)(5 + a_1 + a_2 + a_3 + a_4 + a_5)(5 + a_2 + a_3 + 2a_4 + a_5)(6 + a_1 + a_2 + a_3 + 2a_4 + a_5)$$
$$(7+a_1+a_2+2a_3+2a_4+a_5)(1+a_6)(2+a_5+a_6)(3+a_4+a_5+a_6)(4+a_2+a_4+a_5+a_6)(4+a_3+a_4+a_5+a_6)$$
$$(5+a_1+a_3+a_4+a_5+a_6)(5+a_2+a_3+a_4+a_5+a_6)(6+a_1+a_2+a_3+a_4+a_5+a_6)(6+a_2+a_3+2a_4+a_5+a_6)$$
$$(7 + a_1 + a_2 + a_3 + 2a_4 + a_5 + a_6)(8 + a_1 + a_2 + 2a_3 + 2a_4 + a_5 + a_6)(7 + a_2 + a_3 + 2a_4 + 2a_5 + a_6)$$
$$(8 + a_1 + a_2 + a_3 + 2a_4 + 2a_5 + a_6)(9 + a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 + a_6)(10 + a_1 + a_2 + 2a_3 + 3a_4 + 2a_5 + a_6)$$
$$(11 + a_1 + 2a_2 + 2a_3 + 3a_4 + 2a_5 + a_6).$$

5. Dimension formula of the highest weight $\mathfrak{e}_7$-module

Consider the $\mathcal{E}_7$ root system and its corresponding complex simple exceptional Lie algebra $\mathfrak{e}_7$. The root system is located in the subspace of $\mathbb{R}^8$

$$V_{\mathcal{E}_7} = \{v \in \mathbb{R}^8 : \langle v, e_7 + e_8 \rangle = 0\},$$

and the root system is (cf. [5, page 688])

$$\Delta_{\mathcal{E}_7} = \{\pm e_j \pm e_k : j < k \le 6\} \cup \{\pm(e_7 - e_8)\} \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{n(i)} e_i \in V_{\mathcal{E}_7} : \sum_{i=1}^{8} n(i) \text{ even} \right\}.$$

There are 63 positive roots in $\Delta_{\mathcal{E}_7}$, and we can use the computer algebra system *SageMath* to list all the positive roots and the fundamental weights. The codes are:

```
sage: RSeseven = RootSystem(['E',7]).ambient_space()
sage: RSeseven.fundamental_weights()
RSeseven.positive_roots()
```

The complete lists of the fundamental weights and positive roots are provided in Appendix A.4. For an $\mathfrak{e}_7$-module $\mathscr{V}_\lambda$, we can calculate $\dim \mathscr{V}_\lambda$ by the Weyl dimension formula (1.2). We employ *MATHEMATICA* for the symbolic computation of the Weyl dimension formula, with the corresponding codes attached in Appendix B.4. We summarize the *MATHEMATICA* output in the following theorem.

Theorem 5.1. *Let $\{w_j : j = 1, 2, 3, 4, 5, 6, 7\}$ be the fundamental weights of the $\mathcal{E}_7$ root system, and $\mathscr{V}_\lambda$ the irreducible finite-dimensional $\mathfrak{e}_7$-module with the highest weight $\lambda = \sum_{j=1}^{7} a_j w_j$. The dimension of $\mathscr{V}_\lambda$ is equal to*

$$\begin{aligned}&((1+a_1)(1+a_2)(1+a_3)(2+a_1+a_3)(1+a_4)(2+a_2+a_4)(2+a_3+a_4)(3+a_1+a_3+a_4)(3+a_2+a_3+a_4)\\&(4+a_1+a_2+a_3+a_4)(1+a_5)(2+a_4+a_5)(3+a_2+a_4+a_5)(3+a_3+a_4+a_5)(4+a_1+a_3+a_4+a_5)(4+a_2+a_3+a_4+a_5)\\&(5+a_1+a_2+a_3+a_4+a_5)(5+a_2+a_3+2a_4+a_5)(6+a_1+a_2+a_3+2a_4+a_5)(7+a_1+a_2+2a_3+2a_4+a_5)(1+a_6)\\&(2+a_5+a_6)(3+a_4+a_5+a_6)(4+a_2+a_4+a_5+a_6)(4+a_3+a_4+a_5+a_6)(5+a_1+a_3+a_4+a_5+a_6)(5+a_2+a_3+a_4+a_5+a_6)\\&(6+a_1+a_2+a_3+a_4+a_5+a_6)(6+a_2+a_3+2a_4+a_5+a_6)(7+a_1+a_2+a_3+2a_4+a_5+a_6)(8+a_1+a_2+2a_3+2a_4+a_5+a_6)\\&(7+a_2+a_3+2a_4+2a_5+a_6)(8+a_1+a_2+a_3+2a_4+2a_5+a_6)(9+a_1+a_2+2a_3+2a_4+2a_5+a_6)(10+a_1+a_2+2a_3+3a_4+2a_5+a_6)\\&(11+a_1+2a_2+2a_3+3a_4+2a_5+a_6)(1+a_7)(2+a_6+a_7)(3+a_5+a_6+a_7)(4+a_4+a_5+a_6+a_7)\\&(5+a_2+a_4+a_5+a_6+a_7)(5+a_3+a_4+a_5+a_6+a_7)(6+a_1+a_3+a_4+a_5+a_6+a_7)(6+a_2+a_3+a_4+a_5+a_6+a_7)\\&(7+a_1+a_2+a_3+a_4+a_5+a_6+a_7)(7+a_2+a_3+2a_4+a_5+a_6+a_7)(8+a_1+a_2+a_3+2a_4+a_5+a_6+a_7)\\&(9+a_1+a_2+2a_3+2a_4+a_5+a_6+a_7)(8+a_2+a_3+2a_4+2a_5+a_6+a_7)(9+a_1+a_2+a_3+2a_4+2a_5+a_6+a_7)\\&(10+a_1+a_2+2a_3+2a_4+2a_5+a_6+a_7)(11+a_1+a_2+2a_3+3a_4+2a_5+a_6+a_7)\\&(12+a_1+2a_2+2a_3+3a_4+2a_5+a_6+a_7)(9+a_2+a_3+2a_4+2a_5+2a_6+a_7)(10+a_1+a_2+a_3+2a_4+2a_5+2a_6+a_7)\\&(11+a_1+a_2+2a_3+2a_4+2a_5+2a_6+a_7)(12+a_1+a_2+2a_3+3a_4+2a_5+2a_6+a_7)\\&(13+a_1+2a_2+2a_3+3a_4+2a_5+2a_6+a_7)(13+a_1+a_2+2a_3+3a_4+3a_5+2a_6+a_7)(14+a_1+2a_2+2a_3+3a_4+3a_5+2a_6+a_7)\\&(15+a_1+2a_2+2a_3+4a_4+3a_5+2a_6+a_7)(16+a_1+2a_2+3a_3+4a_4+3a_5+2a_6+a_7)\\&(17+2a_1+2a_2+3a_3+4a_4+3a_5+2a_6+a_7))/194034682781197905456034792184217600000000000.\end{aligned}$$

6. Dimension formula of the highest weight $\mathfrak{e}_8$-module

Consider the $\mathcal{E}_8$ root system and its corresponding complex simple exceptional Lie algebra $\mathfrak{e}_8$. The root system is located in $V_{\mathcal{E}_8} = \mathbb{R}^8$, and the root system is (cf. [5, page 689]):

$$\Delta_{\mathcal{E}_8} = \{\pm e_j \pm e_k : \ j < k\} \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{n(i)} e_i : \ \sum_{i=1}^{8} n(i) \text{ even} \right\}.$$

There are 120 positive roots in $\Delta_{\mathcal{E}_8}$, and we use the computer algebra system *SageMath* to list all the positive roots and the fundamental weights. The codes are:

```
sage: RSeeight = RootSystem(['E',8]).ambient_space()
sage: RSeeight.fundamental_weights()
sage: RSeeight.positive_roots()
```

The complete lists of the fundamental weights and positive roots are provided in Appendix A.5. For an $\mathfrak{e}_8$-module $\mathscr{V}_\lambda$, we can calculate $\dim \mathscr{V}_\lambda$ by the Weyl dimension formula (1.2). We employ *MATHEMATICA* for the symbolic computation of the Weyl dimension formula, with the corresponding codes attached in Appendix B.5. We summarize the *MATHEMATICA* output in the following theorem.

Theorem 6.1. *Let* $\{w_j : \ j = 1, 2, 3, 4, 5, 6, 7, 8\}$ *be the fundamental weights of the* $\mathcal{E}_8$ *root system, and* $\mathscr{V}_\lambda$ *the irreducible finite-dimensional* $\mathfrak{e}_8$*-module with the highest weight* $\lambda = \sum_{j=1}^{8} a_j w_j$*. The dimension of* $\mathscr{V}_\lambda$ *is equal to*
$((1+a_1)(1+a_2)(1+a_3)(2+a_1+a_3)(1+a_4)(2+a_2+a_4)(2+a_3+a_4)(3+a_1+a_3+a_4)(3+a_2+a_3+a_4)(4+a_1+$
$a_2+a_3+a_4)(1+a_5)(2+a_4+a_5)(3+a_2+a_4+a_5)(3+a_3+a_4+a_5)(4+a_1+a_3+a_4+a_5)(4+a_2+a_3+a_4+a_5)(5+$
$a_1+a_2+a_3+a_4+a_5)(5+a_2+a_3+2a_4+a_5)(6+a_1+a_2+a_3+2a_4+a_5)(7+a_1+a_2+2a_3+2a_4+a_5)(1+a_6)(2+$
$a_5+a_6)(3+a_4+a_5+a_6)(4+a_2+a_4+a_5+a_6)(4+a_3+a_4+a_5+a_6)(5+a_1+a_3+a_4+a_5+a_6)(5+a_2+a_3+a_4+a_5+$
$a_6)(6+a_1+a_2+a_3+a_4+a_5+a_6)(6+a_2+a_3+2a_4+a_5+a_6)(7+a_1+a_2+a_3+2a_4+a_5+a_6)(8+a_1+a_2+2a_3+2a_4+$
$a_5+a_6)(7+a_2+a_3+2a_4+2a_5+a_6)(8+a_1+a_2+a_3+2a_4+2a_5+a_6)(9+a_1+a_2+2a_3+2a_4+2a_5+a_6)(10+a_1+$
$a_2+2a_3+3a_4+2a_5+a_6)(11+a_1+2a_2+2a_3+3a_4+2a_5+a_6)(1+a_7)(2+a_6+a_7)(3+a_5+a_6+a_7)(4+a_4+a_5+$
$a_6+a_7)(5+a_2+a_4+a_5+a_6+a_7)(5+a_3+a_4+a_5+a_6+a_7)(6+a_1+a_3+a_4+a_5+a_6+a_7)(6+a_2+a_3+a_4+a_5+a_6+$
$a_7)(7+a_1+a_2+a_3+a_4+a_5+a_6+a_7)(7+a_2+a_3+2a_4+a_5+a_6+a_7)(8+a_1+a_2+a_3+2a_4+a_5+a_6+a_7)(9+a_1+$
$a_2+2a_3+2a_4+a_5+a_6+a_7)(8+a_2+a_3+2a_4+2a_5+a_6+a_7)(9+a_1+a_2+a_3+2a_4+2a_5+a_6+a_7)(10+a_1+a_2+$
$2a_3+2a_4+2a_5+a_6+a_7)(11+a_1+a_2+2a_3+3a_4+2a_5+a_6+a_7)(12+a_1+2a_2+2a_3+3a_4+2a_5+a_6+a_7)(9+a_2+$
$a_3+2a_4+2a_5+2a_6+a_7)(10+a_1+a_2+a_3+2a_4+2a_5+2a_6+a_7)(11+a_1+a_2+2a_3+2a_4+2a_5+2a_6+a_7)(12+$
$a_1+a_2+2a_3+3a_4+2a_5+2a_6+a_7)(13+a_1+2a_2+2a_3+3a_4+2a_5+2a_6+a_7)(13+a_1+a_2+2a_3+3a_4+3a_5+2a_6+$
$a_7)(14+a_1+2a_2+2a_3+3a_4+3a_5+2a_6+a_7)(15+a_1+2a_2+2a_3+4a_4+3a_5+2a_6+a_7)(16+a_1+2a_2+3a_3+$
$4a_4+3a_5+2a_6+a_7)(17+2a_1+2a_2+3a_3+4a_4+3a_5+2a_6+a_7)(1+a_8)(2+a_7+a_8)(3+a_6+a_7+a_8)(4+a_5+a_6+$
$a_7+a_8)(5+a_4+a_5+a_6+a_7+a_8)(6+a_2+a_4+a_5+a_6+a_7+a_8)(6+a_3+a_4+a_5+a_6+a_7+a_8)(7+a_1+a_3+a_4+a_5+$
$a_6+a_7+a_8)(7+a_2+a_3+a_4+a_5+a_6+a_7+a_8)(8+a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8)(8+a_2+a_3+2a_4+a_5+a_6+$
$a_7+a_8)(9+a_1+a_2+a_3+2a_4+a_5+a_6+a_7+a_8)(10+a_1+a_2+2a_3+2a_4+a_5+a_6+a_7+a_8)(9+a_2+a_3+2a_4+2a_5+$
$a_6+a_7+a_8)(10+a_1+a_2+a_3+2a_4+2a_5+a_6+a_7+a_8)(11+a_1+a_2+2a_3+2a_4+2a_5+a_6+a_7+a_8)(12+a_1+a_2+$
$2a_3+3a_4+2a_5+a_6+a_7+a_8)(13+a_1+2a_2+2a_3+3a_4+2a_5+a_6+a_7+a_8)(10+a_2+a_3+2a_4+2a_5+2a_6+a_7+$
$a_8)(11+a_1+a_2+a_3+2a_4+2a_5+2a_6+a_7+a_8)(12+a_1+a_2+2a_3+2a_4+2a_5+2a_6+a_7+a_8)(13+a_1+a_2+2a_3+$
$3a_4+2a_5+2a_6+a_7+a_8)(14+a_1+2a_2+2a_3+3a_4+2a_5+2a_6+a_7+a_8)(14+a_1+a_2+2a_3+3a_4+3a_5+2a_6+a_7+$

$$a_8)(15+a_1+2a_2+2a_3+3a_4+3a_5+2a_6+a_7+a_8)(16+a_1+2a_2+2a_3+4a_4+3a_5+2a_6+a_7+a_8)(17+a_1+2a_2+3a_3+4a_4+3a_5+2a_6+a_7+a_8)(18+2a_1+2a_2+3a_3+4a_4+3a_5+2a_6+a_7+a_8)(11+a_2+a_3+2a_4+2a_5+2a_6+2a_7+a_8)(12+a_1+a_2+a_3+2a_4+2a_5+2a_6+2a_7+a_8)(13+a_1+a_2+2a_3+2a_4+2a_5+2a_6+2a_7+a_8)(14+a_1+a_2+2a_3+3a_4+2a_5+2a_6+2a_7+a_8)(15+a_1+2a_2+2a_3+3a_4+2a_5+2a_6+2a_7+a_8)(15+a_1+a_2+2a_3+3a_4+3a_5+2a_6+2a_7+a_8)(16+a_1+2a_2+2a_3+3a_4+3a_5+2a_6+2a_7+a_8)(17+a_1+2a_2+2a_3+4a_4+3a_5+2a_6+2a_7+a_8)(18+a_1+2a_2+3a_3+4a_4+3a_5+2a_6+2a_7+a_8)(19+2a_1+2a_2+3a_3+4a_4+3a_5+2a_6+2a_7+a_8)(16+a_1+a_2+2a_3+3a_4+3a_5+3a_6+2a_7+a_8)(17+a_1+2a_2+2a_3+3a_4+3a_5+3a_6+2a_7+a_8)(18+a_1+2a_2+2a_3+4a_4+3a_5+3a_6+2a_7+a_8)(19+a_1+2a_2+3a_3+4a_4+3a_5+3a_6+2a_7+a_8)(20+2a_1+2a_2+3a_3+4a_4+3a_5+3a_6+2a_7+a_8)(19+a_1+2a_2+2a_3+4a_4+4a_5+3a_6+2a_7+a_8)(20+a_1+2a_2+3a_3+4a_4+4a_5+3a_6+2a_7+a_8)(21+2a_1+2a_2+3a_3+4a_4+4a_5+3a_6+2a_7+a_8)(21+a_1+2a_2+3a_3+5a_4+4a_5+3a_6+2a_7+a_8)(22+2a_1+2a_2+3a_3+5a_4+4a_5+3a_6+2a_7+a_8)(22+a_1+3a_2+3a_3+5a_4+4a_5+3a_6+2a_7+a_8)(23+2a_1+3a_2+3a_3+5a_4+4a_5+3a_6+2a_7+a_8)(23+2a_1+2a_2+4a_3+5a_4+4a_5+3a_6+2a_7+a_8)(24+2a_1+3a_2+4a_3+5a_4+4a_5+3a_6+2a_7+a_8)(25+2a_1+3a_2+4a_3+6a_4+4a_5+3a_6+2a_7+a_8)(26+2a_1+3a_2+4a_3+6a_4+5a_5+3a_6+2a_7+a_8)(27+2a_1+3a_2+4a_3+6a_4+5a_5+4a_6+2a_7+a_8)(28+2a_1+3a_2+4a_3+6a_4+5a_5+4a_6+3a_7+a_8)(29+2a_1+3a_2+4a_3+6a_4+5a_5+4a_6+3a_7+2a_8))/123897617712810879871619138650110395486291766460317863400253095663136796568899058401280000000000000000000000.$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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A. Fundamental weights and positive roots of the exceptional root systems

We denote the vector $\sum_{j=1}^{l} r_j e_j$ by $(r_1, r_2, \ldots, r_l)$, where $\{e_j \ : \ j = 1, 2, \ldots, l\}$ represents the standard basis of $\mathbb{R}^l$. In our context, l corresponds to 3, 4, 8, 8, and 8 for the root systems of type $\mathcal{G}_2$, $\mathcal{F}_4$, $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{E}_8$, respectively.

A.1. Fundamental weights and positive roots of $\mathcal{G}_2$

The 2 fundamental weights are

```
1: (1, 0, -1), 2: (2, -1, -1).
```

The 6 positive roots are

```
(0, 1, -1), (1, -2, 1), (1, -1, 0), (1, 0, -1), (1, 1, -2), (2, -1, -1).
```

A.2. Fundamental weights and positive roots of $\mathcal{F}_4$

The 4 fundamental weights are

```
1: (1, 1, 0, 0), 2: (2, 1, 1, 0), 3: (3/2, 1/2, 1/2, 1/2), 4: (1, 0, 0, 0).
```

The 24 positive roots are

```
(1, 0, 0, 0),
(0, 1, 0, 0),
(0, 0, 1, 0),
(0, 0, 0, 1),
(1, 1, 0, 0),
```

```
(1, 0, 1, 0),
(1, 0, 0, 1),
(0, 1, 1, 0),
(0, 1, 0, 1),
(0, 0, 1, 1),
(1, -1, 0, 0),
(1, 0, -1, 0),
(1, 0, 0, -1),
(0, 1, -1, 0),
(0, 1, 0, -1),
(0, 0, 1, -1),
(1/2, 1/2, 1/2, 1/2),
(1/2, 1/2, 1/2, -1/2),
(1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2),
(1/2, -1/2, 1/2, 1/2),
(1/2, -1/2, 1/2, -1/2),
(1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, -1/2).
```

A.3. Fundamental weights and positive roots of $\mathcal{E}_6$

The 6 fundamental weights are

```
1: (0, 0, 0, 0, 0, -2/3, -2/3, 2/3),
2: (1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
3: (-1/2, 1/2, 1/2, 1/2, 1/2, -5/6, -5/6, 5/6),
4: (0, 0, 1, 1, 1, -1, -1, 1),
5: (0, 0, 0, 1, 1, -2/3, -2/3, 2/3),
6: (0, 0, 0, 0, 1, -1/3, -1/3, 1/3).
```

The 36 positive roots are

```
(1, 1, 0, 0, 0, 0, 0, 0),
(1, 0, 1, 0, 0, 0, 0, 0),
(1, 0, 0, 1, 0, 0, 0, 0),
(1, 0, 0, 0, 1, 0, 0, 0),
(0, 1, 1, 0, 0, 0, 0, 0),
(0, 1, 0, 1, 0, 0, 0, 0),
(0, 1, 0, 0, 1, 0, 0, 0),
(0, 0, 1, 1, 0, 0, 0, 0),
(0, 0, 1, 0, 1, 0, 0, 0),
(0, 0, 0, 1, 1, 0, 0, 0),
(-1, 1, 0, 0, 0, 0, 0, 0),
(-1, 0, 1, 0, 0, 0, 0, 0),
```

```
(-1, 0, 0, 1, 0, 0, 0, 0),
(-1, 0, 0, 0, 1, 0, 0, 0),
(0, -1, 1, 0, 0, 0, 0, 0),
(0, -1, 0, 1, 0, 0, 0, 0),
(0, -1, 0, 0, 1, 0, 0, 0),
(0, 0, -1, 1, 0, 0, 0, 0),
(0, 0, -1, 0, 1, 0, 0, 0),
(0, 0, 0, -1, 1, 0, 0, 0),
(1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2).
```

A.4. Fundamental weights and positive roots of $\mathcal{E}_7$

The 7 fundamental weights are

```
1: (0, 0, 0, 0, 0, 0, -1, 1),  2: (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1, 1),
3: (-1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -3/2, 3/2),  4: (0, 0, 1, 1, 1, 1, -2, 2),
5: (0, 0, 0, 1, 1, 1, -3/2, 3/2),  6: (0, 0, 0, 0, 1, 1, -1, 1),
7: (0, 0, 0, 0, 0, 1, -1/2, 1/2).
```

The 63 positive roots are

```
(1, 1, 0, 0, 0, 0, 0, 0),
(1, 0, 1, 0, 0, 0, 0, 0),
(1, 0, 0, 1, 0, 0, 0, 0),
(1, 0, 0, 0, 1, 0, 0, 0),
(1, 0, 0, 0, 0, 1, 0, 0),
(0, 1, 1, 0, 0, 0, 0, 0),
(0, 1, 0, 1, 0, 0, 0, 0),
(0, 1, 0, 0, 1, 0, 0, 0),
(0, 1, 0, 0, 0, 1, 0, 0),
```

```
(0, 0, 1, 1, 0, 0, 0, 0),
(0, 0, 1, 0, 1, 0, 0, 0),
(0, 0, 1, 0, 0, 1, 0, 0),
(0, 0, 0, 1, 1, 0, 0, 0),
(0, 0, 0, 1, 0, 1, 0, 0),
(0, 0, 0, 0, 1, 1, 0, 0),
(-1, 1, 0, 0, 0, 0, 0, 0),
(-1, 0, 1, 0, 0, 0, 0, 0),
(-1, 0, 0, 1, 0, 0, 0, 0),
(-1, 0, 0, 0, 1, 0, 0, 0),
(-1, 0, 0, 0, 0, 1, 0, 0),
(0, -1, 1, 0, 0, 0, 0, 0),
(0, -1, 0, 1, 0, 0, 0, 0),
(0, -1, 0, 0, 1, 0, 0, 0),
(0, -1, 0, 0, 0, 1, 0, 0),
(0, 0, -1, 1, 0, 0, 0, 0),
(0, 0, -1, 0, 1, 0, 0, 0),
(0, 0, -1, 0, 0, 1, 0, 0),
(0, 0, 0, -1, 1, 0, 0, 0),
(0, 0, 0, -1, 0, 1, 0, 0),
(0, 0, 0, 0, -1, 1, 0, 0),
(0, 0, 0, 0, 0, 0, -1, 1),
(1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, -1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, -1/2, -1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
```

```
(-1/2, 1/2, -1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2, 1/2).
```

A.5. Fundamental weights and positive roots of $\mathcal{E}_8$

The 8 fundamental weights are

```
1: (0, 0, 0, 0, 0, 0, 0, 2),  2: (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 5/2),
3: (-1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 7/2),  4: (0, 0, 1, 1, 1, 1, 1, 5),
5: (0, 0, 0, 1, 1, 1, 1, 4),  6: (0, 0, 0, 0, 1, 1, 1, 3),
7: (0, 0, 0, 0, 0, 1, 1, 2),  8: (0, 0, 0, 0, 0, 0, 1, 1).
```

The 120 positive roots are

```
(1, 1, 0, 0, 0, 0, 0, 0),
(1, 0, 1, 0, 0, 0, 0, 0),
(1, 0, 0, 1, 0, 0, 0, 0),
(1, 0, 0, 0, 1, 0, 0, 0),
(1, 0, 0, 0, 0, 1, 0, 0),
(1, 0, 0, 0, 0, 0, 1, 0),
(1, 0, 0, 0, 0, 0, 0, 1),
(0, 1, 1, 0, 0, 0, 0, 0),
(0, 1, 0, 1, 0, 0, 0, 0),
(0, 1, 0, 0, 1, 0, 0, 0),
(0, 1, 0, 0, 0, 1, 0, 0),
(0, 1, 0, 0, 0, 0, 1, 0),
(0, 1, 0, 0, 0, 0, 0, 1),
(0, 0, 1, 1, 0, 0, 0, 0),
(0, 0, 1, 0, 1, 0, 0, 0),
(0, 0, 1, 0, 0, 1, 0, 0),
(0, 0, 1, 0, 0, 0, 1, 0),
(0, 0, 1, 0, 0, 0, 0, 1),
(0, 0, 0, 1, 1, 0, 0, 0),
(0, 0, 0, 1, 0, 1, 0, 0),
(0, 0, 0, 1, 0, 0, 1, 0),
(0, 0, 0, 1, 0, 0, 0, 1),
```

```
(0, 0, 0, 0, 1, 1, 0, 0),
(0, 0, 0, 0, 1, 0, 1, 0),
(0, 0, 0, 0, 1, 0, 0, 1),
(0, 0, 0, 0, 0, 1, 1, 0),
(0, 0, 0, 0, 0, 1, 0, 1),
(0, 0, 0, 0, 0, 0, 1, 1),
(-1, 1, 0, 0, 0, 0, 0, 0),
(-1, 0, 1, 0, 0, 0, 0, 0),
(-1, 0, 0, 1, 0, 0, 0, 0),
(-1, 0, 0, 0, 1, 0, 0, 0),
(-1, 0, 0, 0, 0, 1, 0, 0),
(-1, 0, 0, 0, 0, 0, 1, 0),
(-1, 0, 0, 0, 0, 0, 0, 1),
(0, -1, 1, 0, 0, 0, 0, 0),
(0, -1, 0, 1, 0, 0, 0, 0),
(0, -1, 0, 0, 1, 0, 0, 0),
(0, -1, 0, 0, 0, 1, 0, 0),
(0, -1, 0, 0, 0, 0, 1, 0),
(0, -1, 0, 0, 0, 0, 0, 1),
(0, 0, -1, 1, 0, 0, 0, 0),
(0, 0, -1, 0, 1, 0, 0, 0),
(0, 0, -1, 0, 0, 1, 0, 0),
(0, 0, -1, 0, 0, 0, 1, 0),
(0, 0, -1, 0, 0, 0, 0, 1),
(0, 0, 0, -1, 1, 0, 0, 0),
(0, 0, 0, -1, 0, 1, 0, 0),
(0, 0, 0, -1, 0, 0, 1, 0),
(0, 0, 0, -1, 0, 0, 0, 1),
(0, 0, 0, 0, -1, 1, 0, 0),
(0, 0, 0, 0, -1, 0, 1, 0),
(0, 0, 0, 0, -1, 0, 0, 1),
(0, 0, 0, 0, 0, -1, 1, 0),
(0, 0, 0, 0, 0, -1, 0, 1),
(0, 0, 0, 0, 0, 0, -1, 1),
(1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2),
(1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2, 1/2),
(1/2, 1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2, 1/2),
(1/2, 1/2, 1/2, -1/2, -1/2, 1/2, 1/2, 1/2),
(1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
```

```
(1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2),
(1/2, 1/2, -1/2, 1/2, -1/2, 1/2, 1/2, 1/2),
(1/2, 1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2),
(1/2, 1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2),
(1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2, 1/2),
(1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2, 1/2),
(1/2, -1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, -1/2, 1/2, 1/2, 1/2, 1/2),
(1/2, -1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, -1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2),
(1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 1/2),
(1/2, -1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2, 1/2),
(1/2, -1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(1/2, -1/2, -1/2, -1/2, 1/2, -1/2, 1/2, 1/2),
(1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2),
(1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, 1/2, 1/2, -1/2, 1/2, 1/2),
(-1/2, 1/2, 1/2, 1/2, -1/2, 1/2, 1/2, 1/2),
(-1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, -1/2, 1/2, 1/2, 1/2, 1/2),
(-1/2, 1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, -1/2, -1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2),
(-1/2, 1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 1/2),
(-1/2, 1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, 1/2, -1/2, -1/2, 1/2, 1/2),
(-1/2, 1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, 1/2, -1/2, -1/2, 1/2, -1/2, 1/2, 1/2),
(-1/2, 1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2),
(-1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2),
(-1/2, -1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, 1/2, -1/2, 1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, 1/2, -1/2, -1/2, 1/2, 1/2),
```

```
(-1/2, -1/2, 1/2, -1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, -1/2, 1/2, -1/2, 1/2, -1/2, 1/2, 1/2),
(-1/2, -1/2, 1/2, -1/2, -1/2, 1/2, 1/2, 1/2),
(-1/2, -1/2, 1/2, -1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, 1/2, 1/2, 1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, 1/2, 1/2, -1/2, 1/2, 1/2),
(-1/2, -1/2, -1/2, 1/2, -1/2, 1/2, 1/2, 1/2),
(-1/2, -1/2, -1/2, 1/2, -1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2),
(-1/2, -1/2, -1/2, -1/2, 1/2, -1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, -1/2, -1/2, 1/2, -1/2, 1/2),
(-1/2, -1/2, -1/2, -1/2, -1/2, -1/2, 1/2, 1/2).
```

B. *MATHEMATICA* codes for the dimension formulas

The *MATHEMATICA* codes can be downloaded from *GitHub* or *Bitbucket*:
https://github.com/luanyongzhi/Dimensions-G2F4E6E7E8-Modules.git
https://bitbucket.org/luanyongzhi-research/dimensions-g2f4e6e7e8-modules

B.1. MATHEMATICA codes for the $\mathfrak{g}_2$*-module*

We use *MATHEMATICA* to derive a general dimension formula for $\mathscr{V}_\lambda$ with $\lambda = a_1 w_1 + a_2 w_2$. The corresponding codes are provided below:

```
PosRootsGtwo = {{0, 1, −1}, {1, −2, 1}, {1, −1, 0}, {1, 0, −1}, {1, 1, −2}, {2, −1, −1}};
Clear[s]
s = Array[0&, 3];
For[i = 1, i ≤ Length[PosRootsGtwo], s = s + PosRootsGtwo[[i]]; i++];
RhoGtwo = s/2;
FWeightsGtwo = {{1, 0, −1}, {2, −1, −1}};
dim = 1;
Clear[a1, a2]
For[i = 1, i ≤ Length[PosRootsGtwo], dim = dim ∗(Dot[{a1, a2}.
FWeightsGtwo + RhoGtwo, PosRootsGtwo[[i]]]/Dot[RhoGtwo, PosRootsGtwo[[i]]])//Simplify; i++]
dim //Simplify
```

B.2. MATHEMATICA codes for the $\mathfrak{f}_4$*-module*

We use *MATHEMATICA* to derive a general dimension formula for $\mathscr{V}_\lambda$ with $\lambda = \sum_{j=1}^{4} a_j w_j$. The corresponding codes are provided below:

```
PosRootsFfour = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {1, 1, 0, 0}, {1, 0, 1, 0}, {1, 0, 0, 1},
{0, 1, 1, 0}, {0, 1, 0, 1}, {0, 0, 1, 1}, {1, −1, 0, 0}, {1, 0, −1, 0}, {1, 0, 0, −1}, {0, 1, −1, 0}, {0, 1, 0, −1},
{0, 0, 1, −1}, {1/2, 1/2, 1/2, 1/2}, {1/2, 1/2, 1/2, −1/2}, {1/2, 1/2, −1/2, 1/2}, {1/2, 1/2, −1/2, −1/2},
{1/2, −1/2, 1/2, 1/2}, {1/2, −1/2, 1/2, −1/2}, {1/2, −1/2, −1/2, 1/2}, {1/2, −1/2, −1/2, −1/2}};
```

```
Clear[s]
s = Array[0&, 4];
For[i = 1, i ≤ Length[PosRootsFfour], s = s + PosRootsFfour[[i]]; i++];
RhoFfour = s/2;
FWeightsFfour = {{1, 1, 0, 0}, {2, 1, 1, 0}, {3/2, 1/2, 1/2, 1/2}, {1, 0, 0, 0}};
dim = 1;
Clear[a1, a2, a3, a4]
For[i = 1, i ≤ Length[PosRootsFfour],
dim = dim * Dot[{a1,a2,a3,a4}.FWeightsFfour+RhoFfour,PosRootsFfour[[i]]] / Dot[RhoFfour,PosRootsFfour[[i]]] //Simplify; i++]
dim //Simplify
```

B.3. MATHEMATICA codes for the $\mathfrak{e}_6$-module

We use *MATHEMATICA* to derive a general dimension formula for $\mathscr{V}_\lambda$ with $\lambda = \sum_{j=1}^{6} a_j w_j$. The corresponding codes are provided below:

```
PosRootsEsix = {{1, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 1, 0, 0, 0, 0, 0}, {1, 0, 0, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0},
{0, 1, 1, 0, 0, 0, 0, 0}, {0, 1, 0, 1, 0, 0, 0, 0}, {0, 1, 0, 0, 1, 0, 0, 0}, {0, 0, 1, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 1, 0, 0, 0},
{0, 0, 0, 1, 1, 0, 0, 0}, {−1, 1, 0, 0, 0, 0, 0, 0}, {−1, 0, 1, 0, 0, 0, 0, 0}, {−1, 0, 0, 1, 0, 0, 0, 0},
{−1, 0, 0, 0, 1, 0, 0, 0}, {0, −1, 1, 0, 0, 0, 0, 0}, {0, −1, 0, 1, 0, 0, 0, 0}, {0, −1, 0, 0, 1, 0, 0, 0},
{0, 0, −1, 1, 0, 0, 0, 0}, {0, 0, −1, 0, 1, 0, 0, 0}, {0, 0, 0, −1, 1, 0, 0, 0},
{1/2, 1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, 1/2, −1/2, −1/2, −1/2, −1/2, 1/2},
{1/2, 1/2, −1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, −1/2, −1/2, 1/2, −1/2, −1/2, 1/2},
{1/2, −1/2, 1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, −1/2, 1/2, −1/2, 1/2, −1/2, −1/2, 1/2},
{1/2, −1/2, −1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {1/2, −1/2, −1/2, −1/2, −1/2, −1/2, −1/2, 1/2},
{−1/2, 1/2, 1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, 1/2, −1/2, 1/2, −1/2, −1/2, 1/2},
{−1/2, 1/2, −1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, −1/2, −1/2, −1/2, −1/2, −1/2, 1/2},
{−1/2, −1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, 1/2, −1/2, −1/2, −1/2, −1/2, 1/2},
{−1/2, −1/2, −1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, −1/2, −1/2, 1/2, −1/2, −1/2, 1/2}};
Clear[s]
s = Array[0&, 8];
For[i = 1, i ≤ Length[PosRootsEsix], s = s + PosRootsEsix[[i]]; i++];
RhoEsix = s/2;
FWeightsEsix = {{0, 0, 0, 0, 0, −2/3, −2/3, 2/3}, {1/2, 1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2},
{−1/2, 1/2, 1/2, 1/2, 1/2, −5/6, −5/6, 5/6}, {0, 0, 1, 1, 1, −1, −1, 1}, {0, 0, 0, 1, 1, −2/3, −2/3, 2/3},
{0, 0, 0, 0, 1, −1/3, −1/3, 1/3}};
dim = 1;
Clear[a1, a2, a3, a4, a5, a6]
For[i = 1, i ≤ Length[PosRootsEsix],
dim = dim * Dot[{a1,a2,a3,a4,a5,a6}.FWeightsEsix+RhoEsix,PosRootsEsix[[i]]] / Dot[RhoEsix,PosRootsEsix[[i]]] //Simplify; i++
dim //Simplify
```

B.4. MATHEMATICA codes for the $\mathfrak{e}_7$-module

We use *MATHEMATICA* to derive a general dimension formula for $\mathscr{V}_\lambda$ with $\lambda = \sum_{j=1}^{7} a_j w_j$. The corresponding codes are provided below:

```
PosRootsEseven = {{1, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 1, 0, 0, 0, 0, 0}, {1, 0, 0, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0},
{1, 0, 0, 0, 0, 1, 0, 0}, {0, 1, 1, 0, 0, 0, 0, 0}, {0, 1, 0, 1, 0, 0, 0, 0}, {0, 1, 0, 0, 1, 0, 0, 0}, {0, 1, 0, 0, 0, 1, 0, 0},
{0, 0, 1, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 1, 0, 0, 0}, {0, 0, 1, 0, 0, 1, 0, 0}, {0, 0, 0, 1, 1, 0, 0, 0}, {0, 0, 0, 1, 0, 1, 0, 0},
{0, 0, 0, 0, 1, 1, 0, 0}, {−1, 1, 0, 0, 0, 0, 0, 0}, {−1, 0, 1, 0, 0, 0, 0, 0}, {−1, 0, 0, 1, 0, 0, 0, 0},
{−1, 0, 0, 0, 1, 0, 0, 0}, {−1, 0, 0, 0, 0, 1, 0, 0}, {0, −1, 1, 0, 0, 0, 0, 0}, {0, −1, 0, 1, 0, 0, 0, 0},
{0, −1, 0, 0, 1, 0, 0, 0}, {0, −1, 0, 0, 0, 1, 0, 0}, {0, 0, −1, 1, 0, 0, 0, 0}, {0, 0, −1, 0, 1, 0, 0, 0},
{0, 0, −1, 0, 0, 1, 0, 0}, {0, 0, 0, −1, 1, 0, 0, 0}, {0, 0, 0, −1, 0, 1, 0, 0}, {0, 0, 0, 0, −1, 1, 0, 0},
{0, 0, 0, 0, 0, 0, −1, 1}, {1/2, 1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2},
{1/2, 1/2, 1/2, 1/2, −1/2, 1/2, −1/2, 1/2}, {1/2, 1/2, 1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, 1/2, 1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, −1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, 1/2, −1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, −1/2, −1/2, 1/2, −1/2, −1/2, 1/2},
{1/2, 1/2, −1/2, −1/2, −1/2, 1/2, −1/2, 1/2}, {1/2, −1/2, 1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, −1/2, 1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, −1/2, 1/2, −1/2, 1/2, −1/2, −1/2, 1/2},
{1/2, −1/2, 1/2, −1/2, −1/2, 1/2, −1/2, 1/2}, {1/2, −1/2, −1/2, 1/2, 1/2, −1/2, −1/2, 1/2},
{1/2, −1/2, −1/2, 1/2, −1/2, 1/2, −1/2, 1/2}, {1/2, −1/2, −1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, −1/2, −1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, 1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{−1/2, 1/2, 1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, 1/2, −1/2, 1/2, −1/2, −1/2, 1/2},
{−1/2, 1/2, 1/2, −1/2, −1/2, 1/2, −1/2, 1/2}, {−1/2, 1/2, −1/2, 1/2, 1/2, −1/2, −1/2, 1/2},
{−1/2, 1/2, −1/2, 1/2, −1/2, 1/2, −1/2, 1/2}, {−1/2, 1/2, −1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
{−1/2, 1/2, −1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2},
{−1/2, −1/2, 1/2, 1/2, −1/2, 1/2, −1/2, 1/2}, {−1/2, −1/2, 1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
{−1/2, −1/2, 1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, −1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{−1/2, −1/2, −1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, −1/2, −1/2, 1/2, −1/2, −1/2, 1/2},
{−1/2, −1/2, −1/2, −1/2, −1/2, 1/2, −1/2, 1/2}};
Clear[s]
s = Array[0&, 8];
For[i = 1, i ≤ Length[PosRootsEseven], s = s + PosRootsEseven[[i]]; i++];
RhoEseven = s/2;
FWeightsEseven = {{0, 0, 0, 0, 0, 0, −1, 1}, {1/2, 1/2, 1/2, 1/2, 1/2, 1/2, −1, 1},
{−1/2, 1/2, 1/2, 1/2, 1/2, 1/2, −3/2, 3/2}, {0, 0, 1, 1, 1, 1, −2, 2}, {0, 0, 0, 1, 1, 1, −3/2, 3/2},
{0, 0, 0, 0, 1, 1, −1, 1}, {0, 0, 0, 0, 0, 1, −1/2, 1/2}};
dim = 1;
Clear[a1, a2, a3, a4, a5, a6, a7]
For[i = 1, i ≤ Length[PosRootsEseven],
dim = dim * Dot[{a1,a2,a3,a4,a5,a6,a7}.FWeightsEseven+RhoEseven,PosRootsEseven[[i]]] / Dot[RhoEseven,PosRootsEseven[[i]]] //Simplify; i++]
dim //Simplify
```

B.5. *MATHEMATICA codes for the* $\mathfrak{e}_8$*-module*

We use *MATHEMATICA* to derive a general dimension formula for $\mathscr{V}_\lambda$ with $\lambda = \sum_{j=1}^{8} a_j w_j$. The corresponding codes are provided below:

```
PosRootsEeight = {{1, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 1, 0, 0, 0, 0, 0}, {1, 0, 0, 1, 0, 0, 0, 0}, {1, 0, 0, 0, 1, 0, 0, 0},
{1, 0, 0, 0, 0, 1, 0, 0}, {1, 0, 0, 0, 0, 0, 1, 0}, {1, 0, 0, 0, 0, 0, 0, 1}, {0, 1, 1, 0, 0, 0, 0, 0}, {0, 1, 0, 1, 0, 0, 0, 0},
{0, 1, 0, 0, 1, 0, 0, 0}, {0, 1, 0, 0, 0, 1, 0, 0}, {0, 1, 0, 0, 0, 0, 1, 0}, {0, 1, 0, 0, 0, 0, 0, 1}, {0, 0, 1, 1, 0, 0, 0, 0},
{0, 0, 1, 0, 1, 0, 0, 0}, {0, 0, 1, 0, 0, 1, 0, 0}, {0, 0, 1, 0, 0, 0, 1, 0}, {0, 0, 1, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 1, 0, 0, 0},
{0, 0, 0, 1, 0, 1, 0, 0}, {0, 0, 0, 1, 0, 0, 1, 0}, {0, 0, 0, 1, 0, 0, 0, 1}, {0, 0, 0, 0, 1, 1, 0, 0}, {0, 0, 0, 0, 1, 0, 1, 0},
{0, 0, 0, 0, 1, 0, 0, 1}, {0, 0, 0, 0, 0, 1, 1, 0}, {0, 0, 0, 0, 0, 1, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 1}, {−1, 1, 0, 0, 0, 0, 0, 0},
{−1, 0, 1, 0, 0, 0, 0, 0}, {−1, 0, 0, 1, 0, 0, 0, 0}, {−1, 0, 0, 0, 1, 0, 0, 0}, {−1, 0, 0, 0, 0, 1, 0, 0},
{−1, 0, 0, 0, 0, 0, 1, 0}, {−1, 0, 0, 0, 0, 0, 0, 1}, {0, −1, 1, 0, 0, 0, 0, 0}, {0, −1, 0, 1, 0, 0, 0, 0},
{0, −1, 0, 0, 1, 0, 0, 0}, {0, −1, 0, 0, 0, 1, 0, 0}, {0, −1, 0, 0, 0, 0, 1, 0}, {0, −1, 0, 0, 0, 0, 0, 1},
{0, 0, −1, 1, 0, 0, 0, 0}, {0, 0, −1, 0, 1, 0, 0, 0}, {0, 0, −1, 0, 0, 1, 0, 0}, {0, 0, −1, 0, 0, 0, 1, 0},
{0, 0, −1, 0, 0, 0, 0, 1}, {0, 0, 0, −1, 1, 0, 0, 0}, {0, 0, 0, −1, 0, 1, 0, 0}, {0, 0, 0, −1, 0, 0, 1, 0},
{0, 0, 0, −1, 0, 0, 0, 1}, {0, 0, 0, 0, −1, 1, 0, 0}, {0, 0, 0, 0, −1, 0, 1, 0}, {0, 0, 0, 0, −1, 0, 0, 1},
{0, 0, 0, 0, 0, −1, 1, 0}, {0, 0, 0, 0, 0, −1, 0, 1}, {0, 0, 0, 0, 0, 0, −1, 1}, {1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2},
{1/2, 1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, 1/2, 1/2, −1/2, 1/2, −1/2, 1/2},
{1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2, 1/2}, {1/2, 1/2, 1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, 1/2, 1/2, −1/2, 1/2, −1/2, 1/2, 1/2}, {1/2, 1/2, 1/2, −1/2, −1/2, 1/2, 1/2, 1/2},
{1/2, 1/2, 1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, −1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, 1/2, −1/2, 1/2, 1/2, −1/2, 1/2, 1/2}, {1/2, 1/2, −1/2, 1/2, −1/2, 1/2, 1/2, 1/2},
{1/2, 1/2, −1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, −1/2, −1/2, 1/2, 1/2, 1/2, 1/2},
{1/2, 1/2, −1/2, −1/2, 1/2, −1/2, −1/2, 1/2}, {1/2, 1/2, −1/2, −1/2, −1/2, 1/2, −1/2, 1/2},
{1/2, 1/2, −1/2, −1/2, −1/2, −1/2, 1/2, 1/2}, {1/2, −1/2, 1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, −1/2, 1/2, 1/2, 1/2, −1/2, 1/2, 1/2}, {1/2, −1/2, 1/2, 1/2, −1/2, 1/2, 1/2, 1/2},
{1/2, −1/2, 1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {1/2, −1/2, 1/2, −1/2, 1/2, 1/2, 1/2, 1/2},
{1/2, −1/2, 1/2, −1/2, 1/2, −1/2, −1/2, 1/2}, {1/2, −1/2, 1/2, −1/2, −1/2, 1/2, −1/2, 1/2},
{1/2, −1/2, 1/2, −1/2, −1/2, −1/2, 1/2, 1/2}, {1/2, −1/2, −1/2, 1/2, 1/2, 1/2, 1/2, 1/2},
{1/2, −1/2, −1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {1/2, −1/2, −1/2, 1/2, −1/2, 1/2, −1/2, 1/2},
{1/2, −1/2, −1/2, 1/2, −1/2, −1/2, 1/2, 1/2}, {1/2, −1/2, −1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
{1/2, −1/2, −1/2, −1/2, 1/2, −1/2, 1/2, 1/2}, {1/2, −1/2, −1/2, −1/2, −1/2, 1/2, 1/2, 1/2},
{1/2, −1/2, −1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, 1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{−1/2, 1/2, 1/2, 1/2, 1/2, −1/2, 1/2, 1/2}, {−1/2, 1/2, 1/2, 1/2, −1/2, 1/2, 1/2, 1/2},
{−1/2, 1/2, 1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, 1/2, −1/2, 1/2, 1/2, 1/2, 1/2},
{−1/2, 1/2, 1/2, −1/2, 1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, 1/2, −1/2, −1/2, 1/2, −1/2, 1/2},
{−1/2, 1/2, 1/2, −1/2, −1/2, −1/2, 1/2, 1/2}, {−1/2, 1/2, −1/2, 1/2, 1/2, 1/2, 1/2, 1/2},
{−1/2, 1/2, −1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {−1/2, 1/2, −1/2, 1/2, −1/2, 1/2, −1/2, 1/2},
{−1/2, 1/2, −1/2, 1/2, −1/2, −1/2, 1/2, 1/2}, {−1/2, 1/2, −1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
{−1/2, 1/2, −1/2, −1/2, 1/2, −1/2, 1/2, 1/2}, {−1/2, 1/2, −1/2, −1/2, −1/2, 1/2, 1/2, 1/2},
{−1/2, 1/2, −1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2},
{−1/2, −1/2, 1/2, 1/2, 1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, 1/2, 1/2, −1/2, 1/2, −1/2, 1/2},
{−1/2, −1/2, 1/2, 1/2, −1/2, −1/2, 1/2, 1/2}, {−1/2, −1/2, 1/2, −1/2, 1/2, 1/2, −1/2, 1/2},
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```
{−1/2, −1/2, 1/2, −1/2, 1/2, −1/2, 1/2, 1/2}, {−1/2, −1/2, 1/2, −1/2, −1/2, 1/2, 1/2, 1/2},
{−1/2, −1/2, 1/2, −1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, −1/2, 1/2, 1/2, 1/2, −1/2, 1/2},
{−1/2, −1/2, −1/2, 1/2, 1/2, −1/2, 1/2, 1/2}, {−1/2, −1/2, −1/2, 1/2, −1/2, 1/2, 1/2, 1/2},
{−1/2, −1/2, −1/2, 1/2, −1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, −1/2, −1/2, 1/2, 1/2, 1/2, 1/2},
{−1/2, −1/2, −1/2, −1/2, 1/2, −1/2, −1/2, 1/2}, {−1/2, −1/2, −1/2, −1/2, −1/2, 1/2, −1/2, 1/2},
{−1/2, −1/2, −1/2, −1/2, −1/2, −1/2, 1/2, 1/2}};
Clear[s]
s = Array[0&, 8];
For[i = 1, i ≤ Length[PosRootsEeight], s = s + PosRootsEeight[[i]]; i++];
RhoEeight = s/2;
FWeightsEeight = {{0, 0, 0, 0, 0, 0, 0, 2}, {1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 5/2},
{−1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 7/2}, {0, 0, 1, 1, 1, 1, 1, 5}, {0, 0, 0, 1, 1, 1, 1, 4}, {0, 0, 0, 0, 1, 1, 1, 3},
{0, 0, 0, 0, 0, 1, 1, 2}, {0, 0, 0, 0, 0, 0, 1, 1}};
dim = 1;
Clear[a1, a2, a3, a4, a5, a6, a7, a8]
For[i = 1, i ≤ Length[PosRootsEeight],
dim = dim * Dot[{a1,a2,a3,a4,a5,a6,a7,a8}.FWeightsEeight+RhoEeight,PosRootsEeight[[i]]] / Dot[RhoEeight,PosRootsEeight[[i]]] //Simplify; i++]
dim //Simplify
```

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