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Research article

Inverse source problem for multi-term time-fractional diffusion equation with nonlocal boundary conditions

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Abstract: In this paper, we consider an inverse source problem with nonlocal boundary conditions for the heat equation involving multi-term time-fractional derivatives. We determine a source term independent of the space variable, and the temperature distribution from the energy measurement. We reduce the solution of the inverse problem to finding solutions to two problems. The well-posedness of each problem is shown using the generalized Fourier method.

Keywords: inverse source problem; fractional diffusion equation; nonlocal boundary condition; multinomial Mittag-Lefler function; Fourier series
Mathematics Subject Classification: 35A09, 26A33, 45J05, 34K37, 42A16

1. Introduction

For a fixed positive integer m, let α_i and q_i, $(i = 1, ..., m)$ be positive constants such that $1 > \alpha_1 > \cdots > \alpha_m > 0$. We assume $q_1 = 1$ without loss of generality. Consider the inverse problem of finding a pair of functions $\{r, u\}$ such that it satisfies the equation

$$\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u = u_{xx} + r(t) f(x,t), \; (x,t) \in D_T, \tag{1.1}$$

the initial condition

$$u(x,0) = \varphi(x), \; x \in [0,1], \tag{1.2}$$

the nonlocal boundary conditions

$$\begin{aligned} &u(0,t) + u(1,t) = 0, \; t \in [0,T], \\ &a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) = 0, \; t \in [0,T], \end{aligned} \tag{1.3}$$

and the overdetermination condition

$$\int_0^1 u(x,t)dx = E(t),\ t \in [0,T], \tag{1.4}$$

where $T > 0$; $D_T = \{(x,t) :\ 0 < x < 1,\ 0 < t \le T\}$; f, φ and E are given functions; a_0, a_1 and b_1 are real constants such that $|a_1| + |b_1| > 0$, $a_1 + b_1 \neq 0$; $\partial_t^{\alpha_i}$ is the Caputo time-fractional derivative defined in [1] by

$$\partial_t^{\alpha_i} u = \frac{1}{\Gamma(1-\alpha_i)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha_i}},$$

and $\Gamma(\cdot)$ is the Gamma function. Note that, in the case $|a_1| + |b_1| > 0$, $a_1 + b_1 = 0$, problem (1.1)–(1.4) will immediately be incorrect.

For (1.1)–(1.3), the direct problem is the determination of u in $\overline{D}_T$ such that $u(\cdot,t) \in C^2[0,1]$ and $\partial_t^{\alpha_i} u(x,\cdot) \in C(0,T]$ when the initial temperature φ and the source term rf are given and continuous.

If the function $r(t)$, $t \in [0,T]$ is unknown, the inverse problem is formulated as the problem of finding a pair of functions $\{r, u\}$ which satisfy (1.1)–(1.4) with $r \in C[0,T]$, $u(\cdot,t) \in C^2[0,1]$, $\partial_t^{\alpha_i} u(x,\cdot) \in C(0,T]$.

The integral condition (1.4) arises when the data on the boundary cannot be measured directly, but only the average value of the solution can be measured along the boundary [2].

Fractional calculus is used in quantum mechanics [3], biophysics [4], control theory [5], viscoelasticity [6], signal processing [7], biological sciences [8], and many other disciplines.

In [9], the inverse problem for the classical heat equation satisfying the boundary conditions

$$a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0,\ t \in [0,T],$$
$$c_1 u_x(0,t) + d_1 u_x(1,t) + c_0 u(0,t) + d_0 u(1,t) = 0,\ t \in [0,T],$$

was considered. The boundary conditions (1.3) are the particular case of the boundary conditions that considered in [9] when $b_0 = 0$, $c_0 = d_0 = 1$ and $c_1 = d_1 = 0$.

The practical applications of nonlocal boundary value problems span across diverse fields, encompassing chemical diffusion [10], thermal conductivity [11], biological processes [12], and others. Particularly in scenarios like multiphase flows involving liquids, solids, and gases, heat flow is often proportional to the variations in boundary temperatures among distinct phases, alongside the parameters a_0, a_1, b_1 outlined in the nonlocal boundary conditions (1.3). This elucidates the growing importance and extensive utilization of inverse problems featuring nonlocal boundary conditions across various disciplines.

We note several papers devoted to the study the inverse problem for a time-fractional diffusion equation with nonlocal boundary conditions. In [13], the inverse source problem for the time-fractional diffusion equation in two dimensions was considered. [14] focused on determining a time-dependent factor of an unknown source under certain sub-boundary conditions for the time-fractional diffusion equation using nonlocal measurement data. The paper established the existence and uniqueness of the solution to the inverse source problem by applying Lax-Milgram's lemma in appropriate Sobolev spaces. In [15], the inverse source problem for the time-fractional diffusion equation in two dimensional space was considered, where the time fractional derivative is the Hilfer derivative. In [16], the authors considered the problem of determining the distribution and the source term for the time-fractional diffusion equation. Two inverse problems for the time-fractional diffusion equation with a family of nonlocal boundary conditions were discussed in [17].

We also note some papers related to the study of impulsive differential equations and the inverse source problem for a time-fractional diffusion equation. In [18], the study focused on multipoint BVPs concerning a generalized class of impulsive fractional-order nonlinear differential equations. [19] addressed the study of a class of impulsive integro-differential equations, where impulses are not instantaneous. In [20], the estimation of an unknown source term in the time-fractional diffusion equation from measurement data was explored using the alternating direction method of multipliers. The work presented in [21] involved the mathematical analysis of an inverse source problem governed by a time-fractional diffusion equation. The objectives of this research included identifying the source function using additional data via a regularized optimal control approach and determining the regularization parameters through bi-level optimization.

This paper is an extension of the problem considered in [22].

It is well known that the boundary conditions (1.3) are not strongly regular boundary conditions [23]. For this reason, when we solve the problem (1.1)–(1.4) by the Fourier method, then the system of eigenfunctions of the auxiliary spectral problem does not form a basis. Therefore, the application of the Fourier method is impossible and additional research is needed.

2. Preliminaries

The multinomial Mittag–Leffler function is defined as [24]

$$E_{(\beta_1,...,\beta_m),\beta_0}\left(z_1,...,z_m\right)=\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_m=k}\frac{(k;k_1,...,k_m)\prod_{i=1}^m z_i^{k_i}}{\Gamma\left(\beta_0+\sum_{i=1}^m\beta_i k_i\right)},$$

where $0<\beta_0<2,\ 0<\beta_i<1\ (i=1,...,m)$, and $z_i\in\mathbb{C}\ (i=1,...,m)$. Here, $(k;k_1,...,k_m)$ denotes the multinomial coefficient

$$(k;k_1,...,k_m):=\frac{k!}{k_1!\cdots k_m!}\ \text{where}\ k=\sum_{i=1}^m k_i,$$

and $k_i\ (1\le i\le m)$ are non-negative integers.

Lemma 2.1. *[25] Let $0<\beta_0<2,\ 0<\beta_i<1\ (i=1,...,m)$, and $z_i\in\mathbb{C}\ (i=1,...,m)$ be fixed. Then,*

$$\frac{1}{\Gamma\left(\beta_0\right)}+\sum_{i=1}^m z_iE_{(\beta_1,...,\beta_m),\beta_0+\beta_i}\left(z_1,...,z_m\right)=E_{(\beta_1,...,\beta_m),\beta_0}\left(z_1,...,z_m\right).$$

Lemma 2.2. *[25] Let $0<\beta<2$ and $1>\alpha_1>\cdots>\alpha_m>0$ be given. Assume that $\alpha_1\pi/2<\mu<\alpha_1\pi,\ \mu\le|arg(z_1)|\le\pi$, and there exists $K>0$ such that $-K\le z_i<0,\ (i=2,...,m)$. Then, there exists a constant $c>0$ depending on $\mu,\ K,\ \alpha_i\ (i=1,...,m)$, and β such that*

$$\left|E_{(\alpha_1,\alpha_1-\alpha_2,...,\alpha_1-\alpha_m),\beta}\left(z_1,...,z_m\right)\right|\le\frac{c}{1+|z_1|}\le c.$$

Let us denote

$$E_{(\cdot),\alpha_1}(t):=E_{(\alpha_1,\alpha_1-\alpha_2,...,\alpha_1-\alpha_m),\alpha_1}\left(-l_1t^{\alpha_1},-l_2t^{\alpha_1-\alpha_2},...,-l_mt^{\alpha_1-\alpha_m}\right),\ t>0,$$

where $l_1,...,l_m$ are some positive constants.

Lemma 2.3. *[25] Let* $1 > \alpha_1 > \cdots > \alpha_m > 0$. *Then,*

$$\frac{d}{dt}\left(t^{\alpha_1} E_{(\cdot),1+\alpha_1}(t)\right) = t^{\alpha_1 - 1} E_{(\cdot),\alpha_1}(t),\ t > 0.$$

Lemma 2.4. *[26] Let* f_n *be a sequence of functions defined on* $(a, b]$ *for each* $n \in \mathbf{N}$, *such that:*
(1). *For some* $\alpha > 0$, $\partial_t^\alpha f_n(t)$ *exists for all* $n \in \mathbf{N}$, $t \in (a, b]$;
(2). *Both the series* $\sum_{n=1}^{\infty} f_n(t)$ *and* $\sum_{n=1}^{\infty} \partial_t^\alpha f_n(t)$ *are uniformly convergent on the interval* $[a + \varepsilon, b]$ *for any* $\varepsilon > 0$.

Then, $\sum_{n=1}^{\infty} f_n(t)$ *is* $\alpha > 0$ *differentiable, where* $\sum_{n=1}^{\infty} f_n(t)$ *is a series of functions that must satisfy*

$$\partial_t^\alpha \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} \partial_t^\alpha f_n(t).$$

Consider the Volterra integral equation

$$g_0(t) = \int_0^t Q(t, t_1) g_0(t_1) dt_1 + g_1(t),\ 0 \le t \le 1. \tag{2.1}$$

Denote $\Delta := \{(t, t_1) : 0 \le t_1 < t \le 1\}$ and introduce the class S^α of kernels $Q(t, t_1)$ that are defined and continuous on Δ and for $(t, t_1) \in \Delta$ satisfy the inequality

$$|Q(t, t_1)| \le c_0 (t - t_1)^{-\alpha},\ 0 < \alpha < 1,\ c_0 = \text{const} > 0.$$

Lemma 2.5. *[27] Let* $g_1 \in C[0, 1]$ *and* $Q(t, t_1) \in S^\alpha$ *with* $0 \le \alpha < 1$. *Then,* (2.1) *has a unique solution* $g_0 \in C[0, 1]$.

3. Functional relations

The application of the Fourier method to solve problem (1.1)–(1.4) leads to a spectral problem

$$\begin{cases} -z''(x) = \lambda z(x),\ x \in (0, 1), \\ z(0) + z(1) = 0, \\ a_1 z'(0) + b_1 z'(1) + a_0 z(0) = 0, \end{cases} \tag{3.1}$$

where $|a_1| + |b_1| > 0,\ a_1 + b_1 \neq 0$. The system of eigenfunctions of problem (3.1) does not form a basis in $L_2(0, 1)$ [28]. For this reason, we cannot solve problem (1.1)–(1.4) by the Fourier method. Therefore, using the method from [29], we introduce even u_1 and odd u_2 with respect to the variable x parts of the function u as

$$u(x, t) = u_1(x, t) + u_2(x, t),$$

where

$$2u_1(x, t) = u(x, t) + u(1 - x, t),\ 2u_2(x, t) = u(x, t) - u(1 - x, t).$$

At the same time, for all $(x, t) \in D_T$, the relations

$$\begin{aligned} &u_2(x, t) = -u_2(1 - x, t),\ u_{2x}(x, t) = u_{2x}(1 - x, t), \\ &u_1(x, t) = u_1(1 - x, t),\ u_{1x}(x, t) = -u_{1x}(1 - x, t), \end{aligned} \tag{3.2}$$

hold. Equality (3.2) implies boundary relations

$$\begin{aligned} &u_2(0,t) = -u_2(1,t),\ u_{2x}(0,t) = u_{2x}(1,t),\\ &u_1(0,t) = u_1(1,t),\ u_{1x}(0,t) = -u_{1x}(1,t). \end{aligned} \tag{3.3}$$

Substituting u_1 and u_2 into boundary conditions (1.3) and using (3.3), we obtain

$$u_1(0,t) = 0,\ (a_1 + b_1)\, u_{2x}(0,t) + a_0 u_2(0,t) = (b_1 - a_1)\, u_{1x}(0,t). \tag{3.4}$$

Also, substituting u_1 and u_2 into (1.1), (1.2) and (1.4) and using (3.3) and (3.4), we have the first problem of finding a pair of functions $\{r, u_1\}$ in the form

$$\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_1 = u_{1xx} + r(t) f_1(x,t),\ (x,t) \in D_T, \tag{3.5}$$

$$u_1(x,0) = \varphi_1(x),\ x \in [0,1], \tag{3.6}$$

$$u_1(0,t) = 0,\ u_1(1,t) = 0,\ t \in [0,T], \tag{3.7}$$

$$\int_0^1 u_1(x,t)dx = E(t),\ t \in [0,T], \tag{3.8}$$

where

$$2f_1(x,t) = f(x,t) + f(1-x,t),\ 2\varphi_1(x) = \varphi(x) + \varphi(1-x).$$

And, for the function u_2, we have the second problem in the form

$$\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_2 = u_{2xx} + r(t) f_2(x,t),\ (x,t) \in D_T, \tag{3.9}$$

$$u_2(x,0) = \varphi_2(x),\ x \in [0,1], \tag{3.10}$$

$$\begin{aligned} &(a_1 + b_1)\, u_{2x}(0,t) + a_0 u_2(0,t) = (b_1 - a_1)\, u_{1x}(0,t),\ t \in [0,T],\\ &(a_1 + b_1)\, u_{2x}(1,t) - a_0 u_2(1,t) = (b_1 - a_1)\, u_{1x}(0,t),\ t \in [0,T], \end{aligned} \tag{3.11}$$

where

$$2f_2(x,t) = f(x,t) - f(1-x,t),\ 2\varphi_2(x) = \varphi(x) - \varphi(1-x).$$

4. Well-posedness of the first problem

4.1. Existence and uniqueness of the solution of the first problem

Consider the spectral problem

$$\begin{cases} -y''(x) = \lambda y(x),\ x \in (0,1),\\ y(0) = 0,\ y(1) = 0. \end{cases} \tag{4.1}$$

The spectral problem (4.1) has only eigenfunctions

$$y_k(x) = \sqrt{2} \sin k\pi x,\ k = 1, 2, ...,$$

and the eigenvalues are defined by

$$\lambda_k = (k\pi)^2\,,\ k = 1, 2, ...$$

Since (4.1) is the self-adjoint problem, the system of eigenfunctions $\{y_k(x)\}$, $(k = 1, 2, ...)$ forms an orthonormal basis in $L_2(0, 1)$.

Lemma 4.1. *Let* $\varphi_1(x) \in C^4[0, 1]$ *be a function satisfying the conditions*

$$\varphi_1(0) = \varphi_1''(0) = 0,\ \varphi_1(1) = \varphi_1''(1) = 0. \tag{4.2}$$

Then, the following inequality

$$\sum_{k=1}^{\infty} |\lambda_k \varphi_{1k}| \le \widehat{c_1} \|\varphi_1\|_{C^4[0,1]} \le c_1$$

holds, where c_1 *is a constant,* $\varphi_{1k} = (\varphi_1, y_k)\,,\ (k = 1, 2, ...)$.

Proof. By using (4.2), integration by parts four times and the Schwarz and Bessel inequalities, we obtain

$$\sum_{k=1}^{\infty} |\lambda_k \varphi_{1k}| = \sum_{k=1}^{\infty} \left|\lambda_k \frac{\lambda_k}{\lambda_k} \varphi_{1k}\right| \le \left(\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left|\lambda_k^2 \varphi_{1k}\right|^2\right)^{\frac{1}{2}}$$
$$\le \widehat{c_1} \|\varphi_1^{(4)}\|_{L_2[0,1]} \le \widehat{c_1} \|\varphi_1^{(4)}\|_{C[0,1]}.$$

□

Theorem 4.2. *Let the following conditions be satisfied:*
(A_1) $\varphi_1 \in C^4[0, 1],\ \varphi_1(0) = \varphi_1''(0) = 0,\ \varphi_1(1) = \varphi_1''(1) = 0$;
(A_2) $f_1 \in C(\overline{D}_T),\ f_1(\cdot, t) \in C^4[0, 1],\ f_1(0, t) = f_{1xx}(0, t) = 0,\ f_1(1, t) = f_{1xx}(1, t) = 0,\ 0 < \frac{1}{m_0} \le \min\limits_{0\le t\le T} \left|\int_0^1 f_1(x, t)dx\right|$;
(A_3) $E(t) \in C^1[0, T],\ E(0) = \int\limits_0^1 \varphi_1(x)dx$,
where m_0 *is a constant. Then, the inverse problem* (3.5)–(3.8) *has a unique classical solution.*

Proof. **(Existence of the solution of the first problem)** To construct a formal solution of problem (3.5)–(3.8), we will use the Fourier method. Following this method, we seek the solution of (3.5)–(3.8) in a Fourier series as

$$u_1(x, t) = \sum_{k=1}^{\infty} u_{1k}(t) y_k(x),$$

where $u_{1k}(t) = (u_1(\cdot, t), y_k),\ (k = 1, 2, ...)$. For the functions u_{1k} we obtain the Cauchy problem

$$\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_{1k}(t) + \lambda_k u_{1k}(t) = r(t) f_{1k}(t),$$
$$u_{1k}(0) = \varphi_{1k},$$

where $f_{1k}(t) = (f_1(\cdot, t), y_k)\,,\ \varphi_{1k} = (\varphi_1, y_k),\ (k = 1, 2, ...)$. The solution of this Cauchy problem is given in [24] by

$$u_{1k}(t) = \left(t^{\alpha_1 - 1} E_{(\cdot),\alpha_1}(t)\right) * r(t) f_{1k}(t) + \varphi_{1k} \widehat{u_{1k}}(t), \tag{4.3}$$

where

$$E_{(\cdot),\alpha_1}(t) = E_{(\alpha_1,\alpha_1-\alpha_2,...,\alpha_1-\alpha_m),\alpha_1}\left(-\lambda_k t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}, ..., -q_m t^{\alpha_1-\alpha_m}\right),$$
$$\widehat{u}_{1k}(t) = 1 - \lambda_k t^{\alpha_1} E_{(\cdot),1+\alpha_1}(t) - q_2 t^{\alpha_1-\alpha_2} E_{(\cdot),1+\alpha_1-\alpha_2}(t) - \cdots - q_m t^{\alpha_1-\alpha_m} E_{(\cdot),1+\alpha_1-\alpha_m}(t),$$
$$\left(t^{\alpha_1-1}E_{(\cdot),\alpha_1}(t)\right) * r(t) f_{1k}(t) = \int_0^t (t-\tau)^{\alpha_1-1} E_{(\cdot),\alpha_1}(t-\tau) r(\tau) f_{1k}(\tau) d\tau.$$

According to Lemma 2.1, we rewrite (4.3) as

$$u_{1k}(t) = \left(t^{\alpha_1-1}E_{(\cdot),\alpha_1}(t)\right) * r(t) f_{1k}(t) + \varphi_{1k} E_{(\cdot),1}(t). \tag{4.4}$$

Hence, the formal solution of problem (3.5)–(3.8) is expressed via the series

$$u_1(x,t) = \sum_{k=1}^{\infty} \left[\left(t^{\alpha_1-1}E_{(\cdot),\alpha_1}(t)\right) * r(t) f_{1k}(t) + \varphi_{1k} E_{(\cdot),1}(t)\right] y_k(x). \tag{4.5}$$

Now, we get the expression of the term r. Integrating Eq (3.5) between 0 and 1, we obtain

$$\int_0^1 \left[\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_1(x,t)\right] dx = \int_0^1 \left[u_{1xx}(x,t) + r(t) f_1(x,t)\right] dx.$$

By using the overdetermination conditions (3.8) and (3.3), it is easy to deduce that

$$r(t) = h(t)\left(\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} E(t) + 2u_{1x}(0,t)\right), \tag{4.6}$$

where

$$u_{1x}(0,t) = \sqrt{2}\sum_{k=1}^{\infty} k\pi \left(\left(t^{\alpha_1-1}E_{(\cdot),\alpha_1}(t)\right) * r(t) f_{1k}(t) + \varphi_{1k} E_{(\cdot),1}(t)\right).$$
$$h(t) = \left(\int_0^1 f_1(x,t)dx\right)^{-1}.$$

Let us denote

$$L(t) := h(t)\left(\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} E(t) + \sqrt{2}\sum_{k=1}^{\infty} k\pi \varphi_{1k} E_{(\cdot),1}(t)\left(1-(-1)^k\right)\right),$$
$$K(t,\tau) := \sqrt{2}(t-\tau)^{\alpha_1-1}\sum_{k=1}^{\infty} k\pi f_{1k}(\tau) E_{(\cdot),\alpha_1}(t-\tau)\left(1-(-1)^k\right).$$

Then, we obtain the Volterra integral equation of the second kind with respect to r in the form

$$r(t) = h(t)\int_0^t K(t,\tau) r(\tau) d\tau + L(t). \tag{4.7}$$

Before we proceed further, notice that, under assumption (A_2), the series

$$\sum_{k=1}^{\infty} |\lambda_k f_{1k}(t)| \leq \widehat{c_2} \max_{0\leq t\leq T} \|f_1(\cdot,t)\|_{C^4[0,1]} \leq c_2,$$

is uniformly convergent, and for $E(t) \in C^1[0,T]$ the term $\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} E(t)$ is continuous, being the difference of m pieces of continuous functions. According to Lemma 2.2, we estimate the kernel of (4.7) and $L(t)$ in the form

$$|L| \leq m_0\,(m_1 + cc_1) := m_2,\ |K(t,\tau)| \leq m_0 cc_2 (t-\tau)^{\alpha_1-1} := c_0 (t-\tau)^{\alpha_1-1}, \tag{4.8}$$

where m_1 is a bound of $\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} E(t)$ and c, c_1 are constants defined in Lemmas 2.2 and 4.1, respectively. Therefore, the kernel of Eq (4.7) is weakly singular and, by Lemma 2.5, there exists a unique solution $r \in C[0,T]$.

Due to assumptions (A_1) and (A_2) in Theorem 4.2, we have

$$\varphi_{1k} = \frac{\sqrt{2}}{(k\pi)^4} \int_0^1 \varphi_1^{(4)}(x) \sin k\pi x dx := \frac{1}{(k\pi)^4} \varphi_{1k}^{(4)},$$

$$f_{1k}(t) = \frac{\sqrt{2}}{(k\pi)^4} \int_0^1 f_{1x}^{(4)}(x,t) \sin k\pi x dx := \frac{1}{(k\pi)^4} f_{1k}^{(4)}(t).$$

Since the solution u_1 is formally given by series (4.5), we need to show that the series corresponding to $u_1,\ u_{1xx}$, and $\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_1$ converge. Under assumptions (A_1)–(A_3), for all $(x,t) \in \overline{D}_T$, the series corresponding to u_1, u_{1xx} are bounded from above by the series

$$|u_1| \leq c \sum_{k=1}^{\infty} \frac{1}{(k\pi)^4} \left[m_3 \|f_{1k}^{(4)}\|_{C[0,T]} \frac{T^{\alpha_1}}{\alpha_1} + \left|\varphi_{1k}^{(4)}\right| \right],$$

$$|u_{1xx}| \leq c \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} \left[m_3 \|f_{1k}^{(4)}\|_{C[0,T]} \frac{T^{\alpha_1}}{\alpha_1} + \left|\varphi_{1k}^{(4)}\right| \right],$$

where $m_3 := \|r\|_{C[0,T]}$. Obviously, these majorizing series are convergent.

Now we show that $\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_1(x,t)$ are continuous functions on D_T. For this, we first calculate $u_{1k}'(t)$, then we estimate $\partial_t^{\alpha_1} u_{1k}(t)$ on $[\varepsilon, T]$ for all $\varepsilon > 0$. The estimates for $\partial_t^{\alpha_2} u_{1k}(t), ..., \partial_t^{\alpha_m} u_{1k}(t)$ on $[\varepsilon, T]$ are obtained in a similar way.

First, let us get an estimate for u_{1k}'. According to Lemma 2.3, we obtain the estimate

$$\begin{aligned}
|u_{1k}'(t)| \leq & \frac{c}{(k\pi)^4} \left| \int_0^t (t-\tau)^{\alpha_1-1} \right| m_3 \|f_{1k}^{(4)}\|_{C[0,T]} \\
& + \frac{c}{(k\pi)^4} \left|\varphi_{1k}^{(4)}\right| \left| \lambda_k t^{\alpha_1-1} + q_2 t^{\alpha_1-\alpha_2-1} + \cdots + q_m t^{\alpha_1-\alpha_m-1} \right| \\
= & \frac{c}{(k\pi)^4} \frac{t^{\alpha_1-1}}{1-\alpha_1} m_3 \|f_{1k}^{(4)}\|_{C[0,T]} \\
& + \frac{c}{(k\pi)^4} \left|\varphi_{1k}^{(4)}\right| \left| \lambda_k t^{\alpha_1-1} + q_2 t^{\alpha_1-\alpha_2-1} + \cdots + q_m t^{\alpha_1-\alpha_m-1} \right|.
\end{aligned}$$

Then, for $\partial_t^{\alpha_1} u_{1k}$, we have the estimate

$$
\begin{aligned}
|\partial_t^{\alpha_1} u_{1k}(t)| \le & \frac{1}{\Gamma(1-\alpha_1)} \int_0^t \frac{\left|u'_{1k}(\tau)\right|}{(t-\tau)^{\alpha_1}} d\tau \\
\le & \frac{m_3 m_4 \|f_{1k}^{(4)}\|_{C[0,T]}}{\Gamma(1-\alpha_1)(1-\alpha_1)} \int_0^t \frac{\tau^{\alpha_1-1}}{(t-\tau)^{\alpha_1}} d\tau \\
& + \frac{m_4 \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \int_0^t \frac{1}{(t-\tau)^{\alpha_1}} \left|\lambda_k \tau^{\alpha_1-1}\right| d\tau \\
& + \frac{m_4 \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \int_0^t \frac{1}{(t-\tau)^{\alpha_1}} \left|q_2 \tau^{\alpha_1-\alpha_2-1}\right| d\tau \\
& + \cdots \\
& + \frac{m_4 \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \int_0^t \frac{1}{(t-\tau)^{\alpha_1}} \left|q_m \tau^{\alpha_1-\alpha_m-1}\right| d\tau,
\end{aligned} \tag{4.9}
$$

where $m_4 := c/(k\pi)^4$. By applying the change of variable $s = \tau/t$ in (4.9), we have

$$
\begin{aligned}
|\partial_t^{\alpha_1} u_{1k}(t)| \le & \frac{m_3 m_4 \|f_{1k}^{(4)}\|_{C[0,T]}}{\Gamma(1-\alpha_1)(1-\alpha_1)} \int_0^1 s^{\alpha_1-1} (1-s)^{-\alpha_1} ds \\
& + \frac{\lambda_k m_4 \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \int_0^1 (1-s)^{-\alpha_1} s^{\alpha_1-1} ds \\
& + \frac{q_2 m_4 t^{-\alpha_2} \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \int_0^1 (1-s)^{-\alpha_1} s^{\alpha_1-\alpha_2-1} ds \\
& + \cdots \\
& + \frac{m_4 q_m t^{-\alpha_m} \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \int_0^1 (1-s)^{-\alpha_1} s^{\alpha_1-\alpha_m-1} ds \\
= & \frac{m_3 m_4 \|f_{1k}^{(4)}\|_{C[0,T]}}{\Gamma(1-\alpha_1)(1-\alpha_1)} \mathbb{B}(\alpha_1, 1-\alpha_1) + \frac{\lambda_k m_4 \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \mathbb{B}(\alpha_1, 1-\alpha_1) \\
& + \frac{m_4 q_2 t^{-\alpha_2} \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \mathbb{B}(\alpha_1-\alpha_2, 1-\alpha_1) \\
& + \cdots \\
& + \frac{m_4 q_m t^{-\alpha_m} \left|\varphi_{1k}^{(4)}\right|}{\Gamma(1-\alpha_1)} \mathbb{B}(\alpha_1-\alpha_m, 1-\alpha_1),
\end{aligned} \tag{4.10}
$$

where

$$\mathbb{B}(z_1, z_2) = \int_0^1 (1-s)^{z_2-1}\, s^{z_1-1} ds.$$

Consequently, by (4.10), the functions $\sum_{i=1}^m q_i \partial_t^{\alpha_i} u_1(x,t)$ are continuous on D_T. □

Proof. **(Uniqueness of the solution of the first problem)** Let us show that the solution of problem (3.5)–(3.8) is unique. Suppose that there are two pairs of solutions $\{\widehat{r}, \widehat{u_1}\}$ and $\{\tilde{r}, \tilde{u}_1\}$ of the inverse problem (3.5)–(3.8). Then from (4.5) and (4.7), we have

$$\widehat{u_1}(x,t) - \tilde{u}_1(x,t) = \sum_{k=1}^{\infty} \left[\left(t^{\alpha_1 - 1} E_{(\cdot),\alpha_1}(t) \right) * \left(\widehat{r}(t) - \tilde{r}(t) \right) f_{1k}(t) \right] y_k(x), \tag{4.11}$$

and

$$\widehat{r}(t) - \tilde{r}(t) = h(t) \left(\int_0^t K(t,\tau) \left(\widehat{r}(\tau) - \tilde{r}(\tau) \right) d\tau \right). \tag{4.12}$$

Then, (4.12) yields $\widehat{r} = \tilde{r}$. After substituting $\widehat{r} = \tilde{r}$ in (4.11), we have $\widehat{u_1} = \tilde{u}_1$. □

4.2. Continuous dependence of the solution of the first problem on the data

Theorem 4.3. *Let $\mathbb{F}$ be the set of triples $\{\varphi_1, f_1, E\}$ where the functions φ_1, f_1 and E satisfy the assumptions of Theorem 4.2, and*

$$\|\varphi_1\|_{C^4[0,1]} \le M_0, \ \|f_1\|_{C^{4,0}(\overline{D}_T)} \le M_1, \ \|E\|_{C^1[0,T]} \le M_2,$$

for some positive constants M_0, M_1 and M_2. Then, the solution (r, u_1) of the inverse problem (3.5)–(3.8) *depends continuously upon the data on $\mathbb{F}$.*

Proof. Let $\mathbb{F} = \{\varphi_1, f_1, E\}$ and $\overline{\mathbb{F}} = \{\overline{\varphi}_1, \overline{f}_1, \overline{E}\}$ be two sets of data, and $\|\mathbb{F}\| = \|f_1\|_{C^{4,0}(\overline{D}_T)} + \|\varphi_1\|_{C^4[0,1]} + \|E\|_{C^1[0,T]}$, (r, u_1) and $(\overline{r}, \overline{u}_1)$ be the solutions of the inverse problem (3.5)–(3.8) corresponding to the data $\mathbb{F}$ and $\overline{\mathbb{F}}$, respectively.

Let us denote

$$\overline{h}(t) := \left(\int_0^1 \overline{f}_1(x,t) dx \right)^{-1}, \ \psi_k := \sqrt{2} k\pi \left(1 - (-1)^k \right).$$

For the difference $E - \overline{E}$ we have the estimate

$$\sum_{i=1}^m q_i \partial_t^{\alpha_i} \left| E - \overline{E} \right| \le M_3 \|E - \overline{E}\|_{C^1[0,T]}, \ \left| h - \overline{h} \right| \le m_0^2 \|h - \overline{h}\|_{C[0,T]}, \tag{4.13}$$

where

$$M_3 = \frac{q_1}{\Gamma(1-\alpha_1)} \frac{T^{1-\alpha_1}}{1-\alpha_1} + \cdots + \frac{q_m}{\Gamma(1-\alpha_m)} \frac{T^{1-\alpha_m}}{1-\alpha_m}.$$

First, we write the function L, the difference $r - \overline{r}$, and the integral from 0 to t of the kernel of Eq (4.7) as

$$\int_0^t \left(K(t,\tau) - \overline{K}(t,\tau)\right) d\tau = \int_0^t \sum_{k=1}^{\infty} (t-\tau)^{\alpha_1 - 1} \psi_k \left(f_{1k} - \overline{f}_{1k}\right)(\tau) E_{(\cdot),\alpha_1}(t-\tau)\, d\tau, \tag{4.14}$$

$$\begin{aligned} L(t) - \overline{L}(t) =& h(t)\left(\sum_{i=1}^{m} q_i \partial_t^{\alpha_i}\left(E - \overline{E}\right)(t) + \sum_{k=1}^{\infty} \psi_k \left(\varphi_{1k} - \overline{\varphi}_{1k}\right) E_{(\cdot),1}(t)\right) \\ &+ \left(h(t) - \overline{h}(t)\right)\left(\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} \overline{E}(t) + \sum_{k=1}^{\infty} \psi_k \overline{\varphi}_{1k} E_{(\cdot),1}(t)\right), \end{aligned} \tag{4.15}$$

$$\begin{aligned} r(t) - \overline{r}(t) =& L(t) - \overline{L}(t) + \overline{h}(t) \int_0^t \left(K(t,\tau) - \overline{K}(t,\tau)\right) r(\tau) d\tau \\ &+ \overline{h}(t) \int_0^t \overline{K}(t,\tau)\left(r(\tau) - \overline{r}(\tau)\right) d\tau + \left(h(t) - \overline{h}(t)\right) \int_0^t K(t,\tau) r(\tau) d\tau. \end{aligned} \tag{4.16}$$

Then, from Lemma 2.2 and equality (4.15) we obtain

$$\|L - \overline{L}\|_{C[0,T]} \le M_4 \|\varphi_1 - \overline{\varphi}_1\|_{C^4[0,1]} + M_5 \|f_1 - \overline{f}_1\|_{C^{4,0}(\overline{D}_T)} + M_6 \|E - \overline{E}\|_{C^1[0,T]}, \tag{4.17}$$

where $M_4 = m_0^2 (m_1 + c c_1)$, $M_5 = m_0 c \widehat{c_1}, and\ M_6 = m_0 M_3$.

By Lemma 2.2, equality (4.14) we arrive at

$$\|K - \overline{K}\|_{C[0,T] \times C[0,T]} \le M_7 \|f_1 - \overline{f}_1\|_{C^{4,0}(\overline{D}_T)}, \tag{4.18}$$

where $M_7 = m_0 c T^{\alpha_1} / \alpha_1 \left(\widehat{c_2} + m_0 c_2\right)$.

According to inequalities (4.17) and (4.18) from (4.16), we have

$$\|r - \overline{r}\|_{C[0,T]} \le M_9 \left(\|\varphi_1 - \overline{\varphi}_1\|_{C^4[0,1]} + \|f_1 - \overline{f}_1\|_{C^{4,0}(\overline{D}_T)} + \|E - \overline{E}\|_{C^1[0,T]}\right),$$

where

$$M_9 = \max\left(\frac{M_4 + m_3 T M_7}{M_8},\ \frac{M_5}{M_8},\ \frac{M_6}{M_8}\right),\ M_8 = 1 - c_0 \frac{T^{\alpha_1}}{\alpha_1}.$$

From (4.5), a similar estimate can be obtained for $u_1 - \overline{u}_1$. □

5. Well-posedness of the second problem

To begin with, let us rewrite problem (3.9)–(3.11) as

$$\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_2 = u_{2xx} + r(t) f_2(x,t),\ (x,t) \in D_T, \tag{5.1}$$

$$u_2(x,0) = \varphi_2(x),\ x \in [0,1], \tag{5.2}$$

$$\begin{aligned} u_{2x}(0,t) + \alpha u_2(0,t) = \gamma(t),\ t \in [0,T], \\ u_{2x}(1,t) - \alpha u_2(1,t) = \gamma(t),\ t \in [0,T], \end{aligned} \tag{5.3}$$

where

$$\gamma(t) = \sqrt{2}\frac{b_1 - a_1}{b_1 + a_1}\sum_{k=1}^{\infty} k\pi \left[t^{\alpha_1 - 1}E_{(\cdot),\alpha_1}(t) * r(t)f_{1k}(t) + \varphi_{1k}E_{(\cdot),1}(t)\right],$$

$$\alpha = \frac{a_0}{b_1 + a_1}.$$

We search for the solution to problem (5.1)–(5.3) in the form

$$u_2(x,t) = u_0(x,t) + a(x)\gamma(t),$$

where $a(x) = 20x^7 - 70x^6 + 84x^5 - 35x^4 + x$. Consequently, for the unknown function u_0 we have the problem

$$\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} u_0 = u_{0xx} + f_0(x,t),\ (x,t) \in D_T, \tag{5.4}$$

$$u_0(x,0) = \varphi_0(x),\ x \in [0,1], \tag{5.5}$$

$$\begin{aligned} u_{0x}(0,t) + \alpha u_0(0,t) = 0,\ t \in [0,T], \\ u_{0x}(1,t) - \alpha u_0(1,t) = 0,\ t \in [0,T], \end{aligned} \tag{5.6}$$

where

$$f_0(x,t) = f_2(x,t)r(t) - a(x)b(t) + a''(x)\gamma(t),\ \varphi_0(x) = \varphi_2(x) - a(x)\gamma(0),$$

$$b(t) = \sum_{i=1}^{m} q_i \partial_t^{\alpha_i}\gamma(t).$$

The auxiliary spectral problem for the considered direct problem (5.4)–(5.6) is

$$\begin{cases} \mu''(x) + \lambda\mu(x) = 0,\ x \in (0,1), \\ \mu'(0) + \alpha\mu(0) = 0, \\ \mu'(1) - \alpha\mu(1) = 0. \end{cases} \tag{5.7}$$

The spectral problem (5.7) has only eigenfunctions

$$\mu_k(x) = \sqrt{2}\left(\cos(\sqrt{\lambda_k}x) - \frac{\alpha}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}x)\right),\ k = 1,2,...,$$

and the eigenvalues are defined by

$$\tan\sqrt{\lambda_k} = \frac{2\alpha\sqrt{\lambda_k}}{\alpha^2 - \lambda_k},\ k = 1,2,....$$

Since problem (5.7) is self-adjoint, the system of eigenfunctions $\{\mu_k(x)\}$, $(k = 1,2,...)$ forms an orthonormal basis in $L_2(0,1)$. We consider only the case $\alpha > 0$. The case $\alpha < 0$ will be similar.

For sufficiently large k, the asymptotic representation of eigenvalues of problem (5.7) has the form

$$\sqrt{\lambda_k} = k\pi + O\left(\frac{1}{k}\right).$$

Lemma 5.1. *Let $\varphi_0 \in C^4[0,1]$ be a function satisfying the conditions*

$$\begin{aligned} &\varphi_0'(0) + \alpha\varphi_0(0) = 0,\ \varphi_0'''(0) + \alpha\varphi_0''(0) = 0, \\ &\varphi_0'(1) - \alpha\varphi_0(1) = 0,\ \varphi_0'''(1) - \alpha\varphi_0''(1) = 0. \end{aligned} \tag{5.8}$$

Then, the inequality

$$\sum_{k=1}^{\infty} |\lambda_k \varphi_{0k}| \le \widehat{c}_3 \|\varphi_0\|_{C^4[0,1]} \le c_3$$

holds, where c_3 is a constant and $\varphi_{0k} = (\varphi_0, \mu_k)$, $(k = 1, 2, ...)$.

Proof. By using (5.8), integration by parts four times, and the Schwarz and Bessel inequalities, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} |\lambda_k \varphi_{0k}| = \sum_{k=1}^{\infty} \left|\lambda_k \frac{\lambda_k}{\lambda_k} \varphi_{0k}\right| \le \left(\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left|\lambda_k^2 \varphi_{0k}\right|^2\right)^{\frac{1}{2}} \\ \le \widehat{c}_3 \|\varphi_0^{(4)}\|_{L_2[0,1]} \le \widehat{c}_3 \|\varphi_0^{(4)}\|_{C[0,1]}. \end{aligned}$$

□

The class of functions which satisfy the conditions of Lemma 5.1 will be denoted by

$$\Phi \equiv \left\{ \begin{array}{ll} \varphi_0 \in C^4[0,1]: & \varphi_0'(0) + \alpha\varphi_0(0) = 0,\ \varphi_0'''(0) + \alpha\varphi_0''(0) = 0, \\ & \varphi_0'(1) - \alpha\varphi_0(1) = 0,\ \varphi_0'''(1) - \alpha\varphi_0''(1) = 0 \end{array} \right\}.$$

Similarly, as for the inverse problem (3.5)–(3.8), we search for a solution to problem (5.4)–(5.6) by the Fourier method. Then, we have

$$u_0(x,t) = \sum_{k=1}^{\infty} \left[\left(t^{\alpha_1 - 1} E_{(\cdot),\alpha_1}(t)\right) * f_{0k}(t) + \varphi_{0k} E_{(\cdot),1}(t)\right] \mu_k(x), \tag{5.9}$$

where $f_{0k}(t) = (f_0(\cdot,t), \mu_k)$, $\varphi_{0k} = (\varphi_0, \mu_k)$, $(k = 1, 2, ...)$.

Theorem 5.2. *Suppose $\varphi_0 \in \Phi$, $f_0 \in C\left(\overline{D}_T\right)$ and $f_0(\cdot,t) \in \Phi$ for every $t \in [0,T]$. Then,* (5.9) *gives a classical solution u_0 to* (5.4)–(5.6) *and $u_0(\cdot,t) \in C^2[0,1]$, $\partial_t^{\alpha_i} u_0(x,\cdot) \in C(0,T]$, $(i = 1, ..., m)$.*

Proof. Let us denote

$$\varphi_{0k} = \frac{\sqrt{2}}{\lambda_k^2} \int_0^1 \varphi_0^{(4)}(x) \mu_k(x) dx := \frac{1}{\lambda_k^2} \varphi_{0k}^{(4)},$$

$$f_{0k}(t) = \frac{\sqrt{2}}{\lambda_k^2} \int_0^1 f_{0x}^{(4)}(x,t) \mu_k(x) dx := \frac{1}{\lambda_k^2} f_{0k}^{(4)}(t).$$

As in the previous section, we need to show that the series corresponding to u_0, u_{0xx}, and $\sum_{i=1}^m q_i \partial_t^{\alpha_i} u_0$ converge. The series corresponding to u_0 and u_{0xx} are bounded from above by the series

$$|u_0| \le c \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \left[\|f_{0k}^{(4)}\|_{C[0,T]} \frac{T^{\alpha_1}}{\alpha_1} + \left|\varphi_{0k}^{(4)}\right|\right],$$

$$|u_{0xx}| \le c \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left[\|f_{0k}^{(4)}\|_{C[0,T]} \frac{T^{\alpha_1}}{\alpha_1} + \left|\varphi_{0k}^{(4)}\right|\right].$$

According to Lemma 2.3, we obtain the estimate of $\partial_t^{\alpha_1} u_{0k}(t)$ on $[\varepsilon, T]$ for all $\varepsilon > 0$ as

$$\begin{aligned}
|\partial_t^{\alpha_1} u_{0k}(t)| \leq & \frac{c\|f_{0k}^{(4)}\|_{C[0,T]}}{\lambda_k^2 \Gamma(1-\alpha_1)(1-\alpha_1)} \mathbb{B}(\alpha_1, 1-\alpha_1) \\
& + \frac{c\left|\varphi_{0k}^{(4)}\right|}{\lambda_k \Gamma(1-\alpha_1)} \mathbb{B}(\alpha_1, 1-\alpha_1) \\
& + \frac{cq_2 t^{-\alpha_2}\left|\varphi_{0k}^{(4)}\right|}{\lambda_k^2 \Gamma(1-\alpha_1)} \mathbb{B}(\alpha_1-\alpha_2, 1-\alpha_1) \\
& + \cdots \\
& + \frac{cq_m t^{-\alpha_m}\left|\varphi_{0k}^{(4)}\right|}{\lambda_k^2 \Gamma(1-\alpha_1)} \mathbb{B}(\alpha_1-\alpha_m, 1-\alpha_1).
\end{aligned}$$

Therefore, $\partial_t^{\alpha_1} u_{0k}(t)$ represent the continuous function on $[\varepsilon, T]$ for all $\varepsilon > 0$. □

Remark 5.3. *The uniqueness of the solution of problem* (5.4)–(5.6)*, under the conditions of Theorem 5.2 is obtained from the uniqueness of the representation* (5.9).

Lemma 5.4. *[30] Let u_0 satisfy Eq* (5.4) *in $\overline{D}_T$. If $f_0(x,t) \leq 0$ in $\overline{D}_T$, then*

$$u_0(x,t) \leq \max\left\{0, \max_{0\leq x\leq 1} u_0(x,0), \max_{0\leq t\leq T} u_0(0,t), \max_{0\leq t\leq T} u_0(1,t)\right\}.$$

If $f_0(x,t) \geq 0$ in $\overline{D}_T$, then

$$u_0(x,t) \geq \min\left\{0, \min_{0\leq x\leq 1} u_0(x,0), \min_{0\leq t\leq T} u_0(0,t), \min_{0\leq t\leq T} u_0(1,t)\right\}.$$

Theorem 5.5. *The classical solution of problem* (5.4)–(5.6) *depends continuously on $\varphi_0 \in C[0,1]$, $f_0 \in C(\overline{D}_T)$ in the sense that*

$$\|u_0 - \overline{u}_0\|_{C(\overline{D}_T)} \leq \|\varphi_0 - \overline{\varphi}_0\|_{C[0,1]]} + (\alpha+1)\|f_0 - \overline{f}_0\|_{C(\overline{D}_T)}, \tag{5.10}$$

where u_0 and $\overline{u}_0$ are classical solutions of (5.4)–(5.6) *with the data f_0, φ_0, and $\overline{f}_0$, $\overline{\varphi}_0$, respectively.*

Proof. Let u_0 be a classical solution of problem (5.4)–(5.6). We introduce the following:

$$R = \|f_0\|_{C(\overline{D}_T)}, \; M = \|\varphi_0\|_{C[0,1]}.$$

Let us construct the function

$$\omega(x,t) = u_0(x,t) - \frac{Rx^2}{2}.$$

The function ω is the classical solution of problem

$$\begin{aligned}
\sum_{i=1}^{m} q_i \partial_t^{\alpha_i} \omega - \omega_{xx} &= f_0(x,t) - R, \\
\omega(x,0) &= \varphi_0(x) - \frac{Rx^2}{2}, \\
\omega_x(0,t) + \alpha\omega(0,t) &= 0, \\
\omega_x(1,t) - \alpha\omega(1,t) &= \frac{R}{2}(\alpha-1).
\end{aligned}$$

Using the maximum principle, we obtain the estimates

$$\omega(x,t) \leq \max\left\{0,\ \varphi_0(x) - \frac{Rx^2}{2},\ \frac{R}{2}(\alpha-1)\right\}$$

and

$$\begin{aligned} u_0(x,t) &\leq \max\left\{0,\ \varphi_0(x) - \frac{Rx^2}{2},\ \frac{R}{2}(\alpha-1)\right\} + R \\ &\leq \varphi_0(x) - \frac{Rx^2}{2} + \frac{R}{2}(\alpha-1) + R \leq (\alpha+1)R + M. \end{aligned}$$

Similarly, if we introduce the function

$$h(x,t) = u_0(x,t) + \frac{Rx^2}{2}$$

and using the minimum principle we arrive at the opposite estimate

$$u_0(x,t) \geq -(\alpha+1)R - M.$$

Hence, if u_0 is the classical solution of problem (5.4)–(5.6), we have the estimate

$$\|u_0\|_{C(\overline{D}_T)} \leq \|\varphi_0\|_{C[0,1]]} + (\alpha+1)\|f_0\|_{C(\overline{D}_T)}. \tag{5.11}$$

To prove the continuous dependence on the data, we study the difference $g(x,t) = u_0(x,t) - \overline{u}_0(x,t)$. This function is the classical solution of (5.4)–(5.6) with $f_0 - \overline{f}_0$ and $\varphi_0 - \overline{\varphi}_0$ replaced by f_0 and φ_0, respectively. Applying inequality (5.11) to g, we arrive at estimate (5.10). □

As a consequence of Theorem 5.2, we obtain the following main theorem for problem (5.1)–(5.3):

Theorem 5.6. *Let* $\varphi_2 \in C^4[0,1]$, $f_2 \in C\left(\overline{D}_T\right)$, $f_2(\cdot,t) \in C^4[0,1]$ *for every* $t \in [0,T]$, *and the following conditions*

$$\begin{aligned} &\varphi_2'(0) + \alpha\varphi_2(0) = \gamma(0),\ \varphi_2'''(0) + \alpha\varphi_2''(0) = 0, \\ &\varphi_2'(1) - \alpha\varphi_2(1) = \gamma(0),\ \varphi_2'''(1) - \alpha\varphi_2''(1) = 0, \\ &f_{2x}(0,t) + \alpha f_2(0,t) = \frac{b(t)}{r(t)},\ f_{2xxx}(0,t) + \alpha f_{2xx}(0,t) = 0, \\ &f_{2x}(1,t) - \alpha f_2(1,t) = \frac{b(t)}{r(t)},\ f_{2xxx}(1,t) - \alpha f_{2xx}(1,t) = 0 \end{aligned} \tag{5.12}$$

be satisfied. Then, the classical solution u_2 *of problem* (5.1)–(5.3) *exists, is unique and* $u_2(\cdot,t) \in C^2[0,1]$, $\partial_t^{\alpha_i} u_2(x,\cdot) \in C(0,T]$, $(i = 1,...,m)$.

Proof. The solution to problem (5.1)–(5.3) has the form

$$u_2(x,t) = a(x)\gamma(t) + \sum_{k=1}^{\infty}\left[\left(t^{\alpha_1-1}E_{(\cdot),\alpha_1}(t)\right) * f_{0k}(t) + \varphi_{0k}E_{(\cdot),1}(t)\right]\mu_k(x). \tag{5.13}$$

It can be seen from (5.13) that the majorizing series for (5.13) and (5.9) are the same and these series converge when conditions (5.12) are met. The uniqueness comes from the fact that the homogeneous problem (5.1)–(5.3) (that is, when $\varphi_2,\ \gamma,\ rf_2 \equiv 0$) has only a trivial solution.

□

6. The main results

Theorem 6.1. *Let the following conditions be satisfied:*
(H_1) $\varphi \in C^1[0,1],\ \varphi(0)+\varphi(1)=0,\ a_1\varphi'(0)+b_1\varphi'(1)+a_0\varphi(0)=0$;
(H_2) $f \in C\left(\overline{D}_T\right),\ f(\cdot,t)\in C^1[0,1],\ f(0,t)+f(1,t)=0,\ f_x(0,t)+f_x(1,t)+\alpha\left(f(0,t)-f(1,t)\right)=0$;
(H_3) $E \in C^1[0,T],\ E(0)=\int\limits_0^1 \varphi(x)dx.$
Then the classical solution u of problem (1.1)–(1.4) *exists, is unique and* $u(\cdot,t)\in C^2[0,1],\ \partial_t^{\alpha_i}u(x,\cdot)\in C(0,T],\ (i=1,...,m)$.

Proof. The smoothness conditions specified in assumptions (H_1)–(H_3) are derived from Theorems 4.2 and 5.6. To prove the consistency conditions in Theorem 6.1, it is enough for us to prove the following lemma. □

Lemma 6.2. *Let assumptions* (H_1)–(H_3) *be fulfilled. Then, the conditions*

$$\varphi_1(0)=\varphi_1(1)=0,\ \varphi_2'(0)+\alpha\varphi_2(0)=\gamma(0),\ \varphi_2'(1)-\alpha\varphi_2(1)=\gamma(0),$$
$$f_1(0,t)=f_1(1,t)=0,\ f_{2x}(0,t)+\alpha f_2(0,t)=\frac{b(t)}{r(t)},\ f_{2x}(1,t)-\alpha f_2(1,t)=\frac{b(t)}{r(t)},$$
$$E(0)=\int\limits_0^1 \varphi_2(x)dx$$

are satisfied.

Proof. It is easy to see from the representation γ that

$$\gamma(0)=\frac{b_1-a_1}{b_1+a_1}\varphi_1'(0). \tag{6.1}$$

Substituting the expression $\varphi(x)=\varphi_1(x)+\varphi_2(x)$ into $\varphi(0)+\varphi(1)=0$, we have

$$\varphi_1(0)+\varphi_1(1)+\varphi_2(0)+\varphi_2(1)=0. \tag{6.2}$$

Using the conditions $\varphi_2(0)+\varphi_2(1)=0$, $\varphi_1(0)=\varphi_1(1)$ from (6.2), we obtain $\varphi_1(0)=\varphi_1(1)=0$. Also, substituting the expression $\varphi(x)=\varphi_1(x)+\varphi_2(x)$ into $a_1\varphi'(0)+b_1\varphi'(1)+a_0\varphi(0)=0$, we have

$$a_1\left(\varphi_1'(0)+\varphi_2'(0)\right)+b_1\left(\varphi_1'(1)+\varphi_2'(1)\right)+a_0\left(\varphi_1(0)+\varphi_2(0)\right)=0.$$

The conditions $\varphi_1'(0)=-\varphi_1(1),\ \varphi_1(0)=0,\ \varphi_2'(0)=\varphi_2'(1)$ imply that

$$\varphi_2'(0)+\frac{a_0}{b_1+a_1}\varphi_2(0)=\frac{b_1-a_1}{b_1+a_1}\varphi_1'(0),$$

which, from (6.1), becomes $\varphi_2'(0)+\alpha\varphi_2(0)=\gamma(0)$.

Similarly, we get $\varphi_2'(1)-\alpha\varphi_2(1)=\gamma(0)$.

Substituting the expression $f(x,t) = f_1(x,t) + f_2(x,t)$ into $f(0,t) + f(1,t) = 0$, we have

$$f_1(0,t) + f_2(0,t) + f_1(1,t) + f_2(1,t) = 0. \tag{6.3}$$

Then using conditions $f_1(0,t) = f_1(1,t)$, $f_2(0,t) + f_2(1,t) = 0$ from (6.3), we arrive at $f_1(0,t) = f_1(1,t) = 0$.

Applying $f(x,t) = f_1(x,t) + f_2(x,t)$ and the conditions $f_1(0,t) = f_1(1,t)$, $f_2(0,t) + f_2(1,t) = 0$, $f_{1x}(0,t) + f_{1x}(1,t) = 0$, $f_{2x}(0,t) = f_{2x}(1,t)$ from $f_x(0,t) + f_x(1,t) + \alpha\,(f(0,t) - f(1,t)) = 0$, we obtain $f_{2x}(0,t) + \alpha f_2(0,t) = \frac{b(t)}{r(t)}$.

Similarly, we have $f_{2x}(1,t) - \alpha f_2(1,t) = \frac{b(t)}{r(t)}$.

Substituting the expression $\varphi(x) = \varphi_1(x) + \varphi_2(x)$ into (H_3), we obtain $E(0) = \int\limits_0^1 \varphi_2(x)dx$.

The uniqueness of the solution of problem (1.1)–(1.4) comes from the fact that u_1 and u_2 are unique. This completes the proof. □

Remark 6.3. *If we consider the direct problem* (1.1)–(1.3)*, it is easy to see that, for the direct problem* (1.1)–(1.3)*, we have a theorem of existence and uniqueness, which is similar to Theorem 6.1 without assumption* (H_3).

7. Conclusions

In this paper, we considered the inverse source problem with the nonlocal boundary conditions for the heat equation involving multi-term time-fractional derivatives. Since the eigenvalues of the auxiliary spectral problem do not form a basis, we have divided the problem into two sub-problems, one of which is the inverse problem, and the second the direct problem. The well-posedness of the inverse and direct problems are shown by Fourier expansion in terms of eigenfunctions of the corresponding spectral problems. Also, for the well-posedness of the inverse problem, the properties of the Volterra integral equation of the second kind were used. The continuous dependence on the data of the solutions of the inverse and direct problems was proved.

Since a feature of this paper is the nonlocal boundary condition (1.3), performing some numerical tests is more difficult even without the fractional derivative (see for example [31]). This is due to the fact that the sufficient conditions for the existence of a solution are not satisfied using standard methods. It is also difficult to show the convergence of the series that arises when solving (1.1)–(1.4) using the Fourier method.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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