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Research article

Proportional fractional Dirac dynamic system

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Abstract: In this study, considering the proportional fractional derivative, which is a generalization of the conformable fractional derivative, we provided some important spectral properties such as the reality of eigenvalues, the orthogonality of eigenfunctions, the self-adjointness of the operator, the asymptotic estimations of eigenfunctions, and Picone's identity for a proportional Dirac system on an arbitrary time scale. We also presented graphics representing the eigenfunctions of the Dirac system on a time scale, produced by taking advantage of the proportional fractional derivative with some special cases. The main purpose of presenting these graphics was to examine the effect of the proportional fractional derivative on the Dirac system on a time scale, as well as the effect of the eigenvalues, which are meaningful for the subject we were studying for the solution functions.

Keywords: Dirac operator; time scales; proportional fractional derivative
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1. Introduction

Especially in the areas of spectral theory, quantum mechanics, and relativistic quantum field theory, the Dirac system, also known as the Dirac operator, is a key idea in mathematical physics. In the framework of developing a relativistic equation to describe the behavior of spin-1/2 particles like electrons, this idea was first suggested by British scientist Paul Dirac. To comprehend how particles with inherent angular momentum (spin) behave in relativistic conditions, it is essential to grasp how the Dirac system works. The Dirac operator appears in spectral theory as a self-adjoint operator connected to the Dirac equation. Understanding the behavior of relativistic quantum systems requires an in-depth knowledge of eigenvalues and the corresponding eigenvectors of spectra for the Dirac operator.

The classical Dirac system [1–3] can be expanded to a framework that incorporates time scales and makes use of conformable derivatives [4–14] in the Dirac system on time scales. In the study of dynamic systems that display both continuous and discrete responses, this idea is very pertinent.

Time scales are a generalization of real numbers that include continuous and discrete cases, including the well-known instances of real numbers, integers, and rational numbers. Time scale $\mathbb{T}$ is a closed, nonempty subset of $\mathbb{R}$ that is a member of the standard topology of $\mathbb{R}$. One can find details about this theory in previous studies [15–21]. Consequently, the intriguing extension of the Dirac equation combining time scale theory, fractional calculus [22–25], and the Dirac system on time scales with conformable derivatives offers an adaptable tool for modeling and examining systems [26, 27] with a combination of continuous and discrete behaviors. Now, let us talk about the usage areas of the proportional derivative and what it means.

Engineering frequently uses proportional-derivative (PD) control as a form of a control approach to manage the behavior of dynamic systems. This technique involves determining the control action by utilizing both the present error (proportional term) and the rate of change in error (derivative term). The PD control technique can be modified to account for the distinct properties of time scales when used with dynamic systems on time scales. The error in PD control on time scales refers to the difference between the present state of the system and the desired set point. This error is calculated in a way that fits the underlying time scale. For instance, the difference between the desired and actual values at discrete time occurrences will be used in the error computation if the time scale is discrete (e.g., for integers). The proportional term in PD control helps with the control action and is proportionate to the error. As with continuous-time or discrete-time PD control, the proportional term can be calculated on time scales depending on the error at a particular time instance. In PD control, the derivative term considers the rate of change in error.

Applications for PD control on time scales may be found in many disciplines where systems display a combination of continuous and discrete behaviors. This covers a variety of disciplines, including control theory, robotics, process control, and others. PD control on time scales, for instance, can offer efficient control methods in systems with sporadically sampled data or a combination of continuous and discrete dynamics. PD control on time scales extends the conventional PD control strategy to dynamic systems that function on time scales. It entails taking into account the amount and rate of error change while modifying the proportional and derivative terms to reflect the characteristics of time scales. Let us now discuss Picone's identity, which is significant to oscillation theory [28].

For self-adjoint operators, Picone's identity is a key outcome in the field of spectral theory. It is employed to prove various eigenvalue and eigenfunction characteristics of self-adjoint differential operators. A technique for examining the distribution and behavior of eigenvalues with regard to certain inequalities is Picone's identity. It is crucial to understanding the spectral behavior of many mathematical and physical systems because it helps analyze the characteristics of the eigenvalue spectra of differential operators. To prove conclusions concerning the distribution of eigenvalues and the behaviors of eigenfunctions for various self-adjoint operators, which have implications in a variety of domains including quantum mechanics, heat conduction, and elasticity theory, Picone's identity is utilized as a crucial step.

Let us have a look at the proportional Dirac eigenvalue problem on an arbitrary time scale

$$\tau^{\omega} x(t) = \lambda x^{\sigma}(t), \quad t \in [a, b] = \mathfrak{J} \cap \mathbb{T}, \tag{1.1}$$

$$\eta x_1(a) + \beta x_2(a) = 0, \tag{1.2}$$

$$\gamma x_1(b) + \delta x_2(b) = 0. \tag{1.3}$$

The explicit form of the system (1.1) in this case is

$$\begin{aligned}\tau^{\omega}x(t) &= \begin{pmatrix}0 & \mathfrak{D}^{\omega}\\ -\mathfrak{D}^{\omega} & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}q(t) & 0\\ 0 & r(t)\end{pmatrix}\begin{pmatrix}x_1^{\sigma}(t)\\ x_2^{\sigma}(t)\end{pmatrix}\\ &= \begin{pmatrix}\mathfrak{D}^{\omega}x_2(t) + q(t)x_1^{\sigma}(t)\\ -\mathfrak{D}^{\omega}x_1(t) + r(t)x_2^{\sigma}(t)\end{pmatrix},\end{aligned}$$

where $\mathbb{T}$ is an arbitrary time scale, $\mathfrak{D}^{\omega}x$ is the ω-th order proportional delta derivative of x, $\lambda > 0$ is a spectral parameter, σ is the the forward jump operator, $x^{\sigma} = x(\sigma)$, $\eta^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$, $\omega \in [0, 1]$, and $x(t) = \begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$. We suppose that $q, r : \mathfrak{J} \cap \mathbb{T} \to \mathbb{R}$ are continuous functions and

$$L_2^{\omega}\mathfrak{J} = \left\{\phi(t) : \int_a^b \frac{\phi(t)^T\phi(t)\tilde{e}_0(b, \sigma(t))}{\tilde{e}_{\xi}(t, b)}\Delta_{\omega}t < \infty\right\},\ \xi(t) = k_1(\omega, t) - \frac{k_1(w, t)k_0(\omega, t)}{k_0(\omega, t) + \mu(t)k_1(\omega, t)}, \tag{1.4}$$

where $\phi(t) = \begin{pmatrix}\phi_1(t)\\ \phi_2(t)\end{pmatrix}$ and T denotes the transpose throughout the whole research.

By setting $\mathbb{T} = \mathbb{R}$ and $\omega = 1$ in (1.1), we obtain the following general classical Dirac system:

$$\begin{aligned}&y_2'(t) + (V(t) + m)y_1(t) = \lambda y_1(t),\\ &y_1'(t) - (V(t) - m)y_1(t) = -\lambda y_2(t),\end{aligned} \tag{1.5}$$

where $q(t) = V(t) + m$, $r(t) = V(t) - m$. This system serves as the relativistic counterpart of the Schrödinger operator by incorporating the principles of special relativity, known as the Dirac operator in quantum physics, into quantum mechanics:

Under the influence of external potentials or fields, m that denotes the mass of a particle controls the motion of the particle. Within the framework of quantum mechanics, the potential function $V(t)$ generally denotes the potential energy that a particle experiences as a result of its interactions with other particles or external fields. Understanding the behavior of particles depends on this interaction potential, which is a key idea in quantum mechanics.

The spectrum of a system in quantum mechanics is the set of potential eigenvalues for certain operators that describe observables such as energy, momentum, and angular momentum. The spectrum properties of Dirac systems on a time scale provide valuable insights into the behavior of quantum systems with nonclassical dynamics and aid in the determination of eigenvalues and eigenfunctions, which are crucial for comprehending the states and energy levels of the quantum system. It also reveals information about the evolution of the system throughout time. All of them provide insight into the temporal behavior of the quantum system by describing how the eigenvalues alter as the system dynamically develops. One may assess the stability of the quantum system by looking at its spectrum characteristics.

Derivatives illustrate how a system instantly changes in relation to a variable at a particular location in space and time in classical physics. On the other hand, nonlocal behavior, in which characteristics of a particle are not restricted to a particular place, is possible in quantum physics. Fractional derivatives take into account the impact of a particle's whole history of motion or condition in addition to its immediate state, which allows them to capture this nonlocality.

Researching the spectrum characteristics of the Dirac system with the proportional fractional derivatives advances the mathematical physics theory more broadly. For all of these reasons, we hope that our research can pave the way for further interdisciplinary collaborations and discoveries, leading to the development of new mathematical tools and techniques for analyzing complex quantum systems.

Let us briefly describe how our study is organized. We define and explain some basic notations for proportional fractional calculus on $\mathbb{T}$ in Section 2. We establish a few fundamental theorems for the proportional fractional Dirac system on $\mathbb{T}$ in Section 3. We obtain asymptotic estimates of eigenfunctions and Picone's identity for the problem (1.1)–(1.3) in Section 4 using a few techniques. Graphics representing the eigenfunctions of the Dirac system produced by using the proportional fractional derivative on a time scale with some special cases are presented in order to investigate the impact of the eigenvalues on the solution functions, and the effect of the proportional fractional derivative on the Dirac system on a time scale is given in Section 5. Conclusions are given in Section 6.

2. Preliminaries

In this section, we discuss all concepts linked to the required time scale for proportional computations. Let us first define the classic proportional fractional derivative.

Definition 1. *[29] Let $\omega \in [0,1]$. The differential operator $\mathfrak{D}^{\omega}$ is known as a proportional derivative if $\mathfrak{D}^{0}$ is a unit operator and $\mathfrak{D}^{1}$ is a standard differential operator. It is explicitly stated that only $\mathfrak{D}^{0}h(t)=h(t)$ and $\mathfrak{D}^{1}h(t)=h'(t)$ exist for the derivative function $h=h(t)$, which has a proportional operator $\mathfrak{D}^{\omega}$.*

Remark 1. *[29] Based on the use of a proportional-derivative controller with a ϑ controller output at time t, the fundamental concept of the proportional derivative is developed. The algorithm*

$$\vartheta(t)=\kappa_p E(t)+\kappa_d \frac{d}{dt}E(t),$$

is applied by this controller, $\vartheta(t)$.

E denotes the error between the state and process variables in this case, whereas κ_p and κ_d stand for the proportional and derivative benefits, respectively [30].

Definition 2. *[29] Assume that $\omega \in [0,1]$, $\kappa_0, \kappa_1 : [0,1]\times\mathbb{R} \to \mathbb{R}_0^+$ are continuous functions and*

$$\begin{cases} \lim\limits_{\omega\to 0^+} \kappa_0(\omega,t)=0, & \lim\limits_{\omega\to 0^+} \kappa_1(\omega,t)=1, \\ \lim\limits_{\omega\to 1^-} \kappa_0(\omega,t)=1, & \lim\limits_{\omega\to 1^-} \kappa_1(\omega,t)=0, \\ \kappa_0(\omega,t)\neq 0,\ \omega\in(0,1], & \kappa_1(\omega,t)\neq 0,\ \omega\in[0,1), \end{cases} \tag{2.1}$$

hold, where h is the error, κ_1 is a kind of proportional gain κ_p, κ_0 is a type of derivative gain κ_d, and $v=\mathfrak{D}^{\omega}h$ is the controller output, all of which are presented together with the differential operator $\mathfrak{D}^{\omega}$ defined by

$$\mathfrak{D}^{\omega}h(t)=\kappa_1(\omega,t)h(t)+\kappa_0(\omega,t)h'(t), \tag{2.2}$$

in this case.

Now, the proportional fractional delta derivative of a function $h : \mathbb{T} \to \mathbb{R}$ at point $t \in \mathbb{T}^\kappa$ will now be defined on a time scale $\mathbb{T}$. Suppose that $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to \mathbb{R}_0^+$ are continuous functions, and the condition (2.1) is provided in the following expressions.

Definition 3. *[31] Let $h : \mathbb{T} \to \mathbb{R}$ be a function and $\zeta \in \mathbb{T}^\kappa$. If there is a real number $\mathfrak{D}^\omega h(\zeta)$, $\omega \in [0,1]$, such that*

$$\left|\kappa_1(\omega,\zeta) h(\zeta)(\sigma(\zeta)-s) + \kappa_0(\omega,\zeta)\left[h(\sigma(\zeta)) - h(s)\right] - (\mathfrak{D}^\omega h)(\zeta)(\sigma(\zeta)-s)\right| \le \varepsilon\left|\sigma(\zeta)-s\right|, \quad (2.3)$$

for every $\varepsilon > 0$, and for every s in a neighborhood U of point ζ, then that number is known as the ω-th order proportional delta derivative of f at point ζ on $\mathbb{T}$.

With

$$\Omega(\mathbb{T}) = \{h : \mathbb{T} \to \mathbb{R} : \text{For any } t \in \mathbb{T}^\kappa,\ \mathfrak{D}^\omega h(t) \text{ exists and is finite}\},$$

the set of all proportional delta differentiable functions will be shown [31] and $C_{rd}(\mathbb{T})$ will be used to denote the collection of $h : \mathbb{T} \to \mathbb{R}$ rd-continuous functions.

Lemma 1. *[31] If $h, g : \mathbb{T} \to \mathbb{R}$ are proportional delta differentiable at $t \in \mathbb{T}^\kappa$, then the following properties hold:*

(i) $\mathfrak{D}^\omega[\gamma h + \theta g] = \gamma\mathfrak{D}^\omega h + \theta\mathfrak{D}^\omega g$, *all* $\gamma, \theta \in \mathbb{R}$;

(ii) $\mathfrak{D}^\omega[hg] = h^\sigma\mathfrak{D}^\omega g + g\mathfrak{D}^\omega h - h^\sigma g\kappa_1(\omega,.)$;

(iii) $\mathfrak{D}^\omega\left[\frac{h}{g}\right] = \frac{g^\sigma\mathfrak{D}^\omega h - h\mathfrak{D}^\omega g}{gg^\sigma} + \frac{h^\sigma}{g^\sigma}\kappa_1(\omega,.)$, $gg^\sigma \neq 0$.

Definition 4. *[31] Let $\omega \in [0,1]$. $p : \mathbb{T} \to \mathbb{R}$ is regarded as ω-regressive if the condition*

$$1 + \frac{p(\zeta) - \kappa_1(\omega,\zeta)}{\kappa_0(\omega,\zeta)}\mu(\zeta) \neq 0,\ \forall\, \zeta \in \mathbb{T}^\kappa,$$

is satisfied. $\mathcal{R}_\omega$=$\mathcal{R}_\omega(\mathbb{T})$ represents the whole set of ω-regressive and rd-continuous functions on $\mathbb{T}$.

Definition 5. *[31] Let $\omega \in (0,1]$ and $q \in \mathcal{R}_\omega$. Assume q/κ_0, κ_1/κ_0 are delta integrable functions on $\mathbb{T}$, then,*

$$\tilde{e}_q(t,s) = \exp\left[\int_s^t \frac{1}{\mu(\zeta)} Log\left(1 + \frac{q(\zeta) - \kappa_1(\omega,\zeta)}{\kappa_0(\omega,\zeta)}\mu(\zeta)\right)\Delta\zeta\right], \quad (2.4)$$

is a proportional exponential function on $\mathbb{T}$ for the operator $\mathfrak{D}^\omega$, where Log is basic logarithm function. For $\mu(t)$=0, it yields

$$\tilde{e}_q(t,s) = \exp\left[\int_s^t \left(\frac{q(\zeta) - \kappa_1(\omega,\zeta)}{\kappa_0(\omega,\zeta)}\right)\Delta\zeta\right]. \quad (2.5)$$

Lemma 2. *[31] Let $\omega \in (0,1]$ and $q \in \mathcal{R}_\omega$. For fixed $\tau \in \mathbb{T}$,*

$$\mathfrak{D}^\omega\left[\tilde{e}_q(.,\tau)\right] = q(s)\,\tilde{e}_q(.,\tau),$$

and

$$\tilde{e}_q(\sigma(s),\tau) = \left(1 + \frac{q(s) - \kappa_1(\omega,s)}{\kappa_0(\omega,s)}\mu(s)\right)\tilde{e}_q(s,\tau). \quad (2.6)$$

Definition 6. *[31] Assume that $h \in C_{rd}(\mathbb{R})$, $\omega \in (0,1]$, and $t_0 \in \mathbb{T}$, then,*

$$\int \mathfrak{D}^{\omega} h(\zeta)\Delta_{\omega}\zeta = h(\eta) + c\tilde{e}_0(\eta, t_0), \ \forall \eta \in \mathbb{T}, \ c \in \mathbb{R},$$

signifies the indefinite proportional integral (anti-derivative) of h on $[a,b]_{\mathbb{T}}$ according to (2.4)*, whereas*

$$\int_a^t h(\zeta)\tilde{e}_0(t,\sigma(\zeta))\Delta_{\omega}\zeta = \int_a^t \frac{h(\zeta)\tilde{e}_0(t,\sigma(\zeta))}{\kappa_0(\omega,\zeta)}\Delta\zeta, \quad \Delta_{\omega}\zeta = \frac{1}{\kappa_0(\omega,\zeta)}\Delta\zeta, \tag{2.7}$$

defines the definite proportional integral according to Lemma 2.

Lemma 3. *[31] Let $\omega \in (0,1]$, $h \in C_{rd}(\mathbb{T})$, then,*

$$\mathfrak{D}^{\omega}\left[\int_a^t h(\zeta)\tilde{e}_0(t,\sigma(\zeta))\Delta_{\omega}\zeta\right] = h(t). \tag{2.8}$$

Lemma 4. *[31] Let $h, g \in \Omega(\mathbb{T})$.*
(i) $\int_a^t \mathfrak{D}^{\omega}[h(\zeta)]\tilde{e}_0(t,\sigma(\zeta))\Delta_{\omega}\zeta = [h(\zeta)\tilde{e}_0(t,\sigma(\zeta))]_{\zeta=a}^{t}$.
(ii) $\int_a^b h(\zeta)\mathfrak{D}^{\omega}[g(\zeta)]\tilde{e}_0(b,\sigma(\zeta))\Delta_{\omega}\zeta = [h(\zeta)g(\zeta)\tilde{e}_0(b,\sigma(\zeta))]_{\zeta=a}^{b}$

$$-\int_a^b g^{\sigma}(\zeta)\{\mathfrak{D}^{\omega}[h(\zeta)] - \kappa_1(\omega, \zeta)h(\zeta)\}\tilde{e}_0(b,\sigma(\zeta))\Delta_{\omega}\zeta.$$

Theorem 5. *[29] Let $p \in C_{rd}(\mathbb{T}) \cap \mathcal{R}_{\omega}$, $q \in C_{rd}(\mathbb{T})$, $\eta_0 \in \mathbb{T}$, and $x_0 \in \mathbb{R}$. The solution of the proportional type initial value problem*

$$\mathfrak{D}^{\omega}x = p(\eta)x + q(\eta), \quad x(\eta_0) = x_0,$$

is presented with

$$x(\eta) = x_0\tilde{e}_p(\eta,\eta_0) + \int_{\eta_0}^{\eta} q(\zeta)\tilde{e}_g(\sigma(\zeta),\eta)\Delta_{\omega}\zeta, \quad \eta \in \mathbb{T}^{\kappa}, \tag{2.9}$$

where $g = \frac{(p-\kappa_1)(\mu\kappa_1-\kappa_0)}{\kappa_0+\mu(p-\kappa_1)}$.

3. Main results

We present several significant outcomes for the proportional fractional Dirac system on $\mathbb{T}$ in this section. It is generally known that when $\mathbb{T} = \mathbb{R}$ and $\omega = 1$, (1.1)–(1.3) has eigenfunctions that are orthogonal and only real eigenvalues. The conclusions that follow will apply this fundamental consequence to the proportional fractional scenario for the problem (1.1)–(1.3).

Theorem 6. *For operator τ^{ω} and $\omega \in (0,1]$ in* (1.1)*, we assume that*

$$\kappa_0(\omega,t) + \mu(t)\kappa_1(\omega,t) \neq 0.$$

Let $\phi(t) = \begin{pmatrix}\phi_1(t)\\ \phi_2(t)\end{pmatrix}$, and $\Phi(t) = \begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}$ represents eigenfunctions of (1.1)–(1.3)*, then, we obtain*

$$(\phi^{\sigma})^T\tau^{\omega}\Phi - (\tau^{\omega}\phi)^T\Phi^{\sigma} = \frac{\kappa_0 + \kappa_1\mu}{\kappa_0}\mathfrak{D}^{\omega}(W(\phi,\Phi)) + \frac{\kappa_1(\kappa_0 - \kappa_1\mu)}{\kappa_0}W(\phi,\Phi), \tag{3.1}$$

and the Lagrange identity

$$\tilde{e}_\xi(t,b)\mathfrak{D}^\omega\left(\frac{W(\phi,\Phi)}{\tilde{e}_\xi(t,b)}\right) = (\phi^\sigma)^T\tau^\omega\Phi - (\tau^\omega\phi)^T\Phi^\sigma, \quad t,b\in\mathbb{T}^\kappa, \tag{3.2}$$

where $W(\phi,\Phi) = \phi_2\Phi_1^\sigma - \phi_1\Phi_2^\sigma$ *is the Wronskian of* ϕ *and* Φ.

Proof. Using Lemma 1 (ii) and Definition 3,

$$\begin{aligned}\mathfrak{D}^\omega(W(\phi,\Phi)) &= (\mathfrak{D}^\omega\phi_1)\Phi_2^\sigma + \phi_1^\sigma\mathfrak{D}^\omega\Phi_2^\sigma - \kappa_1\phi_1^\sigma\Phi_2^\sigma - (\mathfrak{D}^\omega\Phi_1)\phi_2^\sigma - \Phi_1^\sigma\mathfrak{D}^\omega\phi_2^\sigma + \kappa_1\Phi_1^\sigma\phi_2^\sigma\\ &= (\phi^\sigma)^T\tau^\omega\Phi - (\tau^\omega\phi)^T\Phi^\sigma - \kappa_1 W^\sigma(\phi,\Phi)\\ &= (\phi^\sigma)^T\tau^\omega\Phi - (\tau^\omega\phi)^T\Phi^\sigma - \frac{\kappa_1(\kappa_0-\kappa_1\mu)}{\kappa_0}W(\phi,\Phi) - \frac{\mu\kappa_1}{\kappa_0}W(\phi,\Phi),\end{aligned}$$

is obtained, easily. Thus,

$$\mathfrak{D}^\omega(W(\phi,\Phi)) = \frac{\kappa_0}{\kappa_0+\mu\kappa_1}[(\phi^\sigma)^T\tau^\omega\Phi - (\tau^\omega\phi)^T\Phi^\sigma] - \frac{\kappa_1(\kappa_0-\kappa_1\mu)}{\kappa_0+\mu\kappa_1}W(\phi,\Phi). \tag{3.3}$$

The definition of $\xi(t)$ and Lemma 2 gives us

$$\frac{\tilde{e}_\xi}{\tilde{e}_\xi^\sigma} = \frac{\kappa_0+\kappa_1\mu}{\kappa_0}. \tag{3.4}$$

On the other hand, according to Lemma 1 (iii),

$$\tilde{e}_\xi\mathfrak{D}^\omega\left(\frac{W}{\tilde{e}_\xi}\right) = \frac{\tilde{e}_\xi}{\tilde{e}_\xi^\sigma}(\mathfrak{D}^\omega W - W\xi) + \kappa_1 W. \tag{3.5}$$

If (1.4), (3.3), and (3.4) are substituted into (3.5),

$$\begin{aligned}\tilde{e}_\xi\mathfrak{D}^\omega\left(\frac{W}{\tilde{e}_\xi}\right) &= \frac{\kappa_0+\mu\kappa_1}{\kappa_0}\left[\frac{\kappa_0}{\kappa_0+\kappa_1\mu}((\phi^\sigma)^T\tau^\omega\Phi - (\tau^\omega\phi)^T\Phi^\sigma) - \frac{\kappa_1(\kappa_0-\kappa_1\mu)}{\kappa_0+\mu\kappa_1}W\right]\\ &\quad - \frac{\kappa_0+\mu\kappa_1}{\kappa_0}\left(\kappa_1 - \frac{\kappa_1\kappa_0}{\kappa_1\mu}\right)W + \kappa_1 W\\ &= (\phi^\sigma)^T\tau^\omega\Phi - (\tau^\omega\phi)^T\Phi^\sigma,\end{aligned}$$

is found. $\square$

Definition 7. *Assume that* $\omega\in(0,1]$ *and the condition* $\kappa_0(\omega,t)+\mu(t)\kappa_1(\omega,t)\neq 0$ *is satisfied, then* (1.4) *defines* ξ.

$$<\phi,\Phi>_\omega = \int_a^b \frac{\phi(t)^T\Phi(t)\tilde{e}_0(b,\sigma(t))}{\tilde{e}_\xi(t,b)}\Delta_\omega t, \tag{3.6}$$

denotes the proportional inner product of the functions $\phi,\Phi\in L_2^\omega\mathfrak{I}$, *where* $\phi(t) = \begin{pmatrix}\phi_1(t)\\ \phi_2(t)\end{pmatrix}$ *and* $\Phi(t) = \begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}$.

Lemma 7. *Consider that $\omega \in (0,1]$ and the condition $\kappa_0(\omega,t)+\mu(t)\kappa_1(\omega,t)\neq 0$ is held. The Green's formula is given with*

$$< \phi, \tau^{\omega}\Phi >_{\omega} - < \tau^{\omega}\phi, \Phi >_{\omega} = \frac{W(\phi,\Phi)(t)\tilde{e}_0(b,t)}{\tilde{e}_{\xi}(t,b)}\Big|_a^b. \tag{3.7}$$

Proof. According to Theorem 6,

$$\tilde{e}_{\xi}\mathfrak{D}^{\omega}\Big(\frac{W}{\tilde{e}_{\xi}}\Big) = (\phi^{\sigma})^T\tau^{\omega}\Phi - (\tau^{\omega}\phi)^T\Phi^{\sigma}$$

is valid. If we apply the final equivalence from a to b in terms of the proportional fractional integral of t, we get the result that

$$\int_a^b \mathfrak{D}^{\omega}\Big(\frac{W}{\tilde{e}_{\xi}}\Big)\tilde{e}_0(b,\sigma(t))\Delta_{\omega}t = \int_a^b \frac{(\tau^{\omega}x(t))^T y(t)\tilde{e}_0(b,\sigma(t))}{\tilde{e}_{\xi}(t,b)}\Delta_{\omega}t.$$

□

Green's identity is easily found using Lemma 4 (i).

Theorem 8. *The proportional fractional Dirac operator τ^{ω} is self-adjoint on $L_2^{\omega}\mathfrak{J}$.*

Proof. Let the problem (1.1)–(1.3) have solutions $x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$, $y(t)=\begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix}$. Consequently,

$$\tau^{\omega}x(t) = B\mathfrak{D}^{\omega}(x(t)) + Q(t)x^{\sigma}(t) = \lambda x^{\sigma}(t),$$

$$\tau^{\omega}y(t) = B\mathfrak{D}^{\omega}(y(t)) + Q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t).$$

As a result of taking into account the boundary conditions and the definition of the proportional inner product on $L_2^{\omega}\mathfrak{J}$ and Lemma 4 (ii), we arrive at

$$\begin{aligned}
&< \tau^{\omega}x, y >_{\omega} = \int_a^b \frac{(\tau^{\omega}x(t))^T y(t)\tilde{e}_0(b,\sigma(t))}{\tilde{e}_{\xi}(t,b)}\Delta_{\omega}t\\
&= \int_a^b \mathfrak{D}^{\omega}x_2(t)\frac{\tilde{e}_0(b,\sigma(t))}{\tilde{e}_{\xi}(t,b)}y_1(t)\Delta_{\omega}t - \int_a^b \mathfrak{D}^{\omega}x_1(t)\frac{\tilde{e}_0(b,\sigma(t))}{\tilde{e}_{\xi}(t,b)}y_2(t)\Delta_{\omega}t\\
&+ \int_a^b [q(t)x_1^{\sigma}(t)y_1(t) + r(t)x_2^{\sigma}(t)y_2(t)]\frac{\tilde{e}_0(b,\sigma(t))}{\tilde{e}_{\xi}(t,b)}\Delta_{\omega}t\\
&= \frac{y_1(t)}{\tilde{e}_{\xi}(t,b)}x_2(t)\mid_{t=a}^b - \int_a^b x_2^{\sigma}(t)\bigg(\mathfrak{D}^{\omega}\Big(\frac{y_1(t)}{\tilde{e}_{\xi}(t,b)}\Big) - \kappa_1(\omega,t)\frac{y_1(t)}{\tilde{e}_{\xi}(t,b)}\bigg)\tilde{e}_0(b,\sigma(t))\Delta_{\omega}t\\
&- \frac{y_2(t)}{\tilde{e}_{\xi}(t,b)}x_1(t)\mid_a^b - \int_a^b x_1^{\sigma}(t)\bigg(\mathfrak{D}^{\omega}\Big(\frac{y_2(t)}{\tilde{e}_{\xi}(t,b)}\Big) - \kappa_1(\omega,t)\frac{y_2(t)}{\tilde{e}_{\xi}(t,b)}\bigg)\tilde{e}_0(b,\sigma(t))\Delta_{\omega}t\\
&+ \int_a^b [q(t)x_1^{\sigma}(t)y_1(t) + r(t)x_2^{\sigma}(t)y_2(t)]\frac{\tilde{e}_0(b,\sigma(t))}{\tilde{e}_{\xi}(t,b)}\Delta_{\omega}t\\
&= -\int_a^b \bigg(\frac{x_2(t)\mathfrak{D}^{\omega}y_1(t) - y_1(t)\xi x_2(t)}{\tilde{e}_{\xi}(t,b)}\bigg)\tilde{e}_0(b,\sigma(t))\Delta_{\omega}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b\left(\frac{x_1(t)\mathfrak{D}^\omega y_2(t)-x_1(t)\xi y_2(t)}{\tilde{e}_\xi(t,b)}\right)\tilde{e}_0(b,\sigma(t))\Delta_\omega t\\
&+\int_a^b[q(t)y_1(t)x_1(t)+r(t)y_2(t)x_2(t)]\frac{\tilde{e}_0(b,\sigma(t))}{\tilde{e}_\xi(t,b)}\Delta_\omega t\\
&=< x,\tau^\omega y >_\omega+\int_a^b\left(\frac{y_1(t)x_2(t)-x_1(t)y_2(t)}{\tilde{e}_\xi(t,b)}\right)\xi\tilde{e}_0(b,\sigma(t))\Delta_\omega t,
\end{aligned}
$$

where $t\in\mathfrak{J}$ is right-dense. Since $\xi(t)=0$,

$$
< \tau x, y >_\omega=< x,\tau y >_\omega,
$$

is discovered. This concludes the proof. □

Theorem 9. *The problem* (1.1)–(1.3) *only contains real eigenvalues.*

Proof. Let $\overline{\phi}(t,\lambda)=\begin{pmatrix}\overline{\phi}_1(t,\lambda)\\ \overline{\phi}_2(t,\lambda)\end{pmatrix}$ be an eigenfunction corresponding to the eigenvalue $\overline{\lambda}$ of the problem (1.1)–(1.3), and let $\overline{\lambda}$ be a complex eigenvalue. A straightforward calculation gives us

$$
\begin{aligned}
\mathfrak{D}^\omega(\phi_1\overline{\phi}_2^\sigma-\overline{\phi}_1\phi_2^\sigma)(t,\lambda)&=((\mathfrak{D}^\omega\phi_1)\overline{\phi}_2^\sigma+\phi_1^\sigma\mathfrak{D}^\omega\overline{\phi}_2^\sigma)(t,\lambda)-(\phi_1^\sigma\overline{\phi}_2^\sigma)(t,\lambda)\kappa_1(\omega,t)\\
&-((\mathfrak{D}^\omega\overline{\phi}_1)\phi_2^\sigma-\overline{\phi}_1^\sigma\mathfrak{D}^\omega\overline{\phi}_2^\sigma)(t,\lambda)+(\overline{\phi}_1^\sigma\phi_2^\sigma)(t,\lambda)\kappa_1(\omega,t)\\
&=(-\lambda+r(t))(\phi_2^\sigma\overline{\phi}_2^\sigma)(t,\lambda)+\phi_1^\sigma(t,\lambda)(\overline{\lambda}-q(t))\overline{\phi}_1^\sigma(t,\lambda)\\
&-\kappa_1(\omega,t)(\phi_1^\sigma\overline{\phi}_1^\sigma)(t,\lambda)-(-\overline{\lambda}+r(t))(\overline{\phi}_2^\sigma\phi_2^\sigma)(t,\lambda)\\
&-\overline{\phi}_1^\sigma(t,\lambda)(\lambda-q(t))\phi_1^\sigma(t,\lambda)+\kappa_1(\omega,t)(\overline{\phi}_1^\sigma\phi_2^\sigma)(t,\lambda)\\
&=(\overline{\lambda}-\lambda)(\phi_1^\sigma\overline{\phi}_1^\sigma+\phi_2^\sigma\overline{\phi}_2^\sigma)(t,\lambda)+\kappa_1(\omega,t)(\overline{\phi}_1^\sigma\phi_2^\sigma-\phi_1^\sigma\overline{\phi}_2^\sigma)(t,\lambda)\\
&=(\overline{\lambda}-\lambda)(|\phi_1^\sigma|^2+|\phi_2^\sigma|^2)(t,\lambda)+\kappa_1(\omega,t)(\overline{\phi}_1^\sigma\phi_2^\sigma-\phi_1^\sigma\overline{\phi}_2^\sigma)(t,\lambda).
\end{aligned}
$$

If we take the final equivalence from a to b with regard to t's ω proportional fractional integral, we obtain

$$
\begin{aligned}
\int_a^b\mathfrak{D}^\omega(\phi_1\overline{\phi}_2^\sigma-\overline{\phi}_1\phi_2^\sigma)(t,\lambda)\tilde{e}_0(b,\sigma(t))\Delta_\omega t&=(\overline{\lambda}-\lambda)\int_a^b(|\phi_1^\sigma|^2+|\phi_2^\sigma|^2)(t,\lambda)\tilde{e}_0(b,\sigma(t))\Delta_\omega t\\
&+\int_a^b\kappa_1(\omega,t)(\overline{\phi}_1^\sigma\phi_2^\sigma-\phi_1^\sigma\overline{\phi}_2^\sigma)(t,\lambda)\tilde{e}_0(b,\sigma(t))\Delta_\omega t\\
&=0.
\end{aligned}
$$

If $\kappa_1(\omega,t)=0$ or $(\overline{\phi}_1^\sigma\phi_2^\sigma-\phi_1^\sigma\overline{\phi}_2^\sigma)(t,\lambda)=0$, we arrive at $\overline{\lambda}=\lambda$, concluding the proof. □

Theorem 10. *Eigenfunctions of* (1.1)–(1.3)*,* $\phi(t,\lambda_1)=\begin{pmatrix}\phi_1(t,\lambda_1)\\ \phi_2(t,\lambda_1)\end{pmatrix}$*, and* $\Phi(t,\lambda_2)=\begin{pmatrix}\Phi_1(t,\lambda_2)\\ \Phi_2(t,\lambda_2)\end{pmatrix}$*, which correspond to distinct eigenvalues* λ_1 *and* λ_2*, are orthogonal on* $L_2^\omega\mathfrak{J}$*, i.e.,*

$$
\int_a^b\frac{\phi^T(t,\lambda_1)\Phi(t,\lambda_2)\tilde{e}_0(b,\sigma(t))}{\tilde{e}_\xi(t,b)}\Delta_\omega t=0. \tag{3.8}
$$

Proof. Since $\phi(t,\lambda_1)$ and $\Phi(t,\lambda_2)$ are the solutions of proportional fractional Dirac eigenvalue problem (1.1)–(1.3),

$$\frac{W(\phi,\Phi)(t)\tilde{e}_0(b,t)}{\tilde{e}_\xi(t,b)}\bigg|_a^b =< \phi, \tau^\omega \Phi >_\omega - < \tau^\omega \phi, \Phi >_\omega,$$

then,

$$\begin{aligned} < \phi, \tau^\omega \Phi >_\omega - < \tau^\omega \phi, \Phi >_\omega &= 0, \\ (\lambda_1 - \lambda_2) < \phi, \Phi >_\omega &= 0, \end{aligned}$$

is found by considering Green's identity (3.7). Since $\lambda_1 \neq \lambda_2$, we obtain (3.8). □

4. Eigenfunction estimations for the proportional fractional Dirac system on time scales

The asymptotic estimates of the eigenfunction and Picone's identity of the problem (1.1)–(1.3) on $\mathbb{T}$ are given in this section.

Theorem 11. *If $\phi_1(t,\lambda)$ and $\phi_2(t,\lambda)$ fulfill the equations*

$$\phi_1(t,\lambda) = c_1\Big(\cos_{\frac{1}{1+\mu}}(t,a) - i\sin_{\frac{1}{1+\mu}}(t,a)\Big) + \int_a^t \phi^{(2)}(s)\tilde{e}_{i\gamma+\kappa_1}(s,t)\Delta_w s, \tag{4.1}$$

$$\phi_2(t,\lambda) = c_2\Big(\cos_{\frac{1}{1+\mu}}(t,a) + i\sin_{\frac{1}{1+\mu}}(t,a)\Big) + \int_a^t \phi^{(1)}(s)\tilde{e}_{-i\gamma+\kappa_1}(s,t)\Delta_w s, \tag{4.2}$$

where

$$\phi^{(1)}(t,\lambda) = \cos_{\frac{1}{1+\mu}}(t,a) - i\sin_{\frac{1}{1+\mu}}(t,a) + \int_a^t [\kappa_1(w,s)q(s) - \mathfrak{D}^\omega q(s)]\tilde{e}_{i\gamma+\kappa_1}(s,t)\Delta_w s, \tag{4.3}$$

$$\phi^{(2)}(t,\lambda) = \cos_{\frac{1}{1+\mu}}(t,a) + i\sin_{\frac{1}{1+\mu}}(t,a) + \int_a^t [\mathfrak{D}^\omega r(s) - \kappa_1(w,s)r(s)]\tilde{e}_{-i\gamma+\kappa_1}(s,t)\Delta_w s, \tag{4.4}$$

the solution to the problem (1.1)–(1.3) *is the eigenfunction* $\phi(t,\lambda) = \begin{pmatrix}\phi_1(t,\lambda)\\ \phi_2(t,\lambda)\end{pmatrix}$.

Proof. Let the solution to the problem (1.1)–(1.3) be $\phi(t,\lambda)$. Consequently, the system (1.1) is identical to

$$\mathfrak{D}^\omega \phi_1 = (-\lambda + r)\phi_2^\sigma, \tag{4.5}$$

$$\mathfrak{D}^\omega \phi_2 = (\lambda - q)\phi_1^\sigma, \tag{4.6}$$

where q, r are constants. This gives us

$$\begin{aligned} (\mathfrak{D}^\omega)^2\phi_2 &= \mathfrak{D}^\omega((\lambda - q)\phi_1^\sigma) \\ &= (\kappa_1 q^\sigma - \mathfrak{D}^\omega q)\phi_1^\sigma + (\lambda - q^\sigma)(-\lambda + r)\phi_2^\sigma, \end{aligned}$$

and so

$$(\mathfrak{D}^\omega)^2\phi_2 + (\lambda - q^\sigma)(\lambda - r)\phi_2^\sigma = (\kappa_1 q^\sigma - \mathfrak{D}^\omega q)\phi_1^\sigma. \tag{4.7}$$

When the last equation is solved by using the method in [32], the characteristic equation and its roots,

$$z^2 + (\lambda - q^\sigma)(\lambda - r) = 0,$$

$$z_{1,2} = \mp i\sqrt{(\lambda - q^\sigma)(\lambda - r)} = \mp i\gamma,$$

are obtained, respectively. From the Eq (4.7), we get

$$(\mathfrak{D}^\omega + i\gamma)(\mathfrak{D}^\omega - i\gamma)\phi_2 = (\kappa_1 q^\sigma - D^\omega q)\phi_1.$$

Let $\phi^{(1)} = (\mathfrak{D}^\omega - i\gamma)\phi_2$ and t be right-dense point. In this situation,

$$\mathfrak{D}^\omega \phi^{(1)} = -i\gamma\phi^{(1)} + (\kappa_1 q - \mathfrak{D}^\omega q)\phi_1,$$

can be obtained, and the solution of the last equation is

$$\phi^{(1)}(t) = \cos_{\frac{1}{1+\mu}}(t,a) - i\sin_{\frac{1}{1+\mu}}(t,a) + \int_a^t [\kappa_1(w,s)q(s) - \mathfrak{D}^\omega q(s)]\tilde{e}_{i\gamma+\kappa_1}(s,t)\Delta_w s, \tag{4.8}$$

where $c_1 = 1$. On the other hand, and it is known that

$$(\mathfrak{D}^\omega - i\gamma)\phi_2 = \phi^{(1)}.$$

Thus, $\mathfrak{D}^\omega\phi_2 - i\gamma\phi_2 = \phi^{(1)}$, and it is derived that

$$\phi_2(t) = c_2\tilde{e}_{i\gamma}(t,a) + \int_a^t \phi^{(1)}(s)\tilde{e}_{-i\gamma+\kappa_1}(s,t)\Delta_w s. \tag{4.9}$$

If the above method is repeated considering Eq (4.5),

$$(\mathfrak{D}^\omega)^2\phi_1 = \mathfrak{D}^\omega((-\lambda + r)\phi_2^\sigma),$$

$$\Rightarrow (\mathfrak{D}^\omega)^2\phi_1 + (\lambda - r^\sigma)(\lambda - q)\phi_1 = (\mathfrak{D}^\omega r - \kappa_1 r^\sigma)\phi_2, \tag{4.10}$$

is derived, and its characteristic equation and the roots are

$$z^2 + (\lambda - r^\sigma)(\lambda - q) = 0 \quad \Rightarrow \quad z = \mp i\gamma.$$

Thus, the solution of (4.10) is

$$\phi^{(2)}(t) = \cos_{\frac{1}{1+\mu}}(t,a) + i\sin_{\frac{1}{1+\mu}}(t,a) + \int_a^t [\mathfrak{D}^\omega r(s) - \kappa_1(w,s)r(s)]\tilde{e}_{-i\gamma+\kappa_1}(s,t)\Delta_w s, \tag{4.11}$$

where

$$\phi_1(t) = c_1\Big(\cos_{\frac{1}{1+\mu}}(t,a) - i\sin_{\frac{1}{1+\mu}}(t,a)\Big) + \int_a^t \phi^{(2)}(s)\tilde{e}_{i\gamma+\kappa_1}(s,t)\Delta_w s. \tag{4.12}$$

□

In the context of oscillation theory, Picone's identity is very helpful since it enables one to examine the oscillatory behavior of the solution of a given differential equation. One may find out information about the number of zeros, or oscillations, of the solution in a particular interval by looking at the signs of the various elements in the identity. Based on the signs of specific derivatives and coefficients in the differential equation, Picone's identity aids in determining when and where oscillations occur. We now provide Picone's identity for problem (1.1)–(1.3) on time scales, which is a crucial formula for demonstrating oscillation criteria. The identification of Picone's identity has been the subject of several analyses in the literature [33, 34].

Theorem 12. *(Picone's identity) Let* $\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}, \Phi(t) = \begin{pmatrix} \Phi_1(t) \\ \Phi_2(t) \end{pmatrix}$ *be the solutions of* (1.1). *Thus,*

$$\begin{aligned}-\frac{\phi_1^\sigma}{\phi_2^\sigma}[(\tau^\omega\Phi)^T\phi^\sigma - \lambda(\phi^\sigma)^T\Phi^\sigma] &= \mathfrak{D}^\omega\Big(\frac{\phi_1}{\phi_2}W(\phi,\Phi)\Big) + \kappa_1\frac{\phi_1^\sigma}{\phi_2^\sigma}(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma) \\ &\quad + \frac{1}{\phi_2\phi_2^\sigma}[\lambda((\phi_2^\sigma)^2 + \phi_1\phi_1^\sigma) - r(t)(\phi_2^\sigma)^2 - q(t)\phi_1\phi_1^\sigma]W(\phi,\Phi),\end{aligned}$$

$$\begin{aligned}\mathfrak{D}^\omega\Big(\frac{\phi_1}{\phi_2}W(\phi,\Phi)\Big) &= \Big((-\lambda + r)\frac{\phi_2^\sigma}{\phi_2} - (\lambda - q)\frac{\phi_1\phi_1^\sigma}{\phi_2\phi_2^\sigma}\Big)W(\phi,\Phi) \\ &\quad + \frac{\phi_1^\sigma}{\phi_2^\sigma}[(-\lambda + r^\sigma)\Phi_2^{\sigma\sigma}\phi_2^\sigma - (\lambda - q^\sigma)\phi_1^\sigma\Phi_1^{\sigma\sigma} - \kappa_1(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma)],\end{aligned}$$

where $W(\phi,\Phi) = \phi_2\Phi_1^\sigma - \phi_1\Phi_2^\sigma$.

Proof. Assume that $\phi_2\phi_2^\sigma(t) \neq 0$. Considering the Lagrange's identity, we derive that

$$\begin{aligned}\mathfrak{D}^\omega\Big(\frac{\phi_1}{\phi_2}W(\phi,\Phi)\Big) &= \frac{\phi_1^\sigma}{\phi_2^\sigma}\mathfrak{D}^\omega(W(\phi,\Phi)) + \mathfrak{D}^\omega\Big(\frac{\phi_1}{\phi_2}\Big)W(\phi,\Phi) - \kappa_1\frac{\phi_1^\sigma}{\phi_2^\sigma}W(\phi,\Phi) \\ &= \Big(\frac{(\mathfrak{D}^\omega\phi_1)\phi_2^\sigma - \phi_1(\mathfrak{D}^\omega\phi_2)}{\phi_2\phi_2^\sigma} + \kappa_1\frac{\phi_1^\sigma}{\phi_2^\sigma}\Big)W(\phi,\Phi) \\ &\quad + \frac{\phi_1^\sigma}{\phi_2^\sigma}\mathfrak{D}^\omega(W(\phi,\Phi)) - \kappa_1\frac{\phi_1^\sigma}{\phi_2^\sigma}W(\phi,\Phi) \\ &= \frac{\phi_1^\sigma}{\phi_2^\sigma}[(\tau^W\phi)^T\Phi^\sigma - (\tau^W\Phi)^T\phi^\sigma - \kappa_1(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma)] \\ &\quad + \frac{1}{\phi_2\phi_2^\sigma}((-\lambda + r(t))(\phi_2^\sigma)^2\phi_2^\sigma - (\lambda - q(t))\phi_1^\sigma\phi_1)W(\phi,\Phi) \\ &= \frac{\phi_1^\sigma}{\phi_2^\sigma}[\lambda(\phi^\sigma)^T\Phi^\sigma - (\tau^W\Phi)^T\phi^\sigma - \kappa_1(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma)] \\ &\quad + \frac{1}{\phi_2\phi_2^\sigma}[-\lambda((\phi_2^\sigma)^2 + \phi_1\phi_1^\sigma) + r(t)(\phi_2^\sigma)^2 + q(t)\phi_1\phi_1^\sigma]W(\phi,\Phi)\end{aligned}$$

$$\Rightarrow -\frac{\phi_1^\sigma}{\phi_2^\sigma}[(\tau^W\Phi)^T\phi^\sigma - \lambda(\phi^\sigma)^T\Phi^\sigma] = \mathfrak{D}^\omega\Big(\frac{\phi_1}{\phi_2}W(\phi,\Phi)\Big) + k_1\frac{\phi_1^\sigma}{\phi_2^\sigma}(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma)$$
$$+\frac{1}{\phi_2\phi_2^\sigma}[\lambda((\phi_2^\sigma)^2+\phi_1\phi_1^\sigma) - r(t)(\phi_2^\sigma)^2 - q(t)\phi_1\phi_1^\sigma]W(\phi,\Phi).$$

On the other hand,

$$\mathfrak{D}^\omega\Big(\frac{\phi_1}{\phi_2}W(\phi,\Phi)\Big) = \frac{\phi_1^\sigma}{\phi_2^\sigma}\mathfrak{D}^\omega(W(\phi,\Phi)) + \Big(\frac{(\mathfrak{D}^\omega\phi_1)\phi_2^\sigma - \phi_1\mathfrak{D}^\omega\phi_2}{\phi_2\phi_2^\sigma}\Big)W(\phi,\Phi)$$
$$= \frac{\phi_1^\sigma}{\phi_2^\sigma}[(\mathfrak{D}^\omega\phi_2)\Phi_1^\sigma + \phi_2^\sigma\mathfrak{D}^\sigma\Phi_1^\sigma - \kappa_1\phi_2^\sigma\Phi_1^\sigma - D^\omega(\phi_1)\Phi_2^\sigma - \phi_1^\sigma\mathfrak{D}^\omega\Phi_2^\sigma + \kappa_1\phi_1^\sigma\Phi_2^\sigma]$$
$$+\Big(\frac{(\mathfrak{D}^\omega\phi_1)\phi_2^\sigma - \phi_1\mathfrak{D}^\omega\phi_2}{\phi_2\phi_2^\sigma}\Big)(\phi_2\Phi_1^\sigma - \phi_1\Phi_2^\sigma)$$
$$= \frac{\phi_1^\sigma}{\phi_2^\sigma}[(\lambda-q)\phi_1^\sigma\Phi_1^\sigma + (-\lambda+r^\sigma)\Phi_2^{\sigma\sigma}\phi_2^\sigma - (-\lambda+r)\phi_2^\sigma\Phi_2^\sigma$$
$$-(\lambda-q^\sigma)\Phi_1^{\sigma\sigma}\phi_1^\sigma - \kappa_1\phi_2^\sigma\Phi_1^\sigma + \kappa_1\phi_1^\sigma\Phi_2^\sigma]$$
$$+\Big((-\lambda+r)\frac{\phi_2^\sigma}{\phi_2} - (\lambda-q)\frac{\phi_1^\sigma\phi_1}{\phi_2^\sigma\phi_2}\Big)(\phi_2\Phi_1^\sigma - \phi_1\Phi_2^\sigma)$$
$$= (-\lambda+r)\Big(-\phi_1^\sigma\Phi_2^\sigma + \phi_2^\sigma\Phi_1^\sigma - \frac{\phi_1\phi_2^\sigma\Phi_2^\sigma}{\phi_2}\Big) + (-\lambda+r^\sigma)\phi_1^\sigma\Phi_2^{\sigma\sigma}$$
$$-(\lambda-q)\frac{\phi_1\phi_1^\sigma}{\phi_2\phi_2^\sigma}\Big(\phi_2\Phi_1^\sigma - \phi_1\Phi_2^\sigma - \frac{(\phi_1^\sigma)^2\Phi_1^\sigma}{\phi_2^\sigma}\Big) - (\lambda-q^\sigma)\frac{(\phi_1^\sigma)^2\Phi_1^{\sigma\sigma}}{\phi_2^\sigma}$$
$$-\kappa_1\frac{\phi_1^\sigma}{\phi_1^\sigma}(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma)$$
$$= (-\lambda+r)\Big(-\phi_1^\sigma\Phi_2^\sigma + \frac{\phi_2^\sigma}{\phi_2}W(\phi,\Phi)\Big) + (-\lambda+r^\sigma)\phi_1^\sigma\Phi_2^{\sigma\sigma}$$
$$-(\lambda-q)\frac{\phi_1\phi_1^\sigma}{\phi_2\phi_2^\sigma}\Big(W(\phi,\Phi) - \frac{(\phi_1^\sigma)^2\Phi_1^\sigma}{\phi_2^\sigma}\Big) - (\lambda-q^\sigma)\frac{(\phi_1^\sigma)^2\Phi_1^{\sigma\sigma}}{\phi_2^\sigma}$$
$$-\kappa_1\frac{\phi_1^\sigma}{\phi_2^\sigma}(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma)$$
$$= \Big((-\lambda+r)\frac{\phi_2^\sigma}{\phi_2} - (\lambda-q)\frac{\phi_1\phi_1^\sigma}{\phi_2\phi_2^\sigma}\Big)W(\phi,\Phi)$$
$$+\frac{\phi_1^\sigma}{\phi_2^\sigma}[(-\lambda+r^\sigma)\Phi_2^{\sigma\sigma}\phi_2^\sigma - (\lambda-q^\sigma)\phi_1^\sigma\Phi_1^{\sigma\sigma} - k_1(\phi_2^\sigma\Phi_1^\sigma - \phi_1^\sigma\Phi_2^\sigma)].$$

□

5. Visual results and discussions

This section contains graphics showing the solution functions of the Dirac system on a time scale that is obtained by utilizing the advantages of the proportional fractional derivative. Figure 1 illustrates the variations in the functions $\phi_1(t,\lambda)$ and $\phi_2(t,\lambda)$, the components of the solution $\phi(t,\lambda)$ of the problem (1.1) and (1.2), curve motion for $\omega = 0.8, 0.6, 0.4$ (arbitrary proportional fractional order cases), and $\omega = 1$ (classical case).

On the other hand, it is known that the eigenvalues of the problem (1.1)–(1.3) match the roots of the characteristic equation,

$$\Gamma(\lambda) = \gamma\phi_1(b, \lambda) + \delta\phi_2(b, \lambda).$$

If we replace the asymptotic estimates (4.1) and (4.2) of the eigenfunction $\phi(t, \lambda) = \begin{pmatrix}\phi_1(t, \lambda)\\ \phi_2(t, \lambda)\end{pmatrix}$, then we find the $|\lambda|$ eigenvalues in Table 1, with special choice $\gamma = 1, \delta = 1$ for arbitrary proportional fractional orders. Figures 2 and 3 illustrate how the functions $\phi_1(t, \lambda)$ and $\phi_2(t, \lambda)$ vary depending on these different values of the $|\lambda|$ eigenvalues.

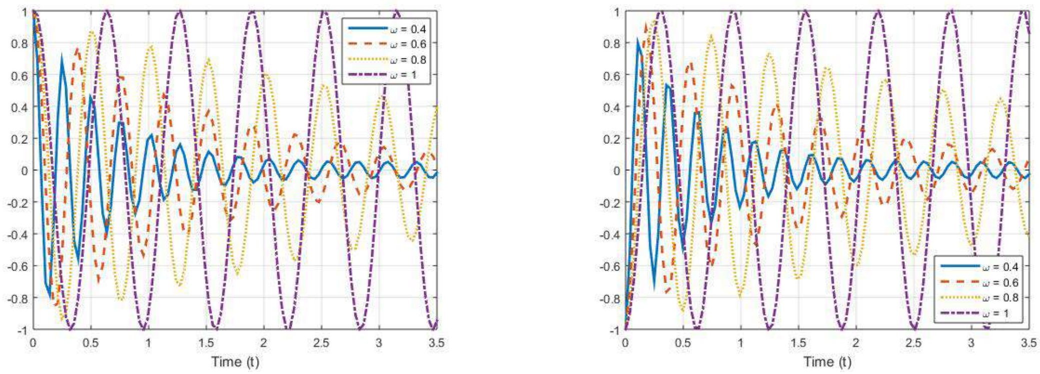

Figure 1. The solution curves of the functions $\phi_1(t, \lambda)$ and $\phi_2(t, \lambda)$ when $\omega = 1, 0.8, 0.6, 0.4$ for the value of $\lambda = 10$ under the condition (1.2), respectively.

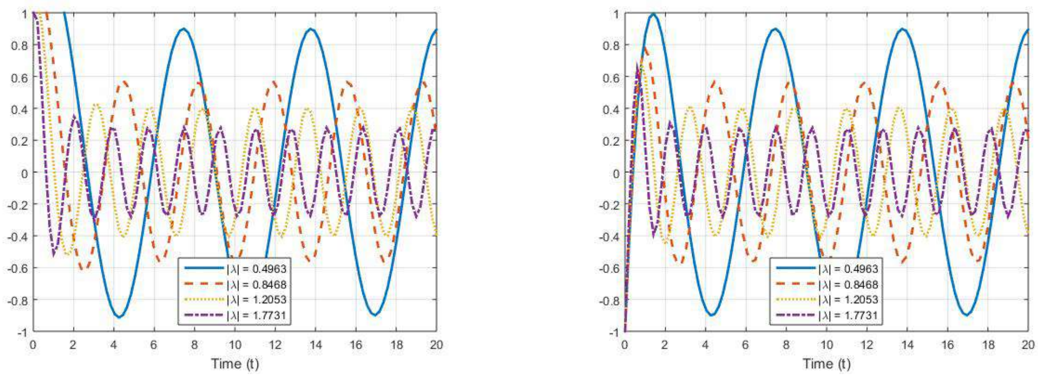

Figure 2. The solution curves of the functions $\phi_1(t, \lambda)$ and $\phi_2(t, \lambda)$ when $\omega = 0.5$ for the value of $|\lambda| = 0.4963, 0.8468, 1.2053, 1.7731$ under the condition (1.3), respectively.

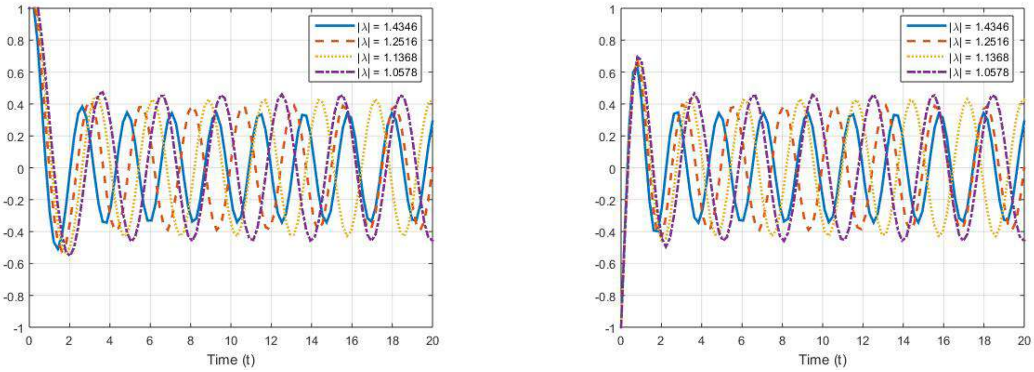

Figure 3. The solution curves of the functions $\phi_1(t, \lambda)$ and $\phi_2(t, \lambda)$ when $\omega = 0.5$ for the value of $|\lambda| = 1.4346, 1.2516, 1.1368, 1.0578$ under the condition (1.3), respectively.

Table 1. Eigenvalues for the problem (1.1)–(1.3).

ω	**Eigenvalue** ($|\lambda|$)	ω	**Eigenvalue** ($|\lambda|$)
0.1	0.1987	0.6	1.4346
0.2	0.4963	0.7	1.2516
0.3	0.8468	0.8	1.1368
0.4	1.2053	0.9	1.0578
0.5	1.7731	1.0	1.0001

For all graphics and the table, it is assumed that the potential functions $q(t), r(t)$ are constants, $a = \mu = 0$, and $\kappa_0(\omega, t) = \omega$, $\kappa_1(\omega, t) = 1 - \omega$, according to various arbitrary order values and eigenvalues on a time scale. Therefore, the main aim of the graphics is to examine the impact of the proportional fractional derivative on the Dirac system over time, as well as the effect of the eigenvalues, which are significant for the issue being studied, on the solution functions. In order to examine both of these cases independently, which are crucial to the current investigation, eigenvalues are left unchanged in some graphics while the values of the proportional derivative are altered in an arbitrary sequence. Likewise, eigenvalues are modified while the derivative order remains unchanged in order to see the impact of the eigenvalues.

6. Conclusions

When compared to other local derivatives, the proportional derivative is seen to have more favorable characteristics. It belongs to the family of local derivatives that includes arbitrary order. It holds significant value, particularly in engineering, because it is founded on control theory. This improved definition of the local derivative is constructed in such a way that $\mathfrak{D}^0$ is a unit operator and $\mathfrak{D}^1$ is a standard differential operator. For alternative selections of the functions $k_0(\omega, t)$ and $k_1(\omega, t)$, multiple special instances can be found in the formulation of the proportional derivative. This is an additional benefit of the proportional derivative because, in practice, one may be able to get better outcomes by making the unique decisions required in accordance with the behavior of the problem being studied. As a result, the proportional derivative is recommended for solving the Dirac dynamic system in this research because of all these benefits. It should be noted that the Dirac system, which is of enormous mathematical and physical relevance, may be addressed and examined with the use of proportional derivatives utilized in control theory, and that doing so can significantly advance scholarship. Since the proportional derivative is a generalization of the conformable fractional derivative, this study, in which the proportional derivative is used in spectral theory, will make a significant contribution to the literature. Using the asymptotic formula of the eigenfunction we obtained, the ideas acquired with eigenfunctions in the classical case ($\omega = 1$) can be generalized in terms of proportional fractional order derivatives. Additionally, we think that the results obtained by substituting $\omega \in [n, n+1], n = 0, 1, ...$ instead of $\omega \in [0, 1]$ may be interesting in that they can be examined over different ranges.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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