AIMS *Mathematics*

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 9926–9950.
DOI: 10.3934/math.2024486
Received: 20 January 2024
Revised: 24 February 2024
Accepted: 05 March 2024
Published: 12 March 2024

Research article

Existence results for a system of sequential differential equations with varying fractional orders via Hilfer-Hadamard sense

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Abstract: We investigate the criteria for both the existence and uniqueness of solutions within a nonlinear coupled system of Hilfer-Hadamard sequential fractional differential equations featuring varying orders. This system is complemented by nonlocal coupled Hadamard fractional integral boundary conditions. The desired outcomes are attained through the application of well-established fixed-point theorems. It is underscored that the fixed-point approach serves as an effective method for establishing both the existence and uniqueness of solutions to boundary value problems. The results obtained are further demonstrated and validated through illustrative examples.

Keywords: Hilfer-Hadamard derivatives; sequential derivatives; Hadamard integrals; existence; uniqueness; fixed point
Mathematics Subject Classification: 34A08, 34B15, 45G15

Abbreviations

The abbreviations used in this manuscript	
BVPs	Boundary Value Problems
HHFDEs	Hilfer-Hadamard Fractional-order Differential Equations
HFIs	Hadamard Fractional Integrals
HHFDs	Hilfer-Hadamard Fractional Derivatives
CFDs	Caputo Fractional Derivatives
HFDs	Hilfer Fractional Derivatives
HFDEs	Hilfer Fractional Differential Equations
HFDs	Hadamard Fractional Derivatives (HFDs)
CHFDs	Caputo-Hadamard Fractional Derivatives (CHFDs)

1. Introduction

This study introduces and investigates a novel nonlinear nonlocal coupled boundary value problem (BVP) encompassing sequential Hilfer-Hadamard fractional-order differential equations (HHFDEs) with varying orders. The problem is formulated as:

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_1,\beta_1} + \mathcal{K}_1\, {}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_1-1,\beta_1})\varrho(\tau) = \rho_1(\tau,\varrho(\tau),\varphi(\tau)), & 1<\psi_1\le 2, \quad \tau\in\mathcal{E}:=[1,\mathfrak{T}], \\ ({}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_2,\beta_2} + \mathcal{K}_2\, {}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_2-1,\beta_2})\varphi(\tau) = \rho_2(\tau,\varrho(\tau),\varphi(\tau)), & 2<\psi_2\le 3, \quad \tau\in\mathcal{E}:=[1,\mathfrak{T}], \end{cases} \tag{1.1}$$

and it is enhanced by nonlocal coupled Hadamard fractional integral (HFI) boundary conditions:

$$\begin{cases} \varrho(1)=0, \quad \varrho(\mathfrak{T}) = \lambda_1\, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_1}\varphi(\eta_1), \\ \varphi(1)=0, \quad \varphi(\eta_2)=0, \quad \varphi(\mathfrak{T}) = \lambda_2\, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_2}\varrho(\eta_3), \quad 1<\eta_1,\eta_2,\eta_3<\mathfrak{T}. \end{cases} \tag{1.2}$$

Here, $\psi_1 \in (1,2]$, $\psi_2 \in (2,3]$, $\beta_1,\beta_2 \in [0,1]$, $\mathcal{K}_1,\mathcal{K}_2 \in \mathbb{R}_+$, $\mathfrak{T} > 1$, $\delta_1,\delta_2 > 0$, $\lambda_1,\lambda_2 \in \mathbb{R}$, ${}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_i,\beta_j}$ denotes the Hilfer-Hadamard fractional derivative (HHFD) operator of order $\psi_i,\beta_j; i = 1,2. j = 1,2.$ ${}^{\mathcal{H}}\mathcal{I}_{1^+}^{\chi}$ is the HFI operator of order $\chi \in \{\delta_1,\delta_2\}$, and $\rho_1,\rho_2 : \mathcal{E}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions. It is noteworthy that this study contributes to the literature by addressing a unique configuration of sequential HHFDEs with distinct orders and coupled HFI boundary conditions. The methodology employed involves the application of the fixed-point approach to establish both existence and uniqueness results for problems (1.1) and (1.2). The conversion of the given problem into an equivalent fixed-point problem is followed by the utilization of the Leray-Schauder alternative and Banach's fixed-point theorem to prove existence and uniqueness results, respectively. The outcomes of this research are novel and enrich the existing body of literature on BVPs involving coupled systems of sequential HHFDEs. Coupled fractional derivatives are essential for modeling systems with non-local interactions and memory effects more accurately than ordinary derivatives. They enable a more precise description of phenomena, such as anomalous diffusion and viscoelasticity, enhancing our understanding of complex physical processes. This improved modeling capability leads to more accurate predictions and insights into real-world phenomena, benefiting various fields ranging from materials science to fluid dynamics and beyond. Over the past few decades, fractional calculus has emerged as a significant and widely explored field within mathematical analysis. The substantial growth observed in this field can be credited to the widespread utilization of fractional calculus methodologies in creating inventive mathematical models to depict diverse phenomena across economics, mechanics, engineering, science, and other domains. References [1–4] provide examples and detailed discussions on this topic.

In the following section, we will present a summary of pertinent scholarly articles related to the discussed problem. The Riemann-Liouville and Caputo fractional derivatives (CFDs), among other fractional derivatives introduced, have drawn a lot of interest due to their applications. The Hilfer fractional derivative (HFD) was introduced by Hilfer in [5]. Its definition includes the Riemann-Liouville and CFDs as special cases for extreme values of the parameter. [6, 7] provided further information about this derivative. [8–12] presented noteworthy results on Hilfer-type initial and boundary value problems (BVPs). A new work [13] explores the Ulam-Hyers stability and existence of solutions for a fully coupled system with integro-multistrip-multipoint boundary conditions and nonlinear sequential Hilfer fractional differential equations (HFDEs). Moreover, [14] investigates a hybrid generalized HFDE boundary value problem.

In 1892, Hadamard proposed the Hadamard fractional derivative (HFD), which is a fractional derivative using a logarithmic function with an arbitrary exponent in its kernel [15]. Later research in [16–20] examined variations such as HHFDs and Caputo-Hadamard fractional derivatives (CHFDs). Importantly, for β values of $\beta = 0$ and $\beta = 1$, respectively, HFDs and CHFDs arise as special examples of the HHFD.

Existence results for an HHFDE with nonlocal integro-multipoint boundary conditions was derived in [21]:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_1^{\alpha,\beta}x(t) = f(t, x(t)), \quad t \in [1, T], \\ x(1) = 0, \quad \displaystyle\sum_{i=1}^{m} \theta_i x(\xi_i) = \lambda {}^{H}\mathcal{I}^{\delta} x(\eta). \end{cases} \tag{1.3}$$

Here, $\alpha \in (1, 2]$, $\beta \in [0, 1]$, $\theta_i, \lambda \in \mathbb{R}$, $\eta, \xi_i \in (1, T)$ $(i = 1, 2, ..., m)$, ${}^{H}\mathcal{I}^{\delta}$ is the HFI of order $\delta > 0$, and $f : [1, T]\times\mathbb{R} \to \mathbb{R}$ is a continuous function. Problem (1.3) represents a non-coupled system, in contrast to problems (1.1)–(1.2), which are coupled systems. Problems (1.1)–(1.2) exhibits nonlocal coupled integral and multi-point boundary conditions involving HFIs, while problem (1.3) incorporates discrete boundary conditions with HFIs. Existence results for nonlocal mixed Hilfer-Hadamard fractional BVPs were developed by the authors of [22]:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_1^{\alpha,\beta}x(t) = f(t, x(t)), \quad t \in [1, T], \\ x(1) = 0, \quad x(T) = \displaystyle\sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i {}^{H}\mathcal{I}^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k {}_{H}\mathcal{D}_1^{\omega_k} x(\mu_k). \end{cases} \tag{1.4}$$

Here, $\alpha \in (1, 2]$, $\beta \in [0, 1]$, $\eta_i, \zeta_i, \lambda_k \in \mathbb{R}$, $\xi_i, \theta_i, \mu_k \in (1, T)$, $(j = 1, 2, ..., m), (i = 1, 2, ..., n), (k = 1, 2, ..., r)$, ${}^{H}\mathcal{I}^{\phi_i}$ is the HFI of order $\phi_i > 0$, ${}_{H}\mathcal{D}_1^{\mu_k}$ is the HFD of order $\mu_k > 0$, and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Problem (1.4) is not a coupled system, while problems (1.1)–(1.2) are coupled systems. Problems (1.1)–(1.2) features nonlocal coupling with integral and multi-point boundary conditions involving HFIs, whereas problem (1.4) incorporates mixed discrete boundary conditions involving HFIs and derivatives. Additionally, [23] investigated a coupled HHFDEs in generalized

Banach spaces. The authors of the aforementioned study [24] successfully derived existence results for a coupled system of HHFDEs with nonlocal coupled boundary conditions:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_1^{\alpha,\beta}u(t) = f(t,u(t),v(t)), \ \ 1<\alpha\le 2, \ \ \tau\in[1,\mathfrak{T}], \\ {}^{\mathcal{HH}}\mathcal{D}_1^{\gamma,\delta}v(t) = g(t,u(t),v(t)), \ \ 1<\gamma\le 2, \ \ \tau\in[1,\mathfrak{T}], \\ u(1)=0, \quad {}^{\mathcal{H}}\mathcal{D}_1^{\varsigma}u(T) = \displaystyle\sum_{i=1}^{m}\int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\varrho_i}u(s)d\mathcal{H}_i(s) + \sum_{i=1}^{n}\int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\sigma_i}v(s)d\mathcal{K}_i(s), \\ v(1)=0, \quad {}^{\mathcal{H}}\mathcal{D}_1^{\vartheta}v(T) = \displaystyle\sum_{i=1}^{p}\int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\eta_i}u(s)d\mathcal{P}_i(s) + \sum_{i=1}^{q}\int_1^T {}^{\mathcal{H}}\mathcal{D}_1^{\theta_i}v(s)d\mathcal{Q}_i(s). \end{cases} \tag{1.5}$$

Here, $\alpha,\gamma \in (1,2]$, $\beta,\delta \in [0,1]$, $\mathfrak{T} > 1$, ${}^{\mathcal{HH}}\mathcal{D}^{\alpha,\beta}$, ${}^{\mathcal{HH}}\mathcal{D}_1^{\gamma,\delta}$ denotes the HHFD operator of order $\alpha,\beta,\gamma,\delta$, ${}^{\mathcal{H}}\mathcal{D}_{1^+}^{\chi}$ is the HFD operator of order $\chi \in \{\varsigma,\vartheta,\varrho_i,\eta_i,\sigma_i,\theta_i\}$, $(i = 1,2,...,m),(i = 1,2,...,n),(i = 1,2,...,p),(i = 1,2,...,q)$, and $f,g : [1,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions. In the boundary conditions, Riemann-Stieltjes integrals are involved with $\mathcal{H}_i,\mathcal{K}_i,\mathcal{P}_i,\mathcal{Q}_i$, $(i = 1,2,...,m),(i = 1,2,...,n),(i = 1,2,...,p),(i = 1,2,...,q)$, which are functions of the bounded variation. Problem (1.5) involves a coupled system of HHFDEs, while problems (1.1)–(1.2) deal with coupled systems of sequential HHFDEs. In problems (1.1)–(1.2), there is nonlocal coupling with integral and multi-point boundary conditions involving HFIs, whereas in problem (1.5), Stieltjes-integral boundary conditions are incorporated, involving HFDs. Within problems (1.1)–(1.2), various fractional orders are involved, while problem (1.5) incorporates a uniform fractional order. The authors [25] conducted an analysis on the coupled system of HHFDEs with nonlocal coupled HFI boundary conditions:

$$\begin{cases} {}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\alpha_1,\beta_1}u(t) = \varrho_1(t,u(t),v(t)), \ \ 1<\alpha_1\le 2, \ \ \tau\in\mathcal{E}:=[1,\mathfrak{T}], \\ {}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\alpha_2,\beta_2}v(t) = \varrho_2(t,u(t),v(t)), \ \ 2<\alpha_2\le 3, \ \ \tau\in\mathcal{E}:=[1,\mathfrak{T}], \\ u(1)=0, \quad u(T) = \lambda_1{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_1}v(\eta_1), \\ v(1)=0, \quad v(\eta_2)=0, \quad v(T) = \lambda_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_2}u(\eta_3), \quad 1<\eta_1,\eta_2,\eta_3<\mathfrak{T}. \end{cases} \tag{1.6}$$

Here, $\alpha_1 \in (1,2]$, $\alpha_2 \in (2,3]$, $\beta_1,\beta_2 \in [0,1]$, $\mathfrak{T} > 1$, $\delta_1,\delta_2 > 0$, $\lambda_1,\lambda_2 \in \mathbb{R}$, ${}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\alpha_i,\beta_j}$ denotes the Hilfer-Hadamard Fractional Derivative (HHFD) operator of order $\alpha_i,\beta_j; i = 1,2. j = 1,2$, ${}^{\mathcal{H}}\mathcal{I}_{1^+}^{\chi}$ is the HFI operator of order $\chi \in \{\delta_1,\delta_2\}$, and $\varrho_1,\varrho_2 : \mathcal{E}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions. Problem (1.6) involves a coupled system of HHFDEs, while problems (1.1)–(1.2) deal with coupled systems of sequential HHFDEs. Despite sharing identical boundary conditions in both (1.1)–(1.2) and (1.6), the auxiliary lemma used in problems (1.1)–(1.2) is entirely different from that in problem (1.6). Therefore, problems (1.1)–(1.2) in the manuscript are distinctly separate from problem (1.6). In problem (1.6), solutions are obtained for the coupled system of HHFDEs, whereas in problems (1.1)–(1.2), solutions are derived for the coupled system of sequential HHFDEs. A two-point boundary value problem for a system of nonlinear sequential HHFDEs was investigated in [26]:

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_1,\beta_1} + \lambda_1{}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_1-1,\beta_1})u(t) = f(t,u(t),v(t)), \ t\in[1,e], \\ ({}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_2,\beta_2} + \lambda_2{}^{\mathcal{HH}}\mathcal{D}_1^{\alpha_2-1,\beta_2})v(t) = g(t,u(t),v(t)), \ t\in[1,e], \\ u(1)=0, \ \ u(e)=\mathscr{A}_1, \ \ v(1)=0, \ \ v(e)=\mathscr{A}_2. \end{cases} \tag{1.7}$$

Here, $\alpha_1, \alpha_2 \in (1, 2]$, $\beta_1, \beta_2 \in [0, 1]$, $\lambda_1, \lambda_2, \mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R}_+$, and $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Within problems (1.1)–(1.2), various fractional orders are involved, while problem (1.7) incorporates a uniform fractional order. Problem (1.7) is characterized by a two-point boundary condition, whereas problems (1.1)–(1.2) incorporates multi-point boundary conditions along with HFIs.

The sections of this document are organized as follows: The fundamental ideas of fractional calculus relating to this research are introduced in Section 2. An auxiliary lemma addressing the linear versions of problems (1.1) and (1.2) is provided in Section 3. The primary findings are presented in Section 4 along with illustrative examples. Finally, Section 5 provides a few recommendations.

2. Preliminaries

Definition 2.1. *For a continuous function $\varphi : [a, \infty) \to \mathbb{R}$, the HFI of order $\delta > 0$ is given by*

$$ {}^{\mathcal{H}}\mathcal{I}^{\delta}_{a^+}\varphi(\tau) = \frac{1}{\Gamma(\delta)} \int_a^{\tau} \left(\log \frac{\tau}{\varpi}\right)^{\delta-1} \frac{\varphi(\varpi)}{\varpi} d\varpi, \tag{2.1} $$

where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. *For a continuous function $\varphi : [a, \infty) \to \mathbb{R}$, the HFD of order $\delta > 0$ is given by*

$$ {}^{\mathcal{H}}\mathcal{D}^{\delta}_{\mathfrak{a}^+}\varphi(\tau) = \mathfrak{p}^n({}^{\mathcal{H}}\mathcal{I}^{n-\delta}_{\mathfrak{a}^+}\varphi)(\tau), \;\; n = [\delta] + 1, \tag{2.2} $$

where $\mathfrak{p}^n = \tau^n \frac{d^n}{d\tau^n}$, and $[\delta]$ represents the integer parts of the real number δ.

Lemma 2.3. *If $\delta, \gamma > 0$ and $0 < \mathfrak{a} < \mathfrak{b} < \infty$, then*

$$ (1) \quad \left({}^{\mathcal{H}}\mathcal{I}^{\delta}_{a^+}\left(\log \frac{\tau}{\mathfrak{a}}\right)^{\gamma-1}\right)(\varrho) = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta)}\left(\log \frac{\varrho}{\mathfrak{a}}\right)^{\gamma+\delta-1}, $$

$$ (2) \quad \left({}^{\mathcal{H}}\mathcal{D}^{\delta}_{a^+}\left(\log \frac{\tau}{\mathfrak{a}}\right)^{\gamma-1}\right)(\varrho) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\delta)}\left(\log \frac{\varrho}{\mathfrak{a}}\right)^{\gamma-\delta-1}. $$

In particular, for $\gamma = 1$, we have $\left({}^{\mathcal{H}}\mathcal{D}^{\delta}_{a^+}\right)(1) = \frac{1}{\Gamma(1-\delta)}\left(\log \frac{\varrho}{\mathfrak{a}}\right)^{-\delta} \neq 0, 0 < \delta < 1$.

Definition 2.4. *For $n - 1 < \delta < n$ and $0 \le \gamma \le 1$, the HHFD of order δ and γ for $\varphi \in \mathcal{L}^1(\mathfrak{a}, \mathfrak{b})$ is defined as*

$$ \begin{aligned} ({}^{\mathcal{H}\mathcal{H}}\mathcal{D}^{\delta,\gamma}_{\mathfrak{a}^+}) &= ({}^{\mathcal{H}}\mathcal{I}^{\gamma(\mathfrak{n}-\delta)}_{\mathfrak{a}^+}\mathfrak{p}^{\mathfrak{n}}\,{}^{\mathcal{H}}\mathcal{I}^{(\mathfrak{n}-\delta)(1-\gamma)}_{\mathfrak{a}^+}\varphi)(\tau) \\ &= ({}^{\mathcal{H}}\mathcal{I}^{\gamma(\mathfrak{n}-\delta)}_{\mathfrak{a}^+}\mathfrak{p}^{\mathfrak{n}}\,{}^{\mathcal{H}}\mathcal{I}^{(\mathfrak{n}-\mathfrak{q})}_{\mathfrak{a}^+}\varphi)(\tau) \\ &= ({}^{\mathcal{H}}\mathcal{I}^{\gamma(\mathfrak{n}-\delta)}_{\mathfrak{a}^+}\,{}^{\mathcal{H}}\mathcal{D}^{\mathfrak{q}}_{\mathfrak{a}^+}\varphi)(\tau), \;\; \mathfrak{q} = \delta + \mathfrak{n}\gamma - \delta\gamma, \end{aligned} $$

where ${}^{\mathcal{H}}\mathcal{I}^{(\cdot)}_{\mathfrak{a}^+}$ and ${}^{\mathcal{H}}\mathcal{D}^{(\cdot)}_{\mathfrak{a}^+}$ are given as defined by (2.1) and (2.2), respectively.

Theorem 2.5. *If* $\varphi \in \mathcal{L}^1(\mathfrak{a}, \mathfrak{b}), 0 < \mathfrak{a} < \mathfrak{b} < \infty$, *and* $\left({}^{\mathcal{H}}\mathcal{I}_{a^+}^{n-\mathfrak{q}}\varphi\right)(\tau) \in \mathcal{AC}_{\mathfrak{p}}^n[\mathfrak{a}, \mathfrak{b}]$, *then*

$$\begin{aligned}{}^{\mathcal{H}}\mathcal{I}_{a^+}^{\delta}\left({}^{\mathcal{HH}}\mathcal{D}_{a^+}^{\delta,\gamma}\varphi\right)(\tau) =& {}^{\mathcal{H}}\mathcal{I}_{a^+}^{\mathfrak{q}}\left({}^{\mathcal{HH}}\mathcal{D}_{a^+}^{\mathfrak{q}}\varphi\right)(\tau)\\ =& \varphi(\tau) - \sum_{j=o}^{n-1}\frac{(\mathfrak{p}^{(n-j-1)}({}^{\mathcal{H}}\mathcal{I}_{a^+}^{\delta}\varphi))(\mathfrak{a})}{\Gamma(\mathfrak{q}-j)}\left(\log\frac{\tau}{\mathfrak{a}}\right)^{\mathfrak{q}-j-1},\end{aligned}$$

where $\delta > 0, 0 \le \gamma \le 1$, *and* $\mathfrak{q} = \delta + n\gamma - \delta\gamma, n = [\delta] + 1$. *Observe that* $\Gamma(\mathfrak{q} - j)$ *exists for all* $j = 1, 2, \cdots, n-1$ *and* $\mathfrak{q} \in [\delta, n]$.

Lemma 2.6. *Let* $\mathfrak{h}_1, \mathfrak{h}_2 \in C(\mathcal{E}, \mathbb{R})$. *Then, the solution to the linear Hilfer-Hadamard coupled BVP is given by:*

$$\begin{cases}({}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_1,\beta_1} + \mathcal{K}_1{}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_1-1,\beta_1})\varrho(\tau) = \mathfrak{h}_1(\tau), \quad 1 < \psi_1 \le 2,\\ ({}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_2,\beta_2} + \mathcal{K}_2{}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_2-1,\beta_2})\varphi(\tau) = \mathfrak{h}_2(\tau), \quad 2 < \psi_2 \le 3,\\ \varrho(1) = 0, \quad \varrho(\mathfrak{T}) = \lambda_1{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_1}\varphi(\eta_1),\\ \varphi(1) = 0, \quad \varphi(\eta_2) = 0, \quad \varphi(\mathfrak{T}) = \lambda_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_2}\varrho(\eta_3), \quad 1 < \eta_1, \eta_2, \eta_3 < \mathfrak{T},\end{cases} \tag{2.3}$$

$$\begin{aligned}\varrho(\tau) =& (\log\tau)^{\gamma_1-2} \times \frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}_{1+}^{\delta_1}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi - \mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi\\ &- \left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi + \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\Bigg\}\\ &+ \mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi - \frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\ &- \Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi - \mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi\\ &- \left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi - \frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\\ &+ \lambda_2\mathcal{I}_{1+}^{\delta_2}\left(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi - \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\right)\Bigg](\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}\\ &- \mathcal{K}_1\int_1^{\tau}\frac{\varrho(\varpi)}{\varpi}d\varpi + \frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\varphi(\tau) =& (\log\tau)^{\gamma_2-2}\log\left(\frac{\tau}{\eta_2}\right) \times \frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}_{1+}^{\delta_1}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi - \mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi\\ &- \left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi + \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\Bigg\}\end{aligned}$$

$$+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\Bigg]\left(\lambda_2{}^{\mathcal{H}}\mathcal{I}_{1+}^{\delta_2}(\log\eta_3)^{\gamma_1-1}\right)$$
$$-\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi$$
$$-\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi$$
$$+\lambda_2\mathcal{I}_{1+}^{\delta_2}\left(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\right)\Bigg](\log\mathfrak{T})^{\gamma_1-1}\Bigg\}$$
$$+\left(\frac{\log\tau}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\tau}\frac{\varphi(\varpi)}{\varpi}d\varpi$$
$$-\left(\frac{\log\tau}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi$$
$$+\frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi, \tag{2.5}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= (\log\mathfrak{T})^{\gamma_1-1},\\
\mathcal{A}_2 &= \lambda_1\frac{\Gamma(\gamma_2-1)(\log\eta_1)^{\delta_1+\gamma_2-2}}{\Gamma(\delta_1+\gamma_2-1)}\left\{\log\eta_2-\frac{\gamma_2-1}{\delta_1+\gamma_2-1}\log\eta_1\right\},\\
\mathcal{B}_1 &= -\lambda_2\frac{\Gamma(\gamma_1)}{\Gamma(\delta_2+\gamma_1)}(\log\eta_3)^{\delta_2+\gamma_1-1},\\
\mathcal{B}_2 &= (\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right),\\
\Delta &= \mathcal{A}_1\mathcal{B}_2-\mathcal{A}_2\mathcal{B}_1.
\end{aligned} \tag{2.6}$$

Proof. From the first equation of (2.3), we have

$$({}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_1,\beta_1}+\mathcal{K}_1{}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_1-1,\beta_1})\varrho(\tau)=\mathfrak{h}_1(\tau), \tag{2.7}$$
$$({}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_2,\beta_2}+\mathcal{K}_2{}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_2-1,\beta_2})\varphi(\tau)=\mathfrak{h}_2(\tau). \tag{2.8}$$

Taking the Hadamard fractional integral of order ψ_1 and ψ_2 on both sides of (2.7) and (2.8), we get

$$\begin{aligned}
({}^{\mathcal{H}}\mathcal{I}_{1^+}^{\psi_1}{}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_1,\beta_1}+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\psi_1}\mathcal{K}_1{}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_1-1,\beta_1})\varrho(\tau)&={}^{\mathcal{H}}\mathcal{I}_{1^+}^{\psi_1}\mathfrak{h}_1(\tau),\\
({}^{\mathcal{H}}\mathcal{I}_{1^+}^{\psi_2}{}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_2,\beta_2}+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\psi_2}\mathcal{K}_2{}^{\mathcal{H}\mathcal{H}}\mathcal{D}_{1^+}^{\psi_2-1,\beta_2})\varphi(\tau)&={}^{\mathcal{H}}\mathcal{I}_{1^+}^{\psi_1}\mathfrak{h}_2(\tau).
\end{aligned} \tag{2.9}$$

Equation (2.9) can be written as follows,

$$\varrho(\tau)=\mathfrak{c}_0(\log\tau)^{\gamma_1-1}+\mathfrak{c}_1(\log\tau)^{\gamma_1-2}-\mathcal{K}_1\int_1^{\tau}\frac{\varrho(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_1)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi. \tag{2.10}$$

$$\begin{aligned}\varphi(\tau) =&\mathfrak{d}_0(\log\tau)^{\gamma_2-1}+\mathfrak{d}_1(\log\tau)^{\gamma_2-2}+\mathfrak{d}_2(\log\tau)^{\gamma_2-3}\\&-\mathcal{K}_2\int_1^\tau\frac{\varphi(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^\tau\left(\log\frac{\tau}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi.\end{aligned}\tag{2.11}$$

Here, $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{d}_0, \mathfrak{d}_1$, and $\mathfrak{d}_2$ are arbitrary constants. Now, using boundary conditions (1.2) together with (2.10) and (2.11), one can get

$$\varrho(\tau) = \mathfrak{c}_0(\log\tau)^{\gamma_1-1}+\frac{\mathfrak{c}_1}{(\log\tau)^{2-\gamma_1}}-\mathcal{K}_1\int_1^\tau\frac{\varrho(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_1)}\int_1^\tau\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi=0,\tag{2.12}$$

$$\begin{aligned}\varphi(\tau) =&\mathfrak{d}_0(\log\tau)^{\gamma_2-1}+\mathfrak{d}_1(\log\tau)^{\gamma_2-2}+\frac{\mathfrak{d}_2}{(\log\tau)^{3-\gamma_2}}\\&-\mathcal{K}_2\int_1^\tau\frac{\varphi(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^\tau\left(\log\frac{\tau}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\\=&0,\end{aligned}\tag{2.13}$$

from which we have $\mathfrak{c}_1 = 0$ and $\mathfrak{d}_2 = 0$. Equations (2.12) and (2.13) can be written as

$$\varrho(\tau) =\mathfrak{c}_0(\log\tau)^{\gamma_1-1}-\mathcal{K}_1\int_1^\tau\frac{\varrho(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_1)}\int_1^\tau\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi,\tag{2.14}$$

$$\varphi(\tau) =\mathfrak{d}_0(\log\tau)^{\gamma_2-1}+\mathfrak{d}_1(\log\tau)^{\gamma_2-2}-\mathcal{K}_2\int_1^\tau\frac{\varphi(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^\tau\left(\log\frac{\tau}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi.\tag{2.15}$$

Using the conditions $\varphi(\eta_2) = 0$ in (2.15), we get

$$\mathfrak{d}_1 = -\frac{1}{(\log\eta_2)^{\gamma_2-2}}\left[\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2}{\varpi}d\varpi+\mathfrak{d}_0(\log\eta_2)^{\gamma_2-1}-\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi\right],\tag{2.16}$$

and substituting the value of $\mathfrak{d}_1$ into (2.15), we obtain

$$\begin{aligned}\varphi(\tau) =&\mathfrak{d}_0(\log\tau)^{\gamma_2-2}\log\left(\frac{\tau}{\eta_2}\right)-\left(\frac{\log\tau}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^\tau\frac{\varphi(\varpi)}{\varpi}d\varpi\\&-\left(\frac{\log\tau}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2}{\varpi}d\varpi+\int_1^\tau\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2}{\varpi}d\varpi.\end{aligned}\tag{2.17}$$

Now, using (2.14) and (2.17) in the conditions:

$$\varrho(\mathfrak{T}) = \lambda_1{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_1}\varphi(\eta_1),$$
$$\varphi(\mathfrak{T}) = \lambda_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_2}\varrho(\eta_3),$$

we find that

$$\begin{cases}\mathfrak{c}_0\mathcal{A}_1+\mathfrak{d}_1\mathcal{A}_2=\mathcal{I}_1,\\ \mathfrak{c}_0\mathcal{B}_1+\mathfrak{d}_1\mathcal{B}_2=\mathcal{I}_2.\end{cases}\tag{2.18}$$

Thus, we get,

$$
\begin{aligned}
\mathfrak{c}_0 =&\frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi\\
&-\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\Bigg\}\\
&+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&-\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi\\
&-\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\\
&+\lambda_2\mathcal{I}^{\delta_2}_{1+}\left(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\right)\Bigg](\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\},
\end{aligned}
\tag{2.19}
$$

and

$$
\begin{aligned}
\mathfrak{d}_0 =&\frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi\\
&-\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\Bigg\}\\
&+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\Bigg]\left(\lambda_2{}^{\mathcal{H}}\mathcal{I}^{\delta_2}_{1+}(\log\eta_3)^{\gamma_1-1}\right)\\
&-\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi\\
&-\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{h_2(\varpi)}{\varpi}d\varpi\\
&+\lambda_2\mathcal{I}^{\delta_2}_{1+}\left(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{h_1(\varpi)}{\varpi}d\varpi\right)\Bigg](\log\mathfrak{T})^{\gamma_1-1}\Bigg\},
\end{aligned}
\tag{2.20}
$$

where Δ is defined in (2.6). By substituting the value of $\mathfrak{c}_0$ obtained from (2.19) into (2.14), and substituting the values of $\mathfrak{d}_0$ and $\mathfrak{d}_1$ obtained from (2.20) and (2.16) into (2.15), the resulting solution is given by (2.4) and (2.5). □

3. Main results

Denote by $\mathcal{X} = \{\varrho(\tau)|\varrho(\tau) \in C([1,\mathfrak{T}],\mathbb{R})$ as the Banach space of all functions (continuous) from $[1,\mathfrak{T}]$ into $\mathbb{R}$ equipped with the norm $\|\varrho\| = \sup_{\tau\in[1,\mathfrak{T}]}|\varrho(\tau)|$. Obviously, $(\mathcal{X},\|\cdot\|)$ is a Banach space and, as a result, the product space $(\mathcal{X}\times\mathcal{X},\|\cdot\|)$ is a Banach space with the norm $\|(\varrho,\varphi)\| = \|\varrho\| + \|\varphi\|$

for $(\varrho, \varphi) \in (\mathcal{X} \times \mathcal{X})$. In view of Lemma 2.6, we define an operator $\Omega : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by

$$\Omega(\varrho, \varphi)(\tau) = \Big(\Omega_1(\varrho, \varphi)(\tau), \Omega_2(\varrho, \varphi)(\tau)\Big), \tag{3.1}$$

where

$$\begin{aligned}
\Omega_1(\varrho, \varphi)(\tau) =&(\log \tau)^{\gamma_1 - 2} \times \frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log \eta_1}{\log \eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi - \mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi \\
&- \left(\frac{\log \eta_1}{\log \eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi \\
&+ \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg\} \\
&+ \mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi - \frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg] \\
&\times (\log \mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right) \\
&- \Bigg[\left(\frac{\log \mathfrak{T}}{\log \eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi - \mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi \\
&- \left(\frac{\log \mathfrak{T}}{\log \eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi \\
&- \frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi \\
&+ \lambda_2\mathcal{I}^{\delta_2}_{1+}\Bigg(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi \\
&- \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg)\Bigg](\log \eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\} \\
&- \mathcal{K}_1\int_1^{\tau}\frac{\varrho(\varpi)}{\varpi}d\varpi + \frac{1}{\Gamma(\psi_1)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\Omega_2(\varrho, \varphi)(\tau) =&(\log \tau)^{\gamma_2 - 2}\log\left(\frac{\tau}{\eta_2}\right) \times \frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log \eta_1}{\log \eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi - \mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi \\
&- \left(\frac{\log \eta_1}{\log \eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi \\
&+ \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg\} \\
&+ \mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi - \frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s, \varrho(\varpi), \varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\lambda_2{}^{\mathcal{H}}\mathcal{I}^{\delta_2}_{1+}(\log\eta_3)^{\gamma_1-1}\right)\\
&-\left[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi\right.\\
&-\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\\
&-\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi+\lambda_2\mathcal{I}^{\delta_2}_{1+}\left(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi\right.\\
&\left.\left.-\frac{1}{\Gamma(\psi_1)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right)\right](\log\mathfrak{T})^{\gamma_1-1}\Bigg\}\\
&+\left(\frac{\log\tau}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\tau}\frac{\varphi(\varpi)}{\varpi}d\varpi\\
&-\left(\frac{\log\tau}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi.
\end{aligned}
$$

We need the following hypotheses in what follows:

($\mathcal{H}_1$) Assume that there exist real constants $\kappa_i,\hat{\kappa}_i\geq 0(i=1,2)$ and $\kappa_0>0,\hat{\kappa}_0>0$ such that, for all $\tau\in[1,\mathfrak{T}],\mathfrak{x}_i\in\mathbb{R},i=1,2,$

$$
\begin{aligned}
|\rho_1(\tau,\varrho,\varphi)|&\leq\kappa_0+\kappa_1|\varrho|+\kappa_2|\varphi|,\\
|\rho_2(\tau,\varrho,\varphi)|&\leq\hat{\kappa}_0+\hat{\kappa}_1|\varrho|+\hat{\kappa}_2|\varphi|.
\end{aligned}
$$

($\mathcal{H}_2$) There exist positive constants $\mathcal{L},\hat{\mathcal{L}}$, such that, for all $\tau\in[1,\mathfrak{T}],\varrho_i,\varphi_i\in\mathbb{R},i=1,2,$

$$
\begin{aligned}
|\rho_1(\tau,\varrho_1,\varrho_2)-\rho_1(\tau,\varphi_1,\varphi_2)|&\leq\mathcal{L}\Big(|\varrho_1-\varphi_1|+|\varrho_2-\varphi_2|\Big),\\
|\rho_2(\tau,\varrho_1,\varrho_2)-\rho_2(\tau,\varphi_1,\varphi_2)|&\leq\hat{\mathcal{L}}\Big(|\varrho_1-\varphi_1|+|\varrho_2-\varphi_2|\Big).
\end{aligned}
$$

Furthermore, we establish the notation:

$$
\begin{aligned}
\mathfrak{W}_1=&\frac{\log\mathfrak{T}^{\gamma_1-2}}{\Delta}\left[\mathcal{K}_1(\log\mathfrak{T})+\frac{(\log\mathfrak{T})^{\psi_1}}{\Gamma(\psi_1+1)}\right](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&+\left[\lambda_2\mathcal{K}_1\frac{(\log\eta_1)^{\delta_2}}{\Gamma(\delta_2+2)}+\lambda_2\frac{(\log\eta_3)^{\delta_2+\psi_1}}{\Gamma(\delta_2+\psi_1+1)}\right](\log\eta_1)^{\gamma_1-2}\log\left(\frac{\eta_1}{\eta_2}\right)\\
&+\mathcal{K}_1(\log\mathfrak{T})+\frac{(\log\mathfrak{T})^{\psi_1}}{\Gamma(\psi_1+1)},
\end{aligned}
\tag{3.3}
$$

$$
\mathfrak{W}_2=\frac{\log\mathfrak{T}^{\gamma_1-2}}{\Delta}\left[\lambda_1\mathcal{K}_2\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}+\lambda_1\mathcal{K}_2\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}\right.
$$

$$+ \lambda_1\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\delta_1+\psi_2+1)}+\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\delta_1+\psi_2+1)}\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)$$

$$+\left[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2(\log\eta_2)+\mathcal{K}_2(\log\mathfrak{T})+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\psi_2+1)}+\frac{(\log\mathfrak{T})^{\psi_2}}{\Gamma(\psi_2+1)}\right]$$

$$\times(\log\eta_1)^{\gamma_1-2}\log\left(\frac{\eta_1}{\eta_2}\right), \tag{3.4}$$

$$\hat{\mathfrak{W}}_1=(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\times\left(\frac{1}{\Delta}\right)\left[\mathcal{K}_1(\log\mathfrak{T})+\frac{(\log\mathfrak{T})^{\psi_1}}{\Gamma(\psi_1+1)}\right]\lambda_2\frac{\Gamma(\gamma_1)}{(\gamma_1+\delta_2)}(\log\eta_3)^{\gamma_1+\delta_2-1}$$

$$+\left[\mathcal{K}_1\lambda_2\frac{(\log\eta_3)^{\delta_2}}{\Gamma(\delta_2+2)}+\lambda_2\frac{(\log\eta_3)^{\psi_1+\delta_2}}{\Gamma(\psi_1+\delta_2+1)}\right](\log\mathfrak{T})^{\gamma_1-1}, \tag{3.5}$$

$$\hat{\mathfrak{W}}_2=(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\times\left(\frac{1}{\Delta}\right)\left[\lambda_1\mathcal{K}_2\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}+\lambda_1\mathcal{K}_2\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}\right.$$

$$\left.+\lambda_1\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2+\delta_1}}{\Gamma(\delta_1+\psi_2+1)}+\frac{(\log\eta_2)^{\psi_2+\delta_1}}{\Gamma(\delta_1+\psi_2+1)}\right]\lambda_2\frac{\Gamma(\gamma_1)}{(\gamma_1+\delta_2)}(\log\eta_3)^{\gamma_1+\delta_2-1}$$

$$+\left[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2(\log\eta_2)+\mathcal{K}_2(\log\mathfrak{T})+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\psi_2+1)}+\frac{(\log\mathfrak{T})^{\psi_2}}{\Gamma(\psi_2+1)}\right]$$

$$\times(\log\mathfrak{T})^{\gamma_1-1}+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2(\log\eta_2)+\mathcal{K}_2(\log\mathfrak{T})+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\psi_2+1)}+\frac{(\log\mathfrak{T})^{\psi_2}}{\Gamma(\psi_2+1)}, \tag{3.6}$$

$$\Phi=\min\{1-[(\mathfrak{W}_1+\hat{\mathfrak{W}}_1)\kappa_1+(\mathfrak{W}_2+\hat{\mathfrak{W}}_2)\hat{\kappa_1}],1-[(\mathfrak{W}_1+\hat{\mathfrak{W}}_1)\kappa_2+(\mathfrak{W}_2+\hat{\mathfrak{W}}_2)\hat{\kappa_2}]\}. \tag{3.7}$$

To demonstrate the existence of solutions for problems (1.1) and (1.2), we employ the following established result.

Lemma 3.1. *The Leray-Schauder alternative. Let* $\mathcal{F}(X)=\{x\in\mathcal{D}:x=kX(x)$ *for some* $0<k<1\}$*, where* $X:\mathcal{D}\to\mathcal{D}$ *is a completely continuous operator. Then, either the set* $F(X)$ *is unbounded or there exists at least one fixed point for operator* X*.*

3.1. Existence results via the Leray-Schauder alternative

We establish an existence result in this section using the Leray-Schauder alternative.

Theorem 3.2. *Presume* $(\mathcal{H}_1)$ *is true. Furthermore, it is presumed that*

$$(\mathfrak{W}_1+\mathfrak{W}_2)\kappa_1+(\hat{\mathfrak{W}}_1+\hat{\mathfrak{W}}_2)\hat{\kappa_1}<1, \tag{3.8}$$

and

$$(\mathfrak{W}_1+\mathfrak{W}_2)\kappa_2+(\hat{\mathfrak{W}}_1+\hat{\mathfrak{W}}_2)\hat{\kappa_2}<1. \tag{3.9}$$

Then, systems (1.1) and (1.2) have at least one solution on $[1,\mathfrak{T}]$*.*

Proof. To demonstrate that Ω, defined by (3.1), has a fixed point, we shall employ the Leray-Schauder alternative. The proof is split into two parts. Step 1, we show that $\Omega:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$, defined by (3.1), is completely continuous (C.C).

First we show that Ω is continuous. Let $\{(\varrho_n,\varphi_n)\}$ be a sequence such that $(\varrho_n,\varphi_n)\to(\varrho,\varphi)$ in $\mathcal{X}\times\mathcal{X}$. Then, for each $\tau\in[1,\mathfrak{T}]$, we have

$$
\begin{aligned}
&|\Omega_1(\varrho_n,\varphi_n)-\Omega_1(\varrho,\varphi)|\\
\le&\left|(\log\tau)^{\gamma_1-2}\right|\times\frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\left|\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi_n(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right|\\
&+\left|\mathcal{K}_2\int_1^{\eta_1}\frac{\varphi_n(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right|\\
&+\left|\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\right|\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\\
&+\left|\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\Bigg\}\\
&+\mathcal{K}_1\left|\int_1^{\mathfrak{T}}\frac{\varrho_n(\varpi)-\varrho(\varpi)}{\varpi}d\varpi\right|\\
&+\frac{1}{\Gamma(\psi_1)}\left|\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\Bigg]\\
&\times(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&+\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\left|\int_1^{\eta_2}\frac{\varphi_n(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right|+\mathcal{K}_2\left|\int_1^{\mathfrak{T}}\frac{\varphi_n(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right|\\
&+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\\
&+\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\\
&+\lambda_2\mathcal{I}^{\delta_2}_{1+}\Bigg(\mathcal{K}_1\left|\int_1^{\eta_3}\frac{\varrho_n(\varpi)-\varrho(\varpi)}{\varpi}d\varpi\right|\\
&+\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\Bigg)\Bigg]\\
&\times(\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}+\mathcal{K}_1\left|\int_1^{\tau}\frac{\varrho_n(\varpi)-\varrho(\varpi)}{\varpi}d\varpi\right|\\
&+\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\\
\le&(\log\tau)^{\gamma_1-2}\times\frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{|\varphi_n(\varpi)-\varphi(\varpi)|}{\varpi}d\varpi
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{K}_2\int_1^{\eta_1}\frac{|\varphi_n(\varpi)-\varphi(\varpi)|}{\varpi}d\varpi\\
&+\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|}{\varpi}d\varpi\Bigg\}\\
&+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{|\varrho_n(\varpi)-\varrho(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{|\rho_1(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|}{\varpi}d\varpi\Bigg]\\
&\times(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&+\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{|\varphi_n(\varpi)-\varphi(\varpi)|}{\varpi}d\varpi+\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{|\varphi_n(\varpi)-\varphi(\varpi)|}{\varpi}d\varpi\\
&+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|}{\varpi}d\varpi\\
&+\lambda_2\mathcal{I}_{1+}^{\delta_2}\Bigg(\mathcal{K}_1\int_1^{\eta_3}\frac{|\varrho_n(\varpi)-\varrho(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{|\rho_1(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|}{\varpi}d\varpi\Bigg)\Bigg]\\
&\times(\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}+\mathcal{K}_1\int_1^{\tau}\frac{|\varrho_n(\varpi)-\varrho(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{|\rho_1(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|}{\varpi}d\varpi.
\end{aligned}
$$

Since ρ_1 is continuous, we get

$$|\rho_1(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|\to 0\quad\text{as }(\varrho_n,\varphi_n)\to(\varrho,\varphi),$$

and

$$|\rho_2(s,\varrho_n(\varpi),\varphi_n(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)|\to 0\quad\text{as }(\varrho_n,\varphi_n)\to(\varrho,\varphi).$$

Then,

$$\|\Omega_1(\varrho_n-\varphi_n)-\Omega_1(\varrho-\varphi)\|\to 0\ \text{ as }\ (\varrho_n,\varphi_n)\to(\varrho,\varphi). \tag{3.10}$$

In the same way, we obtain

$$\|\Omega_2(\varrho_n-\varphi_n)-\Omega_2(\varrho-\varphi)\|\to 0\ \text{ as }\ (\varrho_n,\varphi_n)\to(\varrho,\varphi). \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\|\Omega(\varrho_n - \varphi_n) - \Omega(\varrho - \varphi)\| \to 0 \text{ as } (\varrho_n, \varphi_n) \to (\varrho, \varphi). \tag{3.12}$$

Hence, Ω is continuous. Let us initially establish the complete continuity of the operator $\Omega : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ as defined in (3.1). Evidently, the continuity of the operator Ω in terms of Ω_1 and Ω_2 is a consequence of the continuity of ρ_1 and ρ_2. Subsequently, we proceed to demonstrate that the operator Ω is uniformly bounded.

To achieve this, let $\mathcal{M} \subset \mathcal{X}\times\mathcal{X}$ be a bounded set. Consequently, we can identify positive constants $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfying $\rho_1|(\tau, \varrho(\tau), \varphi(\tau))| \le \mathcal{N}_1$ and $\rho_2|(\tau, \varrho(\tau), \varphi(\tau))| \le \mathcal{N}_2, \forall\ (\varrho, \varphi) \in \mathcal{M}$. Consequently, we obtain

$$\begin{aligned}
\|\Omega_1(\varrho,\varphi)\| =& \sup|\Omega_1(\varrho,\varphi)(\tau)| \\
\le&(\log\tau)^{\gamma_1-2}\times\frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi \\
&-\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi \\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg\} \\
&+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg] \\
&\times(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right) \\
&-\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi \\
&-\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi \\
&-\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi+\lambda_2\mathcal{I}^{\delta_2}_{1+}\Bigg(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi \\
&-\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg)\Bigg](\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\} \\
&-\mathcal{K}_1\int_1^{\tau}\frac{\varrho(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi, \\
\le&\mathcal{N}_1\Bigg\{\frac{\log\mathfrak{T}^{\gamma_1-2}}{\Delta}\Bigg[\mathcal{K}_1(\log\mathfrak{T})+\frac{(\log\mathfrak{T})^{\psi_1}}{\Gamma(\psi_1+1)}\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right) \\
&+\Bigg[\lambda_2\mathcal{K}_1\frac{(\log\eta_1)^{\delta_2}}{\Gamma(\delta_2+2)}+\lambda_2\frac{(\log\eta_3)^{\delta_2+\psi_1}}{\Gamma(\delta_1+\psi_1+1)}\Bigg](\log\eta_1)^{\gamma_1-2}\log\left(\frac{\eta_1}{\eta_2}\right)+\mathcal{K}_1(\log\mathfrak{T})+\frac{(\log\mathfrak{T})^{\psi_1}}{\Gamma(\psi_1+1)}\Bigg\} \\
&+\mathcal{N}_2\Bigg\{\frac{\log\mathfrak{T}^{\gamma_1-2}}{\Delta}\Bigg[\lambda_1\mathcal{K}_2\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}+\lambda_1\mathcal{K}_2\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}
\end{aligned}$$

$$+\lambda_1\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\delta_1+\psi_2+1)}+\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\delta_1+\psi_2+1)}\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)$$
$$+\left[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2(\log\eta_2)+\mathcal{K}_2(\log\mathfrak{T})+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\psi_2+1)}+\frac{(\log\mathfrak{T})^{\psi_2}}{\Gamma(\psi_2+1)}\right]$$
$$\times(\log\eta_1)^{\gamma_1-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}. \tag{3.13}$$

This, considering the notation in (3.3) and (3.4), results in:

$$\|\Omega_1(\varrho,\varphi)\|\leq\mathfrak{W}_1\mathcal{N}_1+\mathfrak{W}_2\mathcal{N}_2. \tag{3.14}$$

Likewise, using the notation of (3.5) and (3.6), we have

$$\|\Omega_2(\varrho,\varphi)\|\leq\hat{\mathfrak{W}}_1\mathcal{N}_1+\hat{\mathfrak{W}}_2\mathcal{N}_2. \tag{3.15}$$

Then, it follows from (3.14) and (3.15) that

$$\|\Omega(\varrho,\varphi)\|\leq(\mathfrak{W}_1+\hat{\mathfrak{W}}_1)\mathcal{N}_1+(\mathfrak{W}_2+\hat{\mathfrak{W}}_2)\mathcal{N}_2. \tag{3.16}$$

This demonstrates that the operator Ω is uniformly bounded.

To establish the equicontinuity of Ω, we consider $\tau_1,\tau_2\in[1,\mathfrak{T}]$ with $\tau_1<\tau_2$. Then, we find that

$$|\Omega_1(\varrho,\varphi)(\tau_2)-\Omega_1(\varrho,\varphi)(\tau_1)|$$
$$\leq(\log\tau_2)^{\gamma_1-2}-(\log\tau_1)^{\gamma_1-2}\times\frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}_{1+}^{\delta_1}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi$$
$$-\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{d\varpi}{\varpi}+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{d\varpi}{\varpi}\Bigg\}$$
$$+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{d\varpi}{\varpi}\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)$$
$$-\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi$$
$$-\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{d\varpi}{\varpi}-\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{d\varpi}{\varpi}$$
$$+\lambda_2\mathcal{I}_{1+}^{\delta_2}\left(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{d\varpi}{\varpi}\right)\Bigg](\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}$$
$$-\mathcal{K}_1\int_{\tau_2}^{\tau_1}\frac{\varrho(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^{\tau_1}\left|\left(\log\frac{\tau_2}{\varpi}\right)^{\psi_1-1}-\left(\log\frac{\tau_1}{\varpi}\right)^{\psi_1-1}\right|\frac{d\varpi}{\varpi}$$
$$+\frac{1}{\Gamma(\psi_2)}\int_{\tau_2}^{\tau_1}\left(\log\frac{\tau_2}{\varpi}\right)^{\psi_1-1}\frac{d\varpi}{\varpi},\ \to 0\ as\ \tau_2\to\tau_1, \tag{3.17}$$

independent of $(\varrho,\varphi)\in\mathcal{M}$. Likewise, it can be shown that $|\Omega_2(\varrho,\varphi)(\tau_2)-\Omega_2(\varrho,\varphi)(\tau_1)|\to 0$ as $\tau_2\to\tau_1$ independent of $(\varrho,\varphi)\in\mathcal{M}$. Thus, the equicontinuity of Ω_1 and Ω_2 implies that the operator Ω is equicontinuous. Hence, the operator Ω is equicontinuous. Therefore, the operator Ω satisfies the conditions for compactness according to Arzela-Ascoli's theorem. Lastly, we confirm the boundedness of the set: $\Theta(\Omega) = \{(\varrho, \varphi) \in \mathcal{X} \times \mathcal{X} : (\varrho, \varphi) = \kappa\Omega(\varrho, \varphi); 0 \leq \kappa \leq 1\}$. Let $(\varrho, \varphi) \in \Theta(\Omega)$. Then $(\varrho, \varphi) = \kappa\Omega(\varrho, \varphi)$, which implies that

$$\varrho(\tau) = \kappa\Omega_1(\varrho, \varphi)(\tau),$$
$$\varphi(\tau) = \kappa\Omega_2(\varrho, \varphi)(\tau),$$

for any $\tau \in [1, \mathfrak{T}]$.

Based on the assumption $(\mathcal{H}_1)$, we obtain:

$$\begin{aligned}
|\varrho(\tau)| \leq{} & (\log\tau)^{\gamma_1-2} \times \frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1 {}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi + \mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi \\
& + \left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{[\hat{\kappa_0} + \hat{\kappa_1}|\varrho| + \hat{\kappa_2}|\varphi|]}{\varpi}d\varpi \\
& + \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{[\hat{\kappa_0} + \hat{\kappa_1}|\varrho| + \hat{\kappa_2}|\varphi|]}{\varpi}d\varpi\Bigg\} \\
& + \mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi - \frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{[\kappa_0 + \kappa_1|\varrho| + \kappa_2|\varphi|]}{\varpi}d\varpi\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right) \\
& + \Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi + \mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi \\
& + \left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{[\hat{\kappa_0} + \hat{\kappa_1}|\varrho| + \hat{\kappa_2}|\varphi|]}{\varpi}d\varpi \\
& + \frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{[\hat{\kappa_0} + \hat{\kappa_1}|\varrho| + \hat{\kappa_2}|\varphi|]}{\varpi}d\varpi \\
& + \lambda_2\mathcal{I}^{\delta_2}_{1+}\left(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi + \frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{[\kappa_0 + \kappa_1|\varrho| + \kappa_2|\varphi|]}{\varpi}d\varpi\right)\Bigg] \\
& \times(\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\} + \mathcal{K}_1\int_1^{\tau}\frac{\varrho(\varpi)}{\varpi}d\varpi \\
& + \frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{[\kappa_0 + \kappa_1|\varrho| + \kappa_2|\varphi|]}{\varpi}d\varpi \\
\leq{} & \mathfrak{W}_1[\kappa_0 + \kappa_1|\varrho| + \kappa_2|\varphi|] + \mathfrak{W}_2[\hat{\kappa_0} + \hat{\kappa_1}|\varrho| + \hat{\kappa_2}|\varphi|],
\end{aligned} \tag{3.18}$$

which implies that

$$\|\varrho\| = \sup_{\tau\in[1,\mathfrak{T}]}|\varrho(\tau)| \leq \mathfrak{W}_1\kappa_0 + \mathfrak{W}_2\hat{\kappa_0} + (\mathfrak{W}_1\kappa_1 + \mathfrak{W}_2\hat{\kappa_1})\|\varrho\| + (\mathfrak{W}_1\kappa_2 + \mathfrak{W}_2\hat{\kappa_2})\|\varphi\|. \tag{3.19}$$

Similarly, one can find that

$$\|\varphi\| \leq \hat{\mathfrak{W}}_1\kappa_0 + \hat{\mathfrak{W}}_2\hat{\kappa_0} + (\hat{\mathfrak{W}}_1\kappa_1 + \hat{\mathfrak{W}}_2\hat{\kappa_1})\|\varrho\| + (\hat{\mathfrak{W}}_1\kappa_2 + \hat{\mathfrak{W}}_2\hat{\kappa_2})\|\varphi\|. \tag{3.20}$$

From (3.19) and (3.20), we obtain

$$\begin{aligned}\|\varrho\| + \|\varphi\| \leq &(\mathfrak{W}_1 + \hat{\mathfrak{W}}_1)\kappa_0 + (\mathfrak{W}_2 + \hat{\mathfrak{W}}_2)\hat{\kappa_0} + (\mathfrak{W}_1 + \hat{\mathfrak{W}}_1)\kappa_1 + (\mathfrak{W}_2 + \hat{\mathfrak{W}}_2)\hat{\kappa_1}\|\varrho\| \\ &+ (\mathfrak{W}_1 + \hat{\mathfrak{W}}_1)\kappa_2 + (\mathfrak{W}_2 + \hat{\mathfrak{W}}_2)\hat{\kappa_2}\|\varphi\|.\end{aligned}$$

Which, by $\|(\varrho,\varphi)\| = \|\varrho\| + \|\varphi\|$, yields

$$\|(\varrho,\varphi)\| \leq \frac{1}{\Phi}[(\mathfrak{W}_1 + \hat{\mathfrak{W}}_1)\kappa_0 + (\mathfrak{W}_2 + \hat{\mathfrak{W}}_2)\hat{\kappa_0}].$$

As a result, $\Theta(\Omega)$ is constrained within bounds. Consequently, the conclusion of Lemma 3.1 is applicable, implying that the operator Ω possesses at least one fixed point. This fixed point indeed corresponds to a solution of problems (1.1) and (1.2). □

In the forthcoming findings, the application of Banach's fixed-point theorem will be utilized to demonstrate the existence of a unique solution for the problems (1.1) and (1.2).

Theorem 3.3. *If condition* $(\mathcal{H}_2)$ *is met, and the inequality*

$$(\mathfrak{W}_1 + \hat{\mathfrak{W}}_1)\mathcal{L}_1 + (\mathfrak{W}_2 + \hat{\mathfrak{W}}_2)\mathcal{L}_2 < 1, \tag{3.21}$$

holds, where $\mathfrak{W}i$ *and* $\hat{\mathfrak{W}}i$ *are defined in (3.3)–(3.6), then problems (1.1) and (1.2) possess unique solutions over the interval* $[1,\mathfrak{T}]$.

Proof. Denoting $\mathfrak{K}1 = \{\sup_{\tau\in[1,\mathfrak{T}]} |\rho_1(\tau,0,0)| < \infty\}$ and $\mathfrak{K}2 = \{\sup_{\tau\in[1,\mathfrak{T}]} |\rho_2(\tau,0,0)| < \infty\}$, it can be inferred from assumption $(\mathcal{H}_1)$ that

$$\begin{aligned}|\rho_1(\tau,\varrho,\varphi)| &\leq \mathcal{L}_1(\|\varrho\| + \|\varphi\|) + \mathfrak{K}_1 \\ &\leq \mathcal{L}_1\|(\varrho,\varphi)\| + \mathfrak{K}_1,\end{aligned}$$

and

$$|\rho_2(\tau,\varrho,\varphi)| \leq \mathcal{L}_2\|(\varrho,\varphi)\| + \mathfrak{K}_2.$$

First, we show that $\Omega\mathcal{B}_{\mathfrak{r}} \subset \mathcal{B}_{\mathfrak{r}}$, where $\mathcal{B}_{\mathfrak{r}} = \{(\varrho,\varphi) \in \mathcal{X}\times\mathcal{X} : \|(\varrho,\varphi)\| \leq \mathfrak{r}\}$, with

$$\mathfrak{r} \geq \frac{(\mathfrak{W}_1 + (\hat{\mathfrak{W}}_1))\mathfrak{K}_1 + (\mathfrak{W}_2 + (\hat{\mathfrak{W}}_2))\mathfrak{K}_2}{1 - (\mathfrak{W}_1 + \hat{(}\mathfrak{W}_{1)})\mathcal{L}_1 + (\mathfrak{W}_2 + (\hat{\mathfrak{W}}_2))\hat{\mathcal{L}}}. \tag{3.22}$$

For $(\varrho,\varphi) \in \mathcal{B}_{\mathfrak{r}}$, we have

$$\begin{aligned}\|\Omega_1(\varrho,\varphi)\| = &\sup_{\tau\in[1,\mathfrak{T}]} |\Omega_1(\varrho,\varphi)(\tau)| \\ \leq &(\log\tau)^{\gamma_1-2} \times \frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}_{1+}^{\delta_1}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi - \mathcal{K}_2\int_1^{\eta_1}\frac{\varphi(\varpi)}{\varpi}d\varpi \\ &- \left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg\}\\
&+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{\varrho(\varpi)}{\varpi}d\varpi-\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg]\\
&\times(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&-\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi(\varpi)}{\varpi}d\varpi-\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{\varphi(\varpi)}{\varpi}d\varpi\\
&-\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\\
&-\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi+\lambda_2\mathcal{I}_{1+}^{\delta_2}\Bigg(\mathcal{K}_1\int_1^{\eta_3}\frac{\varrho(\varpi)}{\varpi}d\varpi\\
&-\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\Bigg)\Bigg](\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}\\
&-\mathcal{K}_1\int_1^{\tau}\frac{\varrho(\varpi)}{\varpi}d\varpi+\frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
\|\Omega(\varrho,\varphi)\|\leq&(\mathcal{L}_1\mathfrak{r}+\mathfrak{K}_1)\Bigg\{\frac{\log\mathfrak{T}^{\gamma_1-2}}{\Delta}\Bigg[\mathcal{K}_1(\log\mathfrak{T})+\frac{(\log\mathfrak{T})^{\psi_1}}{\Gamma(\psi_1+1)}\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&+\Bigg[\lambda_2\mathcal{K}_1\frac{(\log\eta_1)^{\delta_2}}{\Gamma(\delta_2+2)}+\lambda_2\frac{(\log\eta_3)^{\delta_2+\psi_1}}{\Gamma(\delta_1+\psi_1+1)}\Bigg](\log\eta_1)^{\gamma_1-2}\log\left(\frac{\eta_1}{\eta_2}\right)+\mathcal{K}_1(\log\mathfrak{T})\\
&+\frac{(\log\mathfrak{T})^{\psi_1}}{\Gamma(\psi_1+1)}+\frac{\log\mathfrak{T}^{\gamma_1-2}}{\Delta}\Bigg[\lambda_1\mathcal{K}_2\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}+\lambda_1\mathcal{K}_2\frac{(\log\eta_2)^{\delta_1}}{\Gamma(\delta_1+2)}\\
&+\lambda_1\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\delta_1+\psi_2+1)}+\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\delta_1+\psi_2+1)}\Bigg](\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&+\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2(\log\eta_2)+\mathcal{K}_2(\log\mathfrak{T})+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{(\log\eta_2)^{\psi_2}}{\Gamma(\psi_2+1)}+\frac{(\log\mathfrak{T})^{\psi_2}}{\Gamma(\psi_2+1)}\Bigg]\\
&\times(\log\eta_1)^{\gamma_1-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}.
\end{aligned}
$$

Making use of the notation of (3.3)–(3.6), we get

$$
\|\Omega_1(\varrho,\varphi)\|\leq(\mathcal{L}_1\mathfrak{W}_1+\mathcal{L}_2\mathfrak{W}_2)\mathfrak{r}+\mathfrak{W}_1\mathfrak{K}_1+\mathfrak{W}_2\mathfrak{K}_2. \tag{3.24}
$$

Likewise, we can find that

$$
\|\Omega_2(\varrho,\varphi)\|\leq(\mathcal{L}_1\hat{\mathfrak{W}}_1+\mathcal{L}_2\hat{\mathfrak{W}}_2)\mathfrak{r}+\mathfrak{W}_1\hat{\mathfrak{K}}_1+\hat{\mathfrak{W}}_2\mathfrak{K}_2. \tag{3.25}
$$

Then, it follows from (3.24)–(3.25) that

$$\|\Omega(\varrho,\varphi)\| \le \|\Omega_1(\varrho,\varphi)\| + \|\Omega_2(\varrho,\varphi)\| \le \mathfrak{r}.$$

Therefore, $\Omega\mathcal{B}_{\mathfrak{r}} \subset \mathcal{B}_r$ as $(\varrho,\varphi)\in\mathcal{B}_{\mathfrak{r}}$ is an arbitrary element.

To confirm the contraction property of the operator Ω, consider $(\varrho_i,\varphi_j) \in \mathcal{B}_{\mathfrak{r}}$ for $i = 1,2$. Subsequently, we obtain

$$\begin{aligned}
&\|\Omega_1(\varrho_1,\varphi_1) - \Omega_1(\varrho,\varphi)\| \\
\le& \left|(\log\tau)^{\gamma_1-2}\right| \times \frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}_{1+}^{\delta_1}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\left|\mathcal{K}_2\int_1^{\eta_2}\frac{\varphi_1(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right| \\
&+\left|\mathcal{K}_2\int_1^{\eta_1}\frac{\varphi_1(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right| \\
&+\left|\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\right|\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right| \\
&+\left|\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\Bigg\} \\
&+\mathcal{K}_1\left|\int_1^{\mathfrak{T}}\frac{\varrho_1(\varpi)-\varrho(\varpi)}{\varpi}d\varpi\right| \\
&+\frac{1}{\Gamma(\psi_1)}\left|\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\Bigg] \\
&\times(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right) \\
&+\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\left|\int_1^{\eta_2}\frac{\varphi_1(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right| + \mathcal{K}_2\left|\int_1^{\mathfrak{T}}\frac{\varphi_1(\varpi)-\varphi(\varpi)}{\varpi}d\varpi\right| \\
&+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right| \\
&+\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right| \\
&+\lambda_2\mathcal{I}_{1+}^{\delta_2}\Bigg(\mathcal{K}_1\left|\int_1^{\eta_3}\frac{\varrho_1(\varpi)-\varrho(\varpi)}{\varpi}d\varpi\right| \\
&+\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|\Bigg)\Bigg] \\
&\times(\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\} + \mathcal{K}_1\left|\int_1^{\tau}\frac{\varrho_1(\varpi)-\varrho(\varpi)}{\varpi}d\varpi\right| \\
&+\frac{1}{\Gamma(\psi_2)}\left|\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{\rho_1(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_1(s,\varrho(\varpi),\varphi(\varpi))(\varpi)}{\varpi}d\varpi\right|,
\end{aligned}$$

$$
\begin{aligned}
\leq &(\log\tau)^{\gamma_1-2}\times\frac{1}{\Delta}\Bigg\{\Bigg[\lambda_1{}^{\mathcal{H}}\mathcal{I}^{\delta_1}_{1+}\Bigg\{\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{|\varphi_1(\varpi)-\varphi_2(\varpi)|}{\varpi}d\varpi\\
&+\mathcal{K}_2\int_1^{\eta_1}\frac{|\varphi_1(\varpi)-\varphi_2(\varpi)|}{\varpi}d\varpi\\
&+\left(\frac{\log\eta_1}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho_2(\varpi),\varphi_2(\varpi))(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho_2(\varpi),\varphi_2(\varpi))(\varpi)|}{\varpi}d\varpi\Bigg\}\\
&+\mathcal{K}_1\int_1^{\mathfrak{T}}\frac{|\varrho_1(\varpi)-\varrho_2(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_1)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_1-1}\frac{|\rho_1(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_1(s,\varrho_2(\varpi),\varphi_2(\varpi))(\varpi)|}{\varpi}d\varpi\Bigg]\\
&\times(\log\mathfrak{T})^{\gamma_2-2}\log\left(\frac{\mathfrak{T}}{\eta_2}\right)\\
&+\Bigg[\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\mathcal{K}_2\int_1^{\eta_2}\frac{|\varphi_1(\varpi)-\varphi_2(\varpi)|}{\varpi}d\varpi+\mathcal{K}_2\int_1^{\mathfrak{T}}\frac{|\varphi_1(\varpi)-\varphi_2(\varpi)|}{\varpi}d\varpi\\
&+\left(\frac{\log\mathfrak{T}}{\log\eta_2}\right)^{\gamma_2-2}\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho_2(\varpi),\varphi_2(\varpi))(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\mathfrak{T}}\left(\log\frac{\mathfrak{T}}{\varpi}\right)^{\psi_2-1}\frac{|\rho_2(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_2(s,\varrho_2(\varpi),\varphi_2(\varpi))(\varpi)|}{\varpi}d\varpi\\
&+\lambda_2\mathcal{I}^{\delta_2}_{1+}\Bigg(\mathcal{K}_1\int_1^{\eta_3}\frac{|\varrho_1(\varpi)-\varrho_2(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\eta_3}\left(\log\frac{\eta_3}{\varpi}\right)^{\psi_1-1}\frac{|\rho_1(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_1(s,\varrho_2(\varpi),\varphi_2(\varpi))(\varpi)|}{\varpi}d\varpi\Bigg)\Bigg]\\
&\times(\log\eta_1)^{\gamma_2-2}\log\left(\frac{\eta_1}{\eta_2}\right)\Bigg\}+\mathcal{K}_1\int_1^{\tau}\frac{|\varrho_1(\varpi)-\varrho_2(\varpi)|}{\varpi}d\varpi\\
&+\frac{1}{\Gamma(\psi_2)}\int_1^{\tau}\left(\log\frac{\tau}{\varpi}\right)^{\psi_1-1}\frac{|\rho_1(s,\varrho_1(\varpi),\varphi_1(\varpi))(\varpi)-\rho_1(s,\varrho_2(\varpi),\varphi_2(\varpi))(\varpi)|}{\varpi}d\varpi.
\end{aligned}
$$

Which, by $(\mathcal{H}_2)$, yields

$$\|\Omega_1(\varrho_1,\varphi_1)-\Omega_1(\varrho_2,\varphi_2)\|\leq(\mathfrak{W}_1\mathcal{L}_1+\mathfrak{W}_2\mathcal{L}_2)[\|\varrho_1-\varrho_2+\|\varphi_1-\varphi_2\|]. \tag{3.26}$$

Similarly, we can discover that

$$\|\Omega_2(\varrho_1,\varphi_1)-\Omega_1(\varrho_2,\varphi_2)\|\leq(\hat{\mathfrak{W}}_1\mathcal{L}_1+\hat{\mathfrak{W}}_2\mathcal{L}_2)[\|\varrho_1-\varrho_2+\|\varphi_1-\varphi_2\|]. \tag{3.27}$$

Consequently, it follows from (3.26) and (3.27) that

$$\|\Omega(\varrho_1,\varphi_1)-\Omega(\varrho_1,\varphi_1)\|=\|\Omega_1(\varrho_1,\varphi_1)-\Omega_1(\varrho_1,\varphi_1)\|+\|\Omega_2(\varrho_1,\varphi_1)-\Omega_1(\varrho_1,\varphi_1)\|$$

$$\leq ([\mathfrak{W}_1 + \hat{\mathfrak{W}}_1]\mathcal{L}_1 + [\mathfrak{W}_2 + \hat{\mathfrak{W}}_2]\mathcal{L}_2)[||\varrho_1 - \varrho_2 + ||\varphi_1 - \varphi_2||]. \quad (3.28)$$

This, in line with condition (3.21), implies that Ω acts as a contraction. As a result, the operator Ω has a unique fixed point, following the application of the Banach fixed-point theorem. Consequently, there exists a unique solution for problems (1.1) and (1.2) over the interval $[1, \mathfrak{T}]$. □

4. Examples

The sequential fractional differential system under consideration, involving the coupled Hilfer-Hadamard operators, is expressed as:

$$\begin{cases} ({}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_1,\beta_1} + \mathcal{K}_1{}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_1-1,\beta_1})\varrho(\tau) = \rho_1(\tau, \varrho(\tau), \varphi(\tau)), \quad 1 < \psi_1 \leq 2, \quad \tau \in \mathcal{E} := [1, \mathfrak{T}], \\ ({}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_2,\beta_2} + \mathcal{K}_2{}^{\mathcal{HH}}\mathcal{D}_{1^+}^{\psi_2-1,\beta_2})\varphi(\tau) = \rho_2(\tau, \varrho(\tau), \varphi(\tau)), \quad 2 < \psi_2 \leq 3, \quad \tau \in \mathcal{E} := [1, \mathfrak{T}], \end{cases} \quad (4.1)$$

supplemented with nonlocal coupled Hadamard integral boundary conditions:

$$\begin{cases} \varrho(1) = 0, \quad \varrho(\mathfrak{T}) = \lambda_1{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_1}\varphi(\eta_1), \\ \varphi(1) = 0, \quad \varphi(\eta_2) = 0, \quad \varphi(\mathfrak{T}) = \lambda_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\delta_2}\varrho(\eta_3), \quad 1 < \eta_1, \eta_2, \eta_3 < \mathfrak{T}. \end{cases} \quad (4.2)$$

Here, $\psi_1 = \frac{5}{4}, \psi_1 = \frac{3}{2}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{2}, \mathfrak{T} = 10, \delta_1 = \frac{1}{3}, \delta = \frac{3}{4}, \eta_1 = 6, \eta_2 = \frac{4}{3}, \eta_3 = 5, \lambda_1 = 3, \lambda_2 = 2, \gamma_1 = \frac{11}{16}, \gamma_2 = \frac{11}{16}, \mathcal{K}_1 = \frac{1}{7}, \mathcal{K}_2 = \frac{1}{9}, \Delta = 0.114465$ with the given data, and it is found that $\mathfrak{W}_1 = 2.79137199, \mathfrak{W}_2 = 1.574688, \hat{\mathfrak{W}}_1 = 6.799260, \hat{\mathfrak{W}}_2 = 0.91745564$.

In order to demonstrate Theorem 3.2, we use

$$\rho_1(\tau, \varrho(\tau), \varphi(\tau)) = \sqrt{2\tau + 1} + \frac{|\mathrm{u}(\tau)|}{25(1 + |\varrho(\tau)|)} + \frac{\cos\varphi(\tau)}{5\tau + 10}, \quad (4.3)$$

$$\rho_2(\tau, \varrho(\tau), \varphi(\tau)) = e^{-2\tau} + \frac{\tan^{-1}\varrho(\tau)}{30\tau)} + \frac{1}{45}\sin\varphi(\tau).$$

It is evident that condition ($\mathcal{H}1$) is fulfilled with parameter values: $\kappa_0 = \sqrt{3}$, $\kappa_1 = \frac{1}{25}$, $\kappa_2 = \frac{1}{15}$, $\hat{\kappa_0} = \frac{1}{e^2}$, $\hat{\kappa_1} = \frac{1}{30}$, and $\hat{\kappa_2} = \frac{1}{45}$. Moreover, we have

$$(\mathfrak{W}_1 + \mathfrak{W}_2)\frac{1}{25} + (\hat{\mathfrak{W}}_1 + \hat{\mathfrak{W}}_2)\frac{1}{30} \approx 0.4318658333 < 1, \quad (4.4)$$

and

$$(\mathfrak{W}_1 + \mathfrak{W}_2)\frac{1}{15} + (\hat{\mathfrak{W}}_1 + \hat{\mathfrak{W}}_2)\frac{1}{45} \approx 0.462553055 < 1. \quad (4.5)$$

Hence, the assumptions of Theorem 3.2 are satisfied. Consequently, the outcome of Theorem 3.2 is applicable, and therefore, problems (1.1) and (1.2), with ρ_1 and ρ_2 specified in (4.3), possess at least one solution over the interval [1,10].

To demonstrate Theorem 3.3, we take into account

$$\rho_1(\tau, \varrho(\tau), \varphi(\tau)) = \frac{1}{\tau^2 + 4} + \frac{1}{10\sqrt{2\tau + 7}}(\sin\varrho(\tau) + |\varphi(\tau)|), \quad (4.6)$$

$$\rho_2(\tau, \varrho(\tau), \varphi(\tau)) = e^{-2\tau} + \frac{1}{5(\tau+4)}(\tan^{-1}\varrho(\tau) + \cos\varphi(\tau)).$$

Put simply, we discover that $\mathcal{L} = \frac{1}{30}$ and $\hat{\mathcal{L}} = \frac{1}{25}$, and

$$(\mathfrak{W}_1 + \hat{\mathfrak{W}}_1)\frac{1}{30} + (\mathfrak{W}_2 + \hat{\mathfrak{W}}_2)\frac{1}{25} \approx 0.41937332 < 1.$$

Since the conditions of Theorem 3.3 are satisfied, it can be concluded, according to its findings, that problems (1.1) and (1.2), with ρ_1 and ρ_2 defined in (4.6), possess unique solutions over the interval [1,10].

5. Conclusions

We have presented criteria for the existence of solutions to a coupled system of nonlinear sequential HHFDEs with distinct orders, coupled with nonlocal HFI boundary conditions. We derive the expected results using a methodology that uses modern analytical tools. It is imperative to emphasize that the results offered in this specific context are novel and contribute to the corpus of existing literature on the topic. Furthermore, our results encompass cases where the system reduces to one with boundary conditions of the following form: When $\lambda_1 = \lambda_2 = 0$, we get

$$\begin{cases} \varrho(1) = 0, \quad \varrho(\mathfrak{T}) = 0, \\ \varphi(1) = 0, \quad \varphi(\eta_2) = 0, \quad \varphi(\mathfrak{T}) = 0, \quad 1 < \eta_1, \eta_2 < \mathfrak{T}. \end{cases}$$

These cases represent new findings. Looking ahead, our future plans include extending this work to a tripled system of nonlinear sequential HHFDEs with varying orders and integro-multipoint boundary conditions. We also intend to investigate the multivalued analogue of the problem studied in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (grant no. 5979). This study is supported via funding from Prince Sattam bin Abdulaziz University, project number (PSAU/2024/R/1445). M. Manigandan gratefully acknowledges the Center for Computational Modeling, Chennai Institute of Technology, India, vide funding number CIT/CCM/2023/RP-018.

Conflict of interest

The authors declare no conflicts of interest.

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