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Research article

On the axodes of one-parameter spatial movements

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Abstract: In this treatise, several relationships are improved for the axodes of one-parameter spatial movements. Results are devised in some theorems which characterize many kinematical and geometrical properties of the movements employing the geometrical data of the stationary and movable axodes. An example illustrates the application of the formulae derived. Our findings contribute to a greater understanding of the similarities between spatial movements and axodes, with possible applications in fields such as mechanical engineering.

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1. Introduction

In the study of spatial movements (kinematics), the analysis of the trajectory of general rigid body movements relies on two factors: the position and orientation of the movable line. The direction of the movable line describes the shape of a cone. The intersection of the cone and a unit sphere, attached at the vertex of the cone, determines a spherical curve known as the spherical indicatrix or image of the line path. The position of the moveable line, relative to a reference point, is specified by a space curve that is commonly known as the image of the line's path.

However, in spatial movements, it is advisable to take into account the inherent characteristics of the line path while considering ruled surfaces. Moreover, it is established that the instantaneous rotational axis ($\mathcal{ISA}$) of a movable object generates a pair of ruled surfaces, known as the invariable and movable axodes ($\mathcal{AX}$), with the $\mathcal{ISA}$ serving as their ruling (creating) in both the invariable and movable spaces, respectively. The $\mathcal{AX}$ undergoes rolling and sliding motion in a certain direction, ensuring that the tangential contact within the $\mathcal{AX}$ is maintained along the full length of the two matting rulings. These rulings, located in each $\mathcal{AX}$, work together to determine the position of the $\mathcal{ISA}$ at any given moment. It is considered that a certain motion results in a unique pencil of $\mathcal{AX}$, and the same applies in the opposite direction. If the axioms of any motion are fulfilled, it implies that the specific movement may be reconstructed without knowledge of the actual components of the mechanism, their configuration, exact dimensions, or the manner in which they are attached. During the process of synthesis, it has been understood that the $\mathcal{AX}$ plays a crucial role in both the physical mechanism and the actual movement of its components. There are numerous exceptional written works on the subject, including a variety of treatises; for these, refer to [1–8].

Surprisingly, dual numbers have been employed to ponder the movement of a line space. They may well be the most suitable tools for this purpose. In the context of dual number and screw algebra, the E. Study map states that there is a one-to-one correspondence between the pencil of dual points on the dual unit sphere in the dual 3-space $\mathcal{D}^3$ and the pencil of all directed lines in Euclidean 3-space $\mathcal{E}^3$. Using this map: a one-parameter set of points (known as a dual curve) on a unit sphere in dual space corresponds to a one-parameter pencil of directed lines (known as a ruled surface) in three-dimensional Euclidean space $\mathcal{E}^3$ [9–11]. Consequently, numerous researchers have made significant efforts to study the curvature properties of ruled surfaces using various methods [12–23]. However, further investigation is required to gain a deeper understanding.

In this study, we have employed the E. Study map to promptly analyze the kinematic-geometry of one parameter spatial movement. Our research focuses on examining the characteristics of the axodes and comparing them to spherical movements. Consequently, the invariants were deliberated upon and a dual form of the planar Euler-Savary equation was derived. This work aims to elucidate the topic of second-order movement features to foster a comprehensive understanding. Rotations are crucial in various professions, such as in astronomy (the movement of planets) and chemistry (the movement of electrons and the rotation of molecules). The utilization of eigenvector, eigenvalue, and eigenproblem methodologies can provide insights into difficult problems [23].

2. Basic concepts

In this section, we give some conceptualizations that we will employ in this article [1–4, 10, 11]: A directed line ($\mathcal{DL}$) can be identified by a point $\boldsymbol{\alpha} \in \mathbb{L}$ and a normalized orientation vector $\mathbf{r}$ of $\mathbb{L}$, that is, $\|\mathbf{r}\|^2 = 1$. To gain coordinates for $\mathbb{L}$, one composes the moment vector $\mathbf{r}^* = \boldsymbol{\alpha} \times \mathbf{r}$ regarding to the origin point in $\mathcal{E}^3$. If $\boldsymbol{\alpha}$ is offset by any point $\mathbf{y} = \boldsymbol{\alpha} + \nu\mathbf{r}$, $\nu \in \mathbb{R}$ on $\mathbb{L}$, this suggest that $\mathbf{r}^*$ is linearly independent of $\boldsymbol{\alpha}$ on $\mathbb{L}$. The vectors $\mathbf{r}$ and $\mathbf{r}^*$ are not independent of one another; they fulfil that:

$$< \mathbf{r}, \mathbf{r} >=1, \ < \mathbf{r}^*, \mathbf{r} >=0.$$

The six coordinates r_ξ, $r_\xi^*(\xi = 1, 2, 3)$ of $\mathbf{r}$ and $\mathbf{r}^*$ are named the normalized Plűcker coordinates of the line $\mathbb{L}$. Hence, the vectors $\mathbf{r}$ and $\mathbf{r}^*$ locate $\mathbb{L}$.

A dual ($\mathcal{D}$) number $\widehat{r}$ is a number $r + \varepsilon r^*$, where r, r^* in $\mathbb{R}$ and ε is a $\mathcal{D}$ unit with $\varepsilon \neq 0$, and $\varepsilon^2 = 0$. Then, the set

$$\mathcal{D}^3 = \{\widehat{\mathbf{r}}= \mathbf{r} + \varepsilon\mathbf{r}^* =(\widehat{r}_1, \widehat{r}_2, \widehat{r}_3)\}$$

with the inner product

$$< \widehat{\mathbf{r}}, \widehat{\mathbf{r}} >= \widehat{r}_1^2 + \widehat{r}_2^2 + \widehat{r}_3^2$$

forms the $\mathcal{D}$ 3-space $\mathcal{D}^3$. Thereby, a point $\widehat{\mathbf{r}}$ has $\mathcal{D}$ coordinates

$$\widehat{r_i} = (r_i + \varepsilon r_i^*) \in \mathcal{D}.$$

If $\mathbf{r} \neq \mathbf{0}$, the norm $\left\|\widehat{\mathbf{r}}\right\|$ of $\widehat{\mathbf{r}} = \mathbf{r} + \varepsilon\mathbf{r}^*$ is

$$\left\|\widehat{\mathbf{r}}\right\| = \|\mathbf{r}\| + \varepsilon\frac{<\mathbf{r}^*, \mathbf{r}>}{\|\mathbf{r}\|}, \quad \|\mathbf{r}\| \neq 0.$$

So, we may set the $\mathcal{D}$ vector $\widehat{\mathbf{r}}$ as a $\mathcal{D}$ multiplier of a $\mathcal{D}$ unit vector ($\mathcal{DUV}$) in the form

$$\widehat{\mathbf{r}} = \left\|\widehat{\mathbf{r}}\right\|\widehat{\mathbf{s}},$$

where $\widehat{\mathbf{s}}$ is referred to as the axis. The ratio

$$h = \frac{<\mathbf{r}^*, \mathbf{r}>}{\|\mathbf{r}\|^2}$$

is known as the pitch along the axis $\widehat{\mathbf{s}}$. If $h = 0$ and $\|\mathbf{r}\| = 1$, $\widehat{\mathbf{r}}$ is a $\mathcal{DL}$, and when h is definite, $\widehat{\mathbf{r}}$ is an proper screw. When $h \to \infty$, $\widehat{\mathbf{r}}$ is named a pair. A $\mathcal{D}$ vector with norm equal to one is coined a $\mathcal{DUV}$. Hence, every

$$\mathcal{DL} \ \mathbb{L} = (\mathbf{r}, \mathbf{r}^*) \in \mathcal{E}^3 \times \mathcal{E}^3$$

is appeared by $\mathcal{DUV}$

$$\widehat{\mathbf{r}} = \mathbf{r} + \varepsilon\mathbf{r}^* (<\mathbf{r}, \mathbf{r}> = 1, \quad <\mathbf{r}^*, \mathbf{r}> = 0).$$

The $\mathcal{DU}$ sphere in $\mathcal{D}^3$ is expounded by

$$\mathcal{K} = \{\widehat{\mathbf{r}} \in \mathcal{D}^3 \mid \left\|\widehat{\mathbf{r}}\right\|^2 = \widehat{r}_1^2 + \widehat{r}_2^2 + \widehat{r}_3^2 = 1\}.$$

Then, we have the E. Study's map: The set of points of $\mathcal{DU}$ sphere in $\mathcal{D}^3$-space is in bijection with the set of all $\mathcal{DL}$s in $\mathcal{E}^3$ [10, 11].

3. One-parameter dual spherical movement

Let $\mathcal{K}_m$ and $\mathcal{K}_f$ be two $\mathcal{DU}$ spheres. Let $\widehat{\mathbf{0}}$ be the joint center and two orthonormal $\mathcal{D}$ coordinate frames $\{\widehat{\mathbf{0}}; \widehat{\mathbf{e}}_1, \ \widehat{\mathbf{e}}_2, \ \widehat{\mathbf{e}}_3\}$, and $\{\widehat{\mathbf{0}}; \ \widehat{\mathbf{f}}_1, \ \widehat{\mathbf{f}}_2, \ \widehat{\mathbf{f}}_3\}$ be rigidly related with $\mathcal{K}_m$ and $\mathcal{K}_f$, respectively. We set $\{\widehat{\mathbf{0}}; \ \widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_2, \widehat{\mathbf{f}}_3\}$ as invariable, whereas the members of $\{\widehat{\mathbf{0}}; \widehat{\mathbf{e}}_1, \ \widehat{\mathbf{e}}_2, \ \widehat{\mathbf{e}}_3\}$ are functions of a real parameter $t \in \mathbb{R}$ (say the time). Then, we say that $\mathcal{K}_m$ movable with respect to $\mathcal{K}_f$. Such movement is coined a one-parameter $\mathcal{D}$ spherical movement, and is indicated by $\mathcal{K}_m/\mathcal{K}_f$. By setting

$$<\widehat{\mathbf{f}}_\xi, \widehat{\mathbf{e}}_\zeta> = \widehat{\mathcal{A}}_{\xi\zeta}$$

and putting the $\mathcal{D}$ matrix

$$(\mathcal{DM})\widehat{\mathcal{A}} = (\widehat{\mathcal{A}}_{\xi\zeta}),$$

we can set the E. Study map in the matrix sort as follows:

$$\mathcal{K}_m/\mathcal{K}_f : \begin{pmatrix} \widehat{\mathbf{f}}_1 \\ \widehat{\mathbf{f}}_2 \\ \widehat{\mathbf{f}}_3 \end{pmatrix} = \begin{pmatrix} \widehat{\mathcal{A}}_{11} & \widehat{\mathcal{A}}_{12} & \widehat{\mathcal{A}}_{13} \\ \widehat{\mathcal{A}}_{21} & \widehat{\mathcal{A}}_{22} & \widehat{\mathcal{A}}_{23} \\ \widehat{\mathcal{A}}_{31} & \widehat{\mathcal{A}}_{32} & \widehat{\mathcal{A}}_{33} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{e}}_1 \\ \widehat{\mathbf{e}}_2 \\ \widehat{\mathbf{e}}_3 \end{pmatrix}. \tag{3.1}$$

The

$$\mathcal{DM}\,\widehat{\mathcal{A}} := (\widehat{\mathcal{A}}_{\xi\zeta}) = (\mathcal{A}_{\xi\zeta}) + \varepsilon(\mathcal{A}^{*}_{\xi\zeta})$$

has the property that $\widehat{\mathcal{A}}^{-1} = \widehat{\mathcal{A}}^{t}$, which means it is an orthogonal $\mathcal{DM}$. This result indicates that the E. Study map is corresponds with an orthogonal $\mathcal{DM}$. Comparable with the family of real orthogonal matrices, the family of $\mathcal{D}$ orthogonal 3×3 matrices, denoted by $O(\mathcal{D}^{3\times 3})$, form a group with matrix multiplication as the group operation (real orthogonal matrices are subgroup of $\mathcal{D}$ orthogonal matrices). $\mathcal{D}$ orthogonal matrices as understood here then also form a Lie group that is a manifold [6]. The identity element of $O(\mathcal{D}^{3\times 3})$ is the 3×3 unit matrix. Since the center of the $\mathcal{DU}$ sphere in $\mathcal{D}^3$ must remain steady, the transformation group in $\mathcal{D}^3$ (the image of Euclidean movements in $\mathcal{E}^3$) does not hold any translations. Via the E. Study map: If $\mathcal{K}_m$ and $\mathcal{K}_f$ matches to the line spaces $\mathcal{H}_m$ and $\mathcal{H}_f$, respectively, then $\mathcal{K}_m/\mathcal{K}_f$ matches the one-parameter spatial movement $\mathcal{H}_m/\mathcal{H}_f$. Therefore, $\mathcal{H}_m$ is the moveable space against the stationary space $\mathcal{H}_f$ in $\mathcal{E}^3$. Hence, in order to have the Euclidean movements in $\mathcal{D}^3$, we can state the following theorem [10, 11]:

Theorem 3.1. *The Euclidean movements in $\mathcal{E}^3$ are fulfilled in $\mathcal{D}^3$ by 3×3 $\mathcal{D}$ orthogonal matrices $\widehat{\mathcal{A}} = (\widehat{\mathcal{A}}_{\xi\zeta})$, where $\widehat{\mathcal{A}}\widehat{\mathcal{A}}^{t} = \mathcal{I}$, $\widehat{\mathcal{A}}_{\xi\zeta}$ are $\mathcal{D}$ numbers, and $\mathcal{I}$ is the 3×3 unit matrix.*

Via Theorem 3.1, the Lie algebra $\mathcal{L}(O_{D^{3\times 3}})$ of the group $O(\mathcal{D}^{3\times 3})$ of 3×3 $\mathcal{D}$ matrices $\widehat{\mathcal{A}}$ is the algebra of skew-symmetric 3×3 $\mathcal{D}$ matrices. By differentiation of $\widehat{\mathcal{A}}\widehat{\mathcal{A}}^{t} = \mathcal{I}_3$ with respect to $t \in \mathbb{R}$, we obtain

$$\widehat{\mathcal{A}}^{\prime}\widehat{\mathcal{A}}^{t} + (\widehat{\mathcal{A}}^{\prime}\widehat{\mathcal{A}}^{t})^{t} = 0, \tag{3.2}$$

where 0 is the 3×3 zero matrix. We deduce from Eq (3.2) the following identification:

$$\widehat{\boldsymbol{\psi}}(t) := \widehat{\mathcal{A}}^{\prime}\widehat{\mathcal{A}}^{t} = \begin{pmatrix} 0 & -\widehat{\psi}_3 & \widehat{\psi}_2 \\ \widehat{\psi}_3 & 0 & -\widehat{\psi}_1 \\ -\widehat{\psi}_2 & \widehat{\psi}_1 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \\ \widehat{\psi}_3 \end{pmatrix} = \widehat{\boldsymbol{\psi}}(t). \tag{3.3}$$

Consequently, we may write the vectors from $\mathcal{D}^3$ in two ways: as skew-symmetric 3×3 $\mathcal{D}$ matrices or as vectors. In what follows we will use both of these likelihood according to which of the two will be more useful in the specified case. Through the movement $\mathcal{K}_m/\mathcal{K}_f$, the differential velocity vector of a fixed $\mathcal{D}$ point $\widehat{\mathbf{x}}$ on $\mathcal{K}_m$, similar to the real spherical movement [1–5], is

$$\widehat{\mathbf{x}}^{\prime} = \widehat{\boldsymbol{\psi}} \times \widehat{\mathbf{x}}, \tag{3.4}$$

where

$$\widehat{\boldsymbol{\psi}}(t) = \boldsymbol{\psi}(t) + \varepsilon\boldsymbol{\psi}^{*}(t)$$

is the $\mathcal{D}$ screw or angular velocity vector of the movement $\mathcal{K}_m/\mathcal{K}_f$. $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^{*}$, respectively, are the instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of the spatial movement $\mathcal{H}_m/\mathcal{H}_f$.

Let

$$\widehat{\mathbf{r}} = \mathbf{r} + \varepsilon\mathbf{r}^{*}$$

be the $\mathcal{ISA}$ associated with $\widehat{\boldsymbol{\psi}}$. Then,

$$\widehat{\boldsymbol{\psi}} = \psi(1 + \varepsilon h)\widehat{\mathbf{r}}, \tag{3.5}$$

where h is the pitch of the movement $\mathcal{H}_m/\mathcal{H}_f$. By means of Eq (3.1), for $\widehat{\boldsymbol{\psi}}$ in $\mathcal{K}_f$, we have

$$\widehat{\boldsymbol{\psi}}_f = \widehat{\mathcal{A}}'\widehat{\mathcal{A}}^t = \begin{pmatrix} 0 & -\widehat{\psi}_{3f} & \widehat{\psi}_{2f} \\ \widehat{\psi}_{3f} & 0 & -\widehat{\psi}_{1f} \\ -\widehat{\psi}_{2f} & \widehat{\psi}_{1f} & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \widehat{\psi}_{1f} \\ \widehat{\psi}_{2f} \\ \widehat{\psi}_{3f} \end{pmatrix} = \widehat{\boldsymbol{\psi}}_f. \tag{3.6}$$

Once again, the expression of $\widehat{\boldsymbol{\psi}}_m$ in $\mathcal{K}_m$ follows from

$$\widehat{\boldsymbol{\psi}}_m = \widehat{\mathcal{A}}^t\widehat{\mathcal{A}}' = \begin{pmatrix} 0 & -\widehat{\psi}_{3m} & \widehat{\psi}_{2m} \\ \widehat{\psi}_{3m} & 0 & -\widehat{\psi}_{1m} \\ -\widehat{\psi}_{2m} & \widehat{\psi}_{1m} & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \widehat{\psi}_{1m} \\ \widehat{\psi}_{2m} \\ \widehat{\psi}_{3m} \end{pmatrix} = \widehat{\boldsymbol{\psi}}_m. \tag{3.7}$$

Therefore, we have [1–5]:

Definition 3.1. *For a one-parameter $\mathcal{D}$ spherical movement $\mathcal{K}_m/\mathcal{K}_f$, the following holds:*

(i) $\widehat{\boldsymbol{\psi}}_f(t) = \widehat{A}'\widehat{A}^t$ is coined the stationary directed cone.

(ii) $\widehat{\boldsymbol{\psi}}_m(t) = \widehat{A}^t\widehat{A}'$ is coined the movable directed cone.

(iii) $\widehat{\mathbf{r}}_m(t) = \widehat{\boldsymbol{\psi}}_m(t)\left\|\widehat{\boldsymbol{\psi}}_m(t)\right\|^{-1}$ is coined the movable polhode.

(iv) $\widehat{\mathbf{r}}_f(t) = \widehat{\boldsymbol{\psi}}_f(t)\left\|\widehat{\boldsymbol{\psi}}_f(t)\right\|^{-1}$ is coined the invariable polhode.

We will use the subscript i whenever either m or f can be used. This agreement implies that the same quantity must be utilized throughout the entire paper.

Theorem 3.2. *For the curvature functions of the polhodes, we have [10, 11]:*

$$\widehat{p}(t) = \left\|\widehat{\mathbf{r}}_m'\right\| = \left\|\widehat{\mathbf{r}}_f'\right\|. \tag{3.8}$$

Notice that the trajectory of the $\mathcal{ISA}$, which is consists of all oriented lines $\widehat{\mathbf{r}}_f(t)$, is coined the invariable axode. Analogously, $\widehat{\mathbf{r}}_m(t)$ is coined the movable axode $\mathcal{AX}$. From $\widehat{p} = \widehat{p}_i$, it follows that the movable and invariable $\mathcal{AX}$ osculate along the ruling line for every $t \in \mathbb{R}$, that is, the rulings of the $\mathcal{AX}$ gradually turn into one through $\mathcal{H}_m/\mathcal{H}_f$ and the tangent planes synchronize at the matching points. As the movement $\mathcal{H}_m/\mathcal{H}_f$ progresses, the movable $\mathcal{AX}$ rolls and slides over the $\mathcal{ISA}$ (see Figure 1). As an outcome, the following corollary can be specified.

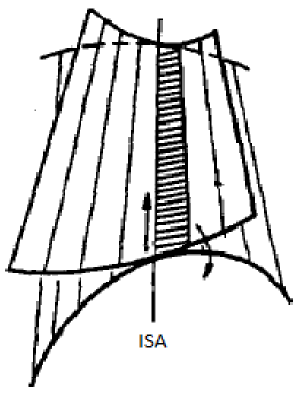

Figure 1. Typical portions of axodes.

Corollary 3.1. *At any instant t, through $\mathcal{H}_m/\mathcal{H}_f$, the π_m osculate with the π_f along the $\mathcal{ISA}$ in the 1st-order and their mutual distribution parameter is*

$$\mu(t) = \frac{p^*}{p}.$$

For further analysis, we recognize the relative Blaschke frame ($\mathcal{RBF}$) as follows:

$$\widehat{\mathbf{r}}_1(t) = \mathbf{r}_1(t) + \varepsilon\mathbf{r}_1^*(t),$$

which is the $\mathcal{ISA}$ for the movement $\mathcal{H}_m/\mathcal{H}_f$, and

$$\begin{aligned}\widehat{\mathbf{r}}_2(t) &:= \mathbf{r}_2(t) + \varepsilon\mathbf{r}_2^*(t)\\ &= \frac{d\mathbf{r}_1}{dt}\left\|\frac{d\mathbf{r}_1}{dt}\right\|^{-1}\end{aligned}$$

as the mutual central normal of $\widehat{\mathbf{r}}_1(t)$ and $\widehat{\mathbf{r}}_1(t+dt)$. A third $\mathcal{DUV}$ is realized as

$$\widehat{\mathbf{r}}_3(t) = \widehat{\mathbf{r}}_1 \times \widehat{\mathbf{r}}_2.$$

The set $\{\widehat{\mathbf{r}}_1(t), \widehat{\mathbf{r}}_2(t), \widehat{\mathbf{r}}_3(t)\}$ so realized will be coined $\mathcal{RBF}$, where $t \in \mathbb{R}$. It is fully appointed by the 1st-order ownerships of $\mathcal{H}_m/\mathcal{H}_f$. For the $\mathcal{RBF}$ with respect to $\mathcal{H}_i$ ($i = m, f$), we have

$$\begin{pmatrix}\widehat{\mathbf{r}}_1' \\ \widehat{\mathbf{r}}_2' \\ \widehat{\mathbf{r}}_3'\end{pmatrix}\Big|_i = \begin{pmatrix}0 & \widehat{p} & 0\\ -\widehat{p} & 0 & \widehat{q}_i\\ 0 & -\widehat{q}_i & 0\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3\end{pmatrix} \tag{3.9}$$

$$= \widehat{\omega}_i \times \begin{pmatrix}\widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3\end{pmatrix}, \quad ('= \frac{d}{dt}), \tag{3.10}$$

where

$$\widehat{\omega}_i(t) = \omega(t) + \varepsilon\omega^*(t) = \widehat{q}_i\widehat{\mathbf{r}}_1 + \widehat{p}\widehat{\mathbf{r}}_3$$

is the relative Darboux vector, and

$$\widehat{p}(t) = p(t) + \varepsilon p^*(t) = \left\|\widehat{\mathbf{r}}_1'\right\|, \quad \widehat{q}_i = q_i + \varepsilon q_i^* = \det(\widehat{\mathbf{r}}_1, \widehat{\mathbf{r}}_1', \widehat{\mathbf{r}}_1'') \tag{3.11}$$

are coined the Blaschke invariants. The tangent of the striction curve ($\mathcal{SC}$) on the $\mathcal{AX}$ is defined by [20–22]

$$\mathbf{c}'(t)\,|_i = \widehat{q}_i(t)\mathbf{r}_1(t) + p(t)\mathbf{r}_3(t). \tag{3.12}$$

On the other hand, due to the spatial three-pole-theorem, we gain the instantaneous screw of $\mathcal{H}_m/\mathcal{H}_f$ as

$$\widehat{\omega}(t) = \widehat{\omega}_f(t) - \widehat{\omega}_m(t). \tag{3.13}$$

The authenticity of this equation is shown in [10, 11]. Therefore,

$$\widehat{\omega}(t) = \omega(t)\widehat{\mathbf{r}}_1 \quad \text{with} \quad \widehat{\omega}(t) := \omega(t) + \varepsilon\omega^*(t) = \widehat{q}_f(t) - \widehat{q}_m(t). \tag{3.14}$$

It is worthy to note that $\widehat{\omega}(t)$ is the $\mathcal{D}$ angular speed of $\mathcal{H}_m/\mathcal{H}_f$. In our mission, we shall set that $\omega^* \neq 0$ to cancel out the pure translational movement. Also, we expel zero divisors $\omega = 0$. Therefore, we work with only non-torsional axodes.

Furthermore, for $\widehat{p}(t) \neq 0$, the relative $\mathcal{D}$ geodesic curvature $\widehat{\gamma}(t)$ of π_i is

$$\widehat{\gamma}(t) := \gamma + \varepsilon(\Gamma - \mu\gamma) = \frac{\widehat{\omega}(t)}{\widehat{p}(t)}, \tag{3.15}$$

where

$$\gamma(t) = \gamma_f(t) - \gamma_m(t), \quad \Gamma(t) = \Gamma_f(t) - \Gamma_m(t), \quad \mu(t) = \frac{p^*(t)}{\widehat{p}(t)}. \tag{3.16}$$

$\gamma(t)$, $\Gamma(t)$, and $\mu(t)$ are coined the relative construction functions of the $\mathcal{AX}$. They are all invariant of the kinematic group and characterize fully the local forms of π_i.

Corollary 3.2. *For all instant $t \in \mathbb{R}$, through $\mathcal{H}_m/\mathcal{H}_f$, the pitch can be written as*

$$\begin{aligned} h(t) &:= \frac{<\boldsymbol{\omega}^*, \boldsymbol{\omega}>}{\|\boldsymbol{\omega}\|^2} \\ &= \frac{\Gamma_f(t) - \Gamma_m(t)}{\gamma_f(t) - \gamma_m(t)}. \end{aligned} \tag{3.17}$$

Furthermore, the $\mathcal{DUV}$

$$\begin{aligned} \widehat{\mathbf{b}}_i(t) &:= \mathbf{b}_i(t) + \varepsilon\mathbf{b}_i^*(t) \\ &= \frac{\widehat{\boldsymbol{\omega}}_i}{\left\|\widehat{\boldsymbol{\omega}}_i\right\|} = \frac{\widehat{q}_i}{\sqrt{\widehat{q}_i^2 + \widehat{p}^2}}\widehat{\mathbf{r}}_1 + \frac{\widehat{p}}{\sqrt{\widehat{q}_i^2 + \widehat{p}^2}}\widehat{\mathbf{r}}_3 \end{aligned} \tag{3.18}$$

is the Disteli-axis of π_i. Let $\widehat{\phi}_i = \phi_i + \varepsilon\phi_i^*$ be the $\mathcal{D}$ radii of curvature among $\widehat{\mathbf{b}}_i$ and $\widehat{\mathbf{r}}_1$. Then,

$$\widehat{\mathbf{b}}_i(t) = \cos\widehat{\phi}\widehat{\mathbf{r}}_1 + \sin\widehat{\phi}\widehat{\mathbf{r}}_3, \tag{3.19}$$

where

$$\cot\widehat{\phi}_i := \cot\phi_i - \varepsilon\phi_i^*(1 + \cot^2\phi_i) = \frac{\widehat{q}_i}{\widehat{p}}. \tag{3.20}$$

Consider two infinitesimally spaced rulings $\widehat{\mathbf{r}}_1(t)$ and $\widehat{\mathbf{r}}_1(t + dt)$. These two rulings are separated by a $\mathcal{D}$ arc-length

$$d\widehat{u} := du + \varepsilon du^* = \left\|\frac{d\mathbf{r}_1}{dt}\right\| dt = \widehat{p}(t)dt. \tag{3.21}$$

From now on, we will take the $\mathcal{D}$ arc length $\widehat{u}$ instead of $t \in \mathbb{R}$. Then $\widehat{\mathbf{r}}_1(\widehat{u})$ is coined a $\mathcal{D}$ arc-length parameter curve. From now on, we shall often not write the $\mathcal{D}$ parameter $\widehat{u}$ explicitly in our formulae.

Let $\{\widehat{\mathbf{t}}_i, \widehat{\mathbf{n}}_i, \widehat{\mathbf{b}}_i\}$ be the mobile Serret-Frenet frame ($\mathcal{SFF}$) along $\widehat{\mathbf{r}}_1(\widehat{u})$. Then,

$$\mathbf{t} + \varepsilon\mathbf{t}^* = \widehat{\mathbf{t}}, \quad \mathbf{n}_i + \varepsilon\mathbf{n}_i^* = \widehat{\mathbf{n}}_i$$

and

$$\mathbf{b}_i + \varepsilon\mathbf{b}_i^* = \widehat{\mathbf{b}}_i$$

are the unit tangent, unit principal normal, and unit binormal vectors of $\widehat{\mathbf{r}}_1(\widehat{u})$. The arc-length derivative of the $\mathcal{SFF}$ is

$$\begin{pmatrix} \dot{\widehat{\mathbf{t}}} \\ \dot{\widehat{\mathbf{n}}}_i \\ \dot{\widehat{\mathbf{b}}}_i \end{pmatrix} = \begin{pmatrix} 0 & \widehat{\kappa_i} & 0 \\ -\widehat{\kappa_i} & 0 & \widehat{\tau_i} \\ 0 & -\widehat{\tau_i} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{t}} \\ \widehat{\mathbf{n}}_i \\ \widehat{\mathbf{b}}_i \end{pmatrix}, \tag{3.22}$$

where

$$\begin{pmatrix} \widehat{\mathbf{t}} \\ \widehat{\mathbf{n}}_i \\ \widehat{\mathbf{b}}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sin\widehat{\phi}_i & 0 & \cos\widehat{\phi}_i \\ \cos\widehat{\phi}_i & 0 & \sin\widehat{\phi}_i \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3 \end{pmatrix}. \tag{3.23}$$

One can display that

$$\left.\begin{aligned} &\widehat{\gamma_i}(\widehat{u}) = \gamma_i + \varepsilon(\Gamma_i - \gamma_i\mu) = \cot\phi - \varepsilon\phi_i^*(1+\cot^2\phi_i), \\ &\widehat{\kappa_i}(\widehat{u}) := \kappa_i + \varepsilon\kappa_i^* = \sqrt{1+\widehat{\gamma}^2} = \tfrac{1}{\sin\widehat{\phi}} = \tfrac{1}{\widehat{\rho_i}(\widehat{u})}, \\ &\widehat{\tau_i}(\widehat{u}) := \tau_i + \varepsilon\tau_i^* = \pm\widehat{\phi}_i' = \pm\tfrac{\widehat{\gamma}_i'}{1+\widehat{\gamma}_i^2} = \tfrac{1}{\widehat{\sigma_i}(\widehat{u})}, \end{aligned}\right\} \tag{3.24}$$

where

$$\widehat{\rho_i} = \rho_i + \varepsilon\rho_i^*$$

and

$$\widehat{\sigma_i} = \sigma_i + \varepsilon\sigma_i^*$$

are the $\mathcal{D}$ radii of curvature and the $\mathcal{D}$ torsion of π_i, respectively.

3.1. Geometrical-kinematical properties of the $\mathcal{AX}$

In this subsection, we consider geometrical-kinematical properties of π_i as follows: Note that the mutual central normal $\widehat{\mathbf{t}}_i(\widehat{\mathbf{r}}_2)$ of π_i is linked with $\widehat{\mathbf{r}}_1$. Its time derivative in $\mathcal{H}_f$ can be derived as [1–5]

$$\widehat{\mathbf{t}}' \,|_f = \widehat{\mathbf{t}}' \,|_m + \widehat{\omega} \times \widehat{\mathbf{t}}, \tag{3.25}$$

where $\widehat{\mathbf{t}}' \,|_m$denotes the time derivative of $\widehat{\mathbf{t}}$ calculated in $\mathcal{H}_m$. Direct computation gives

$$\widehat{\mathbf{t}}' \,|_i = \widehat{p}\dot{\widehat{\mathbf{t}}} \,|_i = \widehat{p\kappa_i}\widehat{\mathbf{n}}_i. \tag{3.26}$$

Considering Eq (3.25) with this, then

$$\widehat{\omega} \times \widehat{\mathbf{t}} = \widehat{p}(\widehat{\kappa}_f\widehat{\mathbf{n}}_f - \widehat{\kappa}_m\widehat{\mathbf{n}}_m). \tag{3.27}$$

It is easily seen from Eq (3.14) that

$$\begin{aligned} < \widehat{\omega}, \widehat{\mathbf{t}} > &= \widehat{\omega} < \widehat{\omega}, \widehat{\mathbf{r}}_1 \\ > &= \frac{\widehat{\omega}}{2}\frac{d}{dt} < \widehat{\mathbf{r}}_1, \widehat{\mathbf{r}}_1 \\ > &= \frac{\widehat{\omega}}{2}\frac{d}{dt}\left\|\widehat{\mathbf{r}}_1\right\|^2 = 0. \end{aligned} \tag{3.28}$$

Hence, we have

$$\begin{aligned}\widehat{\omega}\times(\widehat{\omega}\times\widehat{\mathbf{t}}) &= \left\|\widehat{\mathbf{t}}\right\|^{2}\widehat{\omega}- < \widehat{\omega},\widehat{\mathbf{t}} > \widehat{\mathbf{t}}\\ &= \widehat{\omega}.\end{aligned} \tag{3.29}$$

With the aid of Eqs (3.27) and (3.29), we get

$$\begin{aligned}\widehat{\omega} &= \widehat{\mathbf{t}}\times[\widehat{p}(\widehat{\kappa}_f\widehat{\mathbf{n}}_f-\widehat{\kappa}_m\widehat{\mathbf{n}}_m)]\\ &= \widehat{p}(\widehat{\kappa}_f\widehat{\mathbf{t}}\times\widehat{\mathbf{n}}_f-\widehat{\kappa}_m\widehat{\mathbf{t}}\times\widehat{\mathbf{n}}_m)\\ &= \widehat{p}(\widehat{\kappa}_f\widehat{\mathbf{b}}_f-\widehat{\kappa}_m\widehat{\mathbf{b}}_m).\end{aligned}$$

Theorem 3.3. *For all instant $t\in\mathbb{R}$, through $\mathcal{H}_m/\mathcal{H}_f$, the $\mathcal{D}$ angular velocity is located by the geometry of the axodes and the $\mathcal{D}$ speed of contact due to the equation*

$$\widehat{\omega} = \widehat{p}(\widehat{\kappa}_f\widehat{\mathbf{b}}_f-\widehat{\kappa}_m\widehat{\mathbf{b}}_m). \tag{3.30}$$

It follows from Eqs (3.14) and (3.30) that

$$\widehat{\gamma}\widehat{\mathbf{r}}_1 = \widehat{\kappa}_f\widehat{\mathbf{b}}_f-\widehat{\kappa}_m\widehat{\mathbf{b}}_m. \tag{3.31}$$

Then, from Eqs (3.23) and (3.31), we obtain

$$\cot\widehat{\phi}_f-\cot\widehat{\phi}_m = \widehat{\gamma}_f-\widehat{\gamma}_m. \tag{3.32}$$

This is the $\mathcal{D}$ version of a well-known formula of Euler-Savary from ordinary spherical movements (compared with [1–5]). This version furnishes an engagement for the two $\mathcal{AX}$ in $\mathcal{H}_m/\mathcal{H}_f$. From the real and $\mathcal{D}$ parts of Eq (3.32), respectively, we locate

$$\cot\phi_f-\cot\phi_m = \gamma_f-\gamma_m \tag{3.33}$$

and

$$\frac{\phi_m^*}{\sin^2\phi_f}-\frac{\phi_f^*}{\sin^2\phi_m}+\mu(\cot\phi_f-\cot\phi_m) = \Gamma_f-\Gamma_m. \tag{3.34}$$

Equations (3.33) and (3.34) are new Disteli formulae of spatial movements for the $\mathcal{AX}$. At the same time, Eq (3.33) is a formula of Euler-Savary for the polodes of real spherical movements.

Theorem 3.4. *For all instant $t\in\mathbb{R}$, through $\mathcal{H}_m/\mathcal{H}_f$, the $\mathcal{D}$ angular acceleration is located by the geometry of the $\mathcal{AX}$, the $\mathcal{D}$ speed of contact, and the rate of change of the speed due to the equation*

$$\widehat{\mathbf{a}} = \widehat{\omega}^{'}\mid_f = \widehat{\mathbf{a}}_1+\widehat{\mathbf{a}}_2+\widehat{\mathbf{a}}_3,$$

where

$$\begin{aligned}\widehat{\mathbf{a}}_1 &= \widehat{p}^{'}(\widehat{\kappa}_f\widehat{\mathbf{b}}_f-\widehat{\kappa}_m\widehat{\mathbf{b}}_m),\\ \widehat{\mathbf{a}}_2 &= \widehat{p}^{2}(\dot{\widehat{\kappa}}_f\widehat{\mathbf{b}}_f-\dot{\widehat{\kappa}}_m\widehat{\mathbf{b}}_m),\\ \widehat{\mathbf{a}}_3 &= \widehat{p}^{2}(\widehat{\kappa}_m\widehat{\tau}_m\widehat{\mathbf{n}}_m-\widehat{\kappa}_f\widehat{\tau}_f\widehat{\mathbf{n}}_f)+\widehat{p}\widehat{\omega}\widehat{\mathbf{t}}.\end{aligned}$$

Proof. Differentiating Eq (3.30) with respect to t we obtain

$$\widehat{\mathbf{a}} = \widehat{p}'(\widehat{\kappa}_f\widehat{\mathbf{b}}_f - \widehat{\kappa}_m\widehat{\mathbf{b}}_m) + \widehat{p}^2(\dot{\widehat{\kappa}_f}\widehat{\mathbf{b}}_f - \dot{\widehat{\kappa}_m}\widehat{\mathbf{b}}_m) - \widehat{p}\widehat{\kappa}_m\widehat{\mathbf{b}}'_m\,|_f. \tag{3.35}$$

Making use of the fundamental relation of relative movement

$$\begin{aligned}\widehat{\mathbf{b}}'_m\,|_f &= \widehat{\mathbf{b}}'_m\,|_m + \widehat{\omega}\times\widehat{\mathbf{b}}_m\\ &= \widehat{p}\dot{\widehat{\mathbf{b}}}_m + \widehat{\omega}\widehat{\mathbf{r}}_1\times\widehat{\mathbf{b}}_m\\ &= \widehat{p}\dot{\widehat{\mathbf{b}}}_m - \frac{\widehat{\omega}}{\widehat{\kappa}_m}\widehat{\mathbf{t}}.\end{aligned} \tag{3.36}$$

Substituting Eq (3.36) into Eq (3.35) completes the proof. □

It can be shown that the tangent to the $\mathcal{SC}$ is

$$\mathbf{c}'(u)\,|_i = \widehat{\Gamma}_i\mathbf{r}_1 + \mu(u)\mathbf{r}_3, \quad (' = \frac{d}{du}). \tag{3.37}$$

The curvature κ_i and torsion τ_i of the $\mathcal{SC}$ of π_i can be offered, respectively, by

$$\begin{aligned}\kappa_i &= \frac{\left\|\mathbf{c}'\times\mathbf{c}''\right\|}{\|\mathbf{c}'\|^3}\,|_i\\ &= \frac{1}{(\Gamma_i^2+\mu^2)}\sqrt{(\Gamma_i+\mu\gamma_i)^2 - (\Gamma_i\mu' - \mu\Gamma_i')^2}\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}\tau_i(u) &:= \frac{\det(\mathbf{c}',\mathbf{c}'',\mathbf{c}''')}{\left\|\mathbf{c}'\times\mathbf{c}''\right\|^2}\,|_i\\ &= \frac{\mu+\gamma_i\Gamma_i}{\Gamma_i^2+\mu^2} - \frac{d}{du}(\cot^{-1}\Gamma_i\mu' - \mu\Gamma_i').\end{aligned} \tag{3.39}$$

Letting β_i be the arc length of the $\mathcal{SC}$ of π_i, it follows that

$$d\beta_i = \left\|\mathbf{c}_i'\right\|du = \sqrt{\Gamma_i^2+\mu^2}\,du. \tag{3.40}$$

Making use of the results in [13], for the $\mathcal{SC}$ we may state the following:

Theorem 3.5. *Through $\mathcal{H}_m/\mathcal{H}_f$, the $\mathcal{SC}$ of π_i lies on a sphere of radius $\sqrt{a^2+b^2}$ iff*

$$\frac{1}{\kappa_i} = a\cos\theta + b\sin\theta, \quad \theta = \int_0^{\beta_i}\tau_i d\beta_i,$$

where a and b are constants.

Now, we will examine how the invariants vary when the variable t (or u) varies. Let $\mathbf{y}_i$ denote a point on π_i. Then,

$$\pi_i : \mathbf{y}(u,v)|_i = \mathbf{c}(u)\ |_i + v\mathbf{r}_1(u),\ \ v \in \mathbb{R}. \tag{3.41}$$

The unit normal vector at any point $\mathbf{y}_i(u,v)$ is

$$\mathbf{g}(u)\ |_i = \frac{\mathbf{y}_u \times \mathbf{y}_v}{\|\mathbf{y}_u \times \mathbf{y}_v\|} = \frac{\mu\mathbf{r}_2 - v\mathbf{r}_3}{\sqrt{\mu^2 + v^2}}, \quad \mathbf{y}_t = \frac{\partial \mathbf{y}}{\partial t}. \tag{3.42}$$

The 1st fundamental form $\mathcal{I}$ of π_i is

$$\mathcal{I} = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2, \tag{3.43}$$

where

$$g_{11} = \|\mathbf{y}_u\|^2 = \Gamma_i^2 + \mu^2, \quad g_{12} = \|\mathbf{y}_v\|^2 = \Gamma_i, \quad g_{22} = < \mathbf{y}_v, \mathbf{y}_v >= 1. \tag{3.44}$$

The 2nd fundamental form $\mathcal{II}$ is

$$\mathcal{II} = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2,$$

where h_{11}, h_{12}, and h_{22} are

$$h_{11} = \frac{(\Gamma_i^{'} - \gamma_i)\mu + (\mu^{'} + \gamma^{'} v)}{\sqrt{\mu^2 + v^2}}, \quad h_{12} = \frac{-\mu}{\sqrt{\mu^2 + v^2}}, \quad h_{22} = 0. \tag{3.45}$$

The Gaussian curvature $\mathcal{K}$ and the mean curvature $\mathcal{H}_i$, respectively, are

$$\mathcal{K}_i(u,v) := \mathcal{K}(u,v) = -\frac{\mu^2}{(\mu^2 + v^2)^2} \tag{3.46}$$

and

$$\mathcal{H}_i(u,v) = \frac{(\mu^2 + v^2)\gamma_i + \mu^{'} v + \mu\Gamma_i}{2\,(\mu^2 + v^2)^{\frac{3}{2}}}. \tag{3.47}$$

Hence, we have the following theorem:

Theorem 3.6. *Through $\mathcal{H}_m/\mathcal{H}_f$, the following holds:*
(1) The mean curvatures of the $\mathcal{AX}$ are related as follows:

$$\mathcal{H}_f(u,v) - \mathcal{H}_m(u,v) = \frac{\gamma}{2(\mu^2 + v^2)} + \frac{\mu\Gamma}{2\,[\mu^2 + v^2]^{\frac{3}{2}}}.$$

(2) The mutual Gaussian curvature of the $\mathcal{AX}$ satisfies

$$\mathcal{K}(u,v) + \frac{\sqrt{-\mathcal{K}(u,v)}}{\mu} = \frac{1}{16}\left(\frac{1}{\mathcal{K}(u,v)}\frac{\partial \mathcal{K}(u,v)}{\partial v^2}\right).$$

Due to the values of Γ_i and μ in Eq (36), the geometric characterizations of π_i are as follows:

(a) If $\Gamma_i = 0$, that is, the π_i are binormal surfaces, then

$$\mathbf{c}^{'}(u)\mid_i = \mu(u)\mathbf{r}_3 \tag{3.48}$$

and

$$\left.\begin{array}{l}\kappa_i(u) = \frac{\gamma_i}{\mu}, \quad \mu(u) = \gamma_i(u)(a\cos u + b\sin u) = \frac{1}{\tau_i}, \\ \mathcal{H}_f(u,v) - \mathcal{H}_m(u,v) = \frac{\gamma}{2(\mu^2+v^2)}.\end{array}\right\} \tag{3.49}$$

Hence, we deduce a geometric meaning of μ which is the radii of torsion of the spherical curve $\mathbf{c}(u)$. Further, if $\mu(u)$ is a constant, the π_i are binormal surfaces of a spherical curve of constant torsion.

(b) If $\mu(u) = 0$, that is, the π_i are tangential surfaces, then

$$\mathbf{c}^{'}(u)\mid_i = \Gamma_i(u)\mathbf{r}_1 \tag{3.50}$$

and

$$\left.\begin{array}{l}\Gamma_i(u) = \frac{1}{\kappa_i} = a\cos\int\limits_0^u \gamma_i d\beta_i + b\sin\int\limits_0^u \gamma_i d\beta_i, \ \tau_i(u) = \frac{\gamma_i}{\Gamma_i}, \\ \mathcal{K}_f(u,v) = \mathcal{K}_m(u,v) = 0, \quad \mathcal{H}_f(u,v) - \mathcal{H}_m(u,v) = \frac{\gamma}{2(\mu^2+v^2)}.\end{array}\right\} \tag{3.51}$$

Here $\Gamma_i(u)$ is the radii of curvature of the spherical curve $\mathbf{c}(u)$. Similarly, when $\Gamma_i(u)$ is stationary, the π_i are tangential surfaces of a spherical curve of invariable curvature.

(c) If $\Gamma_i = \mu = 0$, then π_i are circular cones. Then, the $\mathcal{SC}$ degenerates to a point, that is, $\mathbf{c}^{'}(u)\mid_i = \mathbf{0}$. Here, we have

$$\left.\begin{array}{l}\mathcal{K}_f(u,v) = \mathcal{K}_m(u,v) = 0, \\ \mathcal{H}_f(u,v) - \mathcal{H}_m(u,v) = \frac{\gamma}{2v}(\cot\phi_f - \cot\phi_m).\end{array}\right\} \tag{3.52}$$

Hence, from Eqs (3.44) and (3.45), it follows that the π_i are circular cones iff its parametric curves are curvature lines ($g_{12} = h_{12} = 0$).

3.2. Example

Let us explain the above compensations on a straightforward example. Consider the two-parameter dual spherical movement $\mathcal{K}_m/\mathcal{K}_f$ explained by the $\mathcal{DM}$

$$\widehat{\mathcal{A}}(\widehat{\vartheta}) = \begin{pmatrix} \cos^2\widehat{\vartheta} & \sin\widehat{\vartheta} & \sin\widehat{\vartheta}\cos\widehat{\vartheta} \\ -\sin\widehat{\vartheta}\cos\widehat{\vartheta} & \cos\widehat{\vartheta} & -\sin^2\widehat{\vartheta} \\ -\sin\widehat{\vartheta} & 0 & \cos\widehat{\vartheta} \end{pmatrix} \quad \text{with} \quad \widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*. \tag{3.53}$$

Upon substituting into expression (3.6) for $\widehat{\psi}_f$, we attain

$$\widehat{\boldsymbol{\psi}}_f(\widehat{\vartheta}) = \frac{d\widehat{\mathcal{A}}}{d\widehat{\vartheta}}\widehat{\mathcal{A}}^t = \begin{pmatrix} 0 & 1 & \cos\widehat{\vartheta} \\ -1 & 0 & -\sin\widehat{\vartheta} \\ -\cos\widehat{\vartheta} & \sin\widehat{\vartheta} & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \sin\widehat{\vartheta} \\ \cos\widehat{\vartheta} \\ -1 \end{pmatrix} = \widehat{\boldsymbol{\psi}}_f.$$

Similarly, we attain

$$\widehat{\boldsymbol{\psi}}_m(\widehat{\vartheta}) = \widehat{\mathcal{A}}^t \frac{d\widehat{\mathcal{A}}}{d\widehat{\vartheta}} = \begin{pmatrix} 0 & \cos\widehat{\vartheta} & 1 \\ -\cos\widehat{\vartheta} & 0 & -\sin\widehat{\vartheta} \\ -1 & \sin\widehat{\vartheta} & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \sin\widehat{\vartheta} \\ 1 \\ -\cos\widehat{\vartheta} \end{pmatrix} = \widehat{\boldsymbol{\psi}}_m.$$

Then, we find

$$\begin{aligned} \pi_f : \widehat{\mathbf{r}}_f(\widehat{\vartheta}) &= \widehat{\boldsymbol{\psi}}_f(\widehat{\vartheta}) \left\| \widehat{\boldsymbol{\psi}}_f(\widehat{\vartheta}) \right\|^{-1} \\ &= \frac{1}{\sqrt{2}}(\sin\widehat{\vartheta}\widehat{\mathbf{f}}_1 + \cos\widehat{\vartheta}\widehat{\mathbf{f}}_2 - \widehat{\mathbf{f}}_3), \\ \pi_m : \widehat{\mathbf{r}}_m(\widehat{\vartheta}) &= \widehat{\boldsymbol{\psi}}_m(\widehat{\vartheta}) \left\| \widehat{\boldsymbol{\psi}}_m(\widehat{\vartheta}) \right\|^{-1} \\ &= \frac{1}{\sqrt{2}}(\sin\widehat{\vartheta}\widehat{\mathbf{f}}_1 + \widehat{\mathbf{f}}_2 - \cos\widehat{\vartheta}\widehat{\mathbf{f}}_3). \end{aligned} \tag{3.54}$$

Equation (3.54) has only two real parameters ϑ and ϑ^*. Thus, if we choose $\vartheta^* = h\vartheta$, h indicating the pitch of $\mathcal{H}_m/\mathcal{H}_f$ and ϑ as the movement parameter, then Eq (3.54) represents the axodes π_i. It is easily ascertained that

$$\widehat{\mathbf{r}}_f(0) = \widehat{\mathbf{r}}_m(0) = \frac{1}{\sqrt{2}}(\widehat{\mathbf{f}}_2 - \widehat{\mathbf{f}}_3).$$

Hence, π_f and π_m contact along the $\mathcal{ISA}$ at the point $\vartheta = 0$. Consequently, the assumptions of Corollary 3.1 are satisfied. Thus, the Blaschke frame of the invariable axode π_f is

$$\begin{pmatrix} \widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3 \end{pmatrix}\Big|_f = \begin{pmatrix} \frac{\sin\widehat{\vartheta}}{\sqrt{2}} & \frac{\cos\widehat{\vartheta}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \cos\widehat{\vartheta} & -\sin\widehat{\vartheta} & 0 \\ -\frac{\sin\widehat{\vartheta}}{\sqrt{2}} & -\frac{\cos\widehat{\vartheta}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{f}}_1 \\ \widehat{\mathbf{f}}_2 \\ \widehat{\mathbf{f}}_3 \end{pmatrix}. \tag{3.55}$$

If we differentiate these expressions, we find

$$\widehat{p}_f(\vartheta) = \frac{1}{\sqrt{2}}(1 + \varepsilon h), \quad \widehat{q}_f(\vartheta) = \frac{1}{\sqrt{2}}(1 + \varepsilon h). \tag{3.56}$$

For the movable axode π_m, similar discussions show that

$$\begin{pmatrix} \widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3 \end{pmatrix}\Big|_m = \begin{pmatrix} \frac{\sin\widehat{\vartheta}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\cos\widehat{\vartheta}}{\sqrt{2}} \\ \cos\widehat{\vartheta} & 0 & \sin\widehat{\vartheta} \\ \frac{\sin\widehat{\vartheta}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{\cos\widehat{\vartheta}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{f}}_1 \\ \widehat{\mathbf{f}}_2 \\ \widehat{\mathbf{f}}_3 \end{pmatrix}. \tag{3.57}$$

Consequently, we obtain

$$\begin{aligned} \widehat{p}_m(\vartheta) &= -\widehat{q}_m(\vartheta) \\ &= \frac{1}{\sqrt{2}}(1 + \varepsilon h). \end{aligned} \tag{3.58}$$

Hence, combining Eqs (3.56) and (3.58), we have

$$\widehat{p}_f(\vartheta) = \widehat{p}_m(\vartheta), \quad \widehat{q}_f(\vartheta) - \widehat{q}_m(\vartheta) = \sqrt{2}(1 + \varepsilon h). \tag{3.59}$$

which is in a total harmonization with the presumptions of Theorem 3.2. As we see from these equations, the movable axode π_m touches the stationary axode π_f along the $\mathcal{ISA}$ in the 1st-order at any instant $\vartheta \in \mathbb{R}$.

Now, we may calculate the equation of π_i in terms of the point coordinates. Let $\mathbf{y}_i$ be a point on π_i. We can write

$$\pi_i : \mathbf{y}_i(\vartheta, v) = \mathbf{r}_1(\vartheta) \times \mathbf{r}_1^*(\vartheta) + v\mathbf{r}_1(\vartheta), \ (i = f, \ m), \ v \in \mathbb{R}. \tag{3.60}$$

Into Eq (3.60) we substitute from Eqs (3.55) and (3.57) to obtain

$$\pi_f : \mathbf{y}_f(\vartheta, v) = \frac{h\vartheta}{2}(\sin\vartheta, -\cos\vartheta, 1) + \frac{v}{\sqrt{2}}(\sin\vartheta, \cos\vartheta, -1)$$

and

$$\pi_m : \mathbf{y}_m(\vartheta, v) = \frac{h\vartheta}{2}(\sin\vartheta, 1, -\cos\vartheta) + \frac{v}{\sqrt{2}}(\sin\vartheta, 1, -\cos\vartheta).$$

For $h = \sqrt{2}$, $0 \leq \vartheta \leq 2\pi$, $-1 \leq v \leq 1$, and the stationary (movable) axode π_f (π_m) is shown in Figures 2 and 3. The graphs of the movable and stationary axodes are shown in Figure 4.

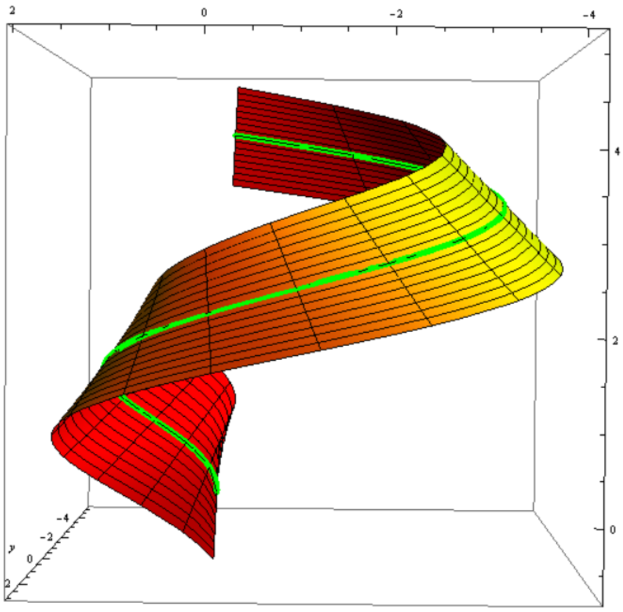

Figure 2. The stationary axode π_f.

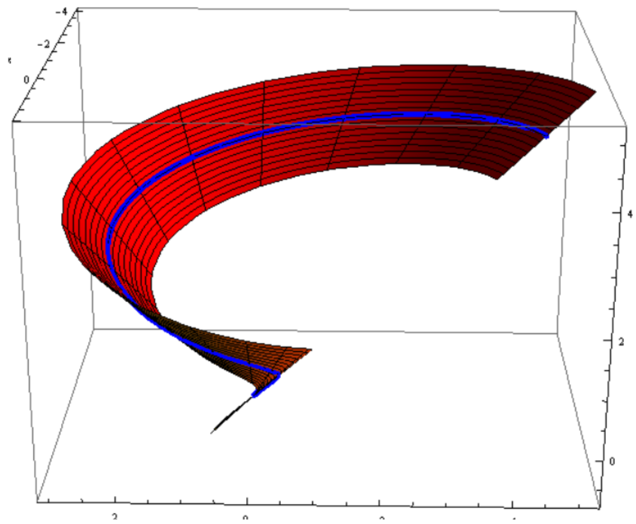

Figure 3. The movable axode π_m.

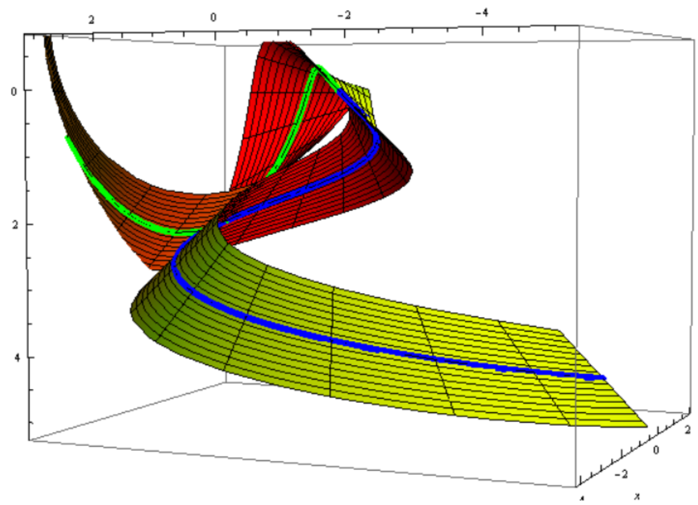

Figure 4. The movable and stationary $\mathcal{AX}$.

4. Conclusions

Some relations are derived for the spatial movement with one parameter. The geometrical properties of spatial movement are derived in the geometrical data of the axodes. The approach applied does not use the tools of instantaneous spherical kinematics [1]. The method offered is based on the E. Study map and dual vector calculus discussed in [2–4, 9–11]. Several theorems, including the dual version of the planer Euler-Savary equation, are obtained, which characterize kinematical and geometrical properties of the movement. Geometrical type relations such as in Theorem 3.6, can be considered as a form of Euler-Savary equation for the axodes. One example shows how we can use the derived formulae to determine the kinematic-geometric properties of the axodes. Take, for instance, rotations, which are essential for a lot of fields, from astronomy (movement of planets) to chemistry (movement of electrons, rotation of molecules). Eigenvector/eiganvalue/eigenproblem approaches may bring light to some difficult problems. Our future research will focus on exploring some implementations of our major findings. We plan to consolidate notions from singularity theory, submanifold theory, and other pertinent results (referenced in [24, 25]) to research favorable avenues within this article.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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