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Research article

Anisotropic Moser-Trudinger type inequality in Lorentz space

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Abstract: Our main purpose in this paper is to obtain the anisotropic Moser-Trudinger type inequality in Lorentz space $L(n, q)$, $1 \leq q \leq \infty$. It can be seen as a generation result of the Moser-Trudinger type inequality in Lorentz space.

Keywords: anisotropic; Moser-Trudinger inequality; Lorentz space
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1. Introduction

Let Ω be a domain with finite measure in Euclidean n-space $\mathbb{R}^n$ with $n \geq 2$. When $1 \leq p < n$, by the Sobolev embedding theorem, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$, $1 \leq q \leq \frac{np}{n-p}$. Moreover, for the critical situation $p = n$, $W_0^{1,n}(\Omega) \subset L^q(\Omega)$, $\forall q \geq 1$. However, we can show by many examples that $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$ [1,2]. For the anisotropic Sobolev inequalities, we refer to [3–6].

In 1971, Moser [2] established Trudinger's inequality

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx \leq C, \tag{1.1}$$

for any $\alpha \leq \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the surface of the unit n-ball. This constant α_n is sharp in the sense that, if $\alpha > \alpha_n$, then the above inequality (1.1) can no longer hold with some C independent of u.

Furthermore, Alvino, Ferone, and Trombetti [7] proved the following Moser-Trudinger type inequality in Lorentz space. They obtained that if

$$\|\nabla u\|_{n,q} \leq 1, \quad 1 < q < \infty, \tag{1.2}$$

then there exists a constant C, depending only on n and q, such that

$$\int_\Omega e^{\beta |u(x)|^{q'}} dx \leq C|\Omega|, \quad \forall \beta \leq \beta_q = (nC_n^{\frac{1}{n}})^{q'}, \tag{1.3}$$

where q' is the conjugate index of q, i.e., $q' = \frac{q}{q-1}$ and C_n is the measure of unit ball in $\mathbb{R}^n$, and the constant β_q is sharp.

There have been many generalizations related to the Moser-Trudinger inequality, see [1, 8–18], etc. These inequalities play a key role in Geometry analysis, calculus of variations and PDEs, see [19–27], etc.

Recently, many authors have intended to establish the Moser-Trudinger type inequality under the anisotropic norm. Let $F \in C^2(\mathbb{R}^n \backslash \{0\})$ be a positive, convex, and homogeneous function, and the polar $F^o(x)$ of which represents a Finsler metric on $\mathbb{R}^n$. By calculating the Euler-Lagrange equation of the minimization problem

$$\min_{u \in W_0^{1,n}(\Omega)} \int_\Omega F^p(\nabla u) dx,$$

we obtain an operator which is called Finsler p-Laplacian operator:

$$\Delta_F u := \sum_{i=1}^n \frac{\partial}{\partial x_i} (F^{p-1}(\nabla u) F_{\xi_i}(\nabla u)).$$

The Finsler p-Laplacian becomes the standard p-Laplacian when F is the Euclidean modulus, as well as the pseudo-p-Laplacian when $F(\xi) = (\sum_{i=1}^n |\xi_i|^p)^{\frac{1}{p}}$. The Finsler p-Laplacian operator has been studied in several papers, see [28–35], etc. More properties of $F(x)$ will be given in Section 2.

The first work involving the anisotropic Moser-Trudinger type inequality was that of Wang and Xia [35]. They replaced the Dirichlet norm $(\int_\Omega |\nabla u|^n dx)^{\frac{1}{n}}$ with the anisotropic norm $(\int_\Omega F^n(\nabla u) dx)^{\frac{1}{n}}$ and proved the following inequality:

$$\sup_{u \in W_0^{1,n}(\Omega), \int_\Omega F^n(\nabla u) dx \le 1} \int_\Omega e^{\lambda |u|^{\frac{n}{n-1}}} dx \le C,$$

where $\lambda \le \lambda_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$, $\kappa_n = |x \in \mathbb{R}^n | F^o(x) \le 1|$ is the volume of the unit Wulff ball in $\mathbb{R}^n$, and the constant λ_n is sharp. Clearly, this is a generation result of (1.1).

Along this line, in this paper we consider the anisotropic Moser-Trudinger type inequality in Lorentz space $L(n, q)$, $1 \le q \le \infty$. The definition and properties of Lorentz space can be seen in Section 2. Now, we state main results in the paper.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \ge 2$, and let $u \in W_0^{1,n}(\Omega)$ be a function such that*

$$\|F(\nabla u)\|_{n,q} \le 1, \quad 1 \le q \le \infty. \tag{1.4}$$

We conclude that:

(i) If $q = 1$, then

$$\|u\|_\infty \le \frac{1}{n \kappa_n^{\frac{1}{n}}} \|F(\nabla u)\|_{n,1}. \tag{1.5}$$

(ii) If $1 < q < \infty$, then there exists a constant C, depending only on n and q, such that

$$\int_\Omega e^{\lambda |u(x)|^{q'}} dx \le C|\Omega|, \quad \forall \lambda \le \bar{\lambda}_q = (n \kappa_n^{\frac{1}{n}})^{q'}, \quad q' = \frac{q}{q-1}. \tag{1.6}$$

What is more, the constant $\bar{\lambda}_q$ is sharp in the sense that, for any $\lambda > \bar{\lambda}_q$, inequality (1.6) can no longer hold with any C independent of u.

(iii) If $q = \infty$, then

$$\int_\Omega e^{\lambda|u(x)|}dx \le C, \quad \forall \lambda < \bar{\lambda}_\infty = n\kappa_n^{\frac{1}{n}}. \tag{1.7}$$

What is more, the constant $\bar{\lambda}_\infty$ is sharp in the sense that, for any $\lambda \ge \bar{\lambda}_\infty$, inequality (1.7) can no longer hold with any C independent of u.

2. Preliminaries

In this section, we provide some preliminaries on the Finsler-Laplacian and Lorentz space.

Let $F(x)$ be a function of class $C^2(\mathbb{R}^n\backslash\{0\})$, which is convex and even. $F(x)$ has positively homogenous of degree 1, i.e., for any $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,

$$F(t\xi) = |t|F(\xi).$$

A classical example is $F(\xi) = (\sum_i |\xi_i|^q)^{\frac{1}{q}}$, $q \ge 1$. We further assume that

$$F(\xi) > 0, \forall \xi \ne 0.$$

By the property of the homogeneity of F, we can find two positive constants $0 < a_1 \le a_2 < \infty$ to have

$$a_1|\xi| \le F(\xi) \le a_2|\xi|, \quad \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

The image of the map $\phi(\xi) = F_\xi(\xi)$, $\xi \in S^{n-1}$, is a smooth and convex hypersurface in $\mathbb{R}^n$, which is called the Wulff shape of F. The support function $F^o(x)$ of $F(x)$ is defined by $F^o(x) := \sup\limits_{\xi \in U}\langle x, \xi\rangle$, where $U = \{x \in \mathbb{R}^n : F(x) \le 1\}$. We can check that $F^o : \mathbb{R}^n \mapsto [0, +\infty)$ is also a function of class $C^2(\mathbb{R}^n\backslash\{0\})$. Besides, $F^o(x)$ is also a convex and homogeneous function. Furthermore, $F^o(x)$ is dual to $F(x)$ in the sense that

$$F^o(x) = \sup_{\xi \ne 0} \frac{\langle x, \xi\rangle}{F(\xi)}, \qquad F(x) = \sup_{\xi \ne 0} \frac{\langle x, \xi\rangle}{F^o(\xi)}.$$

Define

$$\mathcal{W}_r(x_0) = \{x \in \mathbb{R}^n | F^o(x - x_0) \le r\},$$

which is called the Wulff ball of center at x_0 with radius r. Also for convenience, we denote the unit Wulff ball of center at origin as

$$\mathcal{W}_1 := \{x \in \mathbb{R}^n | F^o(x) \le 1\}$$

and

$$\kappa_n = |\mathcal{W}_1|,$$

which is the the volume of $\mathcal{W}_1$.

By the assumptions of $F(x)$, we have some conclusions of the function $F(x)$, see [34, 36–40].

Lemma 2.1. *We have*

(i) $|F(x) - F(y)| \le F(x + y) \le F(x) + F(y)$*;*
(ii) $\frac{1}{C} \le |\nabla F(x)| \le C$, *and* $\frac{1}{C} \le |\nabla F^o(x)| \le C$ *for some* $C > 0$ *and any* $x \ne 0$*;*
(iii) $\langle x, \nabla F(x)\rangle = F(x), \langle x, \nabla F^o(x)\rangle = F^o(x)$ *for any* $x \ne 0$*;*

(iv) $F(\nabla F^o(x)) = 1$, $F^o(\nabla F(x)) = 1$ *for any* $x \neq 0$;
(v) $F^o(x)F_\xi(\nabla F^o(x)) = x$ *for any* $x \neq 0$;
(vi) $F_\xi(t\xi) = sgn(t)F_\xi(\xi)$ *for any* $\xi \neq 0$ *and* $t \neq 0$.

Now, we give the co-area formula and isoperimetric inequality with respect to F. For a domain $\Omega \subset \mathbb{R}^n$, $K \subset \Omega$ and a bounded variation function $u \in BV(\Omega)$, the anisotropic bounded variation of u with respect to F is defined by

$$\int_\Omega |\nabla u|_F = \sup\{\int_\Omega u \operatorname{div}\sigma dx, \sigma \in C_0^1(\Omega; \mathbb{R}^n), F^o(\sigma) \leq 1\},$$

and the anisotropic perimeter of K with respect to F is defined by

$$P_F(K) := \int_\Omega |\nabla \mathcal{X}_K|_F dx,$$

where $\mathcal{X}_K$ is the characteristic function of the set K. Then, we have the co-area formula

$$\int_\Omega |\nabla u|_F = \int_0^\infty P_F(|u| > t) dt \tag{2.2}$$

and the isoperimetric inequality

$$P_F(K) \geq n\kappa_n^{\frac{1}{n}} |K|^{1-\frac{1}{n}}, \tag{2.3}$$

see [33]. Moreover, the equality in (2.3) holds if and only if K is a Wulff ball.

In the following, let $\Omega^\sharp$ be the homothetic Wulff ball in $\mathbb{R}^n$ centered at the origin, which satisfies

$$|\Omega| = |\Omega^\sharp|,$$

where $|\cdot|$ denotes the volume. For a real-valued function $u : \Omega \to \mathbb{R}$, the distribution function $\mu_u(t) : [0, +\infty) \to [0, +\infty]$ of u is defined as

$$\mu_u(t) = |x \in \Omega| |u(x)| > t|, \text{ for } t \geq 0.$$

The decreasing rearrangement u^* of u is defined as

$$u^*(s) = \sup\{t \geq 0 | \mu_u(t) > s\}, \text{ for } s \geq 0.$$

Clearly the support of u^* satisfies $suppu^* \subseteq [0, |\Omega|]$.

Furthermore, the convex symmetrization $u^\sharp$ of u with respect to F is defined as

$$u^\sharp(x) = u^*(\kappa_n F^o(x)^n), \text{ for } x \in \Omega^\sharp.$$

Next, we recall some properties of Lorentz space $L(p, q)$.

A function u belongs to Lorentz space $L(p, q)$, $1 < p < \infty$, $1 \leq q \leq \infty$, if the quantity

$$||u||_{p,q} = \begin{cases} (\int_0^\infty [u^*(t)t^{\frac{1}{p}}]^q \frac{dt}{t})^{\frac{1}{q}}, & \text{if } \quad 1 \leq q < \infty, \\ \sup\limits_{t>0} u^*(t)t^{\frac{1}{p}}, & \text{if } \quad q = \infty, \end{cases} \tag{2.4}$$

is finite. In particular, we note that $L(p,p) = L^p(\Omega)$ and $L(p,\infty) = M^p$, which is called the Marcinkiewicz space. Another important property of Lorentz space is the intermediate property between L^p space. Precisely, for $1 < q < p < r < \infty$, the following conclusion holds:

$$L^r \subset L(p,1) \subset L(p,q) \subset L(p,p) = L^p \subset L(p,r) \subset L(p,\infty) \subset L^q.$$

And, we have

$$\|u\|_{p,r} \le \left(\frac{q}{p}\right)^{\frac{1}{q}-\frac{1}{r}} \|u\|_{p,q}, \quad \text{for} \quad q \le r. \tag{2.5}$$

When $q > p$, it is easy to check that the quantity (2.4) is not a norm. Letting

$$\bar{u}(s) = \frac{1}{s}\int_0^s u^*(t)dt, \quad s \in (0,+\infty),$$

the quantity

$$\|u\|^*_{p,q} = \begin{cases} \left(\int_0^\infty [\bar{u}(t)t^{\frac{1}{p}}]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if} \quad 1 \le q < \infty, \\ \sup\limits_{t>0} \bar{u}(t)t^{\frac{1}{p}}, & \text{if} \quad q = \infty, \end{cases} \tag{2.6}$$

is a norm for any p and q. Besides, it is proved in [41] that quantity (2.6) is equivalent to the quantity (2.4)

$$\|u\|_{p,q} \le \|u\|^*_{p,q} \le C\|u\|_{p,q},$$

where $C \ge 1$ is a constant depending only on p and q. What is more, under the norm (2.6), $L(p,q)$ is a Banach space. We refer to [41–44] for more information involving the Lorentz space $L(p,q)$.

Now, we give a relationship between two nonnegative functions in $L^1(\Omega)$. We say that u is dominated by v, which is written by $u \prec v$, if

$$\begin{cases} \int_0^s u^*(t)dt \le \int_0^s v^*(t)dt, \quad \forall s \in [0,|\Omega|), \\ \int_0^{|\Omega|} u^*(t)dt = \int_0^{|\Omega|} v^*(t)dt. \end{cases} \tag{2.7}$$

Many properties about the relationship are given, for example, in [45]. For later use, we recall the following property:

Lemma 2.2. *[45] The following conclusions are equivalent:*

(i) $u \prec v$;

(ii) for all nonnegative functions $\omega \in L^\infty(\Omega)$,

$$\int_\Omega u(x)\omega(x)dx \le \int_0^{|\Omega|} v^*(s)\omega^*(s)ds, \quad \int_\Omega u(x)dx = \int_\Omega v(x)dx;$$

(iii) for all nonnegative functions $\omega \in L^\infty(\Omega)$,

$$\int_0^{|\Omega|} u^*(s)\omega^*(s)ds \le \int_0^{|\Omega|} v^*(s)\omega^*(s)ds, \quad \int_\Omega u(x)dx = \int_\Omega v(x)dx.$$

Now, we state a key method to construct a function Ψ, which is dominated by a function ψ, see [45]. Let $D(s)$, $s \in [0,|\Omega|]$, be a family of subsets of Ω which have the following properties:

(i) $|D(s)| = s$;
(ii) $D(s_1) \subset D(s_2)$, if $s_1 < s_2$;
(iii) $D(s) = \{x \in \Omega : |u(x)| > t\}, \quad \text{if} \quad s = \mu_u(t)$.

We see that this means that $D(s)$ is the family of the level sets of $|u(x)|$. For a nonnegative function $\psi \in L^1(\Omega)$, we define $\Psi(t)$ as the function such that

$$\int_{D(s)} \psi(x)dx = \int_0^s \Psi(t)dt, \quad s \in [0, |\Omega|]. \tag{2.8}$$

For (2.8), we say that Ψ is built from ψ on the level sets of $|u|$. It is shown in [45] that

$$\Psi \prec \psi. \tag{2.9}$$

3. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. The proof of Theorem 1.1 is an adaptation of ones given in [7]. We first give some key lemmas. Let u be a measurable function in Ω such that

$$g(x) = F(\nabla u) \in L(n, q), \quad 1 \le q \le \infty. \tag{3.1}$$

We let $G(t)$ be the function built from g on the level sets of u, as in (2.8). Then, we have the following result:

Lemma 3.1. *The estimate*

$$u^*(s) \le \frac{1}{n\kappa_n^{\frac{1}{n}}} \int_s^{|\Omega|} G(t) t^{\frac{1}{n}} \frac{dt}{t} \tag{3.2}$$

holds.

Proof. By (2.8) and (3.1), we have

$$-\frac{d}{dt}\int_{|u|>t} F(\nabla u)dx = -\frac{d}{dt}\int_{|u|>t} g(x)dx = -\frac{d}{dt}\int_0^{\mu(t)} G(s)ds = (-\mu'(t))G(\mu(t)),$$

where $\mu(t) = \mu_u(t)$. By the co-area formula (2.2) and isoperimetric inequality (2.3), we have

$$n\kappa_n^{\frac{1}{n}} \mu(t)^{1-\frac{1}{n}} \le -\frac{d}{dt}\int_{|u|>t} F(\nabla u)dx = (-\mu'(t))G(\mu(t)).$$

Then, we get

$$-u^{*\prime}(s) \le \frac{1}{n\kappa_n^{\frac{1}{n}}} \frac{G(s)}{s^{1-\frac{1}{n}}}.$$

Thus, the lemma is obtained by direct integration. □

By Lemma 3.1, for the purpose of the estimate $u(x)$, we can estimate the H-symmetric and decreasing function

$$v(x) = \frac{1}{n\kappa_n^{\frac{1}{n}}} \int_{\kappa_n F^o(x)^n}^{|\Omega|} G(t) t^{\frac{1}{n}} \frac{dt}{t}. \tag{3.3}$$

By the following lemma, we can estimate $u(x)$ by a function involving g^*.

Lemma 3.2. *Let $g \in L^1(\Omega)$. For any nonnegative function G defined in $[0, |\Omega|]$ such that $G \prec g$, we let v be the function defined in (3.3). Then, we obtain*

$$\bar{v}(s) \leq \frac{1}{n\kappa_n^{\frac{1}{n}}}[\int_s^{|\Omega|} g^*(t)t^{\frac{1}{n}}\frac{dt}{t} + \frac{1}{s^{1-\frac{1}{n}}}\int_0^s g^*(t)dt]. \tag{3.4}$$

Proof. By (3.3), we have

$$\begin{aligned}\bar{v}(s) &= \frac{1}{s}\int_0^s v^*(t)dt \\ &= \frac{1}{n\kappa_n^{\frac{1}{n}}}(\int_s^{|\Omega|} G(t)t^{\frac{1}{n}}\frac{dt}{t} + \frac{1}{s}\int_0^s G(t)t^{\frac{1}{n}}dt) \\ &\leq \frac{1}{n\kappa_n^{\frac{1}{n}}}\int_0^{|\Omega|} G(m)h(m,s)dm,\end{aligned}$$

where

$$h(m,s) = \begin{cases} s^{-1+\frac{1}{n}}, & \text{if} \quad 0 \leq m \leq s, \\ m^{-1+\frac{1}{n}}, & \text{if} \quad s < m \leq |\Omega|. \end{cases}$$

Clearly, for any fixed s, $h(m,s)$ is decreasing with respect to m. Then, by Lemma 2.2 and the property $G \prec g$, we obtain (3.4). □

For the aim to prove Theorem 1.1, we need the following lemma proved by Adams [46].

Lemma 3.3. *[46] Let $a(s,t)$ be a nonnegative measurable function in $\mathbb{R} \times [0, \infty)$, and for some $q \in (1, \infty)$, $q' = \frac{q}{q-1}$,*

$$a(s,t) \leq 1, \quad for \quad a.e. \quad 0 < s < t, \tag{3.5}$$

and

$$\sup_{t>0}(\int_{-\infty}^0 a(s,t)^{q'}ds + \int_t^\infty a(s,t)^{q'}ds)^{\frac{1}{q'}} = \nu < +\infty. \tag{3.6}$$

Assume that $\Phi(s) \geq 0$ and

$$\int_{-\infty}^{+\infty} \Phi(s)^q ds \leq 1. \tag{3.7}$$

Then, there exists a constant C, depending only on q and ν such that

$$\int_0^{+\infty} e^{-H(t)}dt \leq C,$$

where

$$H(t) = t - (\int_{-\infty}^{+\infty} a(s,t)\Phi(s)ds)^{q'}.$$

Now, it is sufficient to prove Theorem 1.1.

Proof of Theorem 1.1. We complete the proof by distinguishing three cases.

Case (i) $q = 1$. By Lemma 3.1, we have that

$$\|u\|_\infty \leq u^*(0) \leq \frac{1}{n\kappa_n^{\frac{1}{n}}}\int_0^{|\Omega|} G(t)t^{\frac{1}{n}}\frac{dt}{t}.$$

Then, by $G \prec g$ and Lemma 2.2, we have that

$$||u||_\infty \le \frac{1}{n\kappa_n^{\frac{1}{n}}} \int_0^{|\Omega|} g^*(t) t^{\frac{1}{n}} \frac{dt}{t} = \frac{1}{n\kappa_n^{\frac{1}{n}}} ||F(\nabla u)||_{n,1}.$$

Then, (1.5) holds.
Case (ii) $1 < q < \infty$. By Lemma 3.2, we have

$$\bar{u}(s) \le \frac{1}{n\kappa_n^{\frac{1}{n}}} \Big(\int_s^{|\Omega|} g^*(t) t^{\frac{1}{n}} \frac{dt}{t} + \frac{1}{s^{1-\frac{1}{n}}} \int_0^s g^*(t) dt \Big). \tag{3.8}$$

For the convenience, we denote n' as the conjugate index of n, i.e., $n' = \frac{n}{n-1}$. Then,

$$\begin{aligned}
\bar{u}(|\Omega| e^{-t}) &\le \frac{1}{n\kappa_n^{\frac{1}{n}}} \Big(\int_{|\Omega|e^{-t}}^{|\Omega|} g^*(t) t^{\frac{1}{n}} \frac{dt}{t} + \frac{1}{(|\Omega|e^{-t})^{1-\frac{1}{n}}} \int_0^{|\Omega|e^{-t}} g^*(t) dt \Big) \\
&= \frac{|\Omega|^{\frac{1}{n}}}{n\kappa_n^{\frac{1}{n}}} \Big(\int_0^t g^*(|\Omega|e^{-r}) e^{-\frac{r}{n}} dr + e^{t(1-\frac{1}{n})} \int_t^\infty g^*(|\Omega|e^{-r}) e^{-r} dr \Big) \\
&= \frac{1}{n\kappa_n^{\frac{1}{n}}} \int_{-\infty}^{+\infty} a(s,t) \Phi(s) ds,
\end{aligned}$$

where

$$a(s,t) = \begin{cases} 0, & \text{if} \quad s \le 0, \\ e^{\frac{t-s}{n'}}, & \text{if} \quad t < s < +\infty, \\ 1, & \text{if} \quad 0 < s < t, \end{cases}$$

and

$$\Phi(s) = \begin{cases} |\Omega|^{\frac{1}{n}} g^*(|\Omega|e^{-s}) e^{-\frac{s}{n}}, & \text{if} \quad s \ge 0, \\ 0, & \text{if} \quad s < 0. \end{cases}$$

It is obvious that (3.5) holds. Next, for any $1 < q < \infty$, we obtain

$$\begin{aligned}
&\Big(\int_{-\infty}^0 a(s,t)^{q'} ds + \int_t^\infty a(s,t)^{q'} ds \Big)^{\frac{1}{q'}} \\
&= \Big(\int_t^\infty e^{\frac{q'(t-s)}{n'}} ds \Big)^{\frac{1}{q'}} \\
&= \Big(e^{\frac{tq'}{n'}} \int_t^\infty e^{-\frac{sq'}{n'}} ds \Big)^{\frac{1}{q'}} \\
&= \Big(\frac{n'}{q'} \Big)^{\frac{1}{q'}}.
\end{aligned}$$

Then, we get (3.6) by choosing $\nu = (\frac{n'}{q'})^{\frac{1}{q'}}$.

Finally, by (1.4), we have that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \Phi(s)^q ds &= |\Omega|^{\frac{q}{n}} \int_0^{+\infty} (g^*(|\Omega|e^{-s}) e^{-\frac{s}{n}})^q ds \\
&= \int_0^{|\Omega|} (g^*(t) t^{\frac{1}{n}})^q \frac{dt}{t} \\
&= ||F(\nabla u)||_{n,q}^q \le 1.
\end{aligned}$$

This means that (3.7) holds. Then, by Lemma 3.3, we have

$$\int_0^{+\infty} e^{-t+(\bar{u}(|\Omega|e^{-t})n\kappa_n^{\frac{1}{n}})^{q'}}dt \le C,$$

which means that

$$\int_0^{|\Omega|} e^{(\bar{u}(s)n\kappa_n^{\frac{1}{n}})^{q'}}ds \le C|\Omega|.$$

Furthermore, by the fact $u^*(s) \le \bar{u}(s)$, we obtain

$$\int_\Omega e^{\lambda|u(x)|^{q'}}dx = \int_0^{|\Omega|} e^{\lambda u^*(s)^{q'}}ds \le \int_0^{|\Omega|} e^{\lambda\bar{u}(s)^{q'}}ds \le C|\Omega|, \quad \forall\lambda \le (n\kappa_n^{\frac{1}{n}})^{q'} = \bar{\lambda}_q.$$

Case (iii) $q = \infty$. By (3.8) and (1.4), we obtain

$$\begin{aligned}\bar{u}(s) &\le \frac{1}{n\kappa_n^{\frac{1}{n}}}\left(\int_s^{|\Omega|} g^*(t)t^{\frac{1}{n}}\frac{dt}{t} + \frac{1}{s^{1-\frac{1}{n}}}\int_0^s g^*(t)dt\right)\\ &\le \frac{1}{n\kappa_n^{\frac{1}{n}}}\left(\int_s^{|\Omega|} \frac{1}{t}dt + \frac{1}{s^{1-\frac{1}{n}}}\int_0^s t^{-\frac{1}{n}}dt\right)\\ &= \frac{1}{n\kappa_n^{\frac{1}{n}}}\left(\log\frac{|\Omega|}{s} + \frac{n}{n-1}\right).\end{aligned}$$

It follows that

$$\int_0^{|\Omega|} e^{\lambda\bar{u}(s)}ds \le e^{\frac{\lambda}{(n-1)\kappa_n^{\frac{1}{n}}}}\int_0^{|\Omega|}\left(\frac{|\Omega|}{s}\right)^{\frac{\lambda}{n\kappa_n^{\frac{1}{n}}}}ds.$$

Clearly, the right hand side is finite if and only if $\lambda < n\kappa_n^{\frac{1}{n}} = \bar{\lambda}_\infty$. Then, we get (1.7).

At last, we prove the sharpness of (1.5)–(1.7).

We easily see that equality (1.5) holds if $u(x) = u^\sharp(x)$ and $F(\nabla u) = F(\nabla u)^\sharp \in L(n, 1)$.

The proof of sharpness for (1.6) is more complicated. If $1 < q < \infty$, for any $\lambda > \bar{\lambda}_q$, we will construct a sequence of functions u_k such that $\|F(\nabla u_k)\|_{n,q} \le 1$ and

$$\lim_{k\to\infty}\int_\Omega e^{\lambda|u_k(x)|^{q'}}dx = +\infty. \tag{3.9}$$

Define

$$u_k(x) = \begin{cases} \frac{k^{\frac{1}{q'}}}{n\kappa_n^{\frac{1}{n}}}, & \text{if } \ 0 \le \kappa_n F^o(x)^n \le e^{-k},\\ \frac{1}{n\kappa_n^{\frac{1}{n}}k^{\frac{1}{q}}}\log\left(\frac{1}{\kappa_n F^o(x)^n}\right), & \text{if } \ e^{-k} \le \kappa_n F^o(x)^n \le 1,\\ 0, & \text{if } \ \kappa_n F^o(x)^n > 1.\end{cases} \tag{3.10}$$

Then, by direct calculation, using Lemma 2.1, we have that the decreasing rearrangement of $F(\nabla u_k)$ is

$$F(\nabla u_k)^*(s) = \begin{cases} 0, & \text{if } \ 1 - e^{-k} \le s \le 1,\\ \frac{k^{-\frac{1}{q}}}{(s+e^{-k})^{\frac{1}{n}}}, & \text{if } \ 0 \le s < 1 - e^{-k}.\end{cases}$$

We consider $1 < q < n$, $q = n$, and $n < q < \infty$ separately.
When $1 < q < n$, making the change of variable $m = 1 + se^k$, then

$$\begin{aligned}\|F(\nabla u_k)\|_{n,q} &= (\frac{1}{k}\int_0^{1-e^{-k}} (\frac{s}{s+e^{-k}})^{\frac{q}{n}} \frac{ds}{s})^{\frac{1}{q}} \\ &= (\frac{1}{k}\int_1^{e^k} (1-\frac{1}{m})^{\frac{q}{n}} \frac{dm}{m-1})^{\frac{1}{q}}.\end{aligned}$$

We let

$$\beta_k = \|F(\nabla u_k)\|_{n,q} = (\frac{1}{k}\int_1^{e^k} (1-\frac{1}{m})^{\frac{q}{n}} \frac{dm}{m-1})^{\frac{1}{q}}.$$

Then,

$$\begin{aligned}&\lim_{k\to\infty} \frac{1}{k}\int_1^{e^k} (1-\frac{1}{m})^{\frac{q}{n}} \frac{dm}{m-1} \\ =& \lim_{k\to\infty} \frac{1}{k}\int_1^{e^k} \frac{1}{(m-1)^{1-\frac{q}{n}} m^{\frac{q}{n}}} dm \\ =& \lim_{k\to\infty} \frac{e^k}{(e^k-1)^{1-\frac{q}{n}} (e^k)^{\frac{q}{n}}} \\ =& \lim_{k\to\infty} (\frac{e^k}{e^k-1})^{1-\frac{q}{n}} = 1.\end{aligned}$$

Hence, we have $\lim\limits_{k\to\infty} \beta_k = 1$.
Now, we set

$$v_k(x) = \frac{u_k(x)}{\beta_k}.$$

Clearly, $\|F(\nabla v_k)\|_{n,q} = 1$. However, when $\lambda > \bar{\lambda}_q = (n\kappa_n^{\frac{1}{n}})^{q'}$, as $k \to +\infty$,

$$\begin{aligned}\int_\Omega e^{\lambda |v_k(x)|^{q'}} dx &\geq \int_0^{e^{-k}} \exp[\frac{k\lambda}{\beta_k^{q'} (n\kappa_n^{\frac{1}{n}})^{q'}}] ds \\ &= \exp[k(\frac{\lambda}{\beta_k^{q'} (n\kappa_n^{\frac{1}{n}})^{q'}} - 1)] \\ &\to +\infty.\end{aligned}$$

When $q = n$, the proof is similar to that in [2]. We have

$$\|F(\nabla u_k)\|_{n,n} = \|F(\nabla u_k)\|_n \leq 1. \tag{3.11}$$

Then, when $\lambda > \bar{\lambda}_{n,n} = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$, as $k \to +\infty$,

$$\int_\Omega e^{\lambda |u_k|^{\frac{n}{n-1}}} dx = \int_0^{|\Omega|} e^{\lambda |u_k^*|^{\frac{n}{n-1}}} ds$$

$$\geq \exp[k(\frac{\lambda}{n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}}-1)] \to +\infty.$$

When $n \leq q < \infty$, from (2.5) and (3.11) we have

$$\|F(\nabla u_k)\|_{n,q} \leq \|F(\nabla u_k)\|_n = 1.$$

Then, as the case of $q = n$, it is easy to prove that

$$\int_B e^{\lambda |u_k(x)|^{q'}} dx \to +\infty \quad \text{as} \quad k \to \infty, \quad \text{when} \quad \lambda > \bar{\lambda}_q = (n\kappa_n^{\frac{1}{n}})^{q'}.$$

When $q = \infty$, we construct a function u such that $\|F(\nabla u)\|_{n,\infty} \leq 1$, and for any $\lambda \geq \bar{\lambda}_\infty = n\kappa_n^{\frac{1}{n}}$,

$$\int_\Omega e^{\lambda |u(x)|} dx = +\infty.$$

Let

$$u(x) = \frac{1}{n\kappa_n^{\frac{1}{n}}} \log(\frac{1}{\kappa_n F^o(x)^n}), \quad \forall x \in \mathcal{W}_1.$$

By direct calculation, using Lemma 2.1, we obtain

$$F(\nabla u)^*(s) = \frac{1}{s^{\frac{1}{n}}},$$

and then

$$\|F(\nabla u)\|_{n,\infty} \leq 1.$$

Thus, when $\lambda \geq \bar{\lambda}_\infty = n\kappa_n^{\frac{1}{n}}$, by the co-area formula (2.2), we have

$$\begin{aligned}
\int_{\mathcal{W}_1} e^{\lambda |u(x)|} dx &= \int_0^1 \exp(\frac{\lambda}{n\kappa_n{}^{\frac{1}{n}}} \log(\frac{1}{s})) ds \\
&\geq C \int_0^1 \frac{1}{s} ds = +\infty.
\end{aligned}$$

The proof is completed. □

4. Conclusions

In this paper, we mainly study the anisotropic Moser-Trudinger type inequality in Lorentz space $L(n, q)$, $1 \leq q \leq \infty$. It is a generation result of Moser-Trudinger type inequality in Lorentz space. The extremal function of such inequality is closely related to existence of solutions of Finsler-Liouville type equation. We believe that the sharp inequality will be the key tool to study the existence of solutions for some quasi-linear elliptic equations, such as Finsler-Laplacian equation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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