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Research article

The ruin probability of a discrete risk model with unilateral linear dependent claims

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Abstract: This article focuses on analyzing the finite-time ruin probability within a specific class of discrete risk models. These models incorporate dependent claims, an interest rate component, and stationary noise terms exhibiting semi-heavy-tailed behavior. In this framework, the claim amount follows a unilateral linear dependent process with independent and identically distributed noise terms, while the discount factor is determined by both the interest rate and time. The finite-time ruin probability has been derived under insurance risk conditions resembling the gamma distribution.

Keywords: semi-heavy-tailed; unilateral linear dependence; stationary noise; finite-time ruin probability
Mathematics Subject Classification: 60F99

1. Introduction

Risk theory is a discipline that involves quantitative analysis and the prediction of risks, and it is primarily applied in insurance, risk management, finance, and various other fields. Actuarial mathematics recognizes risk theory as a pivotal subject that has garnered a significant amount of attention [1]. Financial and insurance companies are susceptible to bankruptcy due to substantial losses resulting from a few extraordinary events, which can be characterized by heavy-tailed distributions. As a result, a significant portion of research efforts have been concentrated on assessing the ruin probability in continuous and discrete time risk models, specifically pertaining to insurance risks characterized by heavy-tailed claims [2]. Nonetheless, it is crucial to acknowledge that the impact of small- and medium-sized claims on insurance companies should not be underestimated. Relevant literature addressing this aspect can be found in [3, 4]. Notably, common distributions that capture semi-heavy-tailed behavior include the generalized inverse Gaussian distribution, hyperbolic distribution, generalized Lindley distribution, and others [5–9]. For the convenience of calculation, we assume that the premium is received and the claims are paid at the end of each period. Actually, the premium for each period is paid before the end of the period. This article considers the following discrete risk models:

$$U_0 = y, U_m = U_{m-1}(1 + r) + X_m - Y_m, m = 1, 2, \cdots, \tag{1.1}$$

where $U_0 > 0$ signifies the initial reserve, $r > 0$ represents the interest rate of the insurance company, and $\{Y_m\}_{m\geq 1}$ shows the claim amount for insurance companies from time $m-1$ to time m, which can be considered as insurance risks [10]. $\{X_m\}_{m\geq 1}$ is the premium amount for insurance companies from time $m-1$ to time m, and U_m indicates the surplus of an insurance company at time m. Non-negative random variable sequences $\{Y_m\}_{m\geq 1}$ and $\{X_m\}_{m\geq 1}$ are independent of each other. The above risk model can be transformed into the following equation:

$$U_0 = y, U_m\omega(m, r) = y + \sum_{\zeta=1}^{m}(X_\zeta - Y_\zeta)(1 + r)^{-\zeta}, m = 1, 2, \cdots, \tag{1.2}$$

where $(1 + r)^{-\zeta}$ is the discount factor for time ζ with respect to the initial moment. On this basis, we consider a discrete risk model with a discount factor in the form of a general function:

$$U_0 = y, U_m\omega(m, r) = y + \sum_{\zeta=1}^{m}(X_\zeta - Y_\zeta)\omega(\zeta, r), m = 1, 2, \cdots, \tag{1.3}$$

where $\omega(\zeta, r)$ is the discount factor for periods $\zeta - 1$ to ζ, which can be seen as a positive random function of interest rate and time, with values ranging from 0 to 1.

To characterize the dependent structure of insurance risk, i.e., the claim amount $\{Y_m\}_{m\geq 1}$ with a unilateral linear process [11, 12], let

$$Y_m = \sum_{\zeta=1}^{m} \phi_{m-\zeta}\xi_\zeta + \phi_m\xi_0, m = 1, 2, \cdots, \tag{1.4}$$

where $y_0 \geq 0$, ξ_0 is a non-negative constant, and the noise term $\{\xi_m\}_{m\geq 1}$ is a sequence of non negative random variables that are independent and identically distributed in ξ, with the distribution denoted as W, as shown in the formula (2.5). The coefficients $\phi_{m-\zeta}, \phi_m$ are also non-negative constants satisfying that $\sum_{\zeta=0}^{\infty} \phi_\zeta = \phi \in (0, \infty)$, The bankruptcy probability at time m can be defined as follows:

$$\psi(y, m) = P\{\min_{1\leq\kappa\leq m} U_\kappa < 0 \mid U_0 = y\}. \tag{1.5}$$

In actuarial insurance, discount factors can be categorized as either random or non-random. These factors, including interest rates, inflation, and consumption levels, have a time-dependent nature. When considered together, the formulation of discount factors becomes notably intricate and non-unique. Among these factors, interest rates have received a considerable amount of attention due to their significant influence on discount factors. Regarding the bankruptcy probability for discrete risk models with a constant interest rate, Yang [13] derived the Lundberg inequality and non-exponential upper bounds. Additionally, Yang and Zhang [14] explored upper bounds when the claim amounts satisfy the conditions of an autoregressive process. Tang [15] obtained the form of bankruptcy probability when the claim amount belongs to the sub-exponential distribution family. Yao and Wang [16] considered exponential upper bounds for claim amounts modeled as a moving average process. Furthermore, Wei et al. [17] and Ming et al. [18] investigated the bankruptcy probability for a random interest rate discrete risk model. Yu et al. [19] derived asymptotic estimates of the finite-time bankruptcy probability for a variable interest rate discrete risk model. Linear processes are used to express claim amounts as weighted sums of past innovations. Subsequently, researchers have gradually incorporated linear dependent structures into the insurance risk theory. For instance, Gerber [20] used a linear process, which includes the autoregressive model and the autoregressive moving average model as a special case, to describe the annual gains of an insurance company. Mikosch and Samorodnitsky [21] used a two sided linear process to model single-period net losses, and they studied the asymptotic behavior of the ultimate probability. Peng et al. [11], Guo and Wang [22], and Peng and Wang [23] obtained the ruin probability for a discrete-time risk model with a unilateral linear claim process. Simultaneously, the ruin probability problem of risk models with doubly dependent structures has been discussed by Liu et al. [24], Bai et al. [25], and Jing et al. [26].

The above studies mostly focused on heavy-tailed claims and there is less discussion on the asymptotic formula of the ruin probability with semi-heavy-tailed claims. Gamma-like tailed distribution has been discussed as a form of semi-heavy-tailed distribution. For example, Hashorva and Li [8] studied the asymptotic tail behavior of the reinsured amounts assuming that the claim sizes of an insurance company have a common distribution with a gamma-like tail. Yang and Yuen [27] and Huang et al. [28] discussed the asymptotic for a discrete-time risk model with gamma-like insurance risks. Chen et al. [29] derived some asymptotic formulas for the ruin probability for various scenarios of financial risks with light-tailed or moderately heavy tailed insurance risks. Based on this, we consider a risk model in which the claim amount is a unilaterally linear dependent structure and the noise term follows a semi-heavy-tailed distribution. Moreover, interest rates have a significant impact on the discount factor. Although constant interest rates are easy to calculate, interest rates may change in reality. Therefore, we chose to regard the discount factor as a function of interest rates and time, that includes both constant and variable interest rates. We consider that the claim amount follows a unilateral linear process. A discrete-time risk model with a noise term that follows a gamma-like semi-heavy-tailed distribution is discussed for finite-time ruin probability.

The rest of this paper is organized as follows. In Section 2, we give some definitions and the asymptotic result of the proposed model. Related lemmas and proofs are presented in Sections 3 and 4. We verify the asymptotic behavior through simulation in Section 5. A short conclusion is provided in Section 6.

2. Symbols and main results

For two real numbers y, z and two positive functions $w_1(\cdot), w_2(\cdot)$, all limit symbols in this article follow the assumption that $y \to \infty$.

I. If $\limsup \frac{w_1(y)}{w_2(y)} \leq 1$, then $w_1(y) \lesssim w_2(y)$ or $w_2(y) \gtrsim w_1(y)$.

II. If $\lim \frac{w_1(y)}{w_2(y)} = 1$, then $w_1(y) \sim w_2(y)$.

III. If $\lim \frac{w_1(y)}{w_2(y)} = 0$, then $w_1(y) = o(w_2(y))$.

IV. If $\limsup \frac{w_1(y)}{w_2(y)} < \infty$, then $w_1(y) = O(w_2(y))$.

The heavy-tailed distribution includes many subfamilies. For any $y > 0$, note that $\overline{W}(y) = 1 - W(y)$ is the right-tailed distribution of Y, where $W(y)$ is the distribution function of y. We only considered three types including the long-tailed distribution family ($\mathcal{L}$), the sub exponential distribution family ($\mathcal{S}$), and the regularly varying tailed distribution family ($\mathcal{R}$). At the same time, we describe the relationship between the semi-heavy-tailed distribution family ($\mathcal{SH}$) and these subfamilies.

Definition 1. *[2] $W \in \mathcal{L}$ at $[0, \infty)$ if it satisfies*

$$\lim_{y\to\infty} \frac{\overline{W}(y+z)}{\overline{W}(y)} = 1, \tag{2.1}$$

for any $z > 0$.

Definition 2. *[2] $W \in \mathcal{S}$ at $[0, \infty)$ if it satisfies*

$$\lim_{y\to\infty} \frac{\overline{W^{n*}}(y)}{\overline{W}(y)} = n, \tag{2.2}$$

for any $n = 2, 3, \cdots$, where W^{n*} stands for the n-fold convolution of W itself.

Definition 3. *[2] Ultimately positive measurable function is called regularly varying at infinity with the index $\rho \in \mathbb{R}$ if it satisfies*

$$\lim_{y\to\infty} \frac{\overline{W}(yz)}{\overline{W}(y)} = z^{\rho}, \tag{2.3}$$

for any $z > 0$, that is, $W \in \mathcal{R}_\rho$. $\mathcal{R}_{-\infty}$ includes heavy- and light-tailed distributions.

Definition 4. *[30, 31] $W(\cdot)$ is a semi-heavy-tailed function with the parameter $\gamma > 0$ if it satisfies*

$$\overline{W}(y) = e^{-\gamma y}\overline{G}(y), y \in \mathbb{R},$$

where $G(\cdot)$ is a heavy-tailed distribution. When $G \in \mathcal{R}_\rho, \rho \in \mathbb{R}$, $W(\cdot)$ is called a semi-heavy-tailed distribution with the parameters $(\rho + 1) \in \mathbb{R}$ and $\gamma > 0$, also note that $W \in \mathcal{SH}(\rho + 1, \gamma)$. When $G \in \mathcal{L}$, $W(\cdot)$ satisfies

$$\lim_{y\to\infty} \frac{\overline{W}(y-z)}{\overline{W}(y)} = e^{\gamma z}, \tag{2.4}$$

for any real number z, noting that $W \in \mathcal{SH}(0, \gamma)$.

For any $\lambda > 0$, the slowly varying function satisfies $l(\lambda y) \sim l(y)$. The specific properties are shown in [2, 29, 32]. Moreover, $W(\cdot)$ is defined on $\mathbb{R}$ if there is a slowly varying function $l(\cdot) : (0, \infty) \mapsto (0, \infty)$ such that

$$\overline{W}(y) \sim l(y)y^{\delta-1}e^{-\gamma y}, \tag{2.5}$$

then $W(\cdot)$ is said to have a gamma-like tail with the parameters $\delta > 0$ and $\gamma > 0$. Equation (2.5) can be rewritten in a compact way as $\overline{W}(y) \sim e^{-\gamma y}\overline{G}(y)$, where $\overline{G}(y) = l(y)y^{\delta-1}$ is a regularly varying function with the index $\delta - 1$, that is, $G \in \mathcal{R}_{\delta-1}$ and $W \in \mathcal{SH}(\delta, \gamma)$.

A canonical example of the gamma-like distribution with the parameters $\delta > 0$ and $\gamma > 0$ is the gamma distribution with the corresponding parameters, that is,

$$\overline{W}(y) = \frac{\gamma^{\delta}}{\Gamma(\delta)} \int_{y}^{\infty} z^{\delta-1} e^{-\gamma z} dz, y > 0, \tag{2.6}$$

where $\Gamma(\cdot)$ is the Euler gamma function. For specific properties of gamma-like tailed distributions, we can refer to [8, 30, 31]. In this case, we have

$$\overline{W}(y) \sim \frac{\gamma^{\delta-1}}{\Gamma(\delta)} y^{\delta-1} e^{-\gamma y}. \tag{2.7}$$

Remark 1. *I. Due to the fact that some distributions in the $\mathcal{S}$ family, such as the Weibull distribution and lognormal distribution, also belong to the $\mathcal{R}_{-\infty}$ family, the distribution of the $\mathcal{S} \cap \mathcal{R}_{-\infty}$ family is called a moderate heavy-tailed distribution.*

II. If $W \in \mathcal{SH}(0, \gamma)$ and it satisfies that $E(e^{\gamma Y}) = 2$, then $W \in \mathcal{S}$.

Proof. Note that

$$\begin{aligned} \lim_{y\to\infty} \frac{\overline{W^{2*}}(y)}{\overline{W}(y)} &= \lim_{y\to\infty} \frac{\int_0^{\infty} \overline{W}(y-z)W(dz)}{\overline{W}(y)} = \lim_{y\to\infty} \int_0^{\infty} \frac{\overline{W}(y-z)}{\overline{W}(y)} W(dz) \\ &= \int_0^{\infty} e^{\gamma z} W(dz) = \int_0^{\infty} e^{\gamma y} W(dy) = E(e^{\gamma Y}), \end{aligned}$$

where $E(\cdot)$ denotes the mean of $\{\cdot\}$ with the cumulative distribution function $W(\cdot)$. Based on Definition 2, we complete the proof. □

III. When $\gamma = 0$, $\mathcal{SH}(\delta, 0)$ includes all long-tailed distributions and sub-exponential type distributions.

IV. When $\gamma > 0$, all gamma-like tailed distributions in $\mathcal{SH}(\delta, \gamma)$ are light-tailed and belong to $\mathcal{R}_{-\infty}$ families, that is, $W \in \mathcal{R}_{-\infty}$.

Assume that the sum variable $\sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \eta, r)$ has a common distribution G with a finite upper endpoint

$$\phi_* = \sup\{\phi^* : G(\phi^*) < 1\} < \infty.$$

Denote $p_* = P\{\sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \eta, r) = \phi_*\} \geq 0$ and $L = \frac{p_*}{\phi_*^{\delta-1}}$, where $\delta > 0$. The following results show the asymptotics for the finite time ruin probability in the two cases of $0 < \phi_* < 1$ and $\phi_* = 1$.

Theorem 1. *Let $\{X_m\}_{m\geq1}$ be a sequence of independent and identically distributed non-negative random variables, $\{Y_m\}_{m\geq1}$ be a sequence of non-negative random variables introduced in (1.4), and $\{(X_m, Y_m)\}_{m\geq1}$ be independent and identically distributed random pairs. In the discrete risk model (1.3), if*

(1) function $\omega(\zeta, r)$ satisfies that $\sum_{\zeta=0}^{m} \omega(\zeta, r) < \infty$;

(2) non-negative coefficients denoted by $\{\phi_\zeta\}_{\zeta\geq0}$ satisfy that $0 < \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \eta, r) \leq \phi_* < \infty, \eta = 1, 2, \cdots, m, r > 0$, where ϕ_* is a finite upper endpoint of $\sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \eta, r)$;

(3) $W \in \mathcal{SH}(\delta, \gamma)$ and the parameters $\delta > 0$ and $\gamma > 0$, then it holds that
I. when $0 < \phi_* < 1$ and $E(e^{\gamma \Pi_{m-1}}) < \infty$ for any $m \geq 1$, then

$$\psi(y, m) \sim LE(e^{\gamma \frac{\Pi_{m-1}}{\phi_*}}) l(y) y^{\delta - 1} e^{-\frac{\gamma y}{\phi_*}}; \tag{2.8}$$

II. when $\phi_* = 1$ for any $m \geq 1$, then

$$\psi(y, m) \sim \frac{L^m \gamma^{m-1} \Big(\Gamma(\delta)\Big)^m}{\Gamma(m\delta)} l(y) y^{m\delta - 1} e^{-\gamma y}. \tag{2.9}$$

The assumption that $\phi_* \leq 1$ in relations (2.8) and (2.9) means that the insurer invests all of their surpluses into a risk-free market.

3. Related lemma and proof

Lemma 1. *If $Y_1, Y_2, \cdots, Y_m$ ($m \geq 1$) are mutually independent positive random variables with a gamma-like tailed distribution $\overline{W}_\zeta(y) \sim l_\zeta(y) y^{\delta_\zeta - 1} e^{-\gamma y}$, the parameters $\delta_\zeta > 0$ and $\gamma > 0$, and $l_\zeta(y)$ is a slowly varying function, $\zeta = 1, 2, \cdots, m$, then*

$$P\Big(\sum_{\zeta=1}^{m} Y_\zeta > y\Big) \sim \frac{\gamma^{m-1} \prod_{\zeta=1}^{m} \Gamma(\delta_\zeta)}{\Gamma(\sum_{\zeta=1}^{m} \delta_\zeta)} \Big(\prod_{\zeta=1}^{m} l_\zeta(y)\Big) y^{\sum_{\zeta=1}^{m} \delta_\zeta - 1} e^{-\gamma y}. \tag{3.1}$$

Proof. It can be seen from Lemma 2.1 of [8].

Lemma 2. *Under the conditions of Theorem 1, if $0 < \phi_* \leq 1$, then*

$$P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y\right\} \sim \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y\right\}, \tag{3.2}$$

for any $r > 0$.

Proof. We first prove the upper bound:

$$\begin{aligned}
& P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y\right\} \\
&= P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y, \bigcap_{\varsigma=1}^{m} \Big(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y - L\Big)\right\} \\
&\quad + P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y, \bigcup_{\varsigma=1}^{m} \Big(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y - L\Big)\right\} \\
&\leq P\left\{\bigcup_{\varsigma=1}^{m} \Big(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y - L\Big)\right\} \\
&\quad + P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y, \bigcap_{\varsigma=1}^{m} \Big(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta + \varsigma, r) > y - L\Big)\right\}
\end{aligned}$$

$$:= A_1 + A_2,$$

for any fixed $0 < L < \infty$. If $0 < \phi_* \leq 1$, then

$$P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\} \leq P(\xi_\varsigma > \frac{y}{\phi_*}) \sim o\Big(\overline{W}(y)\Big).$$

According to the Boolean inequality, we have that

$$A_1 \leq \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y-L\right\} + P\left\{\xi_0 \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta, r) > y-L\right\}$$
$$\lesssim \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\} + o\Big(\overline{W}(y)\Big).$$

Now we estimate A_2 and it follows that

$$A_2 = P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y, \bigcap_{\varsigma=1}^{m} \Big(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y-L\Big)\right\}$$
$$\leq P\left\{\bigcap_{\varsigma=1}^{m} \Big(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y-L\Big)\right\}$$
$$\leq \sum_{\zeta=1}^{m} P\Big(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y-L\Big)$$
$$\leq o\Big(\overline{W}(y)\Big).$$

Substituting relations A_1 and A_2 into the following expression, we have that

$$P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\}$$
$$\lesssim \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_j \omega(\zeta+\varsigma, r) > y\right\} + o\Big(\overline{W}(y)\Big)$$
$$\lesssim \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\}.$$

Second, the lower bound of the asymptotic formula is proved. And

$$P\left\{\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\}$$
$$\geq P\left\{\bigcup_{\varsigma=1}^{m} \Big(\xi_\zeta \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\Big)\right\}$$

$$\geq \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\} - P\left\{\bigcap_{\varsigma=1}^{m}\left(\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y-L\right)\right\}$$
$$\geq \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\} - P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y-L\right\}$$
$$\gtrsim \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\} - o\left(\overline{W}(y)\right)$$
$$\gtrsim \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\}.$$

This ends the proof of this lemma when $0 < \phi_* \leq 1$.

Lemma 3. *Under the conditions of Theorem 1, we have that*

$$\sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) + c_0 > y\right\} \sim \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\}, \tag{3.3}$$

holds for any constants $c_0 \geq 0$ and $r > 0$ and the integer $m \geq 1$.

Proof. According to the condition of Theorem 1, $\sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) < \infty$. If $1 \leq \varsigma \leq M$ $(1 \leq M < m)$, the asymptotic equation holds, that is,

$$\sum_{\varsigma=1}^{M} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) + c_0 > y\right\} \sim \sum_{\varsigma=1}^{M} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\}. \tag{3.4}$$

If the asymptotic formula holds at $1 \leq \varsigma \leq m$, then the conclusion can be obtained.

For any $\epsilon > 0$, there exists $z > (1-\epsilon)z$ and a positive number c_1; when $\gamma > 0$, $\lim_{y\to\infty} \frac{\overline{W}(yz)}{\overline{W}(y)} = z^{\delta-1}e^{-\gamma z} \leq c_1 z^{\delta-1} < z^{\delta-1}$. And according to the independence and identically distributed property of $\{\xi_m\}_{m\geq 1}$ and Lemma 2, we can easily obtain that

$$\sum_{\varsigma=M+1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma, r) > y\right\}$$
$$= \sum_{\varsigma=M+1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta, r)\omega(1, r) > y\frac{\omega(1, r)}{\omega(\varsigma, r)}\right\}$$
$$= \sum_{\varsigma=M+1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta, r)\omega(1, r) > y\frac{\omega(1, r)}{\omega(\varsigma, r)}\right\}$$
$$\leq \sum_{\varsigma=M+1}^{m} c_1 l\left(y\frac{\omega(1, r)}{\omega(\varsigma, r)}\right)\left(y\frac{\omega(1, r)}{\omega(\varsigma, r)}\right)^{\alpha-1} e^{-\gamma y\frac{\omega(1,r)}{\omega(\varsigma,r)}}\left(\sum_{\zeta=0}^{m} \phi_i \omega(\zeta, r)\omega(1, r)\right)^{\alpha-1}$$
$$= \sum_{\varsigma=M+1}^{m} c_1 l\left(y\frac{\omega(1, r)}{\omega(\varsigma, r)}\right)\left\{\sum_{\zeta=0}^{m} y\frac{\left(\omega(1, r)\right)^2 \omega(\zeta, r)\phi_\zeta}{\omega(\varsigma, r)}\right\}^{\alpha-1} e^{-\gamma y\frac{\omega(1,r)}{\omega(\varsigma,r)}}$$

$$\lesssim c_1 \sum_{\varsigma=M+1}^{m} \left(\frac{1}{\omega(\varsigma,r)}\right)^{\alpha-1} o\Big(\overline{W}(y)\Big).$$

So

$$\begin{aligned}
&\sum_{\varsigma=1}^{M} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma,r) > y\right\} \\
&\le \sum_{\varsigma=1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma,r) > y\right\} \\
&= \sum_{\varsigma=1}^{M} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma,r) > y\right\} \quad + \sum_{\varsigma=M+1}^{m} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma,r) > y\right\} \\
&\lesssim \sum_{\varsigma=1}^{M} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma,r) > y\right\} + o\Big(\overline{W}(y)\Big) \\
&\lesssim \sum_{\varsigma=1}^{M} P\left\{\xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma,r) > y\right\}.
\end{aligned}$$

Combining this with (3.4), the lemma is proved.

4. The proof of Theorem 1

From (1.5), it can be seen that the ruin probability can be expressed as

$$\begin{aligned}
&\psi(y,m) = P\left\{\min_{1\le\kappa\le m} U_\kappa < 0 \mid U_0 = y\right\} \\
&= P\left\{\min_{1\le\kappa\le m} U_\kappa (1+r)^{-\kappa} < 0 \mid U_0 = y\right\} \\
&= P\left\{\min_{1\le\kappa\le m} \left(y + \sum_{\zeta=1}^{\kappa} (X_\zeta - Y_\zeta)(1+r)^{-\zeta}\right) < 0\right\} \\
&= P\left\{\max_{1\le\kappa\le m} \left(\sum_{\zeta=1}^{m} (Y_\zeta - X_\zeta)(1+r)^{-\zeta}\right) > y\right\} \\
&= P\left\{\max_{1\le\kappa\le m} \left(\sum_{\zeta=1}^{\kappa} (\sum_{\varsigma=1}^{\zeta} \phi_{\zeta-\varsigma}\xi_\varsigma + \phi_\zeta \xi_0 - X_j)(1+r)^{-\zeta}\right) > y\right\} \\
&= P\left\{\max_{1\le\kappa\le m} \left(\sum_{\zeta=1}^{\kappa} \sum_{\varsigma=1}^{\zeta} \phi_{\zeta-\varsigma}\xi_\varsigma \omega(\zeta,r) + \sum_{\zeta=1}^{\kappa} (\xi_0\phi_\zeta - X_\zeta)\omega(\zeta,r)\right) > y\right\}.
\end{aligned}$$

Considering the following relationship

$$\sum_{\varsigma=1}^{m} \xi_\varsigma \sum_{\zeta=0}^{m} \phi_\zeta \omega(\zeta+\varsigma,r) - \sum_{\zeta=1}^{m} X_\zeta \omega(\zeta,r)$$

$$
\begin{aligned}
&= \sum_{\zeta=1}^{m}\sum_{\varsigma=1}^{\zeta}\xi_{\varsigma}\phi_{\zeta-\varsigma}\omega(\zeta,r)-\sum_{\zeta=1}^{m}Y_{\zeta}\omega(\zeta,r)\\
&\le \max_{1\le\kappa\le m}\left\{\sum_{\zeta=1}^{\kappa}\sum_{\varsigma=1}^{\zeta}\phi_{\zeta-\varsigma}\xi_{\varsigma}\omega(\zeta,r)+\sum_{\zeta=1}^{\kappa}(\xi_0\phi_{\zeta}-X_{\zeta})\omega(\zeta,r)\right\}\\
&\le \sum_{\zeta=1}^{m}\sum_{\varsigma=1}^{\zeta}\phi_{\zeta-\varsigma}\xi_{\varsigma}\omega(\zeta,r)+\sum_{\zeta=1}^{m}\varepsilon_0\phi_{\zeta}\omega(\zeta,r)\\
&= \sum_{\varsigma=1}^{m}\xi_{\varsigma}\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+\varsigma,r)+\sum_{\zeta=1}^{m}\xi_0\phi_{\zeta}\omega(\zeta,r),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&P\left\{\sum_{\varsigma=1}^{m}\xi_{\varsigma}\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+\varsigma,r)-\sum_{\zeta=1}^{m}X_{\zeta}\omega(\zeta,r)>y\right\}\\
&\le \Psi(y,m)\\
&\le P\left\{\sum_{\varsigma=1}^{m}\xi_{\varsigma}\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+\varsigma,r)+\sum_{\zeta=1}^{m}\xi_0\phi_{\zeta}\omega(\zeta,r)>y\right\}.
\end{aligned}
$$

According to Lemmas 2 and 3 and the Fatou lemma,

$$
P\left\{\sum_{k=1}^{m}\xi_{\varsigma}\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+\varsigma,r)-\sum_{\zeta=1}^{m}X_{\zeta}\omega(\zeta,r)>y\right\}\sim P\left\{\sum_{\varsigma=1}^{m}\xi_{\varsigma}\sum_{\zeta=0}^{m}\phi_{i}\omega(\zeta+\varsigma,r)>y\right\}.
$$

Let $\Pi_0=0$ and $\Pi_m=P\left\{\sum_{\varsigma=1}^{m}\xi_{\varsigma}\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+\varsigma,r)>y\right\}$; we use mathematical induction to obtain the asymptotic formula of Π_m in two different situations.

(1) When $0<\phi_*<1$, the asymptotic formula of Π_m can be obtained.

First, when $m=1$,

$$
\begin{aligned}
&P(E_1>y)=P\left\{\xi_1\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)>y\right\}\\
&=P\left\{\xi_1\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)>y,\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)\in(0,c_2]\right\}\\
&+P\left\{\xi_1\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)>y,\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)\in(c_2,\phi_*)\right\}\\
&+P\left\{\xi_1\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)>y,\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)=\phi_*\right\}\\
&:=B_1+B_2+B_3,
\end{aligned}
$$

for any constant $0<c_2<\phi_*$; then, for B_1,

$$
\limsup_{y\to\infty}\frac{P\left\{\xi_1\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)>y,\sum_{\zeta=0}^{m}\phi_{\zeta}\omega(\zeta+1,r)\in(0,c_2]\right\}}{P(\xi_1>y)}
$$

$$
\begin{aligned}
&\leq \lim_{y\to\infty} \frac{P\Big(\xi_1 > \frac{y}{\sum_{\zeta=0}^{m}\phi_\zeta\omega(j+1,r)}\Big)}{P(\xi_1 > y)} \leq \lim_{y\to\infty} \frac{P(\xi_1 > \frac{y}{\phi_*})}{P(\xi_1 > y)} \\
&\leq \lim_{y\to\infty} \frac{l(y/\phi_*)(\frac{y}{\phi_*})^{\delta-1}e^{-\gamma y/\phi_*}}{l(x)y^{\delta-1}e^{-\gamma y}} = \lim_{y\to\infty} \frac{e^{-\gamma y(\frac{1}{\phi_*}-1)}}{\phi_*^{\delta-1}} \\
&\longrightarrow 0.
\end{aligned}
$$

For B_2,

$$
\begin{aligned}
&\lim_{c_2\nearrow\phi_*} \limsup_{y\to\infty} \frac{P\left\{\xi_1 \sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) > y, \sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) \in (c_2,\phi_*)\right\}}{P(\varepsilon > y)} \\
&\leq \lim_{c_2\nearrow\phi_*} P\left\{\sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) \in (c_2,\phi_*)\right\}\left(\sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r)\right)^{\delta-1} e^{-\gamma\sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r)} \\
&\longrightarrow 0.
\end{aligned}
$$

For B_3,

$$
\begin{aligned}
&P\left\{\xi_1 \sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) > y, \sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) = \phi_*\right\} \\
&= P\left\{\xi_1 \sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) > y\right\} P\left\{\sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) = \phi_*\right\} \\
&= P\left\{\xi_1 > \frac{y}{\sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r)}\right\} P\left\{\sum_{\zeta=0}^{m}\phi_\zeta\omega(\zeta+1,r) = \phi_*\right\} \\
&\sim p_* l\Big(\frac{y}{\phi_*}\Big)\Big(\frac{y}{\phi_*}\Big)^{\delta-1} e^{-\gamma\frac{y}{\phi_*}} \sim p_* l(y)^{\delta-1} e^{-\gamma\frac{y}{\phi_*}} \\
&= Ll(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}} \in \mathcal{SH}(\delta, \frac{\gamma}{\phi_*}).
\end{aligned}
$$

Combining B_1 and B_2 with B_3, $P(\Pi_1 > y) \sim Ll(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}} \in \mathcal{SH}(\delta, \frac{\gamma}{\phi_*})$ for $m = 1$; the expectation is expressed as follows:

$$
\begin{aligned}
E(\gamma\Pi_1) \quad &= -\int_0^\infty \gamma y dP(\Pi_1 > y) = L\gamma \int_0^\infty l(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}dy \\
&= L\gamma \int_0^\infty P(\Pi_1 > y)dy = L\gamma \int_0^\infty \Big(1 - P(\Pi_1 \leq y)\Big)dy \\
&< \infty.
\end{aligned}
$$

Second, suppose that $m = \kappa \geq 1$, $P(\Pi_\kappa > y) \sim LE(\gamma\frac{\Pi_{\kappa-1}}{\phi_*})l(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}} \in \mathcal{SH}(\delta, \frac{\gamma}{\phi_*})$ and $E(\gamma\Pi_\kappa) < \infty$ hold; then, the result is clearly valid when $m = 1$. We only need to prove that the above result is valid when $m = \kappa + 1$.

First of all,

$$\begin{aligned}
&P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y\right\}\\
&\geq P\Bigg(\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y - \phi_{\kappa+1}\sum_{\zeta=1}^{\kappa}\xi_\zeta\omega(\kappa+1+\zeta,r)\\
&\quad \mid \phi_{\kappa+1}\sum_{\zeta=1}^{\kappa}\xi_\zeta\omega(\kappa+1+\zeta,r) = 0\Bigg)P\Bigg(\phi_{\kappa+1}\sum_{\zeta=1}^{\kappa}\xi_\zeta\omega(\kappa+1+\zeta,r) = 0\Bigg)\\
&= P(\Pi_{\kappa+1} > y) = P\left\{\sum_{\varsigma=1}^{\kappa+1}\xi_\varsigma\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\zeta+\varsigma,r) > y\right\}\\
&= P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y - \phi_{\kappa+1}\sum_{\zeta=1}^{\kappa}\xi_\zeta\omega(\kappa+1+\zeta,r)\right\}\\
&\geq P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y\right\}.
\end{aligned}$$

We need to prove that

$$P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y\right\} \sim LE(e^{\gamma\Pi_\kappa})y^{\alpha-1}l(x)e^{-\gamma\frac{y}{\phi_*}}.$$

Furthermore,

$$\begin{aligned}
&P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y\right\}\\
&= P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, \Pi_\kappa > 0, \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > 0\right\}\\
&+P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, \Pi_n \leq 0, \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > 0\right\}\\
&+P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, \Pi_\kappa > 0, \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) \leq 0\right\}\\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Split the value of Π_n into $\triangle_t$,$\zeta = 1, 2, 3, \cdots$. There exists $0 < (1+\tau)\phi_* < 1$; then,

$$\begin{aligned}
I_1 &= P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, \Pi_\kappa > 0, \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta 1,r) > 0\right\}\\
&= P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, 0 < \Pi_\kappa < \frac{y}{1+\tau}, \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > 0\right\}
\end{aligned}$$

$$
\begin{aligned}
&+P\left\{\Pi_\kappa+\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)>y,\frac{y}{1+\tau}\le\Pi_\kappa\le y,\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)>0\right\}\\
&+P\left\{\Pi_\kappa+\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)>y,\Pi_\kappa>y,\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)>0\right\}\\
&:=I_{11}+I_{12}+I_{13},
\end{aligned}
$$

holds for any $0<\tau<1$. For any $\delta_0>0$ and sufficiently large y, when $0<\Pi_\kappa<\frac{y}{1+\tau}$, we have that

$$
\frac{l(y-\Pi_\kappa)(y-\Pi_\kappa)^{\delta-1}}{l(y)y^{\delta-1}}\le 2\Big(\frac{y-\Pi_\kappa}{y}\Big)^{\delta-1-\delta_0}\le 2\left\{\Big(\frac{\tau}{1+\tau}\Big)^{y-1-\delta_0}\vee 1\right\}.
$$

Based on the independent and identically distributed properties of $\xi_1,\xi_2,\cdots$, we have that

$$
\begin{aligned}
I_{11}&=P\left\{0<\Pi_\kappa<\frac{y}{1+\tau},\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_i\omega(\kappa+\zeta+1,r)>x-\Pi_\kappa\right\}\\
&=P\left\{0<\Pi_\kappa<\frac{y}{1+\tau}\}P\{\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)>y-\Pi_\kappa\right\}\\
&=\sum_{t=1}^{\infty}P\left\{\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)>y-\Pi_\kappa\mid\Pi_\kappa\in\triangle_t\right\}P(\Pi_\kappa\in\triangle_t)\\
&=\sum_{t=1}^{\infty}P\left\{\xi_{\kappa+1}>\frac{y-\Pi_\kappa}{\phi_*}\mid\Pi_\kappa\in\triangle_t\right\}P(\Pi_\kappa\in\triangle_t)P\Big(\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)=\phi_*\Big)\\
&\sim\sum_{t=1}^{\infty}l\Big(\frac{y-\Pi_\kappa}{\phi_*}\Big)\Big(\frac{y-\Pi_\kappa}{\phi_*}\Big)^{\delta-1}e^{-\frac{y-\Pi_\kappa}{\phi_*}}P(\Pi_\kappa\in\triangle_t)p_*\\
&\sim\sum_{t=1}^{\infty}l(y)\Big(\frac{y}{\phi_*}\Big)^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}e^{\gamma\frac{\Pi_\kappa}{\phi_*}}P(\Pi_\kappa\in\triangle_t)p_*\\
&\sim l(y)\Big(\frac{y}{\phi_*}\Big)^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}p_*E\Big(e^{\gamma\frac{\Pi_n}{\phi_*}}\Big)\\
&\sim l(y)\Big(\frac{y}{\phi_*}\Big)^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}p_*E\Big(e^{\gamma\frac{\Pi_\kappa}{\phi_*}}I_{(\Pi_\kappa>0)}\Big)\\
&\sim Ll(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}E\Big(e^{\gamma\frac{\Pi_\kappa}{\phi_*}}I_{(\Pi_\kappa>0)}\Big).
\end{aligned}
$$

When $0<\phi_*<1$,

$$
\begin{aligned}
I_{12}+I_{13}&=P\left\{\xi_{\kappa+1}>\frac{y-\Pi_\kappa}{\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)}>y-\Pi_\kappa,\frac{y}{1+\tau}<\Pi_\kappa\le y\right\}\\
&\quad+P\left\{\xi_{\kappa+1}>y-\Pi_\kappa,\Pi_\kappa>y\right\}\\
&\le 2P\Big(\xi_{\kappa+1}>y-\Pi_\kappa,\Pi_\kappa>\frac{y}{1+\tau}\Big)\le 2P\Big(\Pi_\kappa>\frac{y}{1+\tau}\Big)
\end{aligned}
$$

$$\sim 2LE\Big(e^{\gamma\Pi_{\kappa-1}}\Big)l\Big(\frac{y}{1+\tau}\Big)\Big(\frac{y}{1+\tau}\Big)^{\delta-1}e^{-\gamma\frac{y}{(1+\xi)\phi_*}}$$
$$\sim 2LE\Big(e^{\gamma\Pi_{\kappa-1}}\Big)l(y)\Big(\frac{y}{1+\tau}\Big)^{\delta-1}e^{-\gamma\frac{y}{(1+\xi)\phi_*}}.$$

Combining I_{11}, I_{12} and I_{13}, we can know that

$$I_1 \sim Ll(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}E\Big(e^{\gamma\frac{\Pi_\kappa}{\phi_*}}I_{(\Pi_\kappa>0)}\Big) + 2LE\Big(e^{\gamma\Pi_{\kappa-1}}\Big)l(y)\Big(\frac{x}{1+\tau}\Big)^{\delta-1}e^{-\gamma\frac{y}{(1+\xi)\phi_*}}$$
$$\sim Ll(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}E\Big(e^{\gamma\frac{\Pi_\kappa}{\phi_*}}I_{(\Pi_\kappa>0)}\Big).$$

And

$$I_2 = P\left\{\Pi_\kappa \le 0, \xi_{\kappa+1} > \frac{y-\Pi_\kappa}{\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)} > \frac{y-\Pi_\kappa}{\phi_*} > y-\Pi_\kappa\right\}$$
$$= \sum_{t=1}^{\infty} P\Big(\xi_{\kappa+1} > y-\Pi_\kappa \mid \Pi_\kappa \in \triangle_t\Big)P(\Pi_\kappa \in \triangle_t)$$
$$\sim \sum_{t=1}^{\infty} l(y-\Pi_\kappa)(y-\Pi_\kappa)^{\delta-1}e^{-\gamma(y-\Pi_\kappa)}P(\Pi_\kappa \in \triangle_t)$$
$$\sim \overline{W}(y)\sum_{t=1}^{\infty} e^{\gamma\Pi_\kappa}P(\Pi_\kappa \in \triangle_t).$$

If $\xi < 0$ is split into $\triangle_{t_1}$, $t_1 = 1, 2, \cdots$, then

$$I_3 = P\left\{\Pi_\kappa > 0, \frac{y-\Pi_\kappa}{\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)} < \xi_{\kappa+1} < 0\right\}$$
$$= P\{\Pi_\kappa > 0, \Pi_\kappa > y - \xi_{\kappa+1}\phi_*, \xi_{\kappa+1} < 0\}$$
$$= \sum_{t_1=1}^{\infty} P\{\Pi_\kappa > 0, \Pi_\kappa > y-\xi_{\kappa+1}\phi_* \mid \xi_{\kappa+1} \in \triangle_t\} P(\xi_{\kappa+1} \in \triangle_{t_1})$$
$$\sim \sum_{t_1=1}^{\infty} LE(e^{\gamma\Pi_{\kappa-1}})l(y-\xi_{\kappa+1}\phi_*)(y-\xi_{\kappa+1}\phi_*)^{\delta-1}e^{-\gamma\frac{y-\xi_{\kappa+1}\phi_*}{\phi_*}}P(\xi_{\kappa+1} \in \triangle_{t_1})$$
$$\sim \sum_{t_1=1}^{\infty} LE(e^{\gamma\Pi_{\kappa-1}})l(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}e^{\gamma\frac{\xi_{\kappa+1}}{\phi_*}}P(\xi_{\kappa+1} \in \triangle_{t_1})$$
$$= LE(e^{\gamma\Pi_{\kappa-1}})l(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}E\Big(e^{\gamma\xi_{\kappa+1}}I_{(\xi_{n+1}<0)}\Big)$$
$$= LE(e^{\gamma\Pi_{\kappa-1}})l(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}E\Big(e^{\gamma\xi}I_{(\xi<0)}\Big).$$

Combining I_1, I_2 and I_3, it can be seen that

$$P(\Pi_{\kappa+1} > y) \sim Ll(y)y^{\delta-1}e^{-\gamma\frac{y}{\phi_*}}E(e^{\gamma\frac{\Pi_\kappa}{\phi_*}}) \in \mathcal{SH}(\delta, \frac{\gamma}{\phi_*}),$$

and $E(e^{\gamma\Pi_{\kappa+1}}) < \infty$; so, (2.8) holds.

(2) When $\phi_* = 1$, the asymptotic formula of Π_m can be obtained.

Imitate the process of $0 < \phi_* < 1$ and prove that $P(\Pi_m > y) \sim \frac{L^m\gamma^{m-1}\left(\Gamma(\delta)\right)^m}{\Gamma(m\delta)}\left(l(y)\right)^m y^{m\delta-1}e^{-\gamma y} \in \mathcal{SH}(m\delta,\gamma)$. When $m = 1$, the conclusion is clearly valid. Suppose that the conclusion is valid when $m = \kappa$; we need to prove the asymptotic formula of $P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y\right\}$ when $m = \kappa+1$. Let $S_{\kappa+1} = \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r)$ and $S^c_{m+1} = (S_{\kappa+1} \mid S_\kappa > 0)$,$\Pi^c_\kappa = (\Pi_\kappa \mid \Pi_\kappa > 0)$; then,

$$\begin{aligned}
P(S^c_{\kappa+1} > y) &= P\left\{\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y \mid \xi_{\kappa+1} > 0\right\}\\
&= \frac{P\left(\xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, \xi_{\kappa+1} > 0\right)}{P(\xi_{\kappa+1} > 0)}\\
&\sim \frac{l(y)y^{\alpha-1}e^{-\gamma y}}{\overline{W}(0)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(\Pi^c_\kappa > y) &= P\{\Pi_\kappa > y \mid \Pi_\kappa > 0\}\\
&= \frac{P(\Pi_\kappa > y, \Pi_\kappa > 0)}{P(\Pi_\kappa > 0)} = \frac{P(\Pi_\kappa > y)}{P(\Pi_\kappa > 0)}\\
&\sim \frac{L^\kappa\gamma^{\kappa-1}\left(\Gamma(\delta)\right)^\kappa\left(l(y)\right)^\kappa y^{\kappa\delta-1}e^{-\gamma y}}{\Gamma(\kappa\delta)P(\Pi_\kappa > 0)}\\
&\sim \frac{L^\kappa\gamma^{\kappa-1}\left(\Gamma(\delta)\right)^\kappa l(y)y^{\kappa\delta-1}e^{-\gamma y}}{\Gamma(\kappa\delta)P(\Pi_\kappa > 0)}.
\end{aligned}$$

And

$$\begin{aligned}
&P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y\right\}\\
&= P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, S^c_{\kappa+1} > 0, \Pi^c_\kappa > 0\right\}\\
&+P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, S^c_{\kappa+1} > 0, \Pi^c_\kappa \le 0\right\}\\
&+P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1,r) > y, S^c_{\kappa+1} < 0, \Pi^c_\kappa > 0\right\}\\
&:= J_1 + J_2 + J_3.
\end{aligned}$$

For the tail probability of $\xi_{\kappa+1}$ and Π_n, it follows that

$$P(S_{\kappa+1} > y) \sim \frac{\gamma^{\delta-1}}{\Gamma(\delta)}y^{\delta-1}e^{\gamma y} \sim l_1(y)y^{\delta-1}e^{-\gamma y},$$

$$P(\Pi_\kappa > y) \sim \frac{L^\kappa \gamma^{\kappa-1}\Big(\Gamma(\delta)\Big)^\kappa}{\Gamma(\kappa\delta)}\Big(l(y)\Big)^\kappa y^{\kappa\delta-1}e^{-\gamma y} \sim l_2(y)y^{\kappa\delta-1}e^{-\gamma y}.$$

Then, according to Lemma 1,

$$\begin{aligned}
J_1 &= P(\Pi_\kappa^c + S_{\kappa+1}^c > y)P(\xi_{\kappa+1} > 0)P(\Pi_\kappa > 0)\\
&\sim \frac{\gamma\Gamma(\kappa\delta)\Gamma(\delta)}{\Gamma\Big((\kappa+1)\delta\Big)}\frac{L^\kappa\gamma^{\kappa-1}\Big(\Gamma(\delta)\Big)^\kappa}{\Gamma(\kappa\delta)}l_1(y)\Big(l_2(y)\Big)^\kappa x^{(\kappa+1)\delta-1}e^{-\gamma y}\\
&\sim \frac{L^\kappa\gamma^\kappa\Big(\Gamma(\delta)\Big)^{\kappa+1}}{\Gamma\Big((\kappa+1)\delta\Big)}\Big(l(y)\Big)^{\kappa+1}y^{(\kappa+1)\delta-1}e^{-\gamma y}\\
&\sim \frac{L^\kappa\gamma^\kappa\Big(\Gamma(\delta)\Big)^{\kappa+1}}{\Gamma\Big((\kappa+1)\delta\Big)}l(y)y^{(\kappa+1)\delta-1}e^{-\gamma y}.
\end{aligned}$$

By the control convergence theorem and $\Pi_\kappa \in \mathcal{SH}(\kappa\delta, \gamma)$, we have that

$$\begin{aligned}
J_2 &= P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1, r) > y, S_{\kappa+1}^c > 0, \Pi_\kappa^c \le 0\right\}\\
&= \sum_{t_2=1}^{\infty} P\left\{S_{\kappa+1} > y - \Pi_\kappa, S_{\kappa+1}^c > 0 \mid \Pi_\kappa \in \triangle_{t_2}\right\}P(\Pi_\kappa \in \triangle_{t_2})\\
&\sim \sum_{t_2=1}^{\infty} l(y-\Pi_\kappa)(y-\Pi_\kappa)^{\delta-1}e^{-\gamma(y-\Pi_\kappa)}P(\Pi_\kappa \in \triangle_{t_2})\\
&\sim l(y)y^{\delta-1}e^{-\gamma y}E\Big(e^{\gamma\Pi_\kappa}I_{(\Pi_\kappa\le 0)}\Big)\\
&\sim \Big(l(y)\Big)^{\kappa+1}y^{\kappa\delta+\delta-2}e^{-\gamma y}\\
&\sim o(1)\Big(l(y)\Big)^{\kappa+1}y^{(\kappa+1)\delta-1}e^{-\gamma y}\\
&\sim o(1)l(y)y^{(\kappa+1)\delta-1}e^{-\gamma y}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_3 &= P\left\{\Pi_\kappa + \xi_{\kappa+1}\sum_{\zeta=0}^{\kappa+1}\phi_\zeta\omega(\kappa+\zeta+1, r) > y, S_{\kappa+1}^c < 0, \Pi_\kappa^c > 0\right\}\\
&= \sum_{t_3=1}^{\infty} P\left\{\Pi_\kappa > y - S_{\kappa+1}, \Pi_\kappa^c > 0 \mid S_{\kappa+1} \in \triangle_{t_3}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\sim \sum_{t_3=1}^{\infty} \frac{L^\kappa \gamma^{\kappa-1}\Big(\Gamma(\delta)\Big)^\kappa}{\Gamma(\kappa\delta)} \Big(l(y-S_{\kappa-1})\Big)^\kappa y^{\kappa\delta-1} e^{-\gamma(y-S_{\kappa+1})} P(S_{\kappa+1} \in \triangle_{t_3})\\
&\sim \sum_{t_3=1}^{\infty} \frac{L^\kappa \gamma^{\kappa-1}\Big(\Gamma(\delta)\Big)^\kappa}{\Gamma(\kappa\delta)} \Big(l(y)\Big)^\kappa y^{\kappa\delta-1} e^{-\gamma y} E(e^{\gamma S_{\kappa+1}} I_{(S_{\kappa+1}<0)})\\
&\sim \Big(l(y)\Big)^{\kappa+1} y^{\kappa\delta+\delta-2} e^{-\gamma y} \frac{L^\kappa \gamma^{\kappa-1}\Big(\Gamma(\delta)\Big)^\kappa}{\Gamma(\kappa\delta)}\\
&\sim \Big(l(y)\Big)^{\kappa+1} y^{(\kappa+1)\delta-1} e^{-\gamma y} o(1)\\
&\sim l(y) y^{(\kappa+1)\delta-1} e^{-\gamma y} o(1).
\end{aligned}
$$

Combining J_1, J_2 and J_3, we have that

$$
\begin{aligned}
&P\left\{\Pi_\kappa + \xi_{\kappa+1} \sum_{\zeta=0}^{\kappa+1} \phi_\zeta \omega(\kappa+\zeta+1, r) > y\right\}\\
&= \frac{L^\kappa \gamma^\kappa \Big(\Gamma(\delta)\Big)^{\kappa+1}}{\Gamma\Big((\kappa+1)\delta\Big)} \Big(l(y)\Big)^{\kappa+1} y^{(\kappa+1)\delta-1} e^{-\gamma y}\\
&+2\Big(l(y)\Big)^{\kappa+1} y^{(\kappa+1)\delta-1} e^{-\gamma y} o(1)\\
&\sim \frac{L^\kappa \gamma^\kappa \Big(\Gamma(\delta)\Big)^{\kappa+1}}{\Gamma\Big((\kappa+1)\delta\Big)} \Big(l(y)\Big)^{\kappa+1} y^{(\kappa+1)\delta-1} e^{-\gamma y}\\
&\sim \frac{L^\kappa \gamma^\kappa \Big(\Gamma(\delta)\Big)^{\kappa+1}}{\Gamma\Big((\kappa+1)\delta\Big)} l(y) y^{(\kappa+1)\delta-1} e^{-\gamma y}.
\end{aligned}
$$

So when $m = \kappa + 1$, it follows that

$$
P(\Pi_{\kappa+1} > y) \sim \frac{L^{\kappa+1} \gamma^\kappa \Big(\Gamma(\delta)\Big)^{\kappa+1}}{\Gamma\Big((\kappa+1)\delta\Big)} l(y) y^{(\kappa+1)\delta-1} e^{-\gamma y} \in \mathcal{SH}\Big((\kappa+1)\delta, \gamma\Big).
$$

Thus, (2.9) holds. This completes the proof of Theorem 1.

5. Simulation study

Considering the AR(1) unilateral linear process based on (1.3), sequence $\{\zeta_m\}_{m\geq 1}$ follows the generalized Lindley distribution [9], that is, the random variable $\zeta \sim GL(\alpha, \theta, \gamma)$ has a density function

$$f_\zeta(z) = \frac{\gamma^{\alpha+1}}{(\theta+\gamma)\Gamma(\alpha+1)} z^{\alpha-1}(\alpha+\theta z)e^{-\gamma z}, \alpha, \theta, \gamma > 0, z > 0. \tag{5.1}$$

When $\theta = 0$, it degenerates into a gamma distribution with the parameters α, γ. When $\alpha = \theta$, it degenerates into a Lindley distribution. When $\alpha = 1$ and $\theta = 0$, it degenerates into a standard exponential distribution. At the same time, it is easy to find that $\zeta \in SH(\alpha, \gamma)$ by Definition 4. Assuming that $Y_m = v$, let $v = 20$, $\phi = 1$, $m = 100$ and $\phi_* = 0.5, 1$; simulate $T = 1000$ times. Four groups of the generalized Lindley distribution parameter set (α, θ, γ) were simulated: $(0.8, 0.5, 0.5)$, $(0.8, 0.5, 1)$, $(0.8, 1.5, 1)$,$(1.5, 1.5, 1)$; the process of generating random numbers can be found in [9]. First, random numbers denoted by $w_i, i = 1, 2, \cdots, m$ were generated by applying a uniform distribution, $U(0, 1)$, with sample size n; we then generated random numbers, denoted by $v_{1,i}$ that obey $gamma(\alpha, \gamma)$, and random numbers, denoted by $v_{2,i}$ that obey $gamma(\alpha + 1, \gamma)$. If $w_i \leq \frac{\gamma}{\gamma+\theta}$, then $\varepsilon_i = v_{1i}$. Otherwise, $\zeta_i = v_{2i}, i = 1, 2, \cdots, m$.

Due to mild fluctuations in the interest rate level near a certain benchmark interest rate, in the simulation, for a variable interest rate $r(i)$, when i was odd, we let $r(i) = r_1$; alternatively, when i was an even number, we applied $r(i) = r_2$, $i = 0, 1, 2, \cdots$, where the discount factor $\omega(i, r) = (1 + r(i))^{-i}, i = 0, 1, 2, \cdots$. The values of r_1 and r_2 and the initial reserve x can be found in Tables 1 and 2. For a constant interest rate r, the values were applied as 0.1, 0.15, and 0.2. The simulation results (Sim) for bankruptcy probability and theoretical estimation (Est) are shown in Tables 1 and 2, and ϕ_* is presented in the two tables. We obtained the ratio (Rat) of simulation results to estimated results. When $r_1 = r_2$, it is the bankruptcy probability under constant interest rates; otherwise, it is the bankruptcy probability under variable interest rates.

According to the simulation results shown in Tables 1 and 2, under the same parameter distribution background, if the initial reserve x remains unchanged, when $r_1 = r_2$ is a constant interest rate, the probability of bankruptcy decreases with the increase of the interest rate. When $r_1 \neq r_2$ is the variable interest rate and setting the interest rate as r_1 or r_2 will also cause a change in the bankruptcy probability, the exchange order between the two will correspondingly induce a change in the bankruptcy probability. Under the same parameter settings, if the interest rate r_1 and r_2 remain unchanged, the probability of bankruptcy will also decrease with the increase of the initial reserve x. Therefore, the larger the value of the initial reserve x and the interest rates r_1 and r_2, the smaller the corresponding bankruptcy probability; also, the rate at which its bankruptcy probability decreases will vary. When the parameter settings are different, if the value of the initial reserve x and the interest rates r_1 and r_2 remain unchanged, the bankruptcy probability will mainly be influenced by the values of the parameters α and γ. As α increases, the bankruptcy probability increases, while as γ increases, the bankruptcy probability decreases. Therefore, the tail shape of the claim amount distribution will affect the bankruptcy probability.

From the comparison of the simulation and estimated results in Tables 1 and 2, it can be seen that the larger the initial reserve x, the closer the bankruptcy probability ratio between the two will gradually approach 1, indicating that the estimation is more effective.

Table 1. The ruin probability under the generalized Lindley distribution with different values of (α, θ, γ) ($\phi_* = 0.5$).

x	r_1	r_2	(0.8, 0.5, 0.5)			(0.8, 0.5, 1)			(0.8, 1.5, 1)			(1.5, 1.5, 1)		
			Sim	Est	Rat	Sim	Est	Rat	Sim	Est	Rat	Sim	Est	Rat
5	0.1	0.2	0.1819	0.0942	1.9309	0.1545	0.0368	4.1978	0.3884	0.0918	4.2305	0.2267	0.093	2.4375
	0.05	0.2	0.2685	0.1358	1.9769	0.2348	0.0472	4.9742	0.5405	0.1447	3.7351	0.7085	0.1366	5.1865
	0.15	0.2	0.1893	0.1062	1.7828	0.1135	0.0414	2.7425	0.5724	0.1036	5.5254	0.5322	0.1044	5.0977
	0.15	0.15	0.2065	0.1041	1.9836	0.1631	0.0454	3.5929	0.3741	0.1085	3.4482	0.3719	0.1054	3.5287
	0.2	0.2	0.1397	0.0791	1.7659	0.0871	0.0307	2.8364	2.2572	0.8014	2.8166	0.2597	0.0798	3.255
	0.1	0.05	0.2618	0.1542	1.6977	0.2117	0.0491	4.3112	0.4740	0.147	3.2246	0.8403	0.1503	5.5911
	0.2	0.1	0.1488	0.0877	1.6969	0.1135	0.036	3.1526	0.2590	0.0865	2.994	0.3401	0.0849	4.0058
	0.1	0.1	0.2973	0.1217	2.4429	0.1884	0.0471	3.9999	0.5942	0.1468	4.0474	0.5760	0.1218	4.7291
10	0.1	0.2	0.1470	0.0892	1.6481	0.0513	0.0245	2.0923	0.2889	0.0849	3.4029	0.1270	0.0836	1.5193
	0.05	0.2	0.1804	0.1157	1.5593	0.0825	0.0348	2.3699	0.1642	0.1056	1.5547	0.2657	0.113	2.3514
	0.15	0.2	0.1480	0.1026	1.4428	0.0694	0.0309	2.2449	0.5187	0.105	4.9398	0.1995	0.1011	1.9734
	0.15	0.15	0.1829	0.1029	1.7778	0.0790	0.031	2.5472	0.2163	0.1053	2.054	0.2407	0.1022	2.355
	0.2	0.2	0.1465	0.0863	1.6972	0.0571	0.0227	2.5174	0.1478	0.0836	1.7685	0.1255	0.0754	1.6645
	0.1	0.05	0.1831	0.1165	1.5716	0.1497	0.0377	3.9696	0.2964	0.1152	2.5725	0.2238	0.1293	1.7311
	0.2	0.1	0.1095	0.0812	1.3483	0.0624	0.0244	2.5578	0.1680	0.0838	2.0047	0.1433	0.0807	1.7758
	0.1	0.1	0.1916	0.1061	1.8058	0.0548	0.0311	1.7635	0.3350	0.1068	3.137	0.2673	0.1056	2.5311
30	0.1	0.2	0.1001	0.0693	1.4443	0.0245	0.0148	1.6571	0.0996	0.0688	1.4484	0.0927	0.0653	1.4192
	0.05	0.2	0.0944	0.0794	1.1886	0.0328	0.0244	1.346	0.0843	0.079	1.0666	0.0991	0.0807	1.2283
	0.15	0.2	0.0776	0.0729	1.0651	0.0288	0.0181	1.5938	0.1410	0.0725	1.9443	0.1077	0.0713	1.5108
	0.15	0.15	0.1041	0.0761	1.3683	0.0275	0.0166	1.6555	0.1079	0.0754	1.431	0.1053	0.0764	1.3786
	0.2	0.2	0.0648	0.0528	1.2274	0.0110	0.0104	1.0591	0.0713	0.0561	1.2716	0.0627	0.0544	1.1532
	0.1	0.05	0.1103	0.0875	1.2604	0.0288	0.0257	1.1203	0.1090	0.0834	1.3075	0.1175	0.0852	1.3793
	0.2	0.1	0.0690	0.0635	1.0862	0.0192	0.013	1.4754	0.1099	0.0604	1.819	0.0807	0.0619	1.3041
	0.1	0.1	0.1092	0.0792	1.3786	0.0255	0.019	1.3424	0.1125	0.0775	1.4519	0.1167	0.079	1.4767
100	0.1	0.2	0.0291	0.0337	0.864	0.0134	0.0122	1.0959	0.0395	0.0335	1.1802	0.0411	0.0374	1.099
	0.05	0.2	0.0538	0.0518	1.0382	0.0118	0.0181	0.65	0.0424	0.0524	0.8092	0.0478	0.054	0.8852
	0.15	0.2	0.0427	0.0417	1.0238	0.0154	0.0152	1.0137	0.0599	0.0414	1.4466	0.0439	0.0427	1.0287
	0.15	0.15	0.0530	0.0466	1.1369	0.0165	0.0163	1.0114	0.0604	0.0468	1.2923	0.0478	0.0435	1.0995
	0.2	0.2	0.0334	0.0307	1.0874	0.0080	0.0103	0.7801	0.0301	0.0313	0.9624	0.0293	0.0326	0.8996
	0.1	0.05	0.0658	0.0587	1.1204	0.0182	0.0194	0.9396	0.0361	0.0518	0.6961	0.0758	0.0601	1.2611
	0.2	0.1	0.0334	0.0329	1.0164	0.0124	0.0109	1.1393	0.0329	0.0326	1.0091	0.0419	0.0338	1.2401
	0.1	0.1	0.0548	0.0471	1.1645	0.0238	0.0167	1.4225	0.0647	0.0479	1.3503	0.0562	0.0462	1.2167

Table 2. The ruin probability under the generalized Lindley distribution with different values of (α, θ, γ) $(\phi_* = 1)$.

x	r_1	r_2	$(0.8, 0.5, 0.5)$			$(0.8, 0.5, 1)$			$(0.8, 1.5, 1)$			$(1.5, 1.5, 1)$		
			Sim	Est	Rat	Sim	Est	Rat	Sim	Est	Rat	Sim	Est	Rat
5	0.1	0.2	0.3308	0.1024	3.2309	0.4056	0.0326	4.1988	0.2817	0.0974	2.8927	0.2089	0.0966	2.1624
	0.05	0.2	0.2780	0.1335	2.0822	0.3242	0.0355	2.2295	0.3943	0.1344	2.9337	0.4392	0.1454	3.0206
	0.15	0.2	0.3246	0.1062	3.0566	0.2773	0.0353	2.7023	0.2039	0.1054	1.9345	0.2376	0.1026	2.3149
	0.15	0.15	0.2938	0.1092	2.6906	0.3134	0.0359	2.8939	0.2331	0.1079	2.1603	0.1843	0.1083	1.7013
	0.2	0.2	0.1664	0.0881	1.8892	0.1863	0.0291	2.2258	0.1622	0.0864	1.8768	0.1804	0.0837	2.1548
	0.1	0.05	0.7508	0.1827	4.1092	0.3308	0.0361	1.9725	0.3794	0.1663	2.2816	0.4390	0.1677	2.6177
	0.2	0.1	0.2395	0.0992	2.4147	0.2324	0.0309	2.4593	0.3567	0.0969	3.6809	0.1727	0.0945	1.828
	0.1	0.1	0.4603	0.1129	4.0774	0.3053	0.0349	2.6475	0.2389	0.1124	2.1254	0.4361	0.1153	3.7825
10	0.1	0.2	0.2333	0.0926	2.5198	0.2753	0.027	2.9069	0.2447	0.0913	2.6804	0.1884	0.0947	1.9897
	0.05	0.2	0.2004	0.1257	1.594	0.3013	0.0324	2.2074	0.2312	0.1305	1.772	0.2727	0.1365	1.998
	0.15	0.2	0.1473	0.0922	1.5979	0.2000	0.0284	2.1525	0.1780	0.0924	1.9263	0.2026	0.0929	2.1805
	0.15	0.15	0.2273	0.093	2.4446	0.1756	0.0299	1.7828	0.1807	0.0937	1.9285	0.1513	0.0985	1.5365
	0.2	0.2	0.1447	0.0801	1.8069	0.1422	0.0248	1.7389	0.1291	0.083	1.5555	0.1361	0.0818	1.6636
	0.1	0.05	0.2611	0.1526	1.7108	0.2344	0.0357	1.667	0.2032	0.1335	1.5218	0.2100	0.1406	1.4932
	0.2	0.1	0.1461	0.0839	1.7411	0.1313	0.0256	1.5069	0.2847	0.0836	3.4053	0.1541	0.0871	1.7698
	0.1	0.1	0.3108	0.1089	2.8543	0.2028	0.0321	1.9747	0.1749	0.1063	1.6453	0.1546	0.1027	1.5049
30	0.1	0.2	0.0746	0.0499	1.4952	0.0993	0.0187	2.0217	0.0688	0.0493	1.3949	0.0788	0.0491	1.6043
	0.05	0.2	0.0892	0.0691	1.2916	0.1137	0.0243	1.697	0.0975	0.0664	1.469	0.0920	0.067	1.3725
	0.15	0.2	0.0657	0.051	1.2875	0.0869	0.0223	1.71	0.0643	0.0518	1.2408	0.0800	0.0508	1.5739
	0.15	0.15	0.0814	0.0577	1.4102	0.0710	0.0235	1.2526	0.0851	0.0541	1.5731	0.0818	0.0567	1.4424
	0.2	0.2	0.0469	0.0395	1.1875	0.0628	0.0135	1.6072	0.0396	0.0384	1.0325	0.0579	0.0391	1.4807
	0.1	0.05	0.1006	0.0696	1.4449	0.0718	0.0266	1.0378	0.0773	0.0631	1.2245	0.0895	0.0692	1.2935
	0.2	0.1	0.0718	0.0465	1.5446	0.0723	0.0174	1.5991	0.0508	0.0423	1.1998	0.0539	0.0452	1.193
	0.1	0.1	0.0988	0.0645	1.5324	0.0811	0.0239	1.3769	0.0761	0.0582	1.3072	0.0820	0.0589	1.3916
100	0.1	0.2	0.0392	0.0391	1.0023	0.0668	0.0135	1.7072	0.0393	0.0355	1.1084	0.0358	0.0368	0.9739
	0.05	0.2	0.0538	0.0477	1.1288	0.0436	0.0187	0.9135	0.0606	0.0534	1.1353	0.0432	0.0461	0.9375
	0.15	0.2	0.0422	0.0404	1.0454	0.0426	0.0146	1.0547	0.0324	0.0411	0.7876	0.0636	0.0417	1.5254
	0.15	0.15	0.0377	0.0429	0.878	0.0508	0.0151	1.1853	0.0612	0.0448	1.3665	0.0504	0.0437	1.1534
	0.2	0.2	0.0276	0.0343	0.8037	0.0351	0.0101	1.0228	0.0315	0.0314	1.0047	0.0436	0.0309	1.4098
	0.1	0.05	0.0668	0.0536	1.2471	0.0522	0.0245	0.9748	0.0630	0.0596	1.0569	0.0655	0.0524	1.2504
	0.2	0.1	0.0510	0.0362	1.4079	0.0366	0.0119	1.0097	0.0340	0.034	1.0008	0.0285	0.0312	0.9136
	0.1	0.1	0.0542	0.046	1.178	0.0364	0.0168	0.792	0.0568	0.0477	1.1915	0.0371	0.0438	0.8459

6. Conclusions

This article explored an insurance risk-dependent structure, where the discount factor is modeled as a function of both the interest rate and time. Specifically, we investigated the asymptotic ruin probability of a discrete risk model within a unilaterally dependent structure, incorporating claim stationary noise that exhibits semi-heavy-tailed behavior.

Use of AI tools declaration

We declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

We declare that we have no financial or personal relationship with other people or organizations.

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