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Research article

Existence and controllability of nonlinear evolution equation involving Hilfer fractional derivative with noise and impulsive effect via Rosenblatt process and Poisson jumps

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Abstract: This manuscript explores a new class of Hilfer fractional stochastic differential system, as driven by the Wiener process and Rosenblatt process through the application of non-instantaneous impulsive effects and Poisson jumps. Existence of a mild solution to the considered system is proved. Sufficient conditions for the controllability of the proposed control system are established. To prove our main results, we utilize fractional calculus, stochastic analysis, semigroup theory, and the Sadovskii fixed point theorem. In addition, to illustrate the theoretical findings, we present an example.

Keywords: impulsive effect; fractional calculus; stochastic differential system; control theory; semigroup theory
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1. Introduction

Fractional stochastic differential equations (SDEs) have been used in different fields, such as control theory, thermodynamics, signal processing, and so on (see [1–10]). The theory of impulsive SDEs has been more widely explored than classical order problems. Impulsive SDEs describ several phenomena as sudden updates of software, rhythmic beats, attacks of hackers, influence of the Internet market (see [11–13]). Impulsive actions that begin suddenly and continue for a specific period of time are called non-instantaneous impulses (see [14–16]). There are different forms of controllability such as (exact, local, approximate, null) controllability. In recent years, many authors have been interested in studying the controllability of a fractional stochastic differential system; for example, the approximate controllability of fractional SDEs driven by a Rosenblatt process with non-instantaneous impulses has been investigated in [17]. Approximate controllability and null controllability for conformable SDEs have been studied in [18]. The controllability and stability of fractional stochastic functional systems driven by a Rosenblatt process have been discussed in [19]. The existence of solutions and approximate controllability of fractional nonlocal neutral impulsive SDEs of order $1 < q < 2$ with infinite delay and Poisson jumps have been studied in [20]. Controllability of Prabhakar fractional dynamical systems has been established in [21]. Boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional Brownian motion and Poisson jumps has been studied in [22]. Approximate controllability for Hilfer fractional stochastic non-instantaneous impulsive differential systems with a Rosenblatt process and Poisson jumps has been established in [23]. Controllability of Hilfer fractional differential systems with and without impulsive conditions was studied in [24] by focusing on infinite delay. the existence and controllability of nonlocal mixed Volterra-Fredholm type fractional delay integro-differential equations of order $1 < r < 2$ have been discussed in [25]. Approximate controllability of delayed fractional stochastic differential systems with mixed noise and impulsive effects has been investigated in [26]. Motivated by the works mentioned previously, our objective here was to establish the existence of mild solution and controllability of Hilfer fractional stochastic differential systems driven by Wiener process and Rosenblatt process that are also subject to non-instantaneous impulsive effects and Poisson jumps of the following form:

$$\begin{cases} D_{0+}^{\aleph,\hbar}x(\mathfrak{d}) = -Ax(\mathfrak{d}) + \mathfrak{R}(\mathfrak{d}, x(\mathfrak{d})) + \mathfrak{I}(\mathfrak{d}, x(\mathfrak{d}))\frac{d\omega}{d\mathfrak{d}} + \sigma(\mathfrak{d}, x(\mathfrak{d}))\frac{dZ_H}{d\mathfrak{d}} \\ + \int_{\mathfrak{V}} h(\mathfrak{d}, x(\mathfrak{d}), \mathfrak{v})\tilde{N}(d\mathfrak{d}, d\mathfrak{v}), \quad \mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}], \ \kappa \in [0, \varsigma], \\ x(\mathfrak{d}) = \mathfrak{y}_\kappa(\mathfrak{d}, x(\mathfrak{d})), \quad \mathfrak{d} \in (\mathfrak{d}_\kappa, \wp_\kappa], \ \kappa \in [1, \varsigma], \\ I_{0+}^{(1-\aleph)(1-\hbar)}x(0) = x_0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} D_{0+}^{\aleph,\hbar}x(\mathfrak{d}) = -Ax(\mathfrak{d}) + \mathfrak{R}(\mathfrak{d}, x(\mathfrak{d})) + Bu(\mathfrak{d}) + \mathfrak{I}(\mathfrak{d}, x(\mathfrak{d}))\frac{d\omega}{d\mathfrak{d}} + \sigma(\mathfrak{d}, x(\mathfrak{d}))\frac{dZ_H}{d\mathfrak{d}} \\ + \int_{\mathfrak{V}} h(\mathfrak{d}, x(\mathfrak{d}), \mathfrak{v})\tilde{N}(d\mathfrak{d}, d\mathfrak{v}), \quad \mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}], \ \kappa \in [0, \varsigma], \\ x(\mathfrak{d}) = \mathfrak{y}_\kappa(\mathfrak{d}, x(\mathfrak{d})), \quad \mathfrak{d} \in (\mathfrak{d}_\kappa, \wp_\kappa], \ \kappa \in [1, \varsigma], \\ I_{0+}^{(1-\aleph)(1-\hbar)}x(0) = x_0, \end{cases} \tag{1.2}$$

where

$$D_{0+}^{\aleph,\hbar}f(\mathfrak{d}) = I_{0+}^{\aleph(1-\hbar)}\frac{d}{d\mathfrak{d}}I_{0+}^{(1-\aleph)(1-\hbar)}f(\mathfrak{d}), \ \ I^{\hbar}f(\mathfrak{d}) = \frac{1}{\Gamma(\hbar)}\int_0^{\mathfrak{d}} \frac{f(s)}{(\mathfrak{d}-s)^{1-\hbar}}ds, \ \ \mathfrak{d} > 0$$

is the Hilfer fractional derivative (HFD) of order $0 \le \aleph \le 1$, $0 < \hbar < 1$ [27]. F and $\mathfrak{K}$ are separable Hilbert spaces with $\| \ . \ \|$ and $\langle .,. \rangle$. Additionally, $-A$ generates an analytic semigroup $S(\mathfrak{d})$, $\mathfrak{d} \ge 0$, on F where $\| S(\mathfrak{d}) \| \le M$, $M \ge 1$. Let $\mathfrak{U}$ be a Hilbert space and $u(\cdot)$ be the control function in the Hilbert space of admissible control functions denoted by $(\mathfrak{L}_2(J, \mathfrak{U}))$. $B : \mathfrak{U} \to F$ is a bounded linear operator. $\mathfrak{y}_\kappa$, $\kappa = 1, 2, \ldots, \varsigma$ denotes the non-instantaneous impulsive function. Let $J = (0, T]$ and $0 = \wp_0 < \mathfrak{d}_1 \le \wp_1 \le \mathfrak{d}_2 < \ldots < \wp_{\varsigma-1} < \mathfrak{d}_\varsigma \le \wp_\varsigma \le \mathfrak{d}_{\varsigma+1} = T$. Suppose that $\{\omega(\mathfrak{d})\}_{\mathfrak{d}\ge 0}$ in $\mathfrak{K}$ is an $\mathcal{R}$-Wiener process on $(\Omega, \Upsilon, \{\Upsilon_{\mathfrak{d}}\}_{\mathfrak{d}\ge 0}, P)$ and $\{Z_H(\mathfrak{d})\}_{\mathfrak{d}\ge 0}$ in Hilbert space $\mathfrak{Y}$ is an $\mathcal{R}$-Rosenblatt process with $\frac{1}{2} < H < 1$ on $(\Omega, \Upsilon, \{\Upsilon_{\mathfrak{d}}\}_{\mathfrak{d}\ge 0}, P)$. Then, $\| \ . \ \|$ denotes the norm in F, $\mathfrak{K}$, $\mathfrak{Y}$, $\mathfrak{L}(\mathfrak{K}, F)$ and $\mathfrak{L}(\mathfrak{Y}, F)$. $\mathfrak{L}(\mathfrak{K}, F)$ and $\mathfrak{L}(\mathfrak{Y}, F)$ denote the spaces of all bounded linear operators from $\mathfrak{K}$ into F and $\mathfrak{Y}$ into F.

The given functions are defined as follows: $\mathfrak{R}$ from $(0, T] \times F$ into F, $\mathfrak{I}$ from $(0, T] \times F$ into $\mathfrak{L}(\mathfrak{K}, F)$, σ from $(0, T] \times F$ into $\mathfrak{L}_2^0(\mathfrak{Y}, F)$, h from $(0, T] \times F \times \mathfrak{V}$ into F and $\mathfrak{y}_\kappa$ from $(\mathfrak{d}_\kappa, \wp_\kappa] \times F$ into F.

The contributions of the present work are as follows:

• A new class of Hilfer fractional stochastic differential system driven by a Wiener process and Rosenblatt process through the application of non-instantaneous impulsive effects and Poisson jumps is introduced.

• Existence of a mild solution to the proposed system (1.1) is proved.

• Sufficient conditions for the controllability of the proposed control system (1.2) are established.

• An example is offered to illustrate the obtained results.

2. Preliminaries

The following definitions and lemmas are necessary in order to analyze the suggested problem.

In this paper, a complete probability space is denoted by (Ω, Υ, P) with a normal filtration $\Upsilon_{\mathfrak{d}}, \mathfrak{d} \in [0, T]$. Suppose that $(\mathfrak{V}, F, \ell(d\mathfrak{v}))$ is a σ-finite measurable space. $(p_{\mathfrak{d}})_{\mathfrak{d}\geq 0}$ is defined on (Ω, η, P) with values in $\mathfrak{V}$ and the characteristic measure ℓ. $N(\mathfrak{d}, d\mathfrak{v})$ denotes the counting measure of $p_{\mathfrak{d}}$ such that $\tilde{N}(\mathfrak{d}, \mathbb{G}) := E(N(\mathfrak{d}, \mathbb{G})) = \mathfrak{d}\ell(\mathbb{G}) \ \forall \ \mathbb{G} \in \Psi$. Additionally,$\tilde{N}(\mathfrak{d}, d\mathfrak{v}) := N(\mathfrak{d}, d\mathfrak{v}) - \mathfrak{d}\lambda(d\mathfrak{v})$ i.e., the Poisson martingale measure generated by $p_{\mathfrak{d}}$.

The operator $\mathcal{R} \in \mathfrak{L}(\mathfrak{Y}, \mathfrak{Y})$ is defined by $\mathcal{R}e_n = \lambda_n e_n$ with $Tr(\mathcal{R}) = \sum_{n=1}^{\infty} \lambda_n < \infty$, $\lambda_n \geq 0$ $(n = 1, 2, ...)$ and $\{e_n\}$ is a complete orthonormal basis in $\mathfrak{Y}$.

The $\mathcal{R}$-Rosenblatt process on $\mathfrak{Y}$ is defined by

$$Z_H(\mathfrak{d}) = Z_{\mathcal{R}}(\mathfrak{d}) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(\mathfrak{d}),$$

where $(z_n)_{n\geq 0}$ is a family of real, independent Rosenblatt processes.

Let $\mathfrak{L}_2^0 := \mathfrak{L}_2^0(\mathfrak{Y}, F)$ be the space of all $\mathcal{R}$-Hilbert Schmidt operators $\psi : \mathfrak{Y} \to F$.

$\psi \in \mathfrak{L}(\mathfrak{Y}, F)$ is an $\mathcal{R}$-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathfrak{L}_2^0} := \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \psi e_n\|^2 < \infty.$$

The space $\mathfrak{L}_2^0$ equipped with $\langle \vartheta, \psi \rangle_{\mathfrak{L}_2^0} = \sum_{n=1}^{\infty} \langle \vartheta e_n, \psi e_n \rangle$ is a separable Hilbert space.

Let the function $\phi(s)$; $s \in [0, T]$ exist in $\mathfrak{L}_2^0(\mathfrak{Y}, F)$.

We define the Wiener integral of ϕ with respect to $Z_{\mathcal{R}}$ as follows:

$$\int_0^{\mathfrak{d}} \phi(s) dZ_{\mathcal{R}}(s) = \sum_{n=1}^{\infty} \int_0^{\mathfrak{d}} \sqrt{\lambda} K_H^*(\phi e_n)(\mu_1, \mu_2) dB(\mu_1) dB(\mu_2),$$

where

$$K_H^*(\vartheta)(\mu_1, \mu_2) = \int_{\mu_1 \vee \mu_2}^{T} \vartheta(r) \frac{\partial K}{\partial r}(r, \mu_1, \mu_2) dr \ \ [28].$$

Lemma 2.1. [29] If $\psi : [0, T] \to \mathfrak{L}_2^0(\mathfrak{Y}, F)$ verifies $\int_0^T \|\psi(s)\|^2_{L_2^0} < \infty$, then the following holds:

$$E \left\| \int_0^{\mathfrak{d}} \psi(s) dZ_H(s) \right\|^2 \leq 2H\mathfrak{d}^{2H-1} \int_0^{\mathfrak{d}} \|\psi(s)\|^2_{L_2^0} ds.$$

Let $\mathfrak{L}_2(\Omega, F)$, with $\| x(\cdot) \|^2_{\mathfrak{L}_2(\Omega,F)} = E \| x(.,\omega) \|^2$ as a Banach space.

The Banach space of all continuous maps from $(0,T]$ into $\mathfrak{L}_2(\Omega,F)$ is denoted by $C((0,T],\mathfrak{L}_2(\Omega,F))$.

Here $\bar{C} = \{x : \mathfrak{d}^{(1-\aleph)(1-\hbar)}x(\mathfrak{d}) \in C((0,T],\mathfrak{L}_2(\Omega,F))\}$ with $\| \cdot \|_{\bar{C}} = (\sup_{\mathfrak{d}\in J} E|\mathfrak{d}^{(1-\aleph)(1-\hbar)}x(\mathfrak{d})|^2)^{\frac{1}{2}}$ is a Banach space.

We need the following hypotheses:

$(H1)$ $\mathfrak{R} : (0,T] \times F \to F$ verifies the following:

(i) $\mathfrak{R} : (0,T] \times F \to F$ is continuous;

(ii)$\forall\ q \in N$, $\exists\ f_q(\cdot) : (0,T] \to R^+$ such that (s.t.)

$$\sup_{\|x\|^2 \le q} E\|\mathfrak{R}(\mathfrak{d}, x)\|^2 \le f_q(\mathfrak{d}), \quad q > 0,$$

$s \to (\mathfrak{d} - s)^{\hbar-1} f_q(s) \in L^1((0,T], R^+)$ s.t.

$$\liminf_{q\to\infty} \frac{\int_0^{\mathfrak{d}} (\mathfrak{d} - s)^{\hbar-1} f_q(s)ds}{q} = \Lambda_1 < \infty,\ \mathfrak{d} \in (0,T],\ \Lambda_1 > 0.$$

$(H2)$ $\mathfrak{I} : (0,T] \times F \to \mathfrak{L}(\mathfrak{K}, F)$ verifies the following:

(i) $\forall\ \mathfrak{d} \in (0,T]$, $\mathfrak{I}(\mathfrak{d}, .) : F \to \mathfrak{L}(\mathfrak{K}, F)$ is continuous;

(ii) $\forall\ x \in F$; $\mathfrak{I}(., x) : (0,T] \to \mathfrak{L}(\mathfrak{K}, F)$ is $\Upsilon_{\mathfrak{d}}$-measurable;

(iii) $\forall\ q \in N$, $\exists\ h_q(\cdot) : (0,T] \to R^+$ s.t.

$$\sup_{\|x\|^2 \le q} E \parallel \mathfrak{I}(\mathfrak{d}, x) \parallel^2_{\mathcal{R}} \le h_q(\mathfrak{d}), \quad q > 0,$$

$s \to (\mathfrak{d} - s)^{\hbar-1} h_q(s) \in L^1((0,T], R^+)$ s.t.

$$\liminf_{q\to\infty} \frac{\int_0^{\mathfrak{d}} (\mathfrak{d} - s)^{\hbar-1} h_q(s)ds}{q} = \Lambda_2 < \infty, \quad \mathfrak{d} \in (0,T],\ \Lambda_2 > 0.$$

$(H3)$ $\sigma : (0,T] \times X \to \mathfrak{L}^0_2(\mathfrak{Y}, F)$ verifies the following:

(i) $\forall\ \mathfrak{d} \in J$, the function $\sigma(\mathfrak{d}, .) : F \to \mathfrak{L}^0_2(\mathfrak{Y}, F)$ is continuous and $\forall\ x \in F$; $\sigma(., x) : (0,T] \to \mathfrak{L}^0_2(\mathfrak{Y}, F)$ is $\Upsilon_{\mathfrak{d}}$-measurable;

(ii) $\forall\ q \in N$, $\exists\ \bar{h}_q(\cdot) : (0,T] \to R^+$ s.t.

$$\sup_{\|x\|^2 \le q} E \parallel \sigma(\mathfrak{d}, x) \parallel^2_{L^0_2} \le \bar{h}_q(\mathfrak{d}), \quad q > 0,$$

$s \to (\mathfrak{d} - s)^{\hbar-1} \bar{h}_q(s) \in \mathfrak{L}^1((0,T], R^+)$ s.t.

$$\liminf_{q\to\infty} \frac{\int_0^{\mathfrak{d}} (\mathfrak{d} - s)^{\hbar-1} \bar{h}_q(s)ds}{q} = \Lambda_3 < \infty,\ \mathfrak{d} \in (0,T], \Lambda_3 > 0.$$

$(H4)$ $h : (0,T] \times F \times V \to F$ verifies the following:

(i) $h : (0,T] \times F \times \mathfrak{V} \to F$ is continuous;

(ii) $\forall\ q \in N$, $\exists\, \chi_q(\cdot) : (0,T] \to R^+$ s.t.

$$\sup_{\|x\|^2 \le q} \int_{\mathfrak{V}} E\|h(\mathfrak{d}, x, \mathfrak{v})\|^2 \lambda(d\mathfrak{v}) \le \chi_q(\mathfrak{d}), \quad q > 0,$$

the function $s \to (\mathfrak{d}-s)^{\hbar-1}\chi_q(s) \in L^1((0,\mathfrak{d}],R^+)$, s.t.

$$\lim_{q\to\infty}\inf \frac{\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds}{q} = \Lambda_4 < \infty,\ \mathfrak{d}\in(0,T],\ \Lambda_4>0.$$

$(H5)$ $\mathfrak{y}_\kappa : (\mathfrak{d}_\kappa, \wp_\kappa]\times F \to F$ is continuous and verifies the following:
(i) $\exists\ M_3 > 0$, s.t.

$$E\ \|\mathfrak{y}_\kappa(\mathfrak{d},x)\|^2 \le M_3\ E\ \|x\|^2,\ x\in\mathfrak{X};\ \mathfrak{d}\in(\mathfrak{d}_\kappa,\wp_\kappa],\ \kappa=1,2,\ldots,\varsigma.$$

(ii) $\exists\ M_6 > 0$ s.t.

$$E\ \|\mathfrak{y}_\kappa(\mathfrak{d},x_1)-\mathfrak{y}_\kappa(\mathfrak{d},x_2)\|^2 \le M_4\ E\ \|x_1-x_2\|^2,\ x_1,x_2\in F;\ \mathfrak{d}\in(\mathfrak{d}_\kappa,\wp_\kappa],\ \kappa=1,2,\ldots,\varsigma.$$

Definition 2.1. [30] $x(\mathfrak{d}) : (0,T] \to F$ is a mild solution of (1.1) if $x_0 \in F\ \forall\ s\in[0,T)$, $P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))$ is integrable and

$$x(\mathfrak{d}) = \begin{cases} S_{\aleph,\hbar}(\mathfrak{d})x_0 + \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds + \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{I}(s,x(s))d\omega(s) \\ + \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s) + \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}),\ \mathfrak{d}\in(0,\mathfrak{d}_1] \\ \mathfrak{y}_\kappa(\mathfrak{d},x(\mathfrak{d})),\ \ \mathfrak{d}\in(\mathfrak{d}_\kappa,\wp_\kappa],\ \ \kappa=1,2,\ldots,\varsigma \\ S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)\mathfrak{y}_\kappa(\wp_\kappa,x(\wp_\kappa)) + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds \\ + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{I}(s,x(s))d\omega(s) + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s) \\ + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}), t\in(\wp_\kappa,\mathfrak{d}_{\kappa+1}],\ \kappa=1,2,\ldots,\varsigma \end{cases}$$

where

$$S_{\aleph,\hbar}(\mathfrak{d}) = I_{0+}^{\aleph(1-\hbar)}P_\hbar(\mathfrak{d}),\qquad P_\hbar(\mathfrak{d}) = \mathfrak{d}^{\hbar-1}T_\hbar(\mathfrak{d}),\qquad T_\hbar(\mathfrak{d}) = \int_0^\infty \hbar\varkappa\Psi_\hbar(\varkappa)S(\mathfrak{d}^\hbar\varkappa)d\varkappa, \tag{2.1}$$

$$\Psi_\hbar(\varkappa) = \sum_{n=1}^{\infty}\frac{(-\varkappa)^{n-1}}{(n-1)!\Gamma(1-n\hbar)}, \varkappa\in(0,\infty)$$

and $\int_0^\infty \varkappa^\tau\Psi_\hbar(\varkappa)d\varkappa = \frac{\Gamma(1+\tau)}{\Gamma(1+\hbar\tau)}$ for $\varkappa \ge 0$.

Lemma 2.2. [30] $S_{\aleph,\hbar}$ and $P_\hbar$ have the following conditions:
(i) $\{P_\hbar(\mathfrak{d}) : \mathfrak{d} > 0\}$ is continuous in the uniform operator topology.

$$(ii)\ \ \|P_\hbar(\mathfrak{d})x\| \le \frac{M\mathfrak{d}^{\hbar-1}}{\Gamma(\hbar)}\|x\|,\qquad \|S_{\aleph,\hbar}(\mathfrak{d})x\| \le \frac{M\mathfrak{d}^{(\aleph-1)(1-\hbar)}}{\Gamma(\aleph(1-\hbar)+\hbar)}\|x\|. \tag{2.2}$$

(iii) $\{P_\hbar(\mathfrak{d}) : \mathfrak{d} > 0\}$ and $\{S_{\aleph,\hbar}(\mathfrak{d}) : \mathfrak{d} > 0\}$ are strongly continuous.

3. Existence result

In this section, we prove the existence of a mild solution to the Hilfer fractional evolution system (1.1).

Theorem 3.1. If $(H1)$–$(H5)$ are verified, then the system (1.1) has a mild solution on J s.t.

$$
\begin{aligned}
& 25\left\{\frac{M_3M^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)}+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}[\Lambda_1+Tr(\mathcal{R})\Lambda_2+\Lambda_4]\right. \\
& \left.+\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\Lambda_3\right\}+T^{2(1-\aleph)(1-\hbar)}M_3 \\
< \; & 1 \qquad (3.1)
\end{aligned}
$$

and

$$
\gamma_1=\frac{4M^2T^{2(\aleph-1)(1-\hbar)}M_4}{\Gamma^2(\aleph(1-\hbar)+\hbar)}+M_4+\frac{4M^2T^{2\hbar}}{\Gamma^2(\hbar+1)}<1. \qquad (3.2)
$$

Proof. Define Φ on $\bar{C}$ as follows:

$$
(\Phi x)(\mathfrak{d})=\begin{cases}
S_{\aleph,\hbar}(\mathfrak{d})x_0+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds \\
+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{I}(s,x(s))d\omega(s)+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s) \\
+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}),\ \mathfrak{d}\in(0,\mathfrak{d}_1] \\
\mathfrak{y}_\kappa(\mathfrak{d},x(\mathfrak{d})),\quad \mathfrak{d}\in(\mathfrak{d}_\kappa,\wp_\kappa],\ \kappa=1,2,\ldots,\varsigma, \\
S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)\mathfrak{y}_\kappa(\wp_\kappa,x(\wp_\kappa))+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds \\
+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_0^s\mathfrak{I}(s,x(s))d\omega(s)+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s) \\
+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}),\mathfrak{d}\in(\wp_\kappa,\mathfrak{d}_{\kappa+1}],\ \kappa=1,2,\ldots,\varsigma.
\end{cases}
$$

Set $K_q=\{x\in\bar{C},\ \ \|x\|_{\bar{C}}^2\le q,\ \ q>0\}$.

Clearly, $K_q\subset\bar{C}$ is a bounded closed convex set in $\bar{C}$ $\forall$ q.

From $(H1)$, the Hölder inequality and Lemma 2.2, we get

$$
\begin{aligned}
& E\left\|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds\right\|^2 \\
\le\; & \frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\sup_{\|x\|^2\le q}E\|\mathfrak{R}(s,x(s))\|^2ds \\
\le\; & \frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}f_q(s)ds. \qquad (3.3)
\end{aligned}
$$

From Bochner's theorem, $P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))$ is integrable on J, so Φ is defined on K_q.

From Burkholder Gundy's inequality and $(H2)(ii)$, we get

$$
\begin{aligned}
E\left\|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{I}(s,x(s))d\omega(s)\right\|^2 &\le Tr(\mathcal{R})\frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\sup_{\|x\|^2\le q}E\|\mathfrak{I}(s,x(s))\|_Q^2\,ds \\
&\le Tr(\mathcal{R})\frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}h_q(s)ds. \qquad (3.4)
\end{aligned}
$$

From Burkholder Gundy's inequality and $(H3)(ii)$, we obtain

$$
\begin{aligned}
E\|\int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s)\|^2 &\leq \frac{2HM^2T^{2H+\hbar-1}}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\sup_{\|x\|^2\leq q}E\|\sigma(s,x(s))\|^2_{L_2^0}ds \\
&\leq \frac{2HM^2T^{2H+\hbar-1}}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\bar{h}_q(s)ds. \qquad (3.5)
\end{aligned}
$$

From the Hölder inequality and $(H4)(ii)$, we get

$$
\begin{aligned}
&E\|\int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v})\|^2 \\
\leq\ & \frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\left(\sup_{\|x\|^2\leq q}\int_{\mathfrak{V}} E\|h(s,x(s),\mathfrak{v})\|^2\lambda d\mathfrak{v}\right)ds \\
\leq\ & \frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds. \qquad (3.6)
\end{aligned}
$$

We claim that there exists $q>0$ s.t. $\Phi(K_q)\subseteq K_q$. If it is not true, then $\forall\ q>0$, $\exists\ x_q(\cdot)\in K_q$, but $\Phi(x_q)\notin K_q$, that is $\|(\Phi x_q)(\mathfrak{d})\|^2_{\bar{C}}>q$ for $\mathfrak{d}=\mathfrak{d}(q)\in J$. From (3.3)–(3.6), we have the following for $\mathfrak{d}\in(0,\mathfrak{d}_1]$

$$
\begin{aligned}
\|\Phi x_q\|^2_{\bar{C}} \leq\ & 25\sup_{\mathfrak{d}\in J}\mathfrak{d}^{2(1-\aleph)(1-\hbar)}\Big\{E\|S_{\aleph,\hbar}(\mathfrak{d})x_0\|^2+E\|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(\mathfrak{d},x(\mathfrak{d}))ds\|^2 \\
&+E\|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_0^s\mathfrak{I}(\tau,x(\tau))d\omega(\tau)ds\|^2 \\
&+E\|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s)\|^2 \\
&+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v})\|^2\Big\} \\
\leq\ & 25\Big\{\frac{M^2E\|x(0)\|^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)}+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}f_q(s)ds \\
&+Tr(\mathcal{R})\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}h_q(s)ds \\
&+\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}\bar{h}_q(s)ds \\
&+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds\Big\}. \qquad (3.7)
\end{aligned}
$$

From $(H5)$, we have the following for $\mathfrak{d}\in(\mathfrak{d}_\kappa,\wp_\kappa]$

$$
\|\Phi x_q\|^2_{\bar{C}}\leq\sup_{t\in J}\mathfrak{d}^{2(1-\aleph)(1-\hbar)}E\|\mathfrak{y}_\kappa(\mathfrak{d},x(\mathfrak{d}))\|^2\leq T^{2(1-\aleph)(1-\hbar)}M_3q. \qquad (3.8)
$$

From $(H5)$ and (3.3)–(3.6), we have for $\mathfrak{d}\in(\wp_\kappa,\mathfrak{d}_{\kappa+1}]$

$$\begin{aligned}\| \Phi x_q \|_{\bar{C}}^2 \le\ & 25 \sup_{t\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)} \Big\{ E\|S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)\mathfrak{y}_\kappa(\wp_\kappa, x(\wp_\kappa))\|^2 \\ & +E \left\| \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds \right\|^2 \\ & +E \left\| \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\wp_\kappa}^{s} \mathfrak{I}(\tau,x(\tau))d\omega(\tau)ds \right\|^2 \\ & +E \left\| \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s) \right\|^2 \\ & + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}) \|^2 \Big\} \\ \le\ & 25\Big\{ \frac{qM_3M^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)} + \frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)} \int_{\wp_\kappa}^{T} (T-s)^{\hbar-1} f_q(s)ds \\ & +Tr(\mathcal{R})\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)} \int_{\wp_\kappa}^{T} (T-s)^{\hbar-1} h_q(s)ds \\ & +\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)} \int_{\wp_\kappa}^{T} (T-s)^{\hbar-1} \bar{h}_q(s)ds \\ & +\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)} \int_{\wp_\kappa}^{T} (\mathfrak{d}-s)^{\hbar-1} \chi_q(s)ds \Big\}. \end{aligned} \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) in the inequality $q \le \| (\Phi x_q)(\mathfrak{d}) \|_{\bar{C}}^2$, dividing both sides by q, and taking the limit $q \to +\infty$, we get

$$\begin{aligned} & 25\Big\{ \frac{M_3M^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)} + \frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)} [\Lambda_1 + Tr(\mathcal{R})\Lambda_2 + \Lambda_4] \\ & +\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\Lambda_3 \Big\} + T^{2(1-\aleph)(1-\hbar)}M_3 \\ \ge\ & 1. \end{aligned}$$

From (3.1), this is a contradiction. Hence for $q > 0$, $\Phi(K_q) \subseteq K_q$.

Next we show that Φ has a fixed point on K_q, so (1.1) has a mild solution.

We split Φ into two components Π_1 and Π_2, where

$$(\Pi_1 x)(\mathfrak{d}) = \begin{cases} S_{\aleph,\hbar}(\mathfrak{d})x_0 + \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds, & \mathfrak{d} \in (0, \mathfrak{d}_1] \\ \mathfrak{y}_\kappa(\mathfrak{d}, x(\mathfrak{d})), \quad \mathfrak{d} \in (\mathfrak{d}_\kappa, \wp_\kappa], \quad \kappa = 1,2,\ldots,\varsigma, \\ S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)\mathfrak{y}_\kappa(\wp_\kappa, x(\wp_\kappa)) + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds, & \mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}], \quad \kappa = 1,2,\ldots,\varsigma. \end{cases}$$

$$(\Pi_2 x)(\mathfrak{d}) = \begin{cases} \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_0^s \mathfrak{I}(\tau,x(\tau))d\omega(\tau)ds \\ +\int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s) \\ +\int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}), \quad \mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}], \quad \kappa = 0,1,\ldots,\varsigma, \\ 0, \qquad \textit{otherwise}. \end{cases}$$

We prove that Π_1 satisfies a contraction condition.

Take $x_1,\ x_2 \in K_q$; then, by $(H1)$ and $(H5)$, we have:
for $\mathfrak{d} \in (0, \mathfrak{d}_1]$,

$$\begin{aligned} E \parallel (\Pi_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 &\leq 4E \parallel \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)[\mathfrak{R}(\mathfrak{d}, x_1(\mathfrak{d})) - \mathfrak{R}(\mathfrak{d}, x_2(\mathfrak{d}))]ds \parallel^2 \\ &\leq \frac{4M^2T^{2\hbar}}{\Gamma^2(\hbar+1)} E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2, \end{aligned} \tag{3.10}$$

for $\mathfrak{d} \in (\mathfrak{d}_\kappa, \wp_\kappa]$

$$\begin{aligned} E \parallel (\Pi_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 &\leq E \parallel \mathfrak{y}_\kappa(\mathfrak{d}, x_1(\mathfrak{d})) - \mathfrak{y}_\kappa(\mathfrak{d}, x_2(\mathfrak{d})) \parallel^2 \\ &\leq M_4\ E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 \end{aligned} \tag{3.11}$$

and for $\mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}]$

$$\begin{aligned} & E \parallel (\Pi_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 \\ \leq\ & 4E \parallel S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)(\mathfrak{y}_\kappa(\wp_\kappa, x_1(\wp_\kappa)) - \mathfrak{y}_\kappa(\wp_\kappa, x_2(\wp_\kappa))) \parallel^2 \\ & +4E \parallel \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)[\mathfrak{R}(\mathfrak{d}, x_1(\mathfrak{d})) - \mathfrak{R}(\mathfrak{d}, x_2(\mathfrak{d}))]ds \parallel^2 \\ \leq\ & 4\left[\frac{M^2T^{2(\aleph-1)(1-\hbar)}}{\Gamma^2(\aleph(1-\hbar)+\hbar)} M_4 + \frac{M^2T^{2\hbar}}{\Gamma^2(\hbar+1)}\right] E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 . \end{aligned} \tag{3.12}$$

Combining (3.10), (3.11) and (3.12) , we get

$$\begin{aligned} E \parallel (\Pi_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 &\leq \left[\frac{4M^2T^{2(\aleph-1)(1-\hbar)}M_4}{\Gamma^2(\aleph(1-\hbar)+\hbar)} + M_4 + \frac{4M^2T^{2\hbar}}{\Gamma^2(\hbar+1)}\right] E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 \\ &\leq \gamma_1 E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 . \end{aligned}$$

Taking $\sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)}$, we get

$$\sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)} E \parallel (\Pi_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 \leq \gamma_1 \sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)} E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2,$$

so,

$$\parallel \Pi_1 x_1 - \Pi_1 x_2 \parallel_{\bar{C}}^2 \leq \gamma_1 \parallel x_1 - x_2 \parallel_{\bar{C}}^2 .$$

Hence, Π_1 is a contraction.

We prove that Π_2 is compact.

First, we show that Π_2 is continuous on K_q.

Let $\{x_n\} \subseteq K_q$ with $x_n \to x$ in K_q and rewrite the control function $u(\mathfrak{d}) = u(\mathfrak{d}, x)$. Then, $\forall\ s \in J,\ x_n(s) \to x(s)$, and by $(H2)(i)$, $(H3)(i)$ and $(H4)(i)$, we have that $\mathfrak{I}(s, x_n(s)) \to \mathfrak{I}(s, x(s))$ as $n \to \infty$, $\sigma(s, x_n(s)) \to \sigma(s, x(s))$, as $n \to \infty$ and $h(s, x_n(s), v) \to h(s, x(s), v)$, as $n \to \infty$.

From the dominated convergence theorem, we have

$$
\begin{aligned}
&\| \Pi_2 x_n - \Pi_2 x \|_{\bar{C}}^2 \\
= \ & \sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)} \Big\{ \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s) \int_0^s [\Im(\tau, x_n(\tau)) - \Im(\tau, x(\tau))] d\omega(\tau) ds \\
&+ \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)[\sigma(s, x_n(s)) - \sigma(s, x(s))] dZ_H(s) \\
&+ \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s) \int_{\mathfrak{V}} [h(s, x_n(s), \mathfrak{v}) - h(s, x(s), \mathfrak{v})] \tilde{N}(ds, d\mathfrak{v}) \|^2 \Big\} \to 0,
\end{aligned}
$$

as $n \to \infty$, which is continuous.

We show that $\{\Pi_2 x : x \in K_q\}$ is equicontinuous.

Let $\epsilon > 0$ be small and $\wp_\kappa < \mathfrak{d}_\alpha < \mathfrak{d}_\beta \leq \mathfrak{d}_{\kappa+1}$; then,

$$
\begin{aligned}
&E \| (\Pi_2 x)(\mathfrak{d}_\beta) - (\Pi_2 x)(\mathfrak{d}_\alpha) \|^2 \\
\leq \ & E \| \int_{\wp_\kappa}^{\mathfrak{d}_\alpha - \epsilon} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \int_0^s \Im(\tau, x(\tau)) ds \|^2 \\
&+ E \| \int_{\mathfrak{d}_\alpha - \epsilon}^{\mathfrak{d}_\alpha} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \int_0^s \Im(\tau, x(\tau)) ds \|^2 \\
&+ E \| \int_{\mathfrak{d}_\alpha}^{\mathfrak{d}_\beta} P_\hbar(\mathfrak{d}_\beta - s) \int_0^s \Im(\tau, x(\tau)) d\omega(\tau) ds \|^2 \\
&+ E \| \int_{\wp_\kappa}^{\mathfrak{d}_\alpha - \epsilon} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \sigma(s, x(s)) dZ_H(s) \|^2 \\
&+ E \| \int_{\mathfrak{d}_\alpha - \epsilon}^{\mathfrak{d}_\alpha} (P_\hbar(t_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \sigma(s, x(s)) \|^2 \\
&+ E \| \int_{\mathfrak{d}_\alpha}^{\mathfrak{d}_\beta} P_\hbar(\mathfrak{d}_\beta - s) \sigma(s, x(s)) dZ_H(s) \|^2 \\
&+ E \| \int_{\wp_\kappa}^{\mathfrak{d}_\alpha - \epsilon} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \int_{\mathfrak{V}} h(s, x(s), \mathfrak{v}) \tilde{N}(ds, d\mathfrak{v}) \|^2 \\
&+ E \| \int_{\mathfrak{d}_\alpha - \epsilon}^{\mathfrak{d}_\alpha} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \int_{\mathfrak{V}} h(s, x(s), d\mathfrak{v}) \|^2 \\
&+ E \| \int_{\mathfrak{d}_\alpha}^{\mathfrak{d}_\beta} P_\hbar(\mathfrak{d}_\beta - s) \int_{\mathfrak{V}} h(s, x(s), \mathfrak{v}) \tilde{N}(ds, d\mathfrak{v}) \|^2 .
\end{aligned}
$$

Thus, when $\mathfrak{d}_\beta \to \mathfrak{d}_\alpha$ and $\epsilon \to 0$, then, $E \| (\Pi_2 x)(\mathfrak{d}_\beta) - (\Pi_2 x)(\mathfrak{d}_\alpha) \|^2 \to 0$ independent of $x \in q$. Also, we can show that $\Pi_2 x, \ x \in K_q$ are equicontinuous at $\mathfrak{d} = 0$. Hence Π_2 maps K_q into a family of equicontinuous functions.

In what follows, we show that $V(\mathfrak{d}) = \{(\Pi_2 x)(\mathfrak{d}) : x \in K_q\}$ is relatively compact in K_q. Clearly, $V(0) \in K_q$ is relatively compact.

Let $\wp_\kappa < \mathfrak{d} \leq \mathfrak{d}_{\kappa+1}$ be fixed and $\wp_\kappa < \epsilon < \mathfrak{d}$; we define the following for $x \in K_q, \ \rho > 0$:

$$\begin{aligned}
(\Pi_2^{\epsilon,\rho}x)(\mathfrak{d}) &= \hbar\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_0^s \Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\\
&+\hbar\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\sigma(s,x(s))d\varkappa dZ_H(s)\\
&+\hbar\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v})\\
&= \hbar S(\epsilon^\hbar\rho)\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa-\epsilon^\hbar\rho)\int_0^s \Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\\
&+\hbar S(\epsilon^\hbar\rho)\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa-\epsilon^\hbar\rho)\sigma(s,x(s))d\varkappa dZ_H(s)\\
&+\hbar S(\epsilon^\hbar\rho)\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa-\epsilon^\hbar\rho)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v}).
\end{aligned}$$

Since $S(\epsilon^\hbar\rho),\ \epsilon^\hbar\rho>0$ is a compact operator, it follows that $V^{\epsilon,\rho}(\mathfrak{d})=\{(\Pi_2^{\epsilon,\rho}x)(\mathfrak{d}):x\in K_q\}$ is relatively compact in $F\ \forall\ \wp_\kappa<\epsilon<\mathfrak{d},\ \rho>0$.

Furthermore, we have the following $\forall\ x\in K_q$:

$$\begin{aligned}
&\|\Pi_2 x-\Pi_2^{\epsilon,\rho}x\|_{\bar{C}}^2\\
\le\ &\sup_{\mathfrak{d}\in J}\mathfrak{d}^{2(1-\aleph)(1-\hbar)}\Big\{\hbar^2E\left\|\int_{\wp_\kappa}^{\mathfrak{d}}\int_0^\rho \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_0^s\Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\right\|^2\\
&+\hbar^2E\left\|\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_0^s\Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\right\|^2\\
&+\hbar^2E\left\|\int_{\wp_\kappa}^{\mathfrak{d}}\int_0^\rho \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\sigma(s,x(s))d\varkappa dZ_H(s)\right\|^2\\
&+\hbar^2E\left\|\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\sigma(s,x(s))d\varkappa dZ_H(s)\right\|^2\\
&\hbar^2E\left\|\int_{\wp_\kappa}^{\mathfrak{d}}\int_0^\rho \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v})\right\|^2\\
&+\hbar^2E\left\|\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}\int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v})\right\|^2\Big\}\\
\le\ &16T^{\hbar+2(1-\aleph)(1-\hbar)}\hbar M^2\Big\{\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\int_0^s E\|\Im(\tau,x(\tau))\|_Q^2\,d\tau ds\left(\int_\rho^\infty\varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2\\
&+Tr(\mathcal{R})\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\int_0^s E\|\Im(\tau,x(\tau))\|_Q^2\,d\tau ds\left(\int_\rho^\infty\varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2\\
&+2HT^{2H+-1}\int_{\wp_\kappa}^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}E\|\sigma(s,x(s))\|_{L_2^0}^2\,ds\left(\int_0^\rho\varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2\\
&+2HT^{2H-1}\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}E\|\sigma(s,x(s))\|_{L_2^0}^2\,ds\left(\int_\rho^\infty\varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2
\end{aligned}$$

$$+\int_{\wp_\kappa}^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\int_{\mathfrak{V}}E\parallel h(s,x(s),\mathfrak{v})\parallel^2\lambda d\mathfrak{v}ds\left(\int_0^{\rho}\varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2$$
$$+\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\int_{\mathfrak{V}}E\parallel h(s,x(s),\mathfrak{v})\parallel^2\lambda d\mathfrak{v}ds\left(\int_{\rho}^{\infty}\varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2\Bigg\}\to 0$$

as $\epsilon\to 0^+,\rho\to 0^+$. Thus $V(\mathfrak{d})\in K_q$ is relatively compact.

Therefore, from the Arzela-Ascoli theorem Π_2 is a compact operator. Hence, $\Phi=\Pi_1+\Pi_2$ is a condensing map on K_q, and by the Sadovskii fixed point theorem $\exists$ a fixed point $x(\cdot)$ for Φ on K_q. Thus, (1.1) has a mild solution on J.

4. Controllability result

In this section, we investigate the controllability of the system (1.2).

Definition 4.1. [29] $x(\mathfrak{d}):(0,T]\to F$ is a mild solution of (1.2) if $x_0\in F\ \forall\ s\in[0,T)$, $P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))$ is integrable and

$$x(\mathfrak{d})=\begin{cases}S_{\aleph,\hbar}(\mathfrak{d})x_0+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)Bu(s)ds+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{I}(s,x(s))d\omega(s)\\+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s)+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}),\ \mathfrak{d}\in(0,\mathfrak{d}_1]\\\mathfrak{y}_\kappa(\mathfrak{d},x(\mathfrak{d})),\quad\mathfrak{d}\in(\mathfrak{d}_\kappa,\wp_\kappa],\ \ \kappa=1,2,\ldots,\varsigma,\\S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)\mathfrak{y}_\kappa(\wp_\kappa,x(\wp_\kappa))+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds\\+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)Bu(s)ds+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{I}(s,x(s))d\omega(s)+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s)\\+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}),t\in(\wp_\kappa,\mathfrak{d}_{\kappa+1}],\ \kappa=1,2,\ldots,\varsigma.\end{cases}$$

Definition 4.2. Equation (1.2) is said to be controllable on J, if $\forall\ x_0,x_1\in F$ there exists a control $u\in\mathfrak{L}^2(J,\mathfrak{U})$ s.t. the mild solution $x(\mathfrak{d})$ of (1.2) verifies that $x(T)=x_1$, where x_1 is the preassigned terminal state and T is the time.

To investigate the result, we impose the following condition:

$(H6)$ We define $W:\mathfrak{L}^2(J,\mathfrak{U})\to F$ as follows:

$$Wu=\int_0^T P_\hbar(T-s)Bu(s)ds,$$

which has W^{-1} in $\mathfrak{L}^2(J,\mathfrak{U})\setminus\ker W$, where $\ker W=\{x\in\mathfrak{L}^2(J,\mathfrak{U}):Wx=0\}$ and $\exists\ M_B>0,\ M_w>0$ s.t. $\|B\|^2=M_B,\ \ \|W^{-1}\|^2=M_w$.

Theorem 4.1. If $(H1)$–$(H6)$ are verified, then the control system (1.2) is controllable on J s.t.

$$\left\{1+\frac{36M_wM^2T^{2\hbar}M_B^2}{\hbar^2\Gamma^2(\hbar)}\right\}\left\{\frac{M_3M^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)}+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}[\Lambda_1+Tr(\mathcal{R})\Lambda_2+\Lambda_4]\right.$$
$$\left.+\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\Lambda_3\right\}+T^{2(1-\aleph)(1-\hbar)}M_3+\frac{36M_wM^2T^{2\hbar}E\|x_1\|^2M_B^2}{\hbar^2\Gamma^2(\hbar)}<1,\tag{4.1}$$

and $\gamma_1<1$.

Proof. Define Δ on $\bar{C}$ as follows:

$$(\Delta x)(\eth) = \begin{cases} S_{\aleph,\hbar}(\eth)x_0 + \int_0^{\eth} P_{\hbar}(\eth - s)\mathfrak{R}(s, x(s))ds + \int_0^{\eth} P_{\hbar}(\eth - s)Bu(s)ds \\ + \int_0^{\eth} P_{\hbar}(\eth - s)\mathfrak{I}(s, x(s))d\omega(s) + \int_0^{\eth} P_{\hbar}(\eth - s)\sigma(s, x(s))dZ_H(s) \\ + \int_0^{\eth} P_{\hbar}(\eth - s) \int_{\mathfrak{V}} h(s, x(s), \mathfrak{v})\tilde{N}(ds, d\mathfrak{v}),\ \eth \in (0, \eth_1] \\ \mathfrak{y}_{\kappa}(\eth, x(\eth)),\ \ \eth \in (\eth_{\kappa}, \wp_{\kappa}],\ \ \kappa = 1, 2, \ldots, \varsigma, \\ S_{\aleph,\hbar}(\eth - \wp_{\kappa})\mathfrak{y}_{\kappa}(\wp_{\kappa}, x(\wp_{\kappa})) + \int_{\wp_{\kappa}}^{\eth} P_{\hbar}(\eth - s)\mathfrak{R}(s, x(s))ds + \int_{\wp_{\kappa}}^{\eth} P_{\hbar}(\eth - s)Bu(s)ds \\ + \int_{\wp_{\kappa}}^{\eth} P_{\hbar}(\eth - s) \int_0^s \mathfrak{I}(s, x(s))d\omega(s) + \int_{\wp_{\kappa}}^{\eth} P_{\hbar}(\eth - s)\sigma(s, x(s))dZ_H(s) \\ + \int_{\wp_{\kappa}}^{\eth} P_{\hbar}(\eth - s) \int_{\mathfrak{V}} h(s, x(s), \mathfrak{v})\tilde{N}(ds, d\mathfrak{v}), \eth \in (\wp_{\kappa}, \eth_{\kappa+1}],\ \kappa = 1, 2, \ldots, \varsigma, \end{cases}$$

where

$$u(\eth) = \begin{cases} W^{-1}\{x_1 - S_{\aleph,\hbar}(T)x_0 - \int_0^T P_{\hbar}(T - s)\mathfrak{R}(s, x(s))ds - \int_0^T P_{\hbar}(T - s)\mathfrak{I}(s, x(s))d\omega(s) \\ + \int_0^T P_{\hbar}(T - s)\sigma(s, x(s))dZ_H(s) + \int_0^T P_{\hbar}(T - s) \int_{\mathfrak{V}} h(s, x(s), \mathfrak{v})\tilde{N}(ds, d\mathfrak{v})\}(\eth),\ \eth \in (0, \eth_1], \\ W^{-1}\{x_1 - S_{\aleph,\hbar}(T - \wp_{\kappa})\mathfrak{y}_{\kappa}(\wp_{\kappa}, x(\wp_{\kappa})) + \int_{\wp_{\kappa}}^T P_{\hbar}(T - s)\mathfrak{R}(s, x(s))ds \\ + \int_{\wp_{\kappa}}^T P_{\hbar}(T - s) \int_0^s \mathfrak{I}(s, x(s))d\omega(s) + \int_{\wp_{\kappa}}^T P_{\hbar}(T - s)\sigma(s, x(s))dZ_H(s) \\ + \int_{\wp_{\kappa}}^T P_{\hbar}(T - s) \int_{\mathfrak{V}} h(s, x(s), \mathfrak{v})\tilde{N}(ds, d\mathfrak{v})\}(\eth), \eth \in (\wp_{\kappa}, \eth_{\kappa+1}]. \end{cases}$$

Set $\Re_q = \{x \in \bar{C},\ \ \| x \|_{\bar{C}}^2 \leq q,\ \ q > 0\}$.

Clearly, $\Re_q \subset \bar{C}$ is a bounded closed convex set in $\bar{C}\ \forall\ q$.

From $(H1)$, the Hölder inequality and Lemma 2.2, we get

$$\begin{aligned} E \| \int_0^{\eth} P_{\hbar}(\eth - s)\mathfrak{R}(s, x(s))ds \|^2 &\leq \frac{M^2 T^{\hbar}}{\hbar\Gamma^2(\hbar)} \int_0^{\eth} (\eth - s)^{\hbar - 1} \sup_{\|x\|^2 \leq q} E\|\mathfrak{R}(s, x(s))\|^2 ds \\ &\leq \frac{M^2 T^{\hbar}}{\hbar\Gamma^2(\hbar)} \int_0^{\eth} (\eth - s)^{\hbar - 1} f_q(s)ds. \end{aligned} \tag{4.2}$$

From Bochner's theorem, $P_{\hbar}(\eth - s)\mathfrak{R}(s, x(s))$ is integrable on J, so Δ is defined on $\Re_q$.

From Burkholder Gundy's inequality and $(H2)(ii)$, we get

$$\begin{aligned} E \| \int_0^{\eth} P_{\hbar}(\eth - s)\mathfrak{I}(s, x(s))d\omega(s)) \|^2 &\leq Tr(\mathcal{R})\frac{M^2 T^{\hbar}}{\hbar\Gamma^2(\hbar)} \int_0^{\eth} (\eth - s)^{\hbar - 1} \sup_{\|x\|^2 \leq q} E \| \mathfrak{I}(s, x(s)) \|_Q^2 ds \\ &\leq Tr(\mathcal{R})\frac{M^2 T^{\hbar}}{\hbar\Gamma^2(\hbar)} \int_0^{\eth} (\eth - s)^{\hbar - 1} h_q(s)ds. \end{aligned} \tag{4.3}$$

From Burkholder Gundy's inequality and $(H3)(ii)$, we obtain

$$\begin{aligned} E \| \int_0^{\eth} P_{\hbar}(\eth - s)\sigma(s, x(s))dZ_H(s)\|^2 &\leq \frac{2HM^2 T^{2H + \hbar - 1}}{\hbar\Gamma^2(\hbar)} \int_0^{\eth} (\eth - s)^{\hbar - 1} \sup_{\|x\|^2 \leq q} E\|\sigma(s, x(s))\|_{L_2^0}^2 ds \\ &\leq \frac{2HM^2 T^{2H + \hbar - 1}}{\hbar\Gamma^2(\hbar)} \int_0^{\eth} (\eth - s)^{\hbar - 1} \bar{h}_q(s)ds. \end{aligned} \tag{4.4}$$

From the Hölder inequality and $(H4)(ii)$, we get

$$\begin{aligned}
& E\parallel \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v})\|^2 \\
\leq\ & \frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\left(\sup_{\|x\|^2\leq q}\int_{\mathfrak{V}} E\parallel h(s,x(s),\mathfrak{v})\parallel^2 \lambda d\mathfrak{v}\right)ds \\
\leq\ & \frac{M^2T^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds. \qquad (4.5)
\end{aligned}$$

Also, from the Hölder inequality and $(H1)$–$(H6)$, we get

$$\begin{aligned}
E\|\int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)Bu(s)ds\|^2 &= E\|\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}T_\hbar(\mathfrak{d}-s)Bu(s)ds\|^2 \\
&\leq \frac{M^2T^\hbar M_B}{\hbar\Gamma^2(\hbar)}\int_0^{\mathfrak{d}}(\mathfrak{d}-s)^{\hbar-1}E\|u(s)\|^2ds,
\end{aligned}$$

where, for $\mathfrak{d}\in(0,\mathfrak{d}_1]$

$$\begin{aligned}
E\|u(s)\|^2 \leq\ & M_wE\|x_1\|^2+\frac{M^2M_wE\|x(0)\|^2T^{2(\aleph-1)(1-\hbar)}}{\Gamma^2(\aleph(1-\hbar)+\hbar)} \\
&+\frac{M^2M_wT^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}f_q(s)ds \\
&+Tr(\mathcal{R})\frac{M^2M_wT^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}h_q(s)ds \\
&+\frac{2HM^2M_wT^{2H+\hbar-1}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}\bar{h}_q(s)ds \\
&+\frac{M^2M_wT^\hbar}{\hbar\Gamma^2(\hbar)}\int_0^T(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds,
\end{aligned}$$

and for $\mathfrak{d}\in(\wp_\kappa,\mathfrak{d}_{\kappa+1}]$

$$\begin{aligned}
E\|u(s)\|^2 \leq\ & M_wE\|x_1\|^2+\frac{qM_3M^2M_wT^{2(\aleph-1)(1-\hbar)}}{\Gamma^2(\aleph(1-\hbar)+\hbar)} \\
&+\frac{M^2M_wT^\hbar}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(T-s)^{\hbar-1}f_q(s)ds \\
&+Tr(\mathcal{R})\frac{M^2M_wT^\hbar}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(T-s)^{\hbar-1}h_q(s)ds \\
&+\frac{2HM^2M_wT^{2H+\hbar-1}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(T-s)^{\hbar-1}\bar{h}_q(s)ds \\
&+\frac{M^2M_wT^\hbar}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
E\|\int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)Bu(s)ds\|^2 \le\ & \frac{M_wM^2T^{2\hbar}M_B}{\hbar^2\Gamma^2(\hbar)}\Big\{E\|x_1\|^2+\frac{M^2E\|x(0)\|^2T^{2(\aleph-1)(1-\hbar)}}{\Gamma^2(\aleph(1-\hbar)+\hbar)}\\
&+\frac{M^2T^{\hbar}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}f_q(s)ds\\
&+Tr(\mathcal{R})\frac{M^2T^{\hbar}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}h_q(s)ds\\
&+\frac{2HM^2T^{2H+\hbar-1}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}\bar{h}_q(s)ds\\
&+\frac{M^2T^{\hbar}}{\hbar\Gamma^2(\hbar)}\int_0^T(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds\Big\},\quad \mathfrak{d}\in(0,\mathfrak{d}_1]. \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
E\|\int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)Bu(s)ds\|^2 \le\ & \frac{M_wM^2T^{2\hbar}M_B}{\hbar^2\Gamma^2(\hbar)}\Big\{E\|x_1\|^2+\frac{qM_3M^2T^{2(\aleph-1)(1-\hbar)}}{\Gamma^2(\aleph(1-\hbar)+\hbar)}\\
&+\frac{M^2T^{\hbar}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(T-s)^{\hbar-1}f_q(s)ds\\
&+Tr(\mathcal{R})\frac{M^2T^{\hbar}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(T-s)^{\hbar-1}h_q(s)ds\\
&+\frac{2HM^2T^{2H+\hbar-1}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(T-s)^{\hbar-1}\bar{h}_q(s)ds\\
&+\frac{M^2T^{\hbar}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^T(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds\Big\},\quad \mathfrak{d}\in(\wp_\kappa,\mathfrak{d}_{\kappa+1}]. \qquad (4.7)
\end{aligned}
$$

We claim that there exists $q>0$ s.t. $\Delta(\mathfrak{K}_q)\subseteq\mathfrak{K}_q$. If it is not true, then $\forall\ q>0$, $\exists\ x_q(\cdot)\in\mathfrak{K}_q$, but $\Delta(x_q)\notin\mathfrak{K}_q$, that is $\|\ (\Delta x_q)(\mathfrak{d})\ \|_{\bar{C}}^2>q$ for $\mathfrak{d}=\mathfrak{d}(q)\in J$. From (4.2)–(4.7), we have the following for $\mathfrak{d}\in(0,\mathfrak{d}_1]$

$$
\begin{aligned}
\|\ \Delta x_q\ \|_{\bar{C}}^2 \le\ & 36\sup_{\mathfrak{d}\in J}\mathfrak{d}^{2(1-\aleph)(1-\hbar)}\Big\{E\|S_{\aleph,\hbar}(\mathfrak{d})x_0\|^2+E\ \|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(\mathfrak{d},x(\mathfrak{d}))ds\ \|^2\\
&+E\ \|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)Bu(s)ds\ \|^2\\
&+E\ \|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_0^s\mathfrak{I}(\tau,x(\tau))d\omega(\tau)ds\ \|^2\\
&+E\ \|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s)\ \|^2\\
&+\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v})\ \|^2\ \Big\}\\
\le\ & \Big\{1+\frac{36M_wM^2T^{2\hbar}M_B^2}{\hbar^2\Gamma^2(\hbar)}\Big\}\Big\{\frac{M^2E\|x(0)\|^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)}\\
&+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}f_q(s)ds+Tr(\mathcal{R})\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}h_q(s)ds
\end{aligned}
$$

$$+\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(T-s)^{\hbar-1}\bar{h}_q(s)ds+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_0^T(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds\Big\}$$
$$+\frac{36M_wM^2T^{2\hbar}E\|x_1\|^2M_B^2}{\hbar^2\Gamma^2(\hbar)}. \tag{4.8}$$

From $(\mathfrak{H})$, we have the following for $\mathfrak{d}\in(\mathfrak{d}_\kappa,\wp_\kappa]$

$$\|\Delta x_q\|_{\bar{C}}^2\leq\sup_{t\in J}\mathfrak{d}^{2(1-\aleph)(1-\hbar)}E\|\mathfrak{y}_\kappa(\mathfrak{d},x(\mathfrak{d}))\|^2\leq T^{2(1-\aleph)(1-\hbar)}M_3q. \tag{4.9}$$

From $(\mathfrak{H})$, (4.3)–(4.6) and (4.8), we have the following for $\mathfrak{d}\in(\wp_\kappa,\mathfrak{d}_{\kappa+1}]$

$$\begin{aligned}\|\Delta x_q\|_{\bar{C}}^2 \leq\ & 36\sup_{\mathfrak{d}\in J}\mathfrak{d}^{2(1-\aleph)(1-\hbar)}\Big\{E\|S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)\mathfrak{y}_\kappa(\wp_\kappa,x(\wp_\kappa))\|^2\\
&+E\left\|\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds\right\|^2+E\left\|\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)Bu(s)ds\right\|^2\\
&+E\left\|\int_0^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\wp_\kappa}^{s}\mathfrak{I}(\tau,x(\tau))d\omega(\tau)ds\right\|^2\\
&+E\left\|\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s)\right\|^2\\
&+\int_{\wp_\kappa}^{\mathfrak{d}}P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v})\|^2\Big\}\\
\leq\ & \Big\{1+\frac{36M_wM^2T^{2\hbar}M_B^2}{\hbar^2\Gamma^2(\hbar)}\Big\}\Big\{\frac{qM_3M^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)}\\
&+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^{T}(T-s)^{\hbar-1}f_q(s)ds\\
&+Tr(\mathcal{R})\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^{T}(T-s)^{\hbar-1}h_q(s)ds\\
&+\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^{T}(T-s)^{\hbar-1}\bar{h}_q(s)ds+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\int_{\wp_\kappa}^{T}(\mathfrak{d}-s)^{\hbar-1}\chi_q(s)ds\Big\}\\
&+\frac{36M_wM^2T^{2\hbar}E\|x_1\|^2M_B^2}{\hbar^2\Gamma^2(\hbar)}.\end{aligned} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10) in the inequality $q\leq\|(\Delta x_q)(\mathfrak{d})\|_{\bar{C}}^2$, dividing both sides by q, and taking the limit $q\to+\infty$, we get

$$\begin{aligned}&\Big\{1+\frac{36M_wM^2T^{2\hbar}M_B^2}{\hbar^2\Gamma^2(\hbar)}\Big\}\Big\{\frac{M_3M^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)}+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}[\Lambda_1+Tr(\mathcal{R})\Lambda_2+\Lambda_4]\\
&+\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\Lambda_3\Big\}+T^{2(1-\aleph)(1-\hbar)}M_3+\frac{36M_wM^2T^{2\hbar}E\|x_1\|^2M_B^2}{\hbar^2\Gamma^2(\hbar)}\geq 1.\end{aligned}$$

From (4.1), this is a contradiction. Hence for $q>0$, $\Delta(\mathfrak{K}_q)\subseteq\mathfrak{K}_q$.

Next we show that Δ has a fixed point on $\mathfrak{K}_q$, so (1.2) has a mild solution.

We split Δ into two components Δ_1 and Δ_2, where

$$(\Delta_1 x)(\mathfrak{d}) = \begin{cases} S_{\aleph,\hbar}(\mathfrak{d})x_0 + \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds, & \mathfrak{d} \in (0,\mathfrak{d}_1] \\ \mathfrak{y}_\kappa(\mathfrak{d},x(\mathfrak{d})), \quad \mathfrak{d} \in (\mathfrak{d}_\kappa, \wp_\kappa], \quad \kappa = 1,2,\ldots,\varsigma, \\ S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)\mathfrak{y}_\kappa(\wp_\kappa, x(\wp_\kappa)) + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\mathfrak{R}(s,x(s))ds, & \mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}], \quad \kappa = 1,2,\ldots,\varsigma. \end{cases}$$

$$(\Delta_2 x)(\mathfrak{d}) = \begin{cases} \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)Bu(s)ds + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_0^s \mathfrak{I}(\tau,x(\tau))d\omega(\tau)ds \\ + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\sigma(s,x(s))dZ_H(s) \\ + \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)\int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v}), \quad \mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}], \quad \kappa = 0,1,\ldots,\varsigma, \\ 0, \qquad otherwise. \end{cases}$$

We prove that Δ_1 satisfies a contraction condition.

Take $x_1,\ x_2 \in \mathfrak{K}_\mathfrak{q}$; then, by $(H1)$ and $(H5)$, we have the following:
for $\mathfrak{d} \in (0,\mathfrak{d}_1]$,

$$\begin{aligned} E \parallel (\Delta_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 &\le 4E \parallel \int_0^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)[\mathfrak{R}(\mathfrak{d},x_1(\mathfrak{d})) - \mathfrak{R}(\mathfrak{d},x_2(\mathfrak{d}))]ds \parallel^2 \\ &\le \frac{4M^2T^{2\hbar}}{\Gamma^2(\hbar+1)} E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2, \end{aligned} \tag{4.11}$$

for $\mathfrak{d} \in (\mathfrak{d}_\kappa, \wp_\kappa]$,

$$\begin{aligned} E \parallel (\Delta_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 &\le E \parallel \mathfrak{y}_\kappa(\mathfrak{d},x_1(\mathfrak{d})) - \mathfrak{y}_\kappa(\mathfrak{d},x_2(\mathfrak{d})) \parallel^2 \\ &\le M_4\, E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 \end{aligned} \tag{4.12}$$

and for $\mathfrak{d} \in (\wp_\kappa, \mathfrak{d}_{\kappa+1}]$,

$$\begin{aligned} & E \parallel (\Delta_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 \\ \le\ & 4E \parallel S_{\aleph,\hbar}(\mathfrak{d}-\wp_\kappa)(\mathfrak{y}_\kappa(\wp_\kappa,x_1(\wp_\kappa)) - \mathfrak{y}_\kappa(\wp_\kappa,x_2(\wp_\kappa))) \parallel^2 \\ & +4E \parallel \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)[\mathfrak{R}(\mathfrak{d},x_1(\mathfrak{d})) - \mathfrak{R}(\mathfrak{d},x_2(\mathfrak{d}))]ds \parallel^2 \\ \le\ & 4\left[\frac{M^2T^{2(\aleph-1)(1-\hbar)}}{\Gamma^2(\aleph(1-\hbar)+\hbar)}M_4 + \frac{M^2T^{2\hbar}}{\Gamma^2(\hbar+1)}\right] E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 . \end{aligned} \tag{4.13}$$

Combining (4.11), (4.12) and (4.13) , we get

$$\begin{aligned} E \parallel (\Delta_1 x_1)(\mathfrak{d}) - (\Pi_1 x_2)(\mathfrak{d}) \parallel^2 \quad &\le \left[\frac{4M^2T^{2(\aleph-1)(1-\hbar)}M_4}{\Gamma^2(\aleph(1-\hbar)+\hbar)} + M_4 + \frac{4M^2T^{2\hbar}}{\Gamma^2(\hbar+1)}\right] E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 \\ &\le \gamma_1 E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2 . \end{aligned}$$

Taking $\sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)}$, we get

$$\sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)} E \parallel (\Pi_1 x_1)(\mathfrak{d}) - (\Delta_1 x_2)(\mathfrak{d}) \parallel^2 \le \gamma_1 \sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)} E \parallel x_1(\mathfrak{d}) - x_2(\mathfrak{d}) \parallel^2,$$

so,

$$\| \Delta_1 x_1 - \Delta_1 x_2 \|_{\bar{C}}^2 \leq \gamma_1 \| x_1 - x_2 \|_{\bar{C}}^2 .$$

Hence, Δ_1 is a contraction.

We prove that Δ_2 is compact.

First, we show that Δ_2 is continuous on $\mathfrak{K}_{\mathfrak{q}}$.

Let $\{x_n\} \subseteq K_q$ with $x_n \to x$ in $\mathfrak{K}_{\mathfrak{q}}$ and rewrite the control function $u(\mathfrak{d}) = u(\mathfrak{d}, x)$. Then, $\forall\ s \in J,\ x_n(s) \to x(s)$, and by $(H2)(i)$, $(H3)(i)$ and $(H4)(i)$, we have that $\Im(s, x_n(s)) \to \Im(s, x(s))$ as $n \to \infty$, $\sigma(s, x_n(s)) \to \sigma(s, x(s))$ as $n \to \infty$ and $h(s, x_n(s), v) \to h(s, x(s), v)$ as $n \to \infty$.

From the dominated convergence theorem, we have

$$\begin{aligned}\| \Delta_2 x_n - \Delta_2 x \|_{\bar{C}}^2 &= \sup_{\mathfrak{d}\in J} \mathfrak{d}^{2(1-\aleph)(1-\hbar)} \Big\{ E \| \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s) B(u(s,x_n) - u(s,x)) ds \\ &+ \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s) \int_0^s [\Im(\tau, x_n(\tau)) - \Im(\tau, x(\tau))] d\omega(\tau) ds \\ &+ \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s)[\sigma(s, x_n(s)) - \sigma(s, x(s))] dZ_H(s) \\ &+ \int_{\wp_\kappa}^{\mathfrak{d}} P_\hbar(\mathfrak{d}-s) \int_{\mathfrak{V}} [h(s, x_n(s), \mathfrak{v}) - h(s, x(s), \mathfrak{v})] \tilde{N}(ds, d\mathfrak{v}) \|^2 \Big\} \to 0,\end{aligned}$$

as $n \to \infty$, which is continuous.

We show that $\{\Delta_2 x : x \in \mathfrak{K}_{\mathfrak{q}}\}$ is equicontinuous.

Let $\epsilon > 0$ be small and $\wp_\kappa < \mathfrak{d}_\alpha < \mathfrak{d}_\beta \leq \mathfrak{d}_{\kappa+1}$; then, we have

$$\begin{aligned}& E \| (\Delta_2 x)(\mathfrak{d}_\beta) - (\Delta_2 x)(\mathfrak{d}_\alpha) \|^2 \\ \leq\ & E \| \int_{\wp_\kappa}^{\mathfrak{d}_\alpha - \epsilon} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) Bu(s) ds \|^2 \\ & + E \| \int_{\mathfrak{d}_\alpha - \epsilon}^{\mathfrak{d}_\alpha} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) Bu(s) ds \|^2\ E \| \int_{\mathfrak{d}_\alpha}^{\mathfrak{d}_\beta} P_\hbar(\mathfrak{d}_\beta - s) Bu(s) ds \|^2 \\ & + E \| \int_{\wp_\kappa}^{\mathfrak{d}_\alpha - \epsilon} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \int_0^s \Im(\tau, x(\tau)) ds \|^2 \\ & + E \| \int_{\mathfrak{d}_\alpha - \epsilon}^{\mathfrak{d}_\alpha} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \int_0^s \Im(\tau, x(\tau)) ds \|^2 \\ & + E \| \int_{\mathfrak{d}_\alpha}^{\mathfrak{d}_\beta} P_\hbar(\mathfrak{d}_\beta - s) \int_0^s \Im(\tau, x(\tau)) d\omega(\tau) ds \|^2 \\ & + E \| \int_{\wp_\kappa}^{\mathfrak{d}_\alpha - \epsilon} (P_\hbar(\mathfrak{d}_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \sigma(s, x(s)) dZ_H(s) \|^2 \\ & + E \| \int_{\mathfrak{d}_\alpha - \epsilon}^{\mathfrak{d}_\alpha} (P_\hbar(t_\beta - s) - P_\hbar(\mathfrak{d}_\alpha - s)) \sigma(s, x(s)) \|^2 \\ & + E \| \int_{\mathfrak{d}_\alpha}^{\mathfrak{d}_\beta} P_\hbar(\mathfrak{d}_\beta - s) \sigma(s, x(s)) dZ_H(s) \|^2\end{aligned}$$

$$+E\parallel\int_{\wp_\kappa}^{\mathfrak{d}_\alpha-\epsilon}(P_\hbar(\mathfrak{d}_\beta-s)-P_\hbar(\mathfrak{d}_\alpha-s))\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v})\parallel^2$$
$$+E\parallel\int_{\mathfrak{d}_\alpha-\epsilon}^{\mathfrak{d}_\alpha}(P_\hbar(\mathfrak{d}_\beta-s)-P_\hbar(\mathfrak{d}_\alpha-s))\int_{\mathfrak{V}}h(s,x(s),d\mathfrak{v})\parallel^2$$
$$+E\parallel\int_{\mathfrak{d}_\alpha}^{\mathfrak{d}_\beta}P_\hbar(\mathfrak{d}_\beta-s)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})\tilde{N}(ds,d\mathfrak{v})\parallel^2.$$

Thus, when $\mathfrak{d}_\beta\to\mathfrak{d}_\alpha$ and $\epsilon\to 0$, $E\parallel(\Pi_2x)(\mathfrak{d}_\beta)-(\Delta_2x)(\mathfrak{d}_\alpha)\parallel^2\to 0$, independent of $x\in q$. Also, we can show that $\Pi_2x,\ x\in K_q$ are equicontinuous at $\mathfrak{d}=0$. Hence Δ_2 maps $\mathfrak{K}_\mathfrak{q}$ into a family of equicontinuous functions.

In what follows, we show that $V(\mathfrak{d})=\{(\Delta_2x)(\mathfrak{d}):x\in\mathfrak{K}_\mathfrak{q}\}$ is relatively compact in $\mathfrak{K}_\mathfrak{q}$. Clearly, $V(0)\in\mathfrak{K}_\mathfrak{q}$ is relatively compact.

Let $\wp_\kappa<\mathfrak{d}\le\mathfrak{d}_{\kappa+1}$ be fixed and $\wp_\kappa<\epsilon<\mathfrak{d}$; we define for $x\in\mathfrak{K}_\mathfrak{q},\ \rho>0$:

$$\begin{aligned}(\Delta_2^{\epsilon,\rho}x)(\mathfrak{d}) &= \hbar\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)Bu(s)d\varkappa ds\\ &\quad+\hbar\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_0^s\Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\\ &\quad+\hbar\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\sigma(s,x(s))d\varkappa dZ_H(s)\\ &\quad+\hbar\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v})\\ &= \hbar S(\epsilon^\hbar\rho)\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa-\epsilon^\hbar\rho)\int_0^s\Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\\ &\quad+\hbar S(\epsilon^\hbar\rho)\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa-\epsilon^\hbar\rho)\sigma(s,x(s))d\varkappa dZ_H(s)\\ &\quad+\hbar S(\epsilon^\hbar\rho)\int_{\wp_\kappa}^{\mathfrak{d}-\epsilon}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa-\epsilon^\hbar\rho)\int_{\mathfrak{V}}h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v}).\end{aligned}$$

Since $S(\epsilon^\hbar\rho),\ \epsilon^\hbar\rho>0$ is a compact operator, it follows that $V^{\epsilon,\rho}(\mathfrak{d})=\{(\Delta_2^{\epsilon,\rho}x)(\mathfrak{d}):x\in\mathfrak{K}_\mathfrak{q}\}$ is relatively compact in $F\ \forall\ \wp_\kappa<\epsilon<\mathfrak{d},\ \rho>0$.

Furthermore, we have the following, $\forall\ x\in K_q$:

$$\begin{aligned}&\parallel\Delta_2x-\Delta_2^{\epsilon,\rho}x\parallel_{\mathcal{C}}^2\\ \le&\ 16\sup_{t\in J}\mathfrak{d}^{2(1-\aleph)(1-\hbar)}\Big\{\hbar^2E\parallel\int_{\wp_\kappa}^{\mathfrak{d}}\int_0^\rho\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)Bu(s)d\varkappa ds\parallel^2\\ &+\hbar^2E\parallel\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)Bu(s)d\varkappa ds\parallel^2\\ &+\hbar^2E\parallel\int_{\wp_\kappa}^{\mathfrak{d}}\int_0^\rho\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_0^s\Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\parallel^2\\ &+\hbar^2E\parallel\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}}\int_\rho^\infty\varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\int_0^s\Im(\tau,x(\tau))d\omega(\tau)d\varkappa ds\parallel^2\end{aligned}$$

$$
\begin{aligned}
&+\hbar^2 E \parallel \int_{\wp_\kappa}^{\mathfrak{d}} \int_0^\rho \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\sigma(s,x(s))d\varkappa dZ_H(s) \parallel^2 \\
&+\hbar^2 E \parallel \int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}} \int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa)\sigma(s,x(s))d\varkappa dZ_H(s) \parallel^2 \\
&\hbar^2 E \parallel \int_{\wp_\kappa}^{\mathfrak{d}} \int_0^\rho \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa) \int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v}) \parallel^2 \\
&+\hbar^2 E \parallel \int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}} \int_\rho^\infty \varkappa(\mathfrak{d}-s)^{\hbar-1}\Psi_\hbar(\varkappa)S((\mathfrak{d}-s)^\hbar\varkappa) \int_{\mathfrak{V}} h(s,x(s),\mathfrak{v})d\varkappa\tilde{N}(ds,d\mathfrak{v}) \parallel^2 \Bigg\} \\
\leq\ & 16T^{\hbar+2(1-\aleph)(1-\hbar)}\hbar M^2 \Bigg\{ M_B^2 \int_{\wp_\kappa}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} E \parallel u(s) \parallel^2 ds \left(\int_0^\rho \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \\
&+M_B^2 \int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} E \parallel u(s) \parallel^2 ds \left(\int_\rho^\infty \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \\
&+\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} \int_0^s E \parallel \Im(\tau,x(\tau)) \parallel_Q^2 d\tau ds \left(\int_\rho^\infty \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \\
&+Tr(\mathcal{R}) \int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} \int_0^s E \parallel \Im(\tau,x(\tau)) \parallel_Q^2 d\tau ds \left(\int_\rho^\infty \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \\
&+2HT^{2H+-1} \int_{\wp_\kappa}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} E \parallel \sigma(s,x(s)) \parallel_{L_2^0}^2 ds \left(\int_0^\rho \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \\
&+2HT^{2H-1} \int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} E \parallel \sigma(s,x(s)) \parallel_{L_2^0}^2 ds \left(\int_\rho^\infty \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \\
&+\int_{\wp_\kappa}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} \int_{\mathfrak{V}} E \parallel h(s,x(s),\mathfrak{v}) \parallel^2 \lambda d\mathfrak{v} ds \left(\int_0^\rho \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \\
&+\int_{\mathfrak{d}-\epsilon}^{\mathfrak{d}} (\mathfrak{d}-s)^{\hbar-1} \int_{\mathfrak{V}} E \parallel h(s,x(s),\mathfrak{v}) \parallel^2 \lambda d\mathfrak{v} ds \left(\int_\rho^\infty \varkappa\Psi_\hbar(\varkappa)d\varkappa\right)^2 \Bigg\} \to 0
\end{aligned}
$$

as $\epsilon \to 0^+, \rho \to 0^+$. Thus $V(\mathfrak{d}) \in \Re_q$ is relatively compact.

Therefore, from the Arzela-Ascoli theorem Δ_2 is a compact operator. Hence, $\Delta = \Delta_1 + \Delta_2$ is a condensing map on $\Re_q$, and by the Sadovskii fixed point theorem $\exists$ a fixed point $x(\cdot)$ for Δ on K_q. Thus, (1.2) is controllable on J.

5. Application

Take into account the Hilfer fractional stochastic partial differential system driven by a Wiener process and Rosenblatt process through the application of non-instantaneous impulsive Poisson jumps and control functions as follows:

$$\begin{cases} D_{0+}^{\frac{2}{3},\frac{3}{4}}x(\mathfrak{d},z)+\frac{\partial^2}{\partial z^2}x(\mathfrak{d},z)=\frac{\tan\mathfrak{d}}{1+\tan\mathfrak{d}}x(\mathfrak{d},z)+\eta(\mathfrak{d},z)+e^{-\mathfrak{d}}x(\mathfrak{d},z)\frac{d\omega(\mathfrak{d})}{d\mathfrak{d}} \\ +\frac{\sin\mathfrak{d}}{1+\sin\mathfrak{d}}x(\mathfrak{d},z)\frac{dZ_H(\mathfrak{d})}{d\mathfrak{d}}+\int_{\mathfrak{V}}\bar{h}(\mathfrak{d},x(\mathfrak{d},z),\mathfrak{v})\tilde{N}(d\mathfrak{d},d\mathfrak{v}), \quad \mathfrak{d}\in(0,\frac{2}{3}]\cup(\frac{4}{3},2],\ 0\le z\le\pi, \\ x(\mathfrak{d},0)=x(\mathfrak{d},\pi)=0,\ \mathfrak{d}\in(0,2], \\ x(t,z)=\frac{2}{7}e^{-(\mathfrak{d}-\frac{2}{3})}\frac{|x(\mathfrak{d},z)|}{1+|x(\mathfrak{d},z)|}, \quad \mathfrak{d}\in(\frac{2}{3},\frac{4}{3}],\ 0\le z\le\pi, \\ I_{0+}^{\frac{1}{12}}(x(0,z))=x_0(z), \quad 0\le z\le\pi, \end{cases} \tag{5.1}$$

where $D_{0+}^{\frac{2}{3},\frac{3}{4}}$ is an HFD of order $\aleph=\frac{2}{3}$, $\hbar=\frac{3}{4}$, ω is a Wiener process and Z_H is a Rosenblatt process with parameter $\frac{1}{2}<H<1$.

Suppose that $F=\mathfrak{U}=\mathfrak{K}=\mathfrak{Y}=\mathfrak{L}_2([0,\pi])$ and $A\theta=-(\frac{\partial^2}{\partial z^2})\theta$ with $D(A)=\{\theta\in X:\theta,\ \frac{d\theta}{dz}$ are absolutely continuous, and $(\frac{d^2}{dz^2})\theta\in X,\ \theta(0)=\theta(\pi)=0\}$.

$-A$ generates a strongly continuous semigroup $S(\cdot)$ and has eigenvalues n^2, $n\in N$ with the following associated normalized eigenfunctions

$$e_n=\sqrt{\frac{2}{\pi}}\sin nx, \quad n=1,2,...$$

Then

$$-A\theta=\sum_{n=1}^{\infty}n^2\langle\theta,e_n\rangle e_n,\ \theta\in D(A)$$

and

$$S(\mathfrak{d})\theta=\sum_{n=1}^{\infty}e^{-n^2\mathfrak{d}}\langle\theta,e_n\rangle_{e_n},\ \theta\in X,\ \mathfrak{d}\ge0,$$

with $\|S(\mathfrak{d})\|\le e^{-\mathfrak{d}}\le1$.

$S_{\frac{2}{3},\frac{3}{4}}(\mathfrak{d})$ and $P_{\frac{3}{4}}(\mathfrak{d})$ can be respectively defined by

$$S_{\frac{2}{3},\frac{3}{4}}(\mathfrak{d})x=\frac{3}{4\Gamma(\frac{1}{6})}\int_0^{\mathfrak{d}}\int_0^{\infty}\varkappa(\mathfrak{d}-s)^{\frac{-5}{6}}s^{\frac{-1}{4}}\Psi_{\frac{3}{4}}(\varkappa)S(s^{\frac{3}{4}}\varkappa)xd\varkappa ds,$$

$$P_{\frac{3}{4}}(\mathfrak{d})x=\frac{3}{4}\int_0^{\infty}\varkappa\mathfrak{d}^{\frac{-1}{4}}\Psi_{\frac{3}{4}}(\varkappa)S(s^{\frac{3}{4}}\varkappa)xd\varkappa.$$

Clearly,

$$\|P_{\frac{3}{4}}(\mathfrak{d})\|\le\frac{\mathfrak{d}^{\frac{-1}{4}}}{\Gamma(\frac{3}{4})},\quad \|S_{\frac{2}{3},\frac{3}{4}}(\mathfrak{d})\|\le\frac{\mathfrak{d}^{\frac{-1}{12}}}{\Gamma(\frac{7}{12})}.$$

We define $B=I$ i.e., the identity operator, $\mathfrak{R}(\mathfrak{d},x)=\frac{\tan\mathfrak{d}}{1+\tan\mathfrak{d}}x(\mathfrak{d},z)$, $\mathfrak{I}(\mathfrak{d},x)(z)=e^{-\mathfrak{d}}x(\mathfrak{d},z)$, $\sigma(\mathfrak{d},x)(z)=\frac{\sin\mathfrak{d}}{1+\sin\mathfrak{d}}x(\mathfrak{d},z)$, $h=\bar{h}(\mathfrak{d},x(\mathfrak{d},z),\mathfrak{v})$, and $g_1(\mathfrak{d},x(\mathfrak{d}))=\frac{2}{7}e^{-(\mathfrak{d}-\frac{2}{3})}\frac{|x(\mathfrak{d},\cdot)|}{1+|x(\mathfrak{d},\cdot)|}$.

Therefore, all assumptions of Theorem 4.1 are verified, and

$$\left\{1+\frac{36M_wM^2T^{2\hbar}M_B^2}{\hbar^2\Gamma^2(\hbar)}\right\}\left\{\frac{M_3M^2}{\Gamma^2(\aleph(1-\hbar)+\hbar)}+\frac{M^2T^{1+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}[\Lambda_1+Tr(\mathcal{R})\Lambda_2+\Lambda_4]\right.$$
$$\left.+\frac{2HM^2T^{2H+(1-\hbar)(1-2\aleph)}}{\hbar\Gamma^2(\hbar)}\Lambda_3\right\}+T^{2(1-\aleph)(1-\hbar)}M_3+\frac{36M_wM^2T^{2\hbar}E\|x_1\|^2M_B^2}{\hbar^2\Gamma^2(\hbar)}<1,\quad \gamma_1<1.$$

Thus, (5.1) is controllable on $(0,2]$.

6. Conclusions

In this paper, we established a new class of Hilfer fractional stochastic differential system driven by a Wiener process and Rosenblatt process through the application of non-instantaneous impulsive effects and Poisson jumps. We proved the existence of the mild solution of system (1.1). Sufficient conditions for the controllability of (1.2) were established. Our results were obtained with the aid of fractional calculus, stochastic analysis, semigroup theory and the Sadovskii fixed point theorem. Finally, to explain the results, we offered an example.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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