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Research article

A new type of $\mathcal{R}$-contraction and its best proximity points

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Abstract: In this paper, we aim to overcome the problem given by Abkar et al. [*Abstr. Appl. Anal.*, **2013** (2013), 189567], and so to obtain real generalizations of fixed point results in the literature. In this direction, we introduce a new class of functions, which include $\mathcal{R}$-functions. Thus, we present a new type of $\mathcal{R}$-contraction and weaken $\mathcal{R}$-contractions that have often been studied recently. We also give a new definition of the P-property. Hence, we obtain some best proximity point results, including fixed point results for the new kind of $\mathcal{R}$-contractions. Then, we provide an example to show the effectiveness of our results. Finally, inspired by a nice and interesting technique, we investigate the existence of a best proximity point of the homotopic mappings with the help of our main result.

Keywords: $\mathcal{R}$-contractions; homotopy; best proximity point; fixed point
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1. Introduction and preliminaries

Recently, Khojasteh et al. [16] introduced a new concept of the so-called simulation function, and also $\mathcal{Z}$-contractions by using these functions. Moreover, with the help of these functions, a nice fixed point result was obtained which generalized many famous results [4, 5, 7, 10]. Now, we recall the definition of $\mathcal{Z}$-contractions and a related fixed point result:

Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a function. If ζ satisfies the following conditions, then it is said to be a simulation function:

(ζ_1) $\zeta(0,0) = 0$,

(ζ_2) $\zeta(p,q) < q - p$ for all $q, p > 0$,
(ζ_3) If $\{p_n\}$,$\{q_n\} \subseteq (0,\infty)$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$, then

$$\lim_{n\to\infty} \sup \zeta(p_n, q_n) < 0.$$

Theorem 1 ([16]). *Let $\Upsilon : \Lambda \to \Lambda$ be a mapping on a complete metric space (Λ, d) and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a simulation function. If the mapping Υ is a $\mathcal{Z}$-contraction w.r.t. ζ, that is, it satisfies*

$$\zeta(d(\Upsilon\varkappa, \Upsilon\eta), d(\varkappa, \eta)) \geq 0$$

for each $\varkappa, \eta \in \Lambda$, then Υ has a unique fixed point u in Λ. Also, the Picard sequence $\{\Upsilon^n \varkappa\}$ for any initial point $\varkappa \in \Lambda$ converges to u.

Then, Argoubi et al. [3] noticed that the condition (ζ_1) can be removed because of the fact that it is not used in the proof of Theorem 1. Another approach to these expansion efforts was achieved by Roldán-López-de-Hierro et al. [22] by modifying the condition (ζ_3) as follows:

(ζ_3)′ If $\{p_n\}$,$\{q_n\} \subseteq (0,\infty)$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$ and $p_n < q_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \sup \zeta(p_n, q_n) < 0.$$

Later, surprisingly it was proved that every $\mathcal{Z}$-contraction in the sense of Roldán-López-de-Hierro et al. is a Meir-Keeler contraction. To obtain a real larger family of contractions than the family of Meir-Keeler contractions, Roldán-López-de-Hierro et al. [23] introduced $\mathcal{R}$-functions, and hence $\mathcal{R}$-contractions with the help of these functions:

Definition 1. *Let $\emptyset \neq A \subseteq \mathbb{R}$. If a function $\varrho : A \times A \to \mathbb{R}$ satisfies the following conditions, then it is called an $\mathcal{R}$-function on A:*

(ϱ_1) If $\{p_n\} \subseteq (0,\infty) \cap A$ is a sequence satisfying $\varrho(p_{n+1}, p_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $p_n \to 0$.
(ϱ_2) If $\{p_n\}$,$\{q_n\} \subseteq (0,\infty) \cap A$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = L \geq 0$, $L < p_n$ and $\varrho(p_n, q_n) > 0$ for each $n \in \mathbb{N}$, then we have $L = 0$.

The following property for an $\mathcal{R}$-function ϱ on A is useful in some cases:

(ϱ_3) If $\{p_n\}$,$\{q_n\} \subseteq (0,\infty) \cap A$ are sequences satisfying $\varrho(p_n, q_n) > 0$ for each $n \in \mathbb{N}$ and $q_n \to 0$ as $n \to \infty$, then we get $p_n \to 0$.

Definition 2. *Let $\Upsilon : \Lambda \to \Lambda$ be a mapping on a metric space (Λ, d). If there is an $\mathcal{R}$-function ϱ on A satisfying $ran(d, \Lambda) = \{d(\varkappa, \eta) : \varkappa, \eta \in \Lambda\} \subseteq A$ and*

$$\varrho(d(\Upsilon\varkappa, \Upsilon\eta), d(\varkappa, \eta)) > 0$$

for each $\varkappa, \eta \in \Lambda$ with $\varkappa \neq \eta$, then Υ is called an $\mathcal{R}$-contraction with respect to ϱ.

On the other hand, the best proximity point theory has been considered as a new way of extending the results in fixed point theory. Let P and Q be non-empty subsets of a metric space (Λ, d) and $\Upsilon : P \to Q$ be a mapping. If $P \cap Q = \emptyset$, then Υ cannot have a fixed point. Then, it is natural to investigate the existence of a point $\varkappa \in P$ satisfying $d(\varkappa, \Upsilon\varkappa) = d(P, Q)$ that is said to be a best proximity point of Υ. It is obvious that each best proximity point of Υ is a fixed point of Υ in a special case $P = Q = \Lambda$. Also, a best proximity point of Υ is a solution for the optimization problem $\min_{\varkappa\in P} d(\varkappa, \Upsilon\varkappa)$. There are many papers on this topic in the literature due to these facts [6, 8, 9, 11–13, 15, 18–21, 24–26]. Now, we recall some fundamental notions and definitions about this theory:

Let P and Q be non-empty subsets of a metric space (Λ, d). Denote $P_0, Q_0 \subseteq \Lambda$ by

$$P_0 = \{\varkappa \in P : d(\varkappa, \eta) = d(P, Q) \text{ for some } \eta \in Q\}$$

and

$$Q_0 = \{\eta \in Q : d(\varkappa, \eta) = d(P, Q) \text{ for some } \varkappa \in P\},$$

respectively.

Definition 3 ([18]). *Let $\emptyset \neq P, Q$ be subsets of a metric space (Λ, d). Then, the pair (P, Q) is said to have a P-property if it satisfies*

$$\left.\begin{array}{l} d(\varkappa_1, \eta_1) = d(P, Q) \\ d(\varkappa_2, \eta_2) = d(P, Q) \end{array}\right\} \Rightarrow d(\varkappa_1, \varkappa_2) = d(\eta_1, \eta_2)$$

for all $\varkappa_1, \varkappa_2 \in P$ and $\eta_1, \eta_2 \in Q$.

In this paper, we first modify the condition (ϱ_2) in the definition of the $\mathcal{R}$-function to overcome a problem that will be mentioned in Section 2 in the proof of the main result (Theorem 27) in [23]. Then, considering a weaker condition $(\varrho_1)'$ than, (ϱ_1), we introduce a new concept of the so-called modified $\mathcal{R}$-function. Thus, we extend the family of $\mathcal{R}$-functions. We also change the definition of the P-property by considering a note given by Abkar et al. [2]. Further, we obtain some best proximity point results which are real generalizations of fixed point results for the new kind of $\mathcal{R}$-contractions. Hence, we generalize and unify some famous results in the literature. To show this fact, we provide an interesting example. Finally, inspired a nice and interesting technique used by Vetro et al. [28], we investigate the existence of a best proximity point of the homotopic mappings.

2. Main result

In the proof of the main result (Theorem 27) in [23], we notice that the authors need $L \leq p_n$ for all $n \in \mathbb{N}$ instead of $L < p_n$ for all $n \in \mathbb{N}$ in the condition (ϱ_2) to show the Picard sequence $\{\varkappa_n\}$ is a Cauchy sequence. To overcome this problem, we slightly modify the condition (ϱ_2) as $(\varrho_2)'$. So, in the rest of the paper and in Definition 2, it is reasonable to assume that an $\mathcal{R}$-function on A is a function $\varrho : A \times A \to \mathbb{R}$ satisfying the conditions (ϱ_1) and $(\varrho_2)'$. Also, taking into account the condition $(\varrho_1)'$ which is weaker than (ϱ_1), we introduce a new concept of the so-called modified $\mathcal{R}$-function as follows:

Definition 4. *Let $\emptyset \neq A \subseteq \mathbb{R}$. If a function $\varrho : A \times A \to \mathbb{R}$ satisfies the following conditions, then it is called a modified $\mathcal{R}$-function on A:*

$(\varrho_1)'$ *If $\{p_n\} \subseteq (0, \infty) \cap A$ is a sequence satisfying $\varrho(p_{n+1}, p_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$, then there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} \to 0$ as $k \to \infty$.*

$(\varrho_2)'$ *If $\{p_n\}, \{q_n\} \subseteq (0, \infty) \cap A$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = L \geq 0$, $L \leq p_n$ and $\varrho(p_n, q_n) > 0$ for all $n \in \mathbb{N}$, then we have $L = 0$.*

It is clear that every $\mathcal{R}$-function on A is a modified $\mathcal{R}$-function on A. Using modified $\mathcal{R}$-functions and taking into account the best proximity point theory, we introduce a new type contraction called a generalized $\mathcal{R}$-contraction. Hence, we enlarge the family of $\mathcal{R}$-contractions. Before this new concept, we present the following proposition that is important for our main result.

Proposition 1. *Let $\varrho : A \times A \to \mathbb{R}$ be a modified $\mathcal{R}$-function on A. Then, we have $\varrho(\varkappa, \varkappa) \leq 0$ for all $\varkappa \in (0, \infty) \cap A$.*

Proof. Assume the contrary. That is, there exists a point $\varkappa \in (0, \infty) \cap A$ such that $\varrho(\varkappa, \varkappa) > 0$. If we consider the sequence $p_n = \varkappa$ for all $n \in \mathbb{N}$, then we have $\varrho(p_{n+1}, p_n) > 0$ for all $n \in \mathbb{N}$. Thus, using the condition $(\varrho_1)'$, we conclude that there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} \to 0$ as $k \to \infty$, which contradicts $\varkappa > 0$. □

Now, we state our new concept.

Definition 5. *Let $\Upsilon : P \to Q$ be a mapping on a metric space (Λ, d) where $\emptyset \neq P, Q \subseteq \Lambda$. If there is a modified $\mathcal{R}$-function $\varrho : A \times A \to \mathbb{R}$ on A satisfying $ran(d, P \cup Q) = \{d(\varkappa, \eta) : \varkappa, \eta \in P \cup Q\} \subseteq A$ and*

$$\varrho\left(d(\Upsilon\varkappa, \Upsilon\eta), d(\varkappa, \eta)\right) > 0$$

for each $\varkappa, \eta \in P$ with $\varkappa \neq \eta$, then Υ is called a generalized $\mathcal{R}$-contraction with respect to ϱ.

On the other hand, it has been shown that the existence of a best proximity point under the P-property can be obtained by corresponding fixed point results by Abkar and Gabeleh [2]. Hence, to obtain a real generalization of fixed point results, we modify the definition of the P-property as follows:

Definition 6. *Let $\emptyset \neq P, Q$ be subsets of a metric space (Λ, d). Then, the pair (P, Q) is said to have a generalized P-property if it satisfies*

$$\left.\begin{array}{l} d(\varkappa_1, \eta_1) = d(P, Q) \\ d(\varkappa_2, \eta_2) = d(P, Q) \end{array}\right\} \Rightarrow d(\varkappa_1, \varkappa_2) = d(\eta_1, \eta_2)$$

for all $\varkappa_1, \varkappa_2 \in P$ with $\varkappa_1 \neq \varkappa_2$ and $\eta_1, \eta_2 \in Q$.

Remark 1. *As we know, every nonempty, bounded, closed, and convex pair in a strictly convex and reflexive Banach space Λ has the P-property, and so, it has the generalized P-property. But, there is a reflexive Banach space for which bounded, closed, and convex pairs in this space have the generalized P-property but do not have the P-property. Indeed, if we consider the set $\Lambda = \mathbb{R}^2$ with the maximum norm $\|\cdot\|_\infty$, then $(\Lambda, \|\cdot\|_\infty)$ is a reflexive Banach space, but it is not strictly convex. Suppose that*

$$P = \{(0, 1)\} \text{ and } Q = \{(x, 0) : 0 \leq x \leq 1\}.$$

Then, the pair (P, Q) is a bounded, closed, and convex pair. Also, we have $dist(P, Q) = \inf\{\|\varkappa - \eta\|_\infty : \varkappa \in P, \eta \in Q\} = 1, P_0 = P$ and $Q_0 = Q$. Since the set P is singleton, then the pair (P, Q) has the generalized P-property. But, it does not have the P-property. Indeed, although

$$\begin{aligned} \left\|(0,1) - \left(\frac{1}{2}, 0\right)\right\|_\infty &= 1 = dist(P, Q), \\ \|(0,1) - (0,0)\|_\infty &= 1 = dist(P, Q), \end{aligned}$$

we have

$$\|(0,1) - (0,1)\|_\infty = 0 \neq \frac{1}{2} = \left\|(0,0) - \left(\frac{1}{2}, 0\right)\right\|_\infty.$$

Now, we present our main result:

Theorem 2. *Let $\Upsilon : P \to Q$ be a mapping on a complete metric space (Λ, d) where P and Q are closed subsets of Λ. Assume that $P_0 \neq \emptyset$, $\Upsilon(P_0) \subseteq Q_0$ and the pair (P, Q) has the generalized P-property. Suppose that Υ is a generalized $\mathcal{R}$-contraction with respect to ϱ. If one of the following conditions is satisfied:*

(i) Υ is continuous,
(ii) The function ϱ satisfies the condition (ϱ_3),

then Υ has a unique best proximity point in P.

Proof. Let $\varkappa_0 \in P_0$ be an arbitrary point. Since $\Upsilon\varkappa_0 \in \Upsilon(P_0) \subseteq Q_0$, there exists $\varkappa_1 \in P_0$ satisfying

$$d(\varkappa_1, \Upsilon\varkappa_0) = d(P, Q).$$

Also, since $\Upsilon\varkappa_1 \in \Upsilon(P_0) \subseteq Q_0$, there exists $\varkappa_2 \in P_0$ satisfying

$$d(\varkappa_2, \Upsilon\varkappa_1) = d(P, Q).$$

In this way, we can construct a sequence $\{\varkappa_n\}$ in P_0 such that

$$d(\varkappa_{n+1}, \Upsilon\varkappa_n) = d(P, Q) \tag{2.1}$$

for all $n \in \mathbb{N} \cup \{0\}$. If $\varkappa_{n_0} = \varkappa_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then from (2.1) we have

$$d(\varkappa_{n_0}, \Upsilon\varkappa_{n_0}) = d(P, Q),$$

and so the proof is completed. Hence, we assume that $\varkappa_n \neq \varkappa_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Then, since the pair (P, Q) has the generalized P-property, from (2.1) we get

$$d(\varkappa_n, \varkappa_{n+1}) = d(\Upsilon\varkappa_{n-1}, \Upsilon\varkappa_n) \tag{2.2}$$

for all $n \in \mathbb{N}$. Also, since Υ is a generalized $\mathcal{R}$-contraction w.r.t. ϱ, we obtain

$$\varrho(d(\Upsilon\varkappa_{n-1}, \Upsilon\varkappa_n), d(\varkappa_{n-1}, \varkappa_n)) > 0$$

for all $n \in \mathbb{N}$, and so from (2.2), we get

$$\varrho(d(\varkappa_n, \varkappa_{n+1}), d(\varkappa_{n-1}, \varkappa_n)) > 0 \tag{2.3}$$

for all $n \in \mathbb{N}$. Therefore, if we denote a sequence $\{p_n\}$ by $p_n = d(\varkappa_{n-1}, \varkappa_n)$ for all $n \in \mathbb{N}$, then from (2.3) we have $p_n > 0$ and $\varrho(p_{n+1}, p_n) > 0$ for all $n \in \mathbb{N}$. Also, since $\{p_n\} \subseteq (0, \infty) \cap A$, using the condition $(\varrho_1)'$, we can say that there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that

$$\lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} d(\varkappa_{n_k-1}, \varkappa_{n_k}) = 0. \tag{2.4}$$

Now, we want to show that $\{\varkappa_{n_k}\}$ is a Cauchy sequence. For convenience, let us denote a sequence $\{\eta_k\}$ as $\eta_k = \varkappa_{n_k}$ for all $k \in \mathbb{N}$. Assume the contrary, that is, $\{\eta_k\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and two subsequences of natural numbers $\{k_r\}$, $\{\ell_r\}$ with $\ell_r > k_r \geq r$ such that

$$d(\eta_{k_r}, \eta_{\ell_r}) \geq \varepsilon \tag{2.5}$$

for all $r \in \mathbb{N}$ where ℓ_r is the least integer satisfying (2.5), that is, $d(\eta_{k_r}, \eta_{\ell_r-1}) < \varepsilon$ for all $r \in \mathbb{N}$. Hence, using the triangular inequality we have

$$\begin{aligned} \varepsilon &\leq d(\eta_{k_r}, \eta_{\ell_r}) \\ &\leq d(\eta_{k_r}, \eta_{\ell_r-1}) + d(\eta_{\ell_r-1}, \eta_{\ell_r}) \\ &< \varepsilon + d(\eta_{\ell_r-1}, \eta_{\ell_r}) \end{aligned}$$

for all $r \in \mathbb{N}$. Taking the limit as $r \to \infty$, we get

$$\lim_{r\to\infty} d(\eta_{k_r}, \eta_{\ell_r}) = \varepsilon. \tag{2.6}$$

Also, since

$$\left| d(\eta_{k_r-1}, \eta_{\ell_r-1}) - d(\eta_{k_r}, \eta_{\ell_r}) \right| \leq d(\eta_{k_r-1}, \eta_{k_r}) + d(\eta_{\ell_r-1}, \eta_{\ell_r})$$

for all $r \in \mathbb{N}$, from (2.6) we have

$$\lim_{r\to\infty} d(\eta_{k_r-1}, \eta_{\ell_r-1}) = \varepsilon. \tag{2.7}$$

Because of the fact that Υ is a generalized $\mathcal{R}$-contraction w.r.t. ϱ, we obtain

$$\varrho \left(d\left(\Upsilon\eta_{k_r-1}, \Upsilon\eta_{\ell_r-1} \right), d(\eta_{k_r-1}, \eta_{\ell_r-1}) \right) > 0$$

for all $r \in \mathbb{N}$. Hence, since the pair (P, Q) has the generalized P-property, we have

$$\varrho \left(d(\eta_{k_r}, \eta_{\ell_r}), d(\eta_{k_r-1}, \eta_{\ell_r-1}) \right) > 0 \tag{2.8}$$

for all $r \in \mathbb{N}$. Now, since $\lim_{r\to\infty} d\left(\eta_{k_r-1}, \eta_{\ell_r-1}\right) = \lim_{r\to\infty} d(\eta_{k_r}, \eta_{\ell_r}) = \varepsilon$, taking into account the condition (ϱ_2), from (2.5) and (2.8), we have $\varepsilon = 0$ which is contradiction. Hence, $\{\eta_k\} = \{\varkappa_{n_k}\}$ is a Cauchy sequence in P. Using the equality (2.2), we obtain that $\{\Upsilon\varkappa_{n_k-1}\}$ is a Cauchy sequence in Q, too. Due to the closedness of subsets P and Q of the complete metric space (Λ, d), there exists $\varkappa \in P$ and $\eta \in Q$ such that

$$\lim_{k\to\infty} \varkappa_{n_k} = \varkappa \text{ and } \lim_{k\to\infty} \Upsilon\varkappa_{n_k-1} = \eta. \tag{2.9}$$

From (2.1), taking the limit as $k \to \infty$, we have

$$d(\varkappa, \eta) = d(P, Q). \tag{2.10}$$

Also, we obtain

$$d(\varkappa_{n_k-1}, \varkappa) \leq d(\varkappa_{n_k-1}, \varkappa_{n_k}) + d(\varkappa_{n_k}, \varkappa)$$

for each $k \in \mathbb{N}$. Hence, considering (2.4) and (2.9), we get

$$\lim_{k\to\infty} \varkappa_{n_k-1} = \varkappa. \tag{2.11}$$

If there exists a subsequence of $\{\varkappa_{n_k-1}\}$ whose terms each equal $\varkappa$, then from (2.9) it can be seen that $\eta = \Upsilon\varkappa$. So, from (2.10), the proof is complete. Therefore, suppose that $\varkappa_{n_k-1} \neq \varkappa$ for all $k \in \mathbb{N}$ and for some $r \in \mathbb{N}$ with $k \geq r$. Now, we have the following cases:

Case (i). Suppose that Υ is a continuous mapping. Then, we obtain

$$\lim_{k\to\infty} \Upsilon\varkappa_{n_k-1} = \Upsilon\varkappa,$$

and so $\eta = \Upsilon\varkappa$. From (2.10), we conclude that $\varkappa \in P$ is a best proximity point of Υ.
Case (ii). Suppose that the condition (ϱ_3) is satisfied. Since Υ is a generalized $\mathcal{R}$-contraction mapping, we have

$$\varrho(d(\Upsilon\varkappa_{n_k-1}, \Upsilon\varkappa), d(\varkappa_{n_k-1}, \varkappa)) > 0.$$

Hence, considering the condition (ϱ_3), from (2.11) we have

$$\lim_{k\to\infty} \Upsilon\varkappa_{n_k-1} = \Upsilon\varkappa,$$

and so $\eta = \Upsilon\varkappa$. From (2.10), we conclude that $\varkappa \in P$ is a best proximity point of Υ.

For the uniqueness, suppose that there exists $\varkappa, \eta \in P$ with $\varkappa \neq \eta$ such that

$$d(\varkappa, \Upsilon\varkappa) = d(P, Q)$$

and

$$d(\eta, \Upsilon\eta) = d(P, Q).$$

Hence, considering the generalized P-property we have

$$d(\varkappa, \eta) = d(\Upsilon\varkappa, \Upsilon\eta).$$

Also, because of the fact that Υ is a generalized R-contraction with respect to ζ, we obtain

$$\varrho\left(d(\Upsilon\varkappa, \Upsilon\eta), d(\varkappa, \eta)\right) > 0,$$

which contradicts Proposition 1. Therefore, Υ has a unique best proximity point in P. □

Since every $\mathcal{R}$-function ϱ on A is a modified $\mathcal{R}$-function on A, we obtain the following result, which includes the main result of [23]:

Corollary 1. *Let $\Upsilon : P \to Q$ be a mapping on a complete metric space (Λ, d) where P and Q are closed subsets of Λ. Assume that $P_0 \neq \emptyset$, $\Upsilon(P_0) \subseteq Q_0$ and the pair (P, Q) has the generalized P-property. Suppose that there is an $\mathcal{R}$-function $\varrho : A \times A \to \mathbb{R}$ on A satisfying $ran(d, P \cup Q) = \{d(\varkappa, \eta) : \varkappa, \eta \in P \cup Q\} \subseteq A$ and*

$$\varrho\left(d(\Upsilon\varkappa, \Upsilon\eta), d(\varkappa, \eta)\right) > 0$$

for each $\varkappa, \eta \in P$ with $\varkappa \neq \eta$. If it satisfies one of the following conditions:

(i) Υ is continuous,
(ii) The $\mathcal{R}$-function ϱ satisfies the condition (ϱ_3),

then Υ has a unique best proximity point in P.

The following example shows that Theorem 2 is a real generalization of Corollary 1:

Example 1. *Let $\Lambda = \mathbb{R}^2$ be a complete metric space with the taxi-cab metric d. Consider the closed subsets of Λ*

$$P = \left\{0, \frac{1}{n} : n \in \mathbb{N}\right\} \times \{0\}$$

and

$$Q = \left\{0, \frac{1}{n} : n \in \mathbb{N}\right\} \times \{1\}.$$

Then, $d(P,Q) = 1$, $P_0 = P$, and $Q_0 = Q$. Also, the pair (P,Q) has the generalized P-property. Define the mapping $\Upsilon : P \to Q$ and the function $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\Upsilon(\varkappa, 0) = (0,1)$ and

$$\varrho(p,q) = \begin{cases} 1, & \begin{array}{c} p = \frac{1}{n+1} \text{ and } q = 1 + \frac{1}{n},\ n \geq 1 \\ \text{or} \\ p = 1 + \frac{1}{n} \text{ and } q = \frac{1}{n},\ n \geq 1, \end{array} \\ 0, & \begin{array}{c} p \notin \left\{0, \frac{1}{n+1}\right\} \text{ and } q = 1 + \frac{1}{n},\ n \geq 1 \\ \text{or} \\ p \notin \left\{0, 1 + \frac{1}{n}\right\} \text{ and } q = \frac{1}{n},\ n \geq 1, \end{array} \\ \frac{q}{2} - p, & \text{otherwise}, \end{cases}$$

respectively. Then, it can be easily seen that $\Upsilon(P_0) \subseteq Q_0$ and Υ is a continuous mapping. Let $A = ran(d, P \cup Q)$, that is,

$$A = \left\{0, \frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{\left|\frac{1}{n} - \frac{1}{m}\right| : n, m \in \mathbb{N}\right\} \cup \left\{1 + \left|\frac{1}{n} - \frac{1}{m}\right| : n, m \in \mathbb{N}\right\} \cup \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}.$$

In this case, Υ is a modified $\mathcal{R}$-function on A. Indeed, to show that the condition $(\varrho_1)'$ holds, let us take a sequence $\{p_n\} \subseteq (0,\infty) \cap A$ satisfying $\varrho(p_{n+1}, p_n) > 0$ for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $p_{n_0} = \frac{1}{n_0}$ or $p_{n_0} = 1 + \frac{1}{n_0}$, then we have $p_{n_0+2n} \to 0$ or $p_{n_0+(2n-1)} \to 0$ for all $n \in \mathbb{N}$. Otherwise, since $\varrho(p_{n+1}, p_n) > 0$ for each $n \in \mathbb{N}$, we have

$$\frac{p_n}{2} - p_{n+1} > 0 \tag{2.12}$$

for all $n \in \mathbb{N}$, and so $\{p_n\}$ is decreasing. Hence, there exists $L \geq 0$ such that $p_n \to L$ as $n \to \infty$. Assume that $L > 0$. Taking the limit as $n \to \infty$ in inequality (2.12), we obtain $L \leq \frac{L}{2} < L$, which is contradiction. So, $L = 0$. Similar to $(\varrho_1)'$, it can be shown that the condition $(\varrho_2)'$ holds. Now, we want to show that Υ is a generalized $\mathcal{R}$-contraction w.r.t. ϱ. For this, we have the following conditions:

***Case 1.** Let $\varkappa = (0,0)$, $\eta = \left(\frac{1}{n}, 0\right)$, $n \geq 1$. Then, we have $\Upsilon\varkappa = (0,1)$ and $\Upsilon\eta = (0,1)$. In this case, we obtain*

$$d(\Upsilon\varkappa, \Upsilon\eta) = 0$$

and

$$d(\varkappa, \eta) = \frac{1}{n}.$$

Hence, we get

$$\varrho(d(\Upsilon\varkappa, \Upsilon\eta), d(\varkappa,\eta)) = \varrho\left(0, \frac{1}{n}\right) = \frac{1}{2n} > 0.$$

***Case 2.** Let $\varkappa = \left(\frac{1}{n}, 0\right)$, $\eta = \left(\frac{1}{m}, 0\right)$, $n, m \geq 1$ (without loss of the generality, we assume that $n < m$). Then, we have $\Upsilon\varkappa = (0,1)$ and $\Upsilon\eta = (0,1)$. In this case, we get*

$$d(\Upsilon\varkappa, \Upsilon\eta) = 0$$

and

$$d(\varkappa,\eta)=\left|\frac{1}{n}-\frac{1}{m}\right|=\frac{1}{n}-\frac{1}{m}.$$

Hence, we obtain

$$\varrho(d\left(\Upsilon\varkappa,\Upsilon\eta\right),d(\varkappa,\eta))=\varrho\left(0,\frac{1}{n}-\frac{1}{m}\right)=\frac{1}{2n}-\frac{1}{2m}>0.$$

However, we cannot apply Corollary 1 to this example since ϱ is not an $\mathcal{R}$-function on any subset E of $\mathbb{R}$ satisfying ran$(d, P\cup Q)\subseteq E$. Assume the contrary, that is, there exists a subset E of $\mathbb{R}$ satisfying ran$(d)\subseteq E$ and ϱ is a $\mathcal{R}$-function on E. Now, consider the sequence

$$(p_n)=\left(\frac{1}{2},1+\frac{1}{2},\frac{1}{3},1+\frac{1}{4},\frac{1}{5}\cdots\right)$$

in $(0,\infty)\cap E$. Then, we have $\varrho(p_{n+1},p_n)=1>0$ for all $n\in\mathbb{N}$, but $p_n\nrightarrow 0$, which contradicts the condition (ϱ_1).

If we take $P=Q=\Lambda$ in Corollary 1, we have the main result of [23]:

Corollary 2. *Let $\Upsilon:\Lambda\to\Lambda$ be a mapping on a complete metric space (Λ,d). Suppose there exists an $\mathcal{R}$-function $\varrho:A\times A\to\mathbb{R}$ on A satisfying ran$(d,\Lambda)=\{d(\varkappa,\eta):\varkappa,\eta\in\Lambda\}\subseteq A$ and*

$$\varrho\left(d(\Upsilon\varkappa,\Upsilon\eta),d(\varkappa,\eta)\right)>0$$

for all $\varkappa,\eta\in\Lambda$ with $\varkappa\neq\eta$. If it satisfies one of the following conditions:

(i) Υ is continuous,
(ii) The $\mathcal{R}$-function ϱ satisfies the condition (ϱ_3),

then Υ has a unique fixed point in Λ.

3. Application

Recently, many authors have obtained an application of their fixed point results to homotopy theory because of the close relationship between this topic and other branches of mathematics [1,14,17,27,28]. Therefore, inspired by a nice and interesting technique used by Vetro et al. [28], we present an application of our best proximity point result, Theorem 2, to homotopy theory in this section. In this sense, we show that, under the assumption that one of the homotopic mappings has the best proximity point, another has one, too. Now, we recall the definition of homotopy.

Definition 7. *Let (Λ_1,τ_1) and (Λ_2,τ_2) be topological spaces, $\Upsilon,F:\Lambda_1\to\Lambda_2$ be a continuous mappings. If there exists continuous function $h:\Lambda_1\times[0,1]\to\Lambda_2$ satisfying $h(\varkappa,0)=\Upsilon\varkappa$ and $h(\varkappa,1)=F\varkappa$ for all $\varkappa\in\Lambda_1$, then it is said that Υ and F are homotopic mappings. Also, the mapping h is called a homotopy.*

The following definition is important for the results of this section:

Definition 8. *Let $\emptyset \neq P, Q$ be subsets of a metric space (Λ, d) and $h : P \times [0,1] \to Q$ be a mapping. If $G_d(h) \subseteq (\Lambda \times [0,1] \times \Lambda, d^*)$ is closed, then h is said to be a d-closed mapping, where*

$$G_d(h) = \{(\varkappa, \beta, \eta) : \varkappa \in P, \eta \in \Lambda \text{ and } \beta \in [0,1] \text{ with } d(\eta, h(\varkappa, \beta)) = d(P, Q)\}$$

and

$$d^*((\varkappa_1, \beta_1, \eta_1), (\varkappa_2, \beta_2, \eta_2)) = d(\varkappa_1, \varkappa_2) + |\beta_1 - \beta_2| + d(\eta_1, \eta_2)$$

for all $(\varkappa_1, \beta_1, \eta_1), (\varkappa_2, \beta_2, \eta_2) \in \Lambda \times [0,1] \times \Lambda$.

Note that, in case of $d(P, Q) = 0$, Definition 8 turns to the definition of closed mappings defined from P to Q.

Now, we can present the main result of this section:

Theorem 3. *Let (Λ, d) be a complete metric space, P, Q be nonempty closed subsets of Λ and $\emptyset \neq U \subseteq P$. Assume that the pair (P, Q) has the generalized P-property and $h : P \times [0,1] \to Q$ is a continuous d-closed mapping such that*

(i) $d(\varkappa, h(\varkappa, \lambda)) > d(P, Q)$ for each $\varkappa \in P \backslash U$ and $\lambda \in [0,1]$,

(ii) there exists a modified $\mathcal{R}$-function $\varrho : A \times A \to \mathbb{R}$ on A satisfying $ran(d, P \cup Q) = \{d(\varkappa, \eta) : \varkappa, \eta \in P \cup Q\} \subseteq A$ and

$$\varrho\,(d(h(\varkappa, \lambda), h(\eta, \mu)), d(\varkappa, \eta)) > 0$$

for each $\varkappa, \eta \in P$ with $\varkappa \neq \eta$ and $\lambda, \mu \in [0,1]$,

(iii) for all $\varkappa \in A$, $\beta, r \in [0,1]$, and $\varkappa_0 \in \overline{B}(\varkappa, r) \cap P_0$ there exists $\varkappa_1 \in \overline{B}(\varkappa, r)$ such that $d(\varkappa_1, h(\varkappa_0, \beta)) = d(P, Q)$ where

$$\overline{B}(\varkappa, r) = \{\overline{\varkappa} \in P : d(\varkappa, \overline{\varkappa}) \leq r\}.$$

Then, $h(\cdot, 1)$ has a best proximity point in P if $h(\cdot, 0)$ has a best proximity point in P.

Proof. Consider the following subset:

$$K = \{(\beta, \varkappa) : d(\varkappa, h(\varkappa, \beta)) = d(P, Q)\}.$$

From the hypothesis and the condition (i), there is a point $\varkappa$ in P such that $d(\varkappa, h(x, 0)) = d(P, Q)$, that is, we have $(0, \varkappa) \in K$. Hence, we get $K \neq \emptyset$. Define a partial order on K by

$$(\beta, \varkappa) \preceq (\mu, \eta) \Leftrightarrow \beta \leq \mu \text{ and } d(\varkappa, \eta) \leq \mu - \beta.$$

Now, let L be an arbitrary totally ordered subset of K and $\beta^* = \sup\{\beta : (\beta, \varkappa) \in L\}$. Consider a sequence $\{(\beta_n, \varkappa_n)\}$ in L such that $(\beta_n, \varkappa_n) \preceq (\beta_{n+1}, \varkappa_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $\beta_n \to \beta^*$as $n \to \infty$. In this case, we get

$$d(\varkappa_n, \varkappa_m) \leq \beta_m - \beta_n,$$

for each $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$. Thus, $\{\varkappa_n\} \subseteq P$ is a Cauchy sequence. Then, there is $\varkappa^* \in P$ such that $d(\varkappa_n, \varkappa^*) \to 0$ as $n \to \infty$ since $P \subseteq \Lambda$ is closed and (Λ, d) is a complete metric space. Also, we have $\{(\varkappa_n, \beta_n, \varkappa_n)\} \subseteq G_d(h)$ and

$$\lim_{n\to\infty} d^*\left((\varkappa_n, \beta_n, \varkappa_n), (\varkappa^*, \beta^*, \varkappa^*)\right) = 0).$$

Since h is a d-closed mapping, we get

$$d(\varkappa^*, h(\varkappa^*, \beta^*)) = d(P, Q).$$

From (i), we have $\varkappa^* \in U$, and so $(\beta^*, \varkappa^*) \in K$. Since L is a totally ordered, it satisfies $(\beta, \varkappa) \preceq (\beta^*, \varkappa^*)$ for all $(\beta, \varkappa) \in L$. Hence, $(\beta^*, \varkappa^*)$ is an upper bound of L. Therefore, using the Zorn lemma, we say that K has a maximal element $(\beta_0, \varkappa_0)$ in K. Now, we want to show that $\beta_0 = 1$. Assume the contrary, that is, $\beta_0 < 1$. Then, there is a real number β satisfying $\beta_0 < \beta < 1$. Let $r = \beta - \beta_0$. From (ii), the mapping $H(\cdot, \beta) : \overline{B}(\varkappa_0, r) \to Q$ is a generalized $\mathcal{R}$-contraction. Considering condition (iii) and using Theorem 2, we can say that there exists $\varkappa_\beta \in \overline{B}(\varkappa_0, r)$ such that $d(\varkappa_\beta, h(\varkappa_\beta, \beta)) = d(P, Q)$. From (i), $\varkappa_\beta \in U$, and so $(\beta, \varkappa_\beta) \in K$, which contradicts that $(\beta_0, \varkappa_0)$ is maximal element of K. So, $\beta_0 = 1$ and $h(\cdot, 1)$ has a best proximity point $\varkappa_0$ in P. □

Taking $Q = \Lambda$ in Theorem 3, we obtain the following corollary:

Corollary 3. *Let (Λ, d) be a complete metric space, P be a nonempty closed subset of Λ and $\emptyset \neq U \subseteq P$. Assume that $h : P \times [0, 1] \to \Lambda$ is a continuous closed mapping such that*

(i) $d(\varkappa, h(\varkappa, \lambda)) > 0$ for each $\varkappa \in P \backslash U$ and $\lambda \in [0, 1]$,

(ii) there exists a modified $\mathcal{R}$-function $\varrho : A \times A \to \mathbb{R}$ on A such that $ran(d, \Lambda) = \{d(\varkappa, \eta) : \varkappa, \eta \in \Lambda\} \subseteq A$ and

$$\varrho\left(d(h(\varkappa, \lambda), h(\eta, \mu)), d(\varkappa, \eta)\right) > 0$$

for each $\varkappa, \eta \in P$ with $\varkappa \neq \eta$ and $\lambda, \mu \in [0, 1]$,

(iii) for all $\varkappa \in A$, $\beta, r \in [0, 1]$ and $\varkappa_0 \in \overline{B}(\varkappa, r)$, there exists $\varkappa_1 \in \overline{B}(\varkappa, r)$ such that $\varkappa_1 = h(\varkappa_0, \beta)$.

If $h(\cdot, 0)$ has a fixed point in P, then $h(\cdot, 1)$ has a fixed point P.

Proof. Assume that $h(\cdot, 0)$ has a fixed point $\varkappa$ in P. Therefore, from Theorem 3 we say that there is $\varkappa^* \in P$ satisfying

$$d(\varkappa^*, h(\varkappa^*, 1)) = d(P, \Lambda) = 0.$$

This shows that $\varkappa^*$ is a fixed point of $h(\cdot, 1)$. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

References

1. M. Abbas, H. Iqbal, A. Petrusel, Fixed points for multivalued Suzuki type (θ, R)-contraction mapping with applications, *J. Funct. Space.*, **2019** (2019), 9565804. https://doi.org/10.1155/2019/9565804

2. A. Abkar, M. Gabeleh, A note on some best proximity point theorems proved under P-property, *Abstr. Appl. Anal.*, **2013** (2013), 189567. https://doi.org/10.1155/2013/189567
3. H. Argoubi, B. Samet, C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, *J. Nonlinear Sci. Appl.*, **8** (2015), 1082–1094.
4. H. Aydi, M. Aslam, D. Sagheer, S. Batul, R. Ali, E. Ameer, Kannan-type contractions on new extended-metric spaces, *J. Funct. Space.*, **2021** (2021), 7613684. https://doi.org/10.1155/2021/7613684
5. H. Aydi, A. Felhi, S. Sahmim, On fixed points in quasi partial b-metric spaces and an application to dynamic programming, *Thai J. Math.*, **19** (2021), 407–419.
6. H. Aydi, H. Lakzian, Z. Mitrović, S. Radenović, Best proximity points of $M\Upsilon$-cyclic contractions with property UC, *Numer. Func. Anal. Opt.*, **41** (2020), 871–882. https://doi.org/10.1080/01630563.2019.1708390
7. S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
8. S. Basha, Extensions of Banach's contraction principle, *Numer. Func. Anal. Opt.*, **31** (2010), 569–576. https://doi.org/10.1080/01630563.2010.485713
9. S. Basha, P. Veeramani, Best proximity pairs and best approximations, *Acta Sci. Math.*, **63** (1997), 289–300.
10. M. Berinde, V. Berinde, On a general class of multi-valued weakly Picard mappings, *J. Math. Anal. Appl.*, **326** (2007), 772–782. https://doi.org/10.1016/j.jmaa.2006.03.016
11. M. Gabeleh, Semi-normal structure and best proximity pair results in convex metric spaces, *Banach J. Math. Anal.*, **8** (2014), 214–228. https://doi.org/10.15352/bjma/1396640065
12. M. Gabeleh, Best proximity points for weak proximal contractions, *Bull. Malays. Math. Sci. Soc.*, **38** (2015), 143–154. https://doi.org/10.1007/s40840-014-0009-9
13. M. Gabeleh, O. Otafudu, Markov-Kakutani's theorem for best proximity pairs in Hadamard spaces, *Indagat. Math.*, **28** (2017), 680–693. https://doi.org/10.1016/j.indag.2017.02.004
14. Humaira, M. Sarwar, P. Kumam, Common fixed point results for fuzzy mappings on complex-valued metric spaces with homotopy results, *Symmetry*, **11** (2019), 61. https://doi.org/10.3390/sym11010061
15. M. Imdad, H. Saleh, W. Alfaqih, Best proximity point theorems in metric spaces with applications in partial metric spaces, *TWMS J. Appl. Eng. Math.*, **10** (2020), 190–200.
16. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theory for simulation functions, *Filomat*, **29** (2015), 1189–1194. https://doi.org/10.2298/FIL1506189K
17. D. O'Regan, Topological fixed point theory for compact multifunctions via homotopy and essential maps, *Topol. Appl.*, **265** (2019), 106819. https://doi.org/10.1016/j.topol.2019.106819
18. V. Raj, Best proximity point theorems for non-self mappings, *Fixed Point Theor.*, **14** (2013), 447–454.
19. S. Reich, Approximate selections, best approximations, fixed points and invariant sets, *J. Math. Anal. Appl.*, **62** (1978), 104–113. https://doi.org/10.1016/0022-247X(78)90222-6

20. S. Reich, T. Tuyen, M. Ha, An optimization approach to solving the split feasibility problem in Hilbert spaces, *J. Glob. Optim.*, **79** (2021), 837–852. https://doi.org/10.1007/s10898-020-00964-2
21. S. Reich, R. Zalas, The optimal error bound for the method of simultaneous projections, *J. Approx. Theory*, **223** (2017), 96–107. https://doi.org/10.1016/j.jat.2017.08.005
22. A. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, *J. Comput. Appl. Math.*, **275** (2015), 345–355. https://doi.org/10.1016/j.cam.2014.07.011
23. A. Roldán-López-de-Hierro, N. Shahzad, New fixed point theorem under $\mathcal{R}$-contractions, *Fixed Point Theory A.*, **2015** (2015), 98. https://doi.org/10.1186/s13663-015-0345-y
24. H. Sahin, M. Aslantas, I. Altun, Best proximity and best periodic points for proximal nonunique contractions, *J. Fixed Point Theory Appl.*, **23** (2021), 55. https://doi.org/10.1007/s11784-021-00889-7
25. H. Sahin, A new type of F-contraction and their best proximity point results with homotopy application, *Acta Appl. Math.*, **179** (2022), 9. https://doi.org/10.1007/s10440-022-00496-9
26. H. Sahin, M. Aslantas, A. Nasir, Some extended results for multivalued F-contraction mappings, *Axioms*, **12** (2023), 116. https://doi.org/10.3390/axioms12020116
27. S. Shukla, S. Radenović, C. Vetro, Set-valued Hardy-Rogers type contraction in 0-complete partial metric spaces, *International Journal of Mathematics and Mathematical Sciences*, **2014** (2014), 652925. https://doi.org/10.1155/2014/652925
28. C. Vetro, F. Vetro, A homotopy fixed point theorem in 0-complete partial metric space, *Filomat*, **29** (2015), 2037–2048. https://doi.org/10.2298/FIL1509037V