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Research article

The existence of uniform attractors for the 3D micropolar equations with nonlinear damping term

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Abstract: This paper studies the existence of uniform attractors for 3D micropolar equation with damping term. When $\beta > 3$, with initial data $(u_\tau, \omega_\tau) \in V_1 \times V_2$ and external forces $(f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$, some uniform estimates of the solution in different function spaces are given. Based on these uniform estimates, the $((V_1 \times V_2) \times (\mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)), V_1 \times V_2)$-continuity of the family of processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$ is demonstrated. Meanwhile, the $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform compactness of $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$ is proved. Finally, the existence of a $(V_1 \times V_2, V_1 \times V_2)$-uniform attractor and a $(V_1 \times V_2, \mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega))$-uniform attractor are obtained. Furthermore, the $(V_1\times V_2, V_1\times V_2)$-uniform attractor and the $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform attractor are verified to be the same.

Keywords: uniform attractor; 3D micropolar equation; nonlinear damping term
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1. Introduction

In this paper, we consider the 3D nonlinear damped micropolar equation

$$\begin{cases} u_t + (u \cdot \nabla)u - (\nu + \kappa)\Delta u + \sigma |u|^{\beta-1} u + \nabla p = 2\kappa\nabla \times \omega + f_1(x,t), \\ \omega_t + (u \cdot \nabla)\omega + 4\kappa\omega - \gamma\Delta\omega - \mu\nabla\nabla \cdot \omega = 2\kappa\nabla \times u + f_2(x,t), \\ \nabla \cdot u = 0, \\ u(x,t)|_{t=\tau} = u_\tau(x), \quad \omega(x,t)|_{t=\tau} = \omega_\tau(x), \end{cases} \tag{1.1}$$

where $(x,t) \in \Omega \times [\tau, +\infty)$, $\tau \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^3$ is a bounded domain, $u = u(x,t)$ is the fluid velocity, $\omega = \omega(x,t)$ is the angular velocity, σ is the damping coefficient, which is a positive constant, $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ represent external forces, ν, κ, γ, μ are all positive constants, γ and μ represent the angular viscosities.

Micropolar flow can describe a fluid with microstructure, that is, a fluid composed of randomly oriented particles suspended in a viscous medium without considering the deformation of fluid particles. Since Eringen first published his paper on the model equation of micropolar fluids in 1966 [5], the formation of modern theory of micropolar fluid dynamics has experienced more than 40 years of development. For the 2D case, many researchers have discussed the long time behavior of micropolar equations (such as [2,4,10,24]). It should be mentioned that some conclusions in the 2D case no longer hold for the 3D case due to different structures of the system. In the 3D case, the work of micropolar equations (1.1) with $\sigma = 0$, $f_1 = 0$, and $f_2 = 0$ has attracted a lot of attention (see [6, 14, 19]). Galdi and Rionero [6] proved the existence and uniqueness of solutions of 3D incompressible micropolar equations. In a 3D bounded domain, for small initial data Yamaguchi [19] investigated the existence of a global solution to the initial boundary problem for the micropolar system. In [14], Silva and Cruz et al. studied the L^2-decay of weak solutions for 3D micropolar equations in the whole space $\mathbb{R}^3$. When $f_1 = f_2 = 0$, for the Cauchy problem of the 3D incompressible nonlinear damped micropolar equations, Ye [22] discussed the existence and uniqueness of global strong solutions when $\beta = 3$ and $4\sigma(\nu + \kappa) > 1$ or $\beta > 3$. In [18], Wang and Long showed that strong solutions exist globally for any $1 \leq \beta \leq 3$ when initial data satisfies some certain conditions. Based on [22], Yang and Liu [20] obtained uniform estimates of the solutions for 3D incompressible micropolar equations with damping, and then they proved the existence of global attractors for $3 < \beta < 5$. In [7], Li and Xiao investigated the large time decay of the L^2-norm of weak solutions when $\beta > \frac{14}{5}$, and considered the upper bounds of the derivatives of the strong solution when $\beta > 3$. In [21], for $1 \leq \beta < \frac{7}{3}$, Yang, Liu, and Sun proved the existence of trajectory attractors for 3D nonlinear damped micropolar fluids.

To the best of our knowledge, there are few results on uniform attractors for the three-dimensional micropolar equation with nonlinear damping term. The purpose of this paper is to consider the existence of uniform attractors of system (1.1). When $\omega = 0, \kappa = 0$, system (1.1) is reduced to the Navier-Stokes equations with damping. In recent years, some scholars have studied the three-dimensional nonlinear damped Navier-Stokes equations (see [1,13,15,16,23,25]). In order to obtain the desired conclusion, we will use some proof techniques which have been used in the 3D nonlinear damped Navier Stokes equations. Note that, in [20], for the convenience of discussion the authors choose $\kappa, \mu = \frac{1}{2}, \gamma = 1$, and $\nu = \frac{3}{2}$. In this work, we do not specify these parameters, but only require them to be positive real numbers. More importantly, we obtain the existence of uniform attractors in the case of $\beta > 3$, which undoubtedly expands the range of β when the global attractor exists in [20], i.e., $3 < \beta < 5$. For the convenience of discussion, similar to [3,8,9,11,16], we make some translational compactness assumption on the external forces term in this paper.

The organizational structure of this article is as follows: In Section 2, we give some basic definitions and properties of function spaces and process theory which will be used in this paper. In Section 3, using various Sobolev inequalities and Gronwall inequalities, we make some uniform estimates from the space with low regularity to high regularity on the solution of the equation. Based on these uniform estimates, in Section 4 we prove that the family of processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$ corresponding to (1.1) has uniform attractors $\mathcal{A}_1$ in $V_1 \times V_2$ and $\mathcal{A}_2$ in $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$, respectively. Furthermore, we prove $\mathcal{A}_1 = \mathcal{A}_2$.

2. Preliminaries

We define the usual functional spaces as follows:

$$\begin{aligned}
\mathcal{V}_1 &= \{u \in (C_0^\infty(\Omega))^3 : \mathrm{div} u = 0, \int_\Omega u dx = 0\},\\
\mathcal{V}_2 &= \{\omega \in (C_0^\infty(\Omega))^3 : \int_\Omega \omega dx = 0\},\\
H_1 &= \text{the closure of } \mathcal{V}_1 \text{ in } (L^2(\Omega))^3,\\
H_2 &= \text{the closure of } \mathcal{V}_2 \text{ in } (L^2(\Omega))^3,\\
V_1 &= \text{the closure of } \mathcal{V}_1 \text{ in } (H^1(\Omega))^3,\\
V_2 &= \text{the closure of } \mathcal{V}_2 \text{ in } (H^1(\Omega))^3.
\end{aligned}$$

For H_1 and H_2 we have the inner product

$$(u, \upsilon) = \int_\Omega u \cdot \upsilon dx, \quad \forall u, v \in H_1, \text{or } u, v \in H_2,$$

and norm $\|\cdot\|^2 = \|\cdot\|_2^2 = (\cdot, \cdot)$. In this paper, $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$, and $\|\cdot\|_p$ represents the norm in $\mathbf{L}^p(\Omega)$.

We define operators

$$\begin{aligned}
&Au = -P\Delta u = -\Delta u, \quad A\omega = -\Delta\omega, \ \ \forall(u, \omega) \in H^2 \times H^2,\\
&B(u) = B(u, u) = P((u \cdot \nabla)u), \quad B(u, \omega) = (u \cdot \nabla)\omega, \ \ \forall(u, \omega) \in V_1 \times V_2,\\
&b(u, \upsilon, \omega) = \langle B(u, \upsilon), \omega\rangle = \sum_{i,j=1}^{3} \int_\Omega u_i(D_i\upsilon_j)\omega_j dx, \ \ \forall u \in V_1, \upsilon, \omega \in V_2,
\end{aligned}$$

where P is the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto H_1. $\mathbf{H}^s(\Omega) = (H^s(\Omega))^3$ is the usual Sobolev space, and its norm is defined by $\|\cdot\|_{\mathbf{H}^s} = \| A^{\frac{s}{2}} \cdot \|$; as $s = 2$, $\|\cdot\|_{\mathbf{H}^2} = \| A\cdot \|$.

Let us rewrite system (1.1) as

$$\begin{cases}
u_t + B(u) + (\nu + \kappa)Au + G(u) = 2\kappa\nabla \times \omega + f_1(x, t),\\
\omega_t + B(u, \omega) + 4\kappa\omega + \gamma A\omega - \mu\nabla\nabla \cdot \omega = 2\kappa\nabla \times u + f_2(x, t),\\
\nabla \cdot u = 0,\\
u(x, t)|_{t=\tau} = u_\tau(x), \ \ \omega(x, t)|_{t=\tau} = \omega_\tau(x),
\end{cases} \tag{2.1}$$

where we let $G(u) = P(\sigma|u|^{\beta-1}u)$.

The Poincar$\acute{e}$ inequality [17] gives

$$\sqrt{\lambda_1}\|u\| \le \|\nabla u\|, \ \sqrt{\lambda_2}\|\omega\| \le \|\nabla\omega\|, \forall(u, \omega) \in V_1 \times V_2, \tag{2.2}$$

$$\sqrt{\lambda_1} \ \| \nabla u \| \le \| Au \|, \ \ \sqrt{\lambda_2} \ \| \nabla\omega \| \le \| A\omega \|, \forall(u, \omega) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega), \tag{2.3}$$

where λ_1 is the first eigenvalue of Au, and λ_2 is the first eigenvalue of $A\omega$. Let $\lambda = \min\{\lambda_1, \lambda_2\}$. Then, we have

$$\lambda(\|u\|^2 + \|\omega\|^2) \le \|\nabla u\|^2 + \|\nabla\omega\|^2, \ \forall(u, \omega) \in V_1 \times V_2,$$

$$\lambda(\|\nabla u\|^2 + \|\nabla\omega\|^2) \le \|Au\|^2 + \|A\omega\|^2, \forall (u,\omega) \in \mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega).$$

Agmon's inequality [17] gives

$$\| u \|_\infty \le d_1 \| \nabla u \|^{\frac{1}{2}} \| \Delta u \|^{\frac{1}{2}},\ \forall u \in \mathbf{H}^2(\Omega).$$

The trilinear inequalities [12] give

$$|b(u,v,w)| \le \| u \|_\infty \| \nabla v \| \| w \|, \forall u \in L^\infty(\Omega), v \in V_1 \text{ or } V_2, w \in H_1 \text{ or } H_2, \tag{2.4}$$

$$|b(u,v,w)| \le k \| u \|^{\frac{1}{4}} \| \nabla u \|^{\frac{3}{4}} \| \nabla v \| \| w \|^{\frac{1}{4}} \| \nabla w \|^{\frac{3}{4}}, \forall u,v,w \in V_1 \text{ or } V_2, \tag{2.5}$$

$$|b(u,v,w)| \le k \| \nabla u \| \| \nabla v \|^{\frac{1}{2}} \| Av \|^{\frac{1}{2}} \| w \|, \forall u \in V_1 \text{ or } V_2, v \in \mathbf{H}^2, w \in H_1 \text{ or } H_2. \tag{2.6}$$

Recall that a function $f(t)$ is translation bounded (tr.b.) in $L^2_{\text{loc}}(\mathbb{R};\mathbf{L}^2(\Omega))$ if

$$\| f \|^2_{L^2_b} = \| f \|^2_{L^2_b(\mathbb{R};\mathbf{L}^2(\Omega))} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \| f(t) \|^2\, dt < \infty,$$

where $L^2_b(\mathbb{R};\mathbf{L}^2(\Omega))$ represents the collection of functions that are tr.b. in $L^2_{\text{loc}}(\mathbb{R};\mathbf{L}^2(\Omega))$. We say that $\mathcal{H}(f_0) = \overline{\{f_0(\cdot + t) : t \in \mathbb{R}\}}$ is the shell of f_0 in $L^2_{\text{loc}}(\mathbb{R};\mathbf{L}^2(\Omega))$. If $\mathcal{H}(f_0)$ is compact in $L^2_{\text{loc}}(\mathbb{R};\mathbf{L}^2(\Omega))$, then we say that $f_0(x,t) \in L^2_{\text{loc}}(\mathbb{R};\mathbf{L}^2(\Omega))$ is translation compact (tr.c.). We use $L^2_c(\mathbb{R};\mathbf{L}^2(\Omega))$ to express the collection of all translation compact functions in $L^2_{\text{loc}}(\mathbb{R};\mathbf{L}^2(\Omega))$.

Next, we will provide the existence and uniqueness theorems of the solution of Eq (2.1).

Definition 2.1. *A function pair (u,ω) is said to be a global strong solution to system (2.1) if it satisfies*

$$(u,\omega) \in L^\infty(\tau,T;V_1\times V_2) \cap L^2(\tau,T;\mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega)),$$

$$|u|^{\frac{\beta-1}{2}}\nabla u \in L^2(\tau,T;\mathbf{L}^2(\Omega)),\ \nabla|u|^{\frac{\beta+1}{2}} \in L^2(\tau,T;\mathbf{L}^2(\Omega)),$$

for any given $T > \tau$.

Theorem 2.1. *Suppose $(u_\tau,\omega_\tau) \in V_1\times V_2$ with $\nabla\cdot u_\tau = 0, f_1, f_2 \in L^2_b(\mathbb{R};\mathbf{L}^2(\Omega))$. If $\beta = 3$ and $4\sigma(\nu+\kappa) > 1$ or $\beta > 3$, then there exists a unique global strong solution of (2.1).*

Proof. Since the proof method is similar to that of Theorem 1.2 in [22], we omit it here. □

Let Σ be a metric space. X, Y are two Banach spaces, and $Y \subset X$ is continuous. $\{U_\sigma(t,\tau)\}_{t\ge\tau}, \sigma\in\Sigma$ is a family of processes in Banach space X, i.e., $u(t) = U_\sigma(t,\tau)u_\tau$, $U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \forall t \ge s \ge \tau, \tau\in\mathbb{R}, U_\sigma(\tau,\tau) = I$, where $\sigma\in\Sigma$ is a time symbol space. $\mathcal{B}(X)$ is the set of all bounded subsets of X. $\mathbb{R}^\tau = [\tau,+\infty)$.

For the basic concepts of bi-space uniform absorbing set, uniform attracting set, uniform attractor, uniform compact, and uniform asymptotically compact of the family of processed $\{U_\sigma(t,\tau)\}_{t\ge\tau}, \sigma\in\Sigma$, one can refer to [9, 16].

Let $T(h)$ be a family of operators acting on Σ, satisfying: $T(h)\sigma(s) = \sigma(s+h), \forall s\in\mathbb{R}$. In this paper, we assume that Σ satisfies
$(C1)$ $T(h)\Sigma = \Sigma, \forall h\in\mathbb{R}^+$;
$(C2)$ translation identity:

$$U_\sigma(t+h,\tau+h) = U_{T(h)\sigma}(t,\tau), \quad \forall\sigma\in\Sigma, t\ge\tau, \tau\in\mathbb{R}, h\ge 0.$$

Theorem 2.2. *[3] If the family of processes $\{U_\sigma(t,\tau)\}_{t\geq\tau}, \sigma \in \Sigma$ is (X,Y)-uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact, then it has a (X,Y)-uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma$, $\mathcal{A}_\Sigma$ is compact in Y, and it attracts all bounded subsets of X in the topology of Y.*

In this paper, the letter C represents a positive constant. It may represent different values in different lines, or even in the same line.

3. Uniform estimations of solutions

In this paper, we chose $\mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ as the symbol space. Obviously, $T(t)(\mathcal{H}((f_1^0) \times \mathcal{H}(f_2^0)) = \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$, for all $t \geq 0$. $\{T(t)\}_{t\geq 0}$ is defined by

$$T(h)(f_1(\cdot), f_2(\cdot)) = (f_1(\cdot + h), f_2(\cdot + h)), \quad \forall h \geq 0, (f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0),$$

which is a translation semigroup and is continuous on $\mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$.

Thanks to Theorem 2.1, when $(u_\tau, \omega_\tau) \in V_1 \times V_2$, $f_1, f_2 \in L^2_{\text{loc}}(\mathbb{R}; \mathbf{L}^2(\Omega))$, and $\beta > 3$, we can define a process $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$ in $V_1 \times V_2$ by

$$U_{(f_1,f_2)}(t,\tau)(u_\tau, \omega_\tau) = (u(t), \omega(t)), \ t \geq \tau,$$

where $(u(t), \omega(t))$ is the solution of Eq (1.1) with external forces f_1, f_2 and initial data (u_τ, ω_τ).

Next, let us assume that the external forces $f_1^0(x,t), f_2^0(x,t)$ are tr.c. in $L^2_{\text{loc}}(\mathbb{R}; \mathbf{L}^2(\Omega))$. Then, f_1^0, f_2^0 are tr.b. in $L^2_{\text{loc}}(\mathbb{R}; \mathbf{L}^2(\Omega))$, and

$$\| f_1 \|^2_{L^2_b} = \| f_1 \|^2_{L^2_b(\mathbb{R};\mathbf{L}^2(\Omega))} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \| f_1(s) \|^2 \, ds \leq \| f_1^0 \|^2_{L^2_b} < +\infty, \forall f_1 \in \mathcal{H}(f_1^0),$$

$$\| f_2 \|^2_{L^2_b} = \| f_2 \|^2_{L^2_b(\mathbb{R};\mathbf{L}^2(\Omega))} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \| f_2(s) \|^2 \, ds \leq \| f_2^0 \|^2_{L^2_b} < +\infty, \forall f_2 \in \mathcal{H}(f_2^0).$$

Furthermore, we assume f_1^0, f_2^0 are uniformly bounded in $\mathbf{L}^2(\Omega)$, i.e., there exists a positive constant K, which satisfies

$$\sup_{t\in\mathbb{R}} \| f_1^0(x,t) \| \leq K, \ \sup_{t\in\mathbb{R}} \| f_2^0(x,t) \| \leq K.$$

Meanwhile, we suppose the derivatives $\frac{\mathrm{d}f_1^0}{\mathrm{d}t}$, $\frac{\mathrm{d}f_2^0}{\mathrm{d}t}$, labeled as h_1, h_2, also belong to $L^2_c(\mathbb{R}; \mathbf{L}^2(\Omega))$.

Lemma 3.1. *Suppose $(u_\tau, \omega_\tau) \in V_1 \times V_2$ and $(f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$. If $\beta > 3$ then there exists a time t_0 and constants ρ_1, I_1 such that, for any $t \geq t_0$,*

$$\|u(t)\|^2 + \|\omega(t)\|^2 \leq \rho_1, \tag{3.1}$$

$$\int_t^{t+1} [\|\nabla u(s)\|^2 + \|\nabla \omega(s)\|^2 + \|u(s)\|^{\beta+1}_{\beta+1} + \|\nabla \cdot \omega(s)\|^2] ds \leq I_1. \tag{3.2}$$

Proof. Multiplying $(1.1)_1$ and $(1.1)_2$ with external forces $f_1 \in \mathcal{H}(f_1^0)$, $f_2 \in \mathcal{H}(f_2^0)$ by u and ω, respectively, and integrating the results equations on Ω, using Hölder's inequality, Young's inequality, and Poincaré's inequality, it yields

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \|\omega(t)\|^2) + (\nu + \kappa)\|\nabla u\|^2 + \gamma\|\nabla\omega\|^2 + 4\kappa\|\omega(t)\|^2 + \sigma\|u(t)\|^{\beta+1}_{\beta+1} + \mu\|\nabla \cdot \omega\|^2$$

$$= 4\kappa\int_\Omega \nabla\times u\cdot\omega dx + (f_1, u(t)) + (f_2, \omega(t))$$
$$\leq \kappa\parallel\nabla u\parallel^2 + 4\kappa\parallel\omega\parallel^2 + \frac{\nu\lambda}{2}\parallel u\parallel^2 + \frac{\gamma\lambda}{2}\parallel\omega\parallel^2 + \frac{1}{2\nu\lambda}\parallel f_1\parallel^2 + \frac{1}{2\gamma\lambda}\parallel f_2\parallel^2$$
$$\leq (\frac{\nu}{2}+\kappa)\|\nabla u\|^2 + \frac{\gamma}{2}\|\nabla\omega\|^2 + 4\kappa\|\omega(t)\|^2 + \frac{1}{2\nu\lambda}\parallel f_1\parallel^2 + \frac{1}{2\gamma\lambda}\parallel f_2\parallel^2. \tag{3.3}$$

So, we can obtain that

$$\frac{d}{dt}(\|u(t)\|^2 + \|\omega(t)\|^2) + \nu\|\nabla u\|^2 + \gamma\|\nabla\omega\|^2 + 2\sigma\|u(t)\|_{\beta+1}^{\beta+1} + 2\mu\|\nabla\cdot\omega\|^2 \leq \frac{1}{\nu\lambda}\|f_1(t)\|^2 + \frac{1}{\gamma\lambda}\|f_2(t)\|^2, \tag{3.4}$$

and by Poincaré's inequality, it yields

$$\frac{d}{dt}(\|u(t)\|^2 + \|\omega(t)\|^2) + \lambda\alpha(\|u(t)\|^2 + \|\omega(t)\|^2) \leq \frac{1}{\lambda\alpha}(\|f_1(t)\|^2 + \|f_2(t)\|^2), \tag{3.5}$$

where $\alpha = \min\{\nu, \gamma\}$. So, by Gronwall's inequality, we get

$$\|u(t)\|^2 + \|\omega(t)\|^2 \leq (\|u_\tau\|^2 + \|\omega_\tau\|^2)e^{-\lambda\alpha(t-\tau)} + \frac{1}{\lambda\alpha}\int_\tau^t e^{-\lambda\alpha(t-s)}(\|f_1(s)\|^2 + \|f_2(s)\|^2)ds$$
$$\leq (\|u_\tau\|^2 + \|\omega_\tau\|^2)e^{-\lambda\alpha(t-\tau)} + \frac{1}{\lambda\alpha}[\int_{t-1}^t e^{-\lambda\alpha(t-s)}(\|f_1(s)\|^2 + \|f_2(s)\|^2)ds$$
$$+ \int_{t-2}^{t-1} e^{-\lambda\alpha(t-s)}(\|f_1(s)\|^2 + \|f_2(s)\|^2)ds + ...]$$
$$\leq (\|u_\tau\|^2 + \|\omega_\tau\|^2)e^{-\lambda\alpha(t-\tau)} + \frac{1}{\lambda\alpha}[1 + e^{-\lambda\alpha} + e^{-2\lambda\alpha} + ...](\|f_1\|_{L_b^2}^2 + \|f_2\|_{L_b^2}^2)$$
$$\leq (\|u_\tau\|^2 + \|\omega_\tau\|^2)e^{-\lambda\alpha(t-\tau)} + \frac{1}{\lambda\alpha}(1 - e^{-\lambda\alpha})^{-1}(\|f_1\|_{L_b^2}^2 + \|f_2\|_{L_b^2}^2)$$
$$\leq (\|u_\tau\|^2 + \|\omega_\tau\|^2)e^{-\lambda\alpha(t-\tau)} + \frac{1}{\lambda\alpha}(1 + \frac{1}{\lambda\alpha})(\|f_1\|_{L_b^2}^2 + \|f_2\|_{L_b^2}^2), \quad \forall t \geq \tau.$$

Therefore, there must exists a time $t_0 \geq \tau + \frac{1}{\lambda\alpha}\ln\frac{\lambda^2\alpha^2(\|u_\tau\|^2+\|\omega_\tau\|^2)}{(1+\lambda\alpha)(\|f_1\|_{L_b^2}^2+\|f_2\|_{L_b^2}^2)}$, such that, $\forall t \geq t_0$,

$$\|u(t)\|^2 + \|\omega(t)\|^2 \leq \frac{2}{\lambda\alpha}(1 + \frac{1}{\lambda\alpha})(\|f_1\|_{L_b^2}^2 + \|f_2\|_{L_b^2}^2) \equiv \rho_1. \tag{3.6}$$

Taking $t \geq t_0$, integrating (3.4) from t to $t+1$, and noticing (3.6), we get

$$\int_t^{t+1}[\nu\|\nabla u(s)\|^2 + \gamma\|\nabla\omega(s)\|^2 + 2\sigma\|u(s)\|_{\beta+1}^{\beta+1} + 2\mu\|\nabla\cdot\omega(s)\|^2]ds$$
$$\leq (\|u(t)\|^2 + \|\omega(t)\|^2) + \frac{1}{\nu\lambda}\int_t^{t+1}\|f_1(s)\|^2 ds + \frac{1}{\gamma\lambda}\int_t^{t+1}\|f_2(s)\|^2 ds$$
$$\leq \rho_1 + \frac{1}{\lambda\alpha}(\|f_1\|_{L_b^2}^2 + \|f_2\|_{L_b^2}^2), \quad \forall t \geq t_0. \tag{3.7}$$

Letting $\delta_1 = \min\{\nu, \gamma, 2\sigma, 2\mu\}$, we have

$$\delta_1\int_t^{t+1}[\|\nabla u(s)\|^2 + \|\nabla\omega(s)\|^2 + \|u(s)\|_{\beta+1}^{\beta+1} + \|\nabla\cdot\omega(s)\|^2]ds \leq \rho_1 + \frac{1}{\lambda\alpha}(\|f_1\|_{L_b^2}^2 + \|f_2\|_{L_b^2}^2), \quad \forall t \geq t_0.$$

Letting $I_1 = \frac{1}{\delta_1}(\rho_1 + \frac{1}{\lambda\alpha}(\|f_1\|^2_{L^2_b} + \|f_2\|^2_{L^2_b}))$, we have

$$\int_t^{t+1} [\|\nabla u(s)\|^2 + \|\nabla\omega(s)\|^2 + \|u(s)\|^{\beta+1}_{\beta+1} + \|\nabla\cdot\omega(s)\|^2]ds \le I_1, \quad \forall t \ge t_0.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Assume $\beta > 3$, $(u_\tau, \omega_\tau) \in V_1 \times V_2$ and $(f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$. Then, there exists a time t_2 and a constant ρ_2 such that*

$$\|\nabla u(t)\|^2 + \|\nabla\omega(t)\|^2 + \int_t^{t+1} (\|Au(s)\|^2 + \|A\omega(s)\|^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|^2 + \|\nabla|u|^{\frac{\beta+1}{2}}\|^2)ds \le \rho_2, \tag{3.8}$$

for any $t \ge t_2$.

Proof. Taking the inner product of $-\Delta u$ in H_1 with the first equation of (1.1), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + (\nu+\kappa)\|Au\|^2 + \sigma\||u|^{\frac{\beta-1}{2}}\nabla u\|^2 + \frac{4\sigma(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|^2\\ &= -b(u,u,Au) + 2\kappa\int_\Omega \nabla\times\omega\cdot Audx + (f_1(t), Au).\end{aligned} \tag{3.9}$$

In [18], we find that, when $\beta > 3$,

$$\int_\Omega (u\cdot\nabla u)\cdot\Delta u dx \le \frac{\nu+\kappa}{4}\parallel \Delta u\parallel^2 + \frac{\sigma}{2}\parallel |u|^{\frac{\beta-1}{2}}\nabla u\parallel^2 + C_1\parallel\nabla u\parallel^2, \tag{3.10}$$

where $C_1 = \frac{N^2}{\nu+\kappa} + \frac{N^2}{(\nu+\kappa)(N^{\beta-1}+1)}$, and N is sufficiently large such that

$$N \ge (\frac{2}{\beta-3})^{\frac{1}{\beta-1}} \text{ and } \frac{N^2}{(\nu+\kappa)(N^{\beta-1}+1)} \le \frac{\sigma}{2}.$$

And, because

$$|2\kappa\int_\Omega \nabla\times\omega\cdot Audx| \le \frac{\nu+\kappa}{4}\parallel\Delta u\parallel^2 + \frac{4\kappa^2}{\nu+\kappa}\parallel\nabla\omega\parallel^2, \tag{3.11}$$

$$|(f_1(t), Au)| \le \frac{\nu+\kappa}{4}\parallel\Delta u\parallel^2 + \frac{\parallel f_1(t)\parallel^2}{\nu+\kappa}, \tag{3.12}$$

so combining (3.10)–(3.12) with (3.9), we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla u\|^2 + \frac{\nu+\kappa}{2}\|Au\|^2 + \sigma\||u|^{\frac{\beta-1}{2}}\nabla u\|^2 + \frac{8\sigma(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|^2\\ &\le 2C_1\|\nabla u\|^2 + \frac{8\kappa^2}{\nu+\kappa}\parallel\nabla\omega\parallel^2 + \frac{2\parallel f_1(t)\parallel^2}{\nu+\kappa}\\ &\le C_2(\|\nabla u\|^2 + \|\nabla\omega\|^2 + \|f_1(t)\|^2),\end{aligned} \tag{3.13}$$

where $C_2 = \max\{2C_1, \frac{8\kappa^2}{\nu+\kappa}, \frac{2}{\nu+\kappa}\}$.

Applying uniform Gronwall's inequality to (3.13), we obtaint, $\forall t \geq t_0 + 1 \equiv t_1$,

$$\|\nabla u(t)\|^2 + \int_t^{t+1} \Big(\frac{\nu+\kappa}{2}\|Au(s)\|^2 + \sigma\||u(s)|^{\frac{\beta-1}{2}}\nabla u(s)\|^2 + \frac{8\sigma(\beta-1)}{(\beta+1)^2}\|\nabla|u(s)|^{\frac{\beta+1}{2}}\|^2\Big)ds \leq C_3, \tag{3.14}$$

where C_3 is a positive constant dependent on C_2, I_1, and $\| f_1^0 \|^2_{L_b^2}$.

Taking the inner product of $-\Delta\omega$ in H_2 with the second equation of (1.1), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|^2 + 4\kappa\|\nabla\omega\|^2 + \gamma\|A\omega\|^2 + \mu\|\nabla\nabla\cdot\omega\|^2\\
&= -b(u,\omega,A\omega) + 2\kappa\int_\Omega \nabla\times u\cdot A\omega dx + (f_2(t), A\omega)\\
&\leq \frac{3\gamma}{4}\|A\omega\|^2 + \frac{d_1^2}{\gamma}\|\nabla u\|\|Au\|\|\nabla\omega\|^2 + \frac{4\kappa^2}{\gamma}\|\nabla u\|^2 + \frac{1}{\gamma}\|f_2(t)\|^2.
\end{aligned} \tag{3.15}$$

In the last inequality of (3.15), we used Agmon's inequality and the trilinear inequality. Then,

$$\frac{d}{dt}\|\nabla\omega\|^2 + \frac{\gamma}{2}\|A\omega\|^2 + 2\mu\|\nabla\nabla\cdot\omega\|^2 \leq C_4(\|\nabla u\|\|Au\|\|\nabla\omega\|^2 + \|\nabla u\|^2 + \|f_2(t)\|^2), \tag{3.16}$$

where $C_4 = \max\{\frac{2d_1^2}{\gamma}, \frac{8\kappa^2}{\gamma}, \frac{2}{\gamma}\}$.

By the uniform Gronwall's inequality, we easily obtain that, for $t \geq t_1 + 1 \equiv t_2$,

$$\|\nabla\omega(t)\|^2 + \int_t^{t+1}(\frac{\gamma}{2}\|A\omega(s)\|^2 + 2\mu\|\nabla\nabla\cdot\omega(s)\|^2)ds \leq C_5,\ \text{for } t \geq t_1 + 1 \equiv t_2, \tag{3.17}$$

where C_5 is a positive constant dependent on C_3, C_4, and $\| f_2^0 \|^2_{L_b^2}$.

Adding (3.14) with (3.17) yields

$$\|\nabla u(s)\|^2 + \|\nabla\omega(s)\|^2 + \int_t^{t+1}(\|Au(s)\|^2 + \|A\omega(s)\|^2 + \||u(s)|^{\frac{\beta-1}{2}}\nabla u(s)\|^2 + \|\nabla|u(s)|^{\frac{\beta+1}{2}}\|^2)ds \leq C,$$

for $t \geq t_2$. Hence, Lemma 3.2 is proved. □

Lemma 3.3. *Suppose that $(u_\tau, \omega_\tau) \in V_1 \times V_2$ and $(f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$. Then, for $\beta > 3$, there exists a time t_3 and a constant ρ_3 such that*

$$\|u(t)\|_{\beta+1} + \|\nabla\cdot\omega(t)\|^2 \leq \rho_3, \tag{3.18}$$

for any $t \geq t_3$.

Proof. Multiplying $(1.1)_1$ by u_t, then integrating the equation over Ω, we have

$$\begin{aligned}
&\|u_t\|^2 + \frac{\nu+\kappa}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{\sigma}{\beta+1}\frac{d}{dt}\|u(t)\|_{\beta+1}^{\beta+1}\\
&= -b(u,u,u_t) + 2\kappa\int_\Omega \nabla\times\omega\cdot u_t dx + (f_1(t), u_t)\\
&\leq \frac{1}{2}\|u_t\|^2 + \frac{3d_1^2}{2\sqrt{\lambda_1}}\| \nabla u \|^2\| Au \|^2 + 6\kappa^2\|\nabla\omega\|^2 + \frac{3}{2}\|f_1(t)\|^2.
\end{aligned} \tag{3.19}$$

The trilinear inequality (2.4), Agmon's inequality, and Poincaré's inequality are used in the last inequality of (3.19).

Hence,

$$(\nu+\kappa)\frac{d}{dt}\|\nabla u\|^2+\frac{2\sigma}{\beta+1}\frac{d}{dt}\|u(t)\|_{\beta+1}^{\beta+1}\le C_6(\|\nabla u\|^2\|Au\|^2+\|\nabla\omega\|^2+\|f_1(t)\|^2),\tag{3.20}$$

where $C_6=\max\{\frac{3d_1^2}{\sqrt{\lambda_1}},12\kappa^2,3\}$.

By (3.20), using Lemmas 3.1 and 3.2 and the uniform Gronwall's inequality, we have

$$\|u(t)\|_{\beta+1}\le C,\quad \forall t\ge t_2+1\equiv t_3.\tag{3.21}$$

Similar to (3.19), multiplying $(1.1)_2$ by ω_t and integrating it over Ω, we get

$$\begin{aligned}&\|\omega_t\|^2+2\kappa\frac{d}{dt}\|\omega\|^2+\frac{\gamma}{2}\frac{d}{dt}\|\nabla\omega\|^2+\frac{\mu}{2}\frac{d}{dt}\|\nabla\cdot\omega\|^2=-b(u,\omega,\omega_t)+2\kappa\int_\Omega\nabla\times u\cdot\omega_t dx+(f_2(t),\omega_t)\\&\le\frac{1}{2}\|\omega_t\|^2+\frac{3d_1^2}{2\sqrt{\lambda_1}}\|Au\|^2\|\nabla\omega\|^2+6\kappa^2\|\nabla u\|^2+\frac{3}{2}\|f_2(t)\|^2.\end{aligned}\tag{3.22}$$

Hence,

$$4\kappa\frac{d}{dt}\|\omega\|^2+\gamma\frac{d}{dt}\|\nabla\omega\|^2+\mu\frac{d}{dt}\|\nabla\cdot\omega\|^2\le C_6(\|Au\|^2\|\nabla\omega\|^2+\|\nabla u\|^2+\|f_2(t)\|^2).\tag{3.23}$$

By (3.23), using Lemma 3.2 and the uniform Gronwall's inequality, we infer that

$$\|\nabla\cdot\omega(t)\|^2\le C,\quad\forall t\ge t_3.\tag{3.24}$$

The proof of Lemma 3.3 is finished. □

Lemma 3.4. *Suppose $(u_\tau,\omega_\tau)\in V_1\times V_2$ and $(f_1,f_2)\in\mathcal{H}(f_1^0)\times\mathcal{H}(f_2^0)$. If $\beta>3$, then there exists a time t_4 and a constant ρ_5, such that*

$$\|u_t(s)\|^2+\|\omega_t(s)\|^2\le\rho_5,\tag{3.25}$$

for any $s\ge t_4$.

Proof. Taking the inner products of u_t and ω_t with the first and second equations of (1.1), respectively, and using (3.19) and (3.22), we find

$$\begin{aligned}&\|u_t\|^2+\|\omega_t\|^2+\frac{\nu+\kappa}{2}\frac{d}{dt}\|\nabla u\|^2+\frac{\gamma}{2}\frac{d}{dt}\|\nabla\omega\|^2+2\kappa\frac{d}{dt}\|\omega(t)\|^2+\frac{\sigma}{\beta+1}\frac{d}{dt}\|u(t)\|_{\beta+1}^{\beta+1}+\frac{\mu}{2}\frac{d}{dt}\|\nabla\cdot\omega\|^2\\&=-b(u,u,u_t)-b(u,\omega,\omega_t)+2\kappa\int_\Omega\nabla\times\omega\cdot u_t dx+2\kappa\int_\Omega\nabla\times u\cdot\omega_t dx+(f_1(t),u_t)+(f_2(t),\omega_t)\\&\le\frac{1}{2}(\|u_t\|^2+\|\omega_t\|^2)+C_7(\|f_1(t)\|^2+\|f_2(t)\|^2+\|\nabla u\|^2\\&\quad+\|\nabla\omega\|^2+\|\nabla u\|^2\|Au\|^2+\|\nabla\omega\|^2\|Au\|^2),\end{aligned}\tag{3.26}$$

where $C_7 = \max\{\frac{3d_1^2}{2\sqrt{\lambda_1}}, 6\kappa^2, \frac{3}{2}\}$. The trilinear inequality (2.4), Agmon's inequality, and Poincaré's inequality are used in the last inequality of (3.26).

Integrating (3.26) over $[t, t+1]$ and using Lemmas 3.1–3.3, we get

$$\int_t^{t+1} (\|u_t(s)\|^2 + \|\omega_t(s)\|^2) ds \leq \rho_4, \ \forall t \geq t_3, \tag{3.27}$$

where ρ_4 is a positive constant dependent on $C_7, \rho_2, \rho_3, \| f_1^0 \|^2_{L_b^2}$, and $\| f_2^0 \|^2_{L_b^2}$.

We now differentiate $(2.1)_1$ with respect to t, then take the inner product of u_t with the resulting equation to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \| u_t \|^2 + (\nu + \kappa) \| \nabla u_t \|^2 \\ &= -b(u_t, u, u_t) - \int_\Omega G'(u) u_t \cdot u_t dx + 2\kappa \int_\Omega \nabla \times \omega_t \cdot u_t dx + (f_{1t}, u_t). \end{aligned} \tag{3.28}$$

Then, we differentiate $(2.1)_2$ with respect to t and take the inner product with ω_t to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \| \omega_t \|^2 + 4\kappa \| \omega_t \|^2 + \gamma \| \nabla \omega_t \|^2 + \mu \| \nabla \cdot \omega_t \|^2 \\ &= -b(u_t, \omega, \omega_t) + 2\kappa \int_\Omega \nabla \times u_t \cdot \omega_t dx + (f_{2t}, \omega_t). \end{aligned} \tag{3.29}$$

Adding (3.28) with (3.29), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_t\|^2 + \|\omega_t\|^2) + (\nu + \kappa)\|\nabla u_t\|^2 + \gamma\|\nabla\omega_t\|^2 + 4\kappa\|\omega_t\|^2 + \mu\|\nabla \cdot \omega_t\|^2 \\ &\leq |b(u_t, u, u_t)| + |b(u_t, \omega, \omega_t)| + 2\kappa \int_\Omega \nabla \times \omega_t \cdot u_t dx + 2\kappa \int_\Omega \nabla \times u_t \cdot \omega_t dx \\ &\quad + (f_{1t}, u_t) + (f_{2t}, \omega_t) - \int_\Omega G'(u) u_t \cdot u_t dx \\ &:= \sum_{i=1}^{7} L_i. \end{aligned} \tag{3.30}$$

From Lemma 2.4 in [15], we know that $G'(u)$ is positive definite, so

$$L_7 = -\int_\Omega G'(u) u_t \cdot u_t dx \leq 0. \tag{3.31}$$

For L_1, using the trilinear inequality (2.5) and Lemma 3.2, we have

$$\begin{aligned} L_1 &\leq k\|u_t\|^{\frac{1}{2}}\|\nabla u_t\|^{\frac{3}{2}}\|\nabla u\| \\ &\leq \frac{\nu + \kappa}{4}\|\nabla u_t\|^2 + C\|u_t\|^2\|\nabla u\|^4 \\ &\leq \frac{\nu + \kappa}{4}\|\nabla u_t\|^2 + C\|u_t\|^2, \ \text{for } t \geq t_2. \end{aligned} \tag{3.32}$$

For L_2, by Hölder's inequality, Gagliardo-Nirenberg's inequality, and Young's inequality, we have

$$\begin{aligned} L_2 &\leq C\|u_t\|_4\|\omega_t\|_4\|\nabla\omega\| \\ &\leq C\|u_t\|^{\frac{1}{4}}\|\nabla u_t\|^{\frac{3}{4}}\|\omega_t\|^{\frac{1}{4}}\|\nabla\omega_t\|^{\frac{3}{4}}\|\nabla\omega\| \\ &\leq \frac{\nu+\kappa}{4}\|\nabla u_t\|^2 + \frac{\gamma}{4}\|\nabla\omega_t\|^2 + C(\|u_t\|^2 + \|\omega_t\|^2), \text{ for } t \geq t_2. \end{aligned} \tag{3.33}$$

$$L_3 + L_4 \leq \frac{\nu+\kappa}{4}\|\nabla u_t\|^2 + \frac{\gamma}{2}\|\nabla\omega_t\|^2 + C(\|u_t\|^2 + \|\omega_t\|^2). \tag{3.34}$$

By (3.30)–(3.34), we get

$$\begin{aligned} \frac{d}{dt}(\|u_t\|^2 + \|\omega_t\|^2) &\leq C(\|u_t\|^2 + \|\omega_t\|^2) + (f_{1t}, u_t) + (f_{2t}, \omega_t) \\ &\leq C(\|u_t\|^2 + \|\omega_t\|^2) + \|f_{1t}\|^2 + \|f_{2t}\|^2. \end{aligned} \tag{3.35}$$

Thanks to

$$\int_t^{t+1} \| f_{1t}(s) \|^2 \, ds \leq \| f_{1t} \|^2_{L^2_b} \leq \| h_1 \|^2_{L^2_b}, \int_t^{t+1} \| f_{2t}(s) \|^2 \, ds \leq \| f_{2t} \|^2_{L^2_b} \leq \| h_2 \|^2_{L^2_b},$$

and applying uniform Gronwall's inequality to (3.35), we have for any $s \geq t_3 + 1 \equiv t_4$,

$$\|u_t(s)\|^2 + \|\omega_t(s)\|^2 \leq C. \tag{3.36}$$

Thus, Lemma 3.4 is proved. □

Lemma 3.5. *Suppose $(u_\tau, \omega_\tau) \in V_1 \times V_2$ and $(f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$. Then, for $\beta > 3$, there exists a constant ρ_6 such that*

$$\|Au(t)\|^2 + \|A\omega(t)\|^2 \leq \rho_6, \tag{3.37}$$

for any $t \geq t_4$.

Proof. Taking the inner product of $-\Delta u$ in H_1 with the first equation of (1.1), we have

$$\begin{aligned} &(\nu+\kappa) \| Au \|^2 + \sigma \| |u|^{\frac{\beta-1}{2}} \nabla u \|^2 + \frac{4\sigma(\beta-1)}{(\beta+1)^2} \| \nabla |u|^{\frac{\beta+1}{2}} \|^2 \\ &= -(u_t, Au) - (B(u), Au) + 2\kappa \int_\Omega \nabla \times \omega \cdot Au dx + (f_1(t), Au) \\ &\leq \frac{4(\nu+\kappa)}{6} \| Au \|^2 + \frac{3}{2(\nu+\kappa)} \| u_t \|^2 + \frac{3}{2(\nu+\kappa)} \| B(u) \|^2 \\ &\quad + \frac{6\kappa^2}{\nu+\kappa} \| \nabla\omega \|^2 + \frac{3}{2(\nu+\kappa)} \| f_1(t) \|^2 . \end{aligned} \tag{3.38}$$

Because

$$\frac{3}{2(\nu+\kappa)} \| B(u) \|^2 \leq \frac{3}{2(\nu+\kappa)} \| u \|^2_\infty \| \nabla u \|^2$$

$$\leq \frac{3d_1^2}{2(\nu+\kappa)} \parallel \nabla u \parallel^3 \parallel \Delta u \parallel$$
$$\leq \frac{\nu+\kappa}{6} \parallel Au \parallel^2 + C \parallel \nabla u \parallel^6, \tag{3.39}$$

combining (3.39) with (3.38), we obtain

$$\frac{\nu+\kappa}{6} \parallel Au \parallel^2 \leq \frac{3}{2(\nu+\kappa)} \parallel u_t \parallel^2 + C \parallel \nabla u \parallel^6 + \frac{6\kappa^2}{\nu+\kappa} \parallel \nabla\omega \parallel^2 + \frac{3}{2(\nu+\kappa)} \parallel f_1(t) \parallel^2 . \tag{3.40}$$

From the assumption of $f_1^0(t)$, we can easily get

$$\sup_{t\in\mathbb{R}} \parallel f_1(t) \parallel \leq \sup_{t\in\mathbb{R}} \parallel f_1^0(t) \parallel \leq K, \forall f_1 \in \mathcal{H}(f_1^0). \tag{3.41}$$

By Lemmas 3.2 and 3.4, we obtain

$$\|Au(t)\| \leq C, \text{ for any } t \geq t_4. \tag{3.42}$$

Taking the inner product of $A\omega$ with $(2.1)_2$, we get

$$\gamma\|A\omega\|^2 + 4\kappa \parallel \nabla\omega \parallel^2 + \mu \parallel \nabla\nabla\cdot\omega \parallel^2$$
$$= -(\omega_t, A\omega) - (B(u,\omega), A\omega) + 2\kappa(\nabla\times u, A\omega) + (f_2(t), A\omega)$$
$$\leq \frac{\gamma}{2} \parallel A\omega \parallel^2 + C(\parallel \omega_t \parallel^2 + \parallel B(u,\omega) \parallel^2 + \parallel \nabla u \parallel^2 + \parallel f_2(t) \parallel^2). \tag{3.43}$$

And, by Agmon's inequality,

$$\|B(u,\omega)\|^2 \leq C\|u\|_\infty^2\|\nabla\omega\|^2$$
$$\leq C\|\nabla u\|\|\Delta u\|\|\nabla\omega\|^2$$
$$\leq \|Au\|^2 + C \parallel \nabla u \parallel^2 \parallel \nabla\omega \parallel^4 . \tag{3.44}$$

From the assumption on $f_2^0(t)$, we can easily obtain

$$\sup_{t\in\mathbb{R}} \parallel f_2(t) \parallel \leq \sup_{t\in\mathbb{R}} \parallel f_2^0(t) \parallel \leq K, \forall f_2 \in \mathcal{H}(f_2^0). \tag{3.45}$$

By Lemma 3.2, Lemma 3.4, (3.42), (3.43), (3.44), and (3.45), we get

$$\|A\omega(t)\| \leq C, \text{ for any } t \geq t_4. \tag{3.46}$$

By (3.42) and (3.46), Lemma 3.5 is proved for all $t \geq t_4$. □

Lemma 3.6. *Suppose $(u_\tau, \omega_\tau) \in V_1 \times V_2$ and $(f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$. Then, for $\beta > 3$, there exists a time t_5 and a constant ρ_7 satisfying*

$$\|\nabla u_t(t)\|^2 + \|\nabla\omega_t(t)\|^2 \leq \rho_7, \forall t \geq t_5. \tag{3.47}$$

Proof. In the proof of Lemma 3.4, from (3.30)–(3.34) we can also get

$$\begin{aligned}&\frac{d}{dt}(\| u_t \|^2 + \| \omega_t \|^2) + \frac{\nu+\kappa}{2} \| \nabla u_t \|^2 + \frac{\gamma}{2} \| \nabla\omega_t \|^2 + 2\mu \| \nabla\cdot\omega_t \|^2 \\ &\leq C(\| u_t \|^2 + \| \omega_t \|^2) + \| f_1(t) \|^2 + \| f_2(t) \|^2 .\end{aligned} \tag{3.48}$$

Integrating (3.48) from t to $t+1$, and according to Lemma 3.4, we have

$$\begin{aligned}&\int_t^{t+1}(\|\nabla u_t(s)\|^2 + \|\nabla\omega_t(s)\|^2 + \|\nabla\cdot\omega_t(s)\|^2)ds \\ &\leq C(\|u_t(t)\|^2 + \|\omega_t(t)\|^2 + \int_t^{t+1}(\|u_t(s)\|^2 + \|\omega_t(s)\|^2)ds + \int_t^{t+1}\|f_{1t}(s)\|^2 ds + \int_t^{t+1}\|f_{2t}(s)\|^2 ds) \\ &\leq C + \|h_1\|^2_{L_b^2} + \|h_2\|^2_{L_b^2} \\ &\leq C,\ \forall t \geq t_4.\end{aligned} \tag{3.49}$$

By Lemma 3.5, we get

$$\|u(t)\|_{H^2} + \|\omega(t)\|_{H^2} \leq C, \forall t \geq t_4. \tag{3.50}$$

So, by Lemma 3.2, applying Agmon's inequality, we get

$$\|u(t)\|_\infty + \|\omega(t)\|_\infty \leq C, \forall t \geq t_4. \tag{3.51}$$

Taking the derivative of $(2.1)_1$ and $(2.1)_2$ with respect to t, then multiplying by Au_t and $A\omega_t$, respectively, and integrating the resulting equations over Ω, we then have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u_t\|^2 + \|\nabla\omega_t\|^2) + (\nu+\kappa)\|Au_t\|^2 + \gamma\|A\omega_t\|^2 + 4\kappa\|\nabla\omega_t\|^2 + \mu \| \nabla\nabla\cdot\omega_t \|^2 \\ &\leq |b(u_t, u, Au_t)| + |b(u, u_t, Au_t)| + |b(u, \omega_t, A\omega_t)| + |b(u_t, \omega, A\omega_t)| \\ &\quad + 2\kappa\int_\Omega |\nabla\times\omega_t\cdot Au_t|dx + 2\kappa\int_\Omega |\nabla\times u_t\cdot A\omega_t|dx + |\int_\Omega G'(u)u_t\cdot Au_t dx| \\ &\quad + (f_{1t}, Au_t) + (f_{2t}, A\omega_t) \\ &:= \sum_{i=1}^{9} J_i.\end{aligned} \tag{3.52}$$

For J_1, J_2, using (2.6) and Lemmas 3.2 and 3.5, we have

$$\begin{aligned}J_1 &\leq k\|\nabla u_t\|\|\nabla u\|^{\frac{1}{2}}\|Au\|^{\frac{1}{2}}\|Au_t\| \\ &\leq \frac{\nu+\kappa}{5}\|Au_t\|^2 + C\|\nabla u_t\|^2,\ \forall t \geq t_4,\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}J_2 &\leq k\|\nabla u\|\|\nabla u_t\|^{\frac{1}{2}}\|Au_t\|^{\frac{1}{2}}\|Au_t\| \\ &\leq k\|\nabla u\|\|\nabla u_t\|^{\frac{1}{2}}\|Au_t\|^{\frac{3}{2}}\end{aligned}$$

$$\leq \frac{\nu+\kappa}{5}\|Au_t\|^2 + C\|\nabla u_t\|^2, \ \forall t \geq t_4. \tag{3.54}$$

For J_3 and J_4, similar to (3.53) and (3.54), we get

$$J_3 \leq k\|\nabla u\|\|\nabla \omega_t\|^{\frac{1}{2}}\|A\omega_t\|^{\frac{1}{2}}\|A\omega_t\|$$
$$\leq \frac{\gamma}{4}\|A\omega_t\|^2 + C\|\nabla \omega_t\|^2, \ \forall t \geq t_4, \tag{3.55}$$
$$J_4 \leq k\|\nabla u_t\|\|\nabla \omega\|^{\frac{1}{2}}\|A\omega\|^{\frac{1}{2}}\|A\omega_t\|$$
$$\leq \frac{\gamma}{4}\|A\omega_t\|^2 + C\|\nabla u_t\|^2, \ \forall t \geq t_4. \tag{3.56}$$

For J_5, J_6, and J_7, applying Hölder's inequality and Young's inequality, we have

$$J_5 + J_6 \leq \frac{\nu+\kappa}{5}\|Au_t\|^2 + \frac{\gamma}{4}\|A\omega_t\|^2 + C(\|\nabla u_t\|^2 + \|\nabla \omega_t\|^2), \tag{3.57}$$

and thanks to (3.51),

$$J_7 \leq C\|u\|_\infty^{\beta-1}\|u_t\|\|Au_t\|$$
$$\leq \frac{\nu+\kappa}{5}\|Au_t\|^2 + C\|u_t\|^2, \ \forall t \geq t_4. \tag{3.58}$$

For J_8 and J_9, we have

$$J_8 \leq \frac{\nu+\kappa}{5}\|Au_t\|^2 + C\|f_{1t}\|^2, \tag{3.59}$$
$$J_9 \leq \frac{\gamma}{4}\|A\omega_t\|^2 + C\|f_{2t}\|^2. \tag{3.60}$$

By (3.52)–(3.60), we obtain

$$\frac{d}{dt}(\|\nabla u_t\|^2 + \|\nabla \omega_t\|^2) \leq C(\|\nabla u_t\|^2 + \|\nabla \omega_t\|^2) + C\|u_t\|^2 + C(\|f_{1t}\|^2 + \|f_{2t}\|^2). \tag{3.61}$$

Then, by (3.27), (3.49), and using the uniform Gronwall's lemma, we get

$$\|\nabla u_t(s)\|^2 + \|\nabla \omega_t(s)\|^2 \leq C, \ \forall s \geq t_4 + 1 \equiv t_5. \tag{3.62}$$

Thus, Lemma 3.6 is proved. □

4. Existence of uniform attractors

In this section, we consider the existence of the $(V_1 \times V_2, V_1 \times V_2)$-uniform (w.r.t. $(f_1, f_2) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$) attractor and the $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform attractor for $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}, f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$.

Lemma 4.1. *Suppose $\beta > 3$. The family of processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$, corresponding to (2.1) is $((V_1 \times V_2) \times (\mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)), V_1 \times V_2)$-continuous for $\tau \geq t_5$.*

Proof. Let $\tau_n \subset [\tau, +\infty)$ be a time sequence, $U_{(f_1^{(n)}, f_2^{(n)})}(t,\tau)(u_{\tau_n}, \omega_{\tau_n}) = (u^{(n)}(t), \omega^{(n)}(t))$, $U_{(f_1,f_2)}(t,\tau)(u_\tau, \omega_\tau) = (u(t), \omega(t))$ and

$$\begin{aligned}(\bar{u}^{(n)}(t), \bar{\omega}^{(n)}(t)) &= (u(t) - u^{(n)}(t), \omega(t) - \omega^{(n)}(t))\\ &= U_{(f_1,f_2)}(t,\tau)(u_\tau, \omega_\tau) - U_{(f_1^{(n)}, f_2^{(n)})}(t,\tau)(u_{\tau_n}, \omega_{\tau_n}).\end{aligned}$$

It is evident that $\bar{u}^{(n)}(t)$ is the solution of

$$\frac{\partial \bar{u}^{(n)}(t)}{\partial t} + B(u) - B(u^{(n)}(t)) + (\nu + \kappa)A\bar{u}^{(n)} + G(u) - G(u^{(n)}) = 2\kappa \nabla \times \bar{\omega}^{(n)} + (f_1 - f_1^{(n)}), \tag{4.1}$$

and $\bar{\omega}^{(n)}(t)$ is the solution of the following system

$$\frac{\partial \bar{\omega}^{(n)}(t)}{\partial t} + B(u,\omega) - B(u^{(n)}, \omega^{(n)}) + 4\kappa\bar{\omega}^{(n)} + \gamma A\bar{\omega}^{(n)} - \mu \nabla\nabla \cdot \bar{\omega}^{(n)} = 2\kappa \nabla \times \bar{u}^{(n)} + (f_2 - f_2^{(n)}), \tag{4.2}$$

for each n.

Taking the inner product of (4.1) with $A\bar{u}^{(n)}$ in H_1, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla \bar{u}^{(n)}\|^2 + b(u,u,A\bar{u}^{(n)}) - b(u^{(n)}, u^{(n)}, A\bar{u}^{(n)}) + (\nu+\kappa) \parallel A\bar{u}^{(n)} \parallel^2 + (G(u) - G(u^{(n)}), A\bar{u}^{(n)})\\ &= 2\kappa(\nabla \times \bar{\omega}^{(n)}, A\bar{u}^{(n)}) + (f_1 - f_1^{(n)}, A\bar{u}^{(n)}).\end{aligned} \tag{4.3}$$

Taking the inner product of (4.2) with $A\bar{\omega}^{(n)}$ in H_2, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla \bar{\omega}^{(n)}\|^2 + b(u,\omega,A\bar{\omega}^{(n)}) - b(u^{(n)}, \omega^{(n)}, A\bar{\omega}^{(n)}) + 4\kappa\|\nabla\bar{\omega}^{(n)}\|^2 + \gamma \parallel A\bar{\omega}^{(n)} \parallel^2 + \mu \parallel \nabla\nabla \cdot \bar{\omega}^{(n)} \parallel^2\\ &= 2\kappa(\nabla \times \bar{u}^{(n)}, A\bar{\omega}^{(n)}) + (f_2 - f_2^{(n)}, A\bar{\omega}^{(n)}).\end{aligned} \tag{4.4}$$

Combining (4.3) with (4.4), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla \bar{u}^{(n)}\|^2 + \|\nabla \bar{\omega}^{(n)}\|^2) + b(u,u,A\bar{u}^{(n)}) - b(u^{(n)}, u^{(n)}, A\bar{u}^{(n)}) + (\nu+\kappa)\|A\bar{u}^{(n)}\|^2\\ &+ (G(u) - G(u^{(n)}), A\bar{u}^{(n)}) + b(u,\omega,A\bar{\omega}^{(n)}) - b(u^{(n)}, \omega^{(n)}, A\bar{\omega}^{(n)})\\ &+ 4\kappa \parallel \nabla\bar{\omega}^{(n)} \parallel^2 + \gamma \parallel A\bar{\omega}^{(n)} \parallel^2 + \mu \parallel \nabla\nabla \cdot \bar{\omega}^{(n)} \parallel^2\\ &= 2\kappa(\nabla \times \bar{\omega}^{(n)}, A\bar{u}^{(n)}) + 2\kappa(\nabla \times \bar{u}^{(n)}, A\bar{\omega}^{(n)}) + (f_1 - f_1^{(n)}, A\bar{u}^{(n)}) + (f_2 - f_2^{(n)}, A\bar{\omega}^{(n)}).\end{aligned} \tag{4.5}$$

Due to

$$b(u,u,A\bar{u}^{(n)}) - b(u^{(n)}, u^{(n)}, A\bar{u}^{(n)}) = b(\bar{u}^{(n)}, u, A\bar{u}^{(n)}) + b(u^{(n)}, \bar{u}^{(n)}, A\bar{u}^{(n)}), \tag{4.6}$$

$$b(u,\omega,A\bar{\omega}^{(n)}) - b(u^{(n)}, \omega^{(n)}, A\bar{\omega}^{(n)}) = b(\bar{u}^{(n)}, \omega, A\bar{\omega}^{(n)}) + b(u^{(n)}, \bar{\omega}^{(n)}, A\bar{\omega}^{(n)}), \tag{4.7}$$

and

$$\begin{aligned}|b(\bar{u}^{(n)}, u, A\bar{u}^{(n)})| &\le k\|\nabla\bar{u}^{(n)}\|\|\nabla u\|^{\frac{1}{2}}\|Au\|^{\frac{1}{2}}\|A\bar{u}^{(n)}\|\\ &\le \frac{\nu+k}{5}\|A\bar{u}^{(n)}\|^2 + C\|\nabla\bar{u}^{(n)}\|^2\|\nabla u\|\|Au\|,\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
|b(u^{(n)}, \bar{u}^{(n)}, A\bar{u}^{(n)})| &\le k\|\nabla u^{(n)}\|\|\nabla \bar{u}^{(n)}\|^{\frac{1}{2}}\|A\bar{u}^{(n)}\|^{\frac{1}{2}}\|A\bar{u}^{(n)}\| \\
&\le \frac{\nu+k}{5}\|A\bar{u}^{(n)}\|^2 + C\|\nabla u^{(n)}\|^4\|\nabla \bar{u}^{(n)}\|^2, && (4.9)\\
b(\bar{u}^{(n)}, \omega, A\bar{\omega}^{(n)}) &\le k\|\nabla \bar{u}^{(n)}\|\|\nabla \omega\|^{\frac{1}{2}}\|A\omega\ \|^{\frac{1}{2}}\ \|A\bar{\omega}^{(n)}\| \\
&\le \frac{\gamma}{4}\|A\bar{\omega}^{(n)}\|^2 + C\|\nabla \bar{u}^{(n)}\|^2\|\nabla\omega\|\|A\omega\|, && (4.10)\\
b(u^{(n)}, \bar{\omega}^{(n)}, A\bar{\omega}^{(n)}) &\le k\|\nabla u^{(n)}\|\|\nabla \bar{\omega}^{(n)}\|^{\frac{1}{2}}\|A\bar{\omega}^{(n)}\|^{\frac{1}{2}}\|A\bar{\omega}^{(n)}\| \\
&\le \frac{\gamma}{4}\|A\bar{\omega}^{(n)}\|^2 + C\parallel \nabla u^{(n)}\parallel^4\parallel \nabla\bar{\omega}^{(n)}\parallel^2, && (4.11)\\
2\kappa|(\nabla\times\bar{\omega}^{(n)}, A\bar{u}^{(n)})| &\le 2\kappa\|A\bar{u}^{(n)}\|\|\nabla\bar{\omega}^{(n)}\| \\
&\le \frac{\nu+k}{5}\|A\bar{u}^{(n)}\|^2 + C\|\nabla\bar{\omega}^{(n)}\|^2, && (4.12)\\
2\kappa|(\nabla\times\bar{u}^{(n)}, A\bar{\omega}^{(n)})\| &\le 2\kappa\parallel A\bar{\omega}^{(n)}\|\|\nabla\bar{u}^{(n)}\| \\
&\le \frac{\gamma}{4}\|A\bar{\omega}^{(n)}\|^2 + C\|\nabla\bar{u}^{(n)}\|^2, && (4.13)\\
|(f_1 - f_1^{(n)}, A\bar{u}^{(n)})| &\le \frac{\nu+k}{5}\|A\bar{u}^{(n)}\|^2 + \frac{5}{4(\nu+\kappa)}\|f_1 - f_1^{(n)}\|^2, && (4.14)\\
|(f_2 - f_2^{(n)}, A\bar{\omega}^{(n)})| &\le \frac{\gamma}{4}\|A\bar{\omega}^{(n)}\|^2 + \frac{1}{\gamma}\|f_2 - f_2^{(n)}\|^2, && (4.15)\\
\|G(u) - G(u^{(n)})\|^2 &= \int_\Omega \left|\sigma|u|^{\beta-1}u - \sigma|u^{(n)}|^{\beta-1}u^{(n)}\right|^2 dx \\
&\le C\int_\Omega [|u|^{\beta-1}|\bar{u}^{(n)}| + \left||u|^{\beta-1} - |u^{(n)}|^{\beta-1}\right|\cdot|u^{(n)}|]^2 dx \\
&\le C\int_\Omega |u|^{2(\beta-1)}|\bar{u}^{(n)}|^2 dx + C\int_\Omega [|u|^{\beta-2} + |u^{(n)}|^{\beta-2}]^2|u^{(n)}|^2|\bar{u}^{(n)}|^2 dx \\
&\le C[\parallel u\parallel_\infty^{2(\beta-1)} + (\parallel u\parallel_\infty^{2(\beta-2)} + \parallel u^{(n)}\parallel_\infty^{2(\beta-2)})\parallel u^{(n)}\parallel_\infty^2]\parallel \nabla\bar{u}^{(n)}\parallel^2, && (4.16)
\end{aligned}
$$

where $\bar{u}^{(n)}(t) = u(t) - u^n(t)$. In the above inequality, we used the fact that

$$
|x^p - y^p| \le cp(|x|^{p-1} + |y|^{p-1})|x - y|
$$

for any $x, y \ge 0$, where c is an absolute constant.

Therefore,

$$
\begin{aligned}
(G(u) - G(u^{(n)}), A\bar{u}^{(n)}) &\le \frac{\nu+\kappa}{5}\|A\bar{u}^{(n)}\|^2 + \frac{5}{4(\nu+\kappa)}\|G(u) - G(u^{(n)})\|^2 \\
&\le C[\parallel u\parallel_\infty^{2(\beta-1)} + (\parallel u\parallel_\infty^{2(\beta-2)} + \parallel u^{(n)}\parallel_\infty^{2(\beta-2)})\parallel u^{(n)}\parallel_\infty^2]\parallel \nabla\bar{u}^{(n)}\parallel^2 \\
&\quad + \frac{\nu+k}{5}\parallel A\bar{u}^{(n)}\parallel^2 . && (4.17)
\end{aligned}
$$

By (4.5)–(4.15) and (4.17), we obtain

$$
\begin{aligned}
\frac{d}{dt}(\|\nabla\bar{u}^{(n)}\|^2 + \|\nabla\bar{\omega}^{(n)}\|^2) \le{}& C[\parallel u\parallel_\infty^{2(\beta-1)} + (\parallel u\parallel_\infty^{2(\beta-2)} + \parallel u^{(n)}\parallel_\infty^{2(\beta-2)})\parallel u^{(n)}\parallel_\infty^2 \\
&+ \|\nabla u\|\|Au\| + \|\nabla u^{(n)}\|^4 + \|\nabla\omega\|\|A\omega\| + 1]
\end{aligned}
$$

$$\cdot (\|\nabla \bar{u}^{(n)}\|^2 + \|\nabla \bar{\omega}^{(n)}\|^2) + \frac{5}{2(\nu+\kappa)}\|f_1 - f_1^{(n)}\|^2 + \frac{2}{\gamma}\|f_2 - f_2^{(n)}\|^2. \tag{4.18}$$

Using Gronwall's inequality in (4.18) yields

$$\begin{aligned} \|\nabla \bar{u}^{(n)}\|^2 + \|\nabla \bar{\omega}^{(n)}\|^2 \le & \Big(\|\nabla \bar{u}_\tau^{(n)}\|^2 + \|\nabla \bar{\omega}_\tau^{(n)}\|^2 + \frac{5}{2(\nu+\kappa)}\int_\tau^t \|f_1 - f_1^{(n)}\|^2 ds \\ & + \frac{2}{\gamma}\int_\tau^t \|f_2 - f_2^{(n)}\|^2 ds\Big) \\ & \cdot \exp\Big\{C\int_\tau^t [\| u \|_\infty^{2(\beta-1)} + (\| u \|_\infty^{2(\beta-2)} + \| u^{(n)} \|_\infty^{2(\beta-2)}) \| u^{(n)} \|_\infty^2 \\ & + \|\nabla u\|\|Au\| + \|\nabla u^{(n)}\|^4 + \|\nabla \omega\|\|A\omega\| + 1]ds\Big\}. \end{aligned} \tag{4.19}$$

From Lemmas 3.2 and 3.5, and using Agmon's inequality, we know that

$$\| u \|_\infty < C, \| u^{(n)} \|_\infty < C, \forall t \ge t_5.$$

So, from Lemmas 3.2–3.5, we have

$$\begin{aligned} & \exp\Big\{C\int_\tau^t [\| u \|_\infty^{2(\beta-1)} + (\| u \|_\infty^{2(\beta-2)} + \| u^{(n)} \|_\infty^{2(\beta-2)}) \| u^{(n)} \|_\infty^2 + \|\nabla u\|\|Au\| \\ & + \|\nabla u^{(n)}\|^4 + \|\nabla \omega\|\|A\omega\| + 1]ds\Big\} < +\infty, \end{aligned}$$

for any given t and τ, $t \ge \tau$, $\tau \ge t_5$.

Thus, from (4.19), we have that $\{U_{(f_1,f_2)}(t,\tau)\}_{t\ge\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ is $((V_1 \times V_2) \times (\mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)), V_1 \times V_2)$-continuous, for $\tau \ge t_5$. □

By Lemma 3.5, the fact of compact imbedding $\mathbf{H}^2 \times \mathbf{H}^2 \hookrightarrow V_1 \times V_2$, and Theorem 3.1 in [16], we have the following theorems.

Theorem 4.1. *Suppose $\beta > 3$. The family of processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\ge\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ with respect to problem (1.1) has a $(V_1 \times V_2, V_1 \times V_2)$ uniform attractor $\mathcal{A}_1$. Moreover,*

$$\mathcal{A}_1 = \bigcup_{(f_1,f_2)\in\mathcal{H}(f_1^0)\times\mathcal{H}(f_2^0)} \mathcal{K}_{(f_1,f_2)}(0), \tag{4.20}$$

where $\mathcal{K}_{(f_1,f_2)}(0)$ is the section at $t = 0$ of kernel $\mathcal{K}_{(f_1,f_2)}$ of the processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\ge\tau}$.

Theorem 4.2. *Suppose $\beta > 3$. The family of processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\ge\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ with respect to problem (1.1) has a $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform attractor $\mathcal{A}_2$. $\mathcal{A}_2$ is compact in $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$, and it attracts every bounded subset of $V_1 \times V_2$ in the topology of $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$.*

Proof. By Theorem 2.2, we only need to prove that $\{U_{(f_1,f_2)}(t,\tau)\}_{t\ge\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ acting on $V_1 \times V_2$ is $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform (*w.r.t.* $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$) asymptotically compact.

Thanks to Lemma 3.5, we know that $B = \{(u \times \omega) \in \mathbf{H}^2 \times \mathbf{H}^2 : \|Au\|^2 + \|A\omega\|^2 \le C\}$ is a bounded $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform absorbing set of $\{U_{(f_1,f_2)}(t,\tau)\}_{t\ge\tau}$. Then, we just need to prove that, for any $\tau_n \in \mathbb{R}$, any $t \to +\infty$, and $(u_{\tau_n}, \omega_{\tau_n}) \in B$, $\{(u_n(t), \omega_n(t))\}_{n=0}^{\infty}$ is precompact in $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$, where$(u_n(t), \omega_n(t)) = U_{(f_1,f_2)}(t, \tau_n)(u_{\tau_n}, \omega_{\tau_n})$.

Because $V_1 \hookrightarrow H_1, V_2 \hookrightarrow H_2$ are compact, from Lemma 3.6 we obtain that $\{\frac{d}{dt}u_n(t)\}_{n=0}^{\infty}$, $\{\frac{d}{dt}\omega_n(t)\}_{n=0}^{\infty}$ are precompact in H_1 and H_2, respectively.

Next, we will prove $\{u_n(t)\}_{n=0}^{\infty}$, $\{\omega_n(t)\}_{n=0}^{\infty}$ are Cauchy sequences in $\mathbf{H}^2(\Omega)$. From (2.1), we have

$$
\begin{aligned}
(\nu + \kappa)(Au_{n_k}(t) - Au_{n_j}(t)) = &-\frac{d}{dt}u_{n_k}(t) + \frac{d}{dt}u_{n_j}(t) - B(u_{n_k}(t)) + B(u_{n_j}(t)) \\
&- G(u_{n_k}(t)) + G(u_{n_j}(t)) + 2\kappa\nabla \times \omega_{n_k}(t) - 2\kappa\nabla \times \omega_{n_j}(t). \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
\gamma(A\omega_{n_k}(t) - A\omega_{n_j}(t)) - \mu\nabla\nabla \cdot \omega_{n_k}(t) + \mu\nabla\nabla \cdot \omega_{n_j}(t) = &-\frac{d}{dt}\omega_{n_k}(t) + \frac{d}{dt}\omega_{n_j}(t) - B(u_{n_k}(t), \omega_{n_k}(t)) \\
&+ B(u_{n_j}(t), \omega_{n_j}(t)) - 4\kappa\omega_{n_k}(t) \\
&+ 4\kappa\omega_{n_j}(t) + 2\kappa\nabla \times u_{n_k}(t) - 2\kappa\nabla \times u_{n_j}(t). \qquad (4.22)
\end{aligned}
$$

Multiplying (4.21) by $Au_{n_k}(t) - Au_{n_j}(t)$, we obtain

$$
\begin{aligned}
(\nu + \kappa) \| Au_{n_k}(t) - Au_{n_j}(t) \|^2 \leq &\| \frac{d}{dt}u_{n_k}(t) - \frac{d}{dt}u_{n_j}(t) \| \cdot \| Au_{n_k}(t) - Au_{n_j}(t) \| + \| B(u_{n_k}(t)) - B(u_{n_j}(t)) \| \\
&\cdot \| Au_{n_k}(t) - Au_{n_j}(t) \| + \| G(u_{n_k}(t)) - G(u_{n_j}(t)) \| \cdot \| Au_{n_k}(t) - Au_{n_j}(t) \| \\
&+ 2\kappa \| \nabla\omega_{n_k}(t) - \nabla\omega_{n_j}(t) \| \cdot \| Au_{n_k}(t) - Au_{n_j}(t) \| \\
\leq &\frac{4(\nu + \kappa)}{5} \| Au_{n_k}(t) - Au_{n_j}(t) \|^2 + \frac{5}{4(\nu + \kappa)} \| \frac{d}{dt}u_{n_k}(t) - \frac{d}{dt}u_{n_j}(t) \|^2 \\
&+ \frac{5}{4(\nu + \kappa)} \| B(u_{n_k}(t)) - B(u_{n_j}(t)) \|^2 + \frac{5}{4(\nu + \kappa)} \| G(u_{n_k}(t)) - G(u_{n_j}(t)) \|^2 \\
&+ \frac{5\kappa^2}{\nu + \kappa} \| \nabla\omega_{n_k}(t) - \nabla\omega_{n_j}(t) \|^2,
\end{aligned}
$$

so we have

$$
\begin{aligned}
\frac{\nu + \kappa}{5} \| Au_{n_k}(t) - Au_{n_j}(t) \|^2 \leq &\frac{5}{4(\nu + \kappa)} \| \frac{d}{dt}u_{n_k}(t) - \frac{d}{dt}u_{n_j}(t) \|^2 \\
&+ \frac{5}{4(\nu + \kappa)} \| B(u_{n_k}(t)) - B(u_{n_j}(t)) \|^2 \\
&+ \frac{5}{4(\nu + \kappa)} \| G(u_{n_k}(t)) - G(u_{n_j}(t)) \|^2 \\
&+ \frac{5\kappa^2}{\nu + \kappa} \| \nabla\omega_{n_k}(t) - \nabla\omega_{n_j}(t) \|^2 . \qquad (4.23)
\end{aligned}
$$

Multiplying (4.22) by $A\omega_{n_k}(t) - A\omega_{n_j}(t)$ we obtain

$$
\begin{aligned}
&\gamma \| A\omega_{n_k}(t) - A\omega_{n_j}(t) \|^2 + \mu \| \nabla\nabla \cdot (\omega_{n_k}(t) - \omega_{n_j}(t)) \|^2 \\
\leq &\| \frac{d}{dt}\omega_{n_k}(t) - \frac{d}{dt}\omega_{n_j}(t) \| \cdot \| A\omega_{n_k}(t) - A\omega_{n_j}(t) \| + \| B(u_{n_k}(t), \omega_{n_k}(t)) - B(u_{n_j}(t), \omega_{n_j}(t)) \| \\
&\cdot \| A\omega_{n_k}(t) - A\omega_{n_j}(t) \| + 4\kappa \| \omega_{n_k}(t) - \omega_{n_j}(t) \| \cdot \| A\omega_{n_k}(t) - A\omega_{n_j}(t) \| \\
&+ 2\kappa \| \nabla u_{n_k}(t) - \nabla u_{n_j}(t) \| \cdot \| A\omega_{n_k}(t) - A\omega_{n_j}(t) \|
\end{aligned}
$$

$$\leq \frac{4\gamma}{5} \parallel A\omega_{n_k}(t) - A\omega_{n_j}(t) \parallel^2 + \frac{5}{4\gamma} \parallel \frac{d}{dt}\omega_{n_k}(t) - \frac{d}{dt}\omega_{n_j}(t) \parallel^2$$
$$+ \frac{5}{4\gamma} \parallel B(u_{n_k}(t), \omega_{n_k}(t)) - B(u_{n_j}(t), \omega_{n_j}(t)) \parallel^2 + \frac{20\kappa^2}{\gamma} \parallel \omega_{n_k}(t) - \omega_{n_j}(t)) \parallel^2$$
$$+ \frac{5\kappa^2}{\gamma} \parallel \nabla u_{n_k}(t) - \nabla u_{n_j}(t) \parallel^2,$$

so we get

$$\frac{\gamma}{5} \parallel A\omega_{n_k}(t) - A\omega_{n_j}(t) \parallel^2 + \mu \parallel \nabla\nabla \cdot (\omega_{n_k}(t) - \omega_{n_j}(t)) \parallel^2$$
$$\leq \frac{5}{4\gamma} \parallel \frac{d}{dt}\omega_{n_k}(t) - \frac{d}{dt}\omega_{n_j}(t) \parallel^2 + \frac{5}{4\gamma} \parallel B(u_{n_k}(t), \omega_{n_k}(t)) - B(u_{n_j}(t), \omega_{n_j}(t)) \parallel^2$$
$$+ \frac{20\kappa^2}{\gamma} \parallel \omega_{n_k}(t) - \omega_{n_j}(t) \parallel^2 + \frac{5\kappa^2}{\gamma} \parallel \nabla u_{n_k}(t) - \nabla u_{n_j}(t) \parallel^2 . \tag{4.24}$$

Combining (4.23) with (4.24), we have

$$\frac{\nu + \kappa}{5} \parallel Au_{n_k}(t) - Au_{n_j}(t) \parallel^2 + \frac{\gamma}{5} \parallel A\omega_{n_k}(t) - A\omega_{n_j}(t) \parallel^2$$
$$\leq \frac{5}{4(\nu + \kappa)} \parallel \frac{d}{dt}u_{n_k}(t) - \frac{d}{dt}u_{n_j}(t) \parallel^2 + \frac{5}{4(\nu + \kappa)} \parallel B(u_{n_k}(t)) - B(u_{n_j}(t)) \parallel^2$$
$$+ \frac{5}{4(\nu + \kappa)} \parallel G(u_{n_k}(t)) - G(u_{n_j}(t)) \parallel^2 + \frac{5\kappa^2}{\nu + \kappa} \parallel \nabla\omega_{n_k}(t) - \nabla\omega_{n_j}(t) \parallel^2$$
$$+ \frac{5}{4\gamma} \parallel \frac{d}{dt}\omega_{n_k}(t) - \frac{d}{dt}\omega_{n_j}(t) \parallel^2 + \frac{5}{4\gamma} \parallel B(u_{n_k}(t), \omega_{n_k}(t)) - B(u_{n_j}(t), \omega_{n_j}(t)) \parallel^2$$
$$+ \frac{20\kappa^2}{\gamma} \parallel \omega_{n_k}(t) - \omega_{n_j}(t) \parallel^2 + \frac{5\kappa^2}{\gamma} \parallel \nabla u_{n_k}(t) - \nabla u_{n_j}(t) \parallel^2 . \tag{4.25}$$

Because $V_2 \hookrightarrow H_2$ is compact, from Lemma 3.2 we know that $\{\omega_n(t)\}_{n=0}^{\infty}$ is precompact in H_2. And, using the compactness of embedding $\mathbf{H}^2(\Omega) \hookrightarrow V_1, \mathbf{H}^2(\Omega) \hookrightarrow V_2$ and Lemma 3.5, we have that $\{u_n(t)\}_{n=0}^{\infty}, \{\omega_n(t)\}_{n=0}^{\infty}$ are precompact in V_1 and V_2, respectively. Considering $V_1 \hookrightarrow H_1, V_2 \hookrightarrow H_2$ are compact, from Lemma 3.6 we know that $\{\frac{d}{dt}u_n(t)\}_{n=0}^{\infty}$, $\{\frac{d}{dt}\omega_n(t)\}_{n=0}^{\infty}$ are precompact in H_1 and H_2, respectively.

Using (2.6), we have

$$\parallel B(u_{n_k}(t)) - B(u_{n_j}(t)) \parallel^2$$
$$\leq C(\parallel B(u_{n_k}(t), u_{n_k}(t) - u_{n_j}(t)) \parallel^2 + \parallel B(u_{n_k}(t) - u_{n_j}(t), u_{n_j}(t)) \parallel^2)$$
$$\leq C(\parallel \nabla u_{n_k}(t) \parallel^2 \parallel \nabla(u_{n_k}(t) - u_{n_j}(t)) \parallel \parallel A(u_{n_k}(t) - u_{n_j}(t)) \parallel$$
$$+ \parallel \nabla(u_{n_k}(t) - u_{n_j}(t)) \parallel^2 \parallel \nabla u_{n_j}(t) \parallel \parallel Au_{n_j}(t) \parallel) \to 0, \text{as } k, j \to +\infty, \tag{4.26}$$

and

$$\parallel B(u_{n_k}(t), \omega_{n_k}(t)) - B(u_{n_j}(t), \omega_{n_j}(t)) \parallel^2$$
$$\leq C(\parallel B(u_{n_k}(t), \omega_{n_k}(t) - \omega_{n_j}(t)) \parallel^2 + \parallel B(u_{n_k}(t) - u_{n_j}(t), \omega_{n_j}(t)) \parallel^2)$$

$$\leq C(\| \nabla u_{n_k}(t) \|^2 \| \nabla(\omega_{n_k}(t) - \omega_{n_j}(t)) \| \| A(\omega_{n_k(t)} - \omega_{n_j}(t)) \| \\ + \| \nabla(u_{n_k}(t) - u_{n_j}(t)) \|^2 \| \nabla\omega_{n_j}(t) \| \| A\omega_{n_j}(t) \|) \to 0, \text{ as } k, j \to +\infty. \tag{4.27}$$

From the proof of Lemma 4.2 in [15], we have

$$\| G(u_{n_k}(t)) - G(u_{n_j}(t)) \|^2 \leq C \| u_{n_k}(t) - u_{n_j}(t) \|^2 \to 0, \text{ as } k, j \to +\infty. \tag{4.28}$$

Taking into account (4.25)–(4.28), we have

$$\frac{\nu + \kappa}{5} \| Au_{n_k}(t) - Au_{n_j}(t) \|^2 + \frac{\gamma}{5} \| A\omega_{n_k}(t) - A\omega_{n_j}(t) \|^2 \to 0, \text{ as } k, j \to +\infty. \tag{4.29}$$

(4.29) indicates that the processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$ are uniformly asymptotically compact in $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$. So, by Theorem 2.2, it has a $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform attractor $\mathcal{A}_2$. □

Theorem 4.3. *Suppose $\beta > 3$. The $(V_1 \times V_2, V_1 \times V_2)$-uniform attractor $\mathcal{A}_1$ of the family of processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ is actually the $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform attractor $\mathcal{A}_2$, i.e., $\mathcal{A}_1 = \mathcal{A}_2$.*

Proof. First, we will prove $\mathcal{A}_1 \subset \mathcal{A}_2$. Because $\mathcal{A}_2$ is bounded in $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$, and the embedding $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \hookrightarrow V_1 \times V_2$ is continuous, $\mathcal{A}_2$ is bounded in $V_1 \times V_2$. From Theorem 4.2, we know that $\mathcal{A}_2$ attracts uniformly all bounded subsets of $V_1 \times V_2$, so $\mathcal{A}_2$ is a bounded uniform attracting set of $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ in $V_1 \times V_2$. By the minimality of $\mathcal{A}_1$, we have $\mathcal{A}_1 \subset \mathcal{A}_2$.

Now, we will prove $\mathcal{A}_2 \subset \mathcal{A}_1$. First, we will prove $\mathcal{A}_1$ is a $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniformly attracting set of $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$. That is to say, we will prove

$$\lim_{t\to+\infty} \Big(\sup_{(f_1,f_2)\in\mathcal{H}(f_1^0)\times\mathcal{H}(f_2^0)} \operatorname{dist}_{\mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega)}(U_{(f_1,f_2)}(t,\tau)B, \mathcal{A}_1)\Big) = 0, \tag{4.30}$$

for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(V_1 \times V_2)$.

If we suppose (4.30) is not valid, then there must exist some $\tau \in \mathbb{R}$, $B \in \mathcal{B}(V_1 \times V_2)$, $\varepsilon_0 > 0$, $(f_1^{(n)}, f_2^{(n)}) \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$, and $t_n \to +\infty$, when $n \to +\infty$, such that, for all $n \geq 1$,

$$\operatorname{dist}_{\mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega)}(U_{(f_1^{(n)},f_2^{(n)})}(t_n,\tau)B, \mathcal{A}_1) \geq 2\varepsilon_0. \tag{4.31}$$

This shows that there exists $(u_n, \omega_n) \in B$ such that

$$\operatorname{dist}_{\mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega)}(U_{(f_1^{(n)},f_2^{(n)})}(t_n,\tau)(u_n,\omega_n), \mathcal{A}_1) \geq \varepsilon_0. \tag{4.32}$$

In the light of Theorem 4.2, $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ has a $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform attractor $\mathcal{A}_2$ which attracts any bounded subset of $V_1 \times V_2$ in the topology of $\mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$. Therefore, there exists $(\zeta, \eta) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$ and a subsequence of $U_{(f_1^{(n)},f_2^{(n)})}(t_n,\tau)(u_n,\omega_n)$ such that

$$U_{(f_1^{(n)},f_2^{(n)})}(t_n,\tau)(u_n,\omega_n) \to (\zeta, \eta) \quad \text{strongly in } \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega). \tag{4.33}$$

On the other side, the processes $\{U_{(f_1,f_2)}(t,\tau)\}_{t\geq\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$ have a $(V_1 \times V_2, V_1 \times V_2)$-uniform attractor $\mathcal{A}_1$, which attracts uniformly any bounded subsets of $V_1 \times V_2$ in the topology of $V_1 \times V_2$. So, there exists $(u, \omega) \in V_1 \times V_2$ and a subsequence of $U_{(f_1^{(n)},f_2^{(n)})}(t_n,\tau)(u_n,\omega_n)$ such that

$$U_{(f_1^{(n)},f_2^{(n)})}(t_n,\tau)(u_n,\omega_n) \to (u, \omega) \text{ strongly in } V_1 \times V_2. \tag{4.34}$$

From (4.33) and (4.34), we have $(u, \omega) = (\zeta, \eta)$, so (4.33) can also be written as

$$U_{(f_1^{(n)}, f_2^{(n)})}(t_n, \tau)(u_n, \omega_n) \to (u, \omega) \text{ strongly in } \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega). \tag{4.35}$$

And, from Theorem 4.1, we know that $\mathcal{A}_1$ attracts B, so

$$\lim_{n\to+\infty} \text{dist}_{V_1\times V_2}(U_{(f_1^{(n)}, f_2^{(n)})}(t_n, \tau)(u_n, \omega_n), \mathcal{A}_1) = 0. \tag{4.36}$$

By (4.34), (4.36), and the compactness of $\mathcal{A}_1$ in $V_1 \times V_2$, we have $(u, \omega) \in \mathcal{A}_1$. Considering (4.35), we have

$$\begin{aligned} &\lim_{n\to+\infty} \text{dist}_{\mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega)}(U_{(f_1^{(n)}, f_2^{(n)})}(t_n, \tau)(u_n, \omega_n), \mathcal{A}_1) \\ \le &\lim_{n\to+\infty} \text{dist}_{\mathbf{H}^2(\Omega)\times\mathbf{H}^2(\Omega)}(U_{(f_1^{(n)}, f_2^{(n)})}(t_n, \tau)(u_n, \omega_n), (u, \omega)) \\ = &\ 0, \end{aligned}$$

which contradicts (4.32). Therefore, $\mathcal{A}_1$ is a $(V_1 \times V_2, \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega))$-uniform attractor of $\{U_{(f_1,f_2)}(t, \tau)\}_{t\ge\tau}$, $f_1 \times f_2 \in \mathcal{H}(f_1^0) \times \mathcal{H}(f_2^0)$, and by the minimality of $\mathcal{A}_2$, we have $\mathcal{A}_2 \subset \mathcal{A}_1$. □

5. Conclusions

In this paper, we discussed the existence of uniform attractors of strong solutions for 3D incompressible micropolar equations with nonlinear damping. Based on some translation-compactness assumption on the external forces, and when $\beta > 3$, we made a series of uniform estimates on the solutions in various functional spaces. According to these uniform estimates, we proved the existence of uniform attractors for the process operators corresponding to the solution of the equation in $V_1 \times V_2$ and $\mathbf{H}^2 \times \mathbf{H}^2$, and verified that the uniform attractors in $V_1 \times V_2$ and $\mathbf{H}^2 \times \mathbf{H}^2$ are actually the same.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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