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Research article

One kind two-term exponential sums weighted by third-order character

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Abstract: We give a correction to the previous paper of S. F. Cao and T. T. Wang (2021). Additionally, provide a conversion formula of the power mean of a certain exponential sums weighted by a Dirichlet character χ_3 and some of its applications.

Keywords: exponential sums; Gauss sums; power mean; calculating formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let $q, m, n \in \mathbf{Z}^+$ with $q > 2$ and $m > n \geq 1$. For any $u, v \in \mathbf{Z}$, we are concerned with the two-term exponential sums

$$\mathcal{G}(u, v, m, n; q) = \sum_{j=1}^{q} \mathrm{e}_q\left(u j^m + v j^n\right),$$

where $\mathrm{e}_q(x) = \exp(2\pi \mathrm{i} x/q)$ and $\mathrm{i}^2 = -1$.

For convenience, the following letters and symbols are commonly used in this paper and should be interpreted in the following sense unless otherwise stated.

- χ is Dirichlet character.
- χ_k is k-order Dirichlet character.
- $\phi(a)$ is Euler function.
- α is uniquely determined by $4p = \alpha^2 + 27\beta^2$ and $\alpha \equiv 1 \bmod 3$.
- $\tau(\chi)$ is Gauss sums defined by

$$\tau(\chi) = \sum_{s=1}^{q} \chi(s)\mathrm{e}_q(s).$$

The mean value calculation and upper bound estimation of exponential sums has always been a classical problem in analytic number theory. As a special kind of exponential sums, Gauss sums have had an important effect on both cryptography and analytic number theory. Analytic number theory and cryptography will benefit greatly from any significant advancements made in this area. In this paper, we will estimate and calculate the fourth power mean value of two-term exponential sums weighted by a character χ_3. In this field, many scholars have investigated the results of $\mathcal{G}(u, v, m, n; q)$ in various forms, and obtained many meaningful results, see [3,5,7,8,11,15]. For instance, Zhang and Zhang [9] obtained the power mean about $\mathcal{G}(u, v, 3, 1; p)$

$$\sum_{u=1}^{p-1}\left|\sum_{i=1}^{p} \mathrm{e}_p\left(ui^3 + vi\right)\right|^4 = \begin{cases} 2p^3 - p^2 & \text{if} \quad 3 \nmid p-1, \\ 2p^3 - 7p^2 & \text{if} \quad 3 \mid p-1, \end{cases}$$

where p is an odd prime and v is not divisible by p.

Wang and Zhang [6] obtained the eighth power mean of $\mathcal{G}(u, v, 3, 1; p)$

$$\sum_{u=1}^{p-1}\left|\sum_{i=1}^{p} \mathrm{e}_p\left(ui^3 + vi\right)\right|^8 = \begin{cases} 7(2p^5 - 3p^4) & \text{if} \quad 6 \mid p-5, \\ 14p^5 - 75p^4 - 8p^3\alpha^2 & \text{if} \quad 6 \mid p-1. \end{cases}$$

In addition, Zhang and Han [12] shown the power mean of $\mathcal{G}(1, v, 3, 1; p)$

$$\sum_{v=1}^{p-1}\left|\sum_{i=1}^{p} \mathrm{e}_p\left(i^3 + vi\right)\right|^6 = 5p^4 - 8p^3 - p^2, \tag{1.1}$$

where p is an odd prime with $3 \nmid \phi(p)$.

But if $3 \mid \phi(p)$, whether there exists an exact formula for (1.1). Consider the mean of the simplest

$$\sum_{v=1}^{p-1}\left|\sum_{i=1}^{p} \mathrm{e}_p\left(i^3 + vi\right)\right|^4. \tag{1.2}$$

It is worth mentioning that, Zhang and Zhang [10] studied the power mean of the exponential sums weight by χ_2, one has the identities

$$\sum_{u=1}^{p-1}\chi_2(u)\left|\sum_{i=1}^{p} \mathrm{e}_p\left(ui^3 + i\right)\right|^4 = \begin{cases} p^2(\zeta + 3) & \text{if} \quad 6 \mid p-5, \\ p^2(\zeta - 3) & \text{if} \quad 6 \mid p-1, \end{cases}$$

where $\zeta = \sum_{t=1}^{p-1}\left(\frac{t - 1 + \bar{t}}{p}\right)$ with $\zeta \in \mathbf{Z}$ satisfies inequality $|\zeta| \leq 2\sqrt{p}$.

Cao and Wang (see Lemma 3 in [2]) proved the following conclusion, that is, if p is a prime with $3 \mid \phi(p)$, then for any χ_3 mod p, one has the identity

$$\sum_{u=1}^{p-1}\chi_3(u)\left(\sum_{i=1}^{p} \mathrm{e}_p\left(ui^3 + i\right)\right)^4 = (\overline{\chi_3}(3) - 3p - \overline{\chi_3}(3)p)\tau^2\left(\overline{\chi_3}\right) - \alpha p\tau(\chi_3).$$

Unfortunately, this lemma is incorrect, there is a calculation error in it. It is precisely because of the computational error in this lemma that the main result in the whole text is wrong.

The following year, Zhang and Meng [16] studied the power mean of $\mathcal{G}(u, 1, 3, 1; p)$ weighted by χ_2. In this paper, We intend to correct the error in [2] and give a correct conclusion. At the same time, as an application, we give an exact result for (1.2). That is, it will prove these two conclusions:

Theorem 1. *If p is a prime with* $3 \mid \phi(p)$*, then we have*

$$\sum_{u=1}^{p-1} \chi_3(u) \left| \sum_{i=1}^{p} e_p\left(ui^3 + i\right) \right|^4 = -\alpha p \tau(\chi_3) - 3p \cdot \tau^2(\overline{\chi_3}).$$

Theorem 2. *If p is a prime with* $3 \mid \phi(p)$*, then we have*

$$\sum_{v=1}^{p-1} \left| \sum_{i=1}^{p} e_p\left(i^3 + vi\right) \right|^4 = 2p^3 - p^2 - 3pA_k^2 - p\alpha A_k,$$

where $A_k = \omega^k \left[\frac{\alpha p}{2} + \left(\left(\frac{\alpha p}{2}\right)^2 - p^3 \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} + \omega^{-k} \left[\frac{\alpha p}{2} - \left(\left(\frac{\alpha p}{2}\right)^2 - p^3 \right)^{\frac{1}{2}} \right]^{\frac{1}{3}}$*,* $k = 1, 2$ *or* 3 *is dependent on p, and* $\omega = \frac{-1+\sqrt{3}i}{2}$.

Corollary 1. *If p is a prime with* $3 \mid \phi(p)$*, then we have the asymptotic formula*

$$\sum_{v=1}^{p-1} \left| \sum_{i=1}^{p} e_p\left(i^3 + vi\right) \right|^4 = 2p^3 + O\left(p^2\right).$$

Corollary 2. *If p is a prime with* $3 \mid \phi(p)$*, then for any integer l, we have recursive formula*

$$\begin{aligned} V_l(p) &= \sum_{u=1}^{p-1} \vartheta^l(u) \left| \sum_{j=1}^{p} e_p\left(uj^3 + j\right) \right|^4 \\ &= \alpha p \sum_{u=1}^{p-1} \vartheta^{l-3}(u) \left| \sum_{j=1}^{p} e_p\left(uj^3 + j\right) \right|^4 + 3p \sum_{u=1}^{p-1} \vartheta^{l-2}(u) \left| \sum_{j=1}^{p} e_p\left(uj^3 + j\right) \right|^4 \\ &= \alpha p V_{l-3}(p) + 3p V_{l-2}(p), \end{aligned}$$

when l take 1–3*, the following equations hold*

$$\begin{aligned} V_1(p) = \sum_{u=1}^{p-1} \vartheta(u) \left| \sum_{j=1}^{p} e_p\left(uj^3 + j\right) \right|^4 &= -5\alpha p^2, \\ V_2(p) = \sum_{u=1}^{p-1} \vartheta^2(u) \left| \sum_{j=1}^{p} e_p\left(uj^3 + j\right) \right|^4 &= 4p^4 - 20p^3 - \alpha^2 p^2, \\ V_3(p) = \sum_{u=1}^{p-1} \vartheta^3(u) \left| \sum_{j=1}^{p} e_p\left(uj^3 + j\right) \right|^4 &= 2\alpha p^4 - 22\alpha p^3, \end{aligned}$$

where $\vartheta(u) = \sum_{i=1}^{p} e_p\left(ui^3\right)$.

In fact, with the third-order linear recursive formula in Corollary 2 and its three initial values $V_1(p)$, $V_2(p)$ and $V_3(p)$, we can easily give the general term formula for the sequence $\{V_l(p)\}$.

Corollary 3. *If p is a prime with $3 \mid \phi(p)$, then we have*

$$\sum_{u=1}^{p-1} \left| \frac{\sum_{j=1}^{p} e_p\left(uj^3 + j\right)}{\sum_{i=1}^{p} e_p\left(ui^3\right)} \right|^4 = \frac{54p^3}{\alpha^4} - \frac{p^2}{\alpha^2} - \frac{27p^2}{\alpha^4} + \frac{2p}{\alpha^2} - 1.$$

2. Some lemmas

Before starting our proofs of main results, we present the proofs of several key equations in preparation for the next chapter. The properties of Gauss sums and reduced (complete) residue systems are used repeatedly in the proof. In addition, we will refer to the basic contents of number theory in references [1] and [14].

Lemma 1. *If p is a prime with $3 \mid \phi(p)$, then*

$$\tau^3(\chi_3) + \tau^3\left(\overline{\chi_3}\right) = \alpha p. \tag{2.1}$$

Proof. This is consequence of [4] or [13], herein we omit it. □

Lemma 2. *If p is a prime with $3 \mid \phi(p)$, then*

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p} \overline{\chi_3}\left(i^3 + j^3 - s^3 - 1\right) = p(\alpha - 3) + 3\tau^3\left(\overline{\chi_3}\right).$$

Proof. Recall that $\tau(\chi_3) \cdot \tau\left(\overline{\chi_3}\right) = p$ and (2.1), we have

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p} \overline{\chi_3}\left(i^3 + j^3 - s^3 - 1\right) \\
=\ & \frac{1}{\tau(\chi_3)} \sum_{t=1}^{p-1} \chi_3(t) \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p} \mathrm{e}_p\left(t\left(i^3 + j^3 - s^3 - 1\right)\right) \\
=\ & \frac{1}{\tau(\chi_3)} \sum_{t=1}^{p-1} \chi_3(t) \mathrm{e}_p(-t) \left(\sum_{i=1}^{p} \mathrm{e}_p\left(it^3\right)\right)^2 \left(\sum_{s=1}^{p} \mathrm{e}_p\left(-st^3\right)\right) \\
=\ & \frac{1}{\tau(\chi_3)} \sum_{t=1}^{p-1} \chi_3(t) \mathrm{e}_p(-t) \left(1 + \sum_{i=1}^{p-1} \left(1 + \chi_3(i) + \overline{\chi_3}(i)\right) \mathrm{e}_p(it)\right)^3 \\
=\ & \frac{1}{\tau(\chi_3)} \sum_{t=1}^{p-1} \chi_3(t) \mathrm{e}_p(-t) \left(\overline{\chi_3}(t) \cdot \tau(\chi_3) + \chi_3(t) \cdot \tau\left(\overline{\chi_3}\right)\right)^3 \\
=\ & \frac{1}{\tau(\chi_3)} \sum_{t=1}^{p-1} \chi_3(t) \mathrm{e}_p(-t) \left[\tau^3(\chi_3) + \tau^3\left(\overline{\chi_3}\right) + 3p\left(\overline{\chi_3}(t) \cdot \tau(\chi_3) + \chi_3(t) \cdot \tau\left(\overline{\chi_3}\right)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha p+\frac{3p}{\tau(\chi_3)}\cdot\tau(\chi_3)\cdot\sum_{t=1}^{p-1}\mathrm{e}_p(-t)+\frac{3p}{\tau(\chi_3)}\cdot\tau\left(\overline{\chi_3}\right)\cdot\sum_{t=1}^{p-1}\chi_3^2(t)\mathrm{e}_p(-t)\\
&= p(\alpha-3)+\frac{3p\cdot\tau^2\left(\overline{\chi_3}\right)}{\tau(\chi_3)}=p(\alpha-3)+3\tau^3\left(\overline{\chi_3}\right).
\end{aligned}
$$

This completes the proof. □

Lemma 3. *If p is a prime with $3\mid\phi(p)$, then*

$$
\tau(\overline{\chi_3}\chi_2)=\frac{\overline{\chi_3}(2)\cdot\tau^2(\chi_3)\cdot\tau(\chi_2)}{p}.
$$

Proof. Recall that $\tau(\chi_3)\cdot\tau\left(\overline{\chi_3}\right)=p$, we obtain

$$
\begin{aligned}
\sum_{i=1}^{p}\chi_3(i^2-1) &= \sum_{i=1}^{p}\chi_3(i^2+2i)\\
&= \frac{1}{\tau(\overline{\chi_3})}\sum_{j=1}^{p-1}\overline{\chi_3}(j)\sum_{i=1}^{p-1}\chi_3(i)\mathrm{e}_p(j(i+2))\\
&= \frac{\tau(\chi_3)}{\tau(\overline{\chi_3})}\sum_{j=1}^{p-1}\chi_3(j)\mathrm{e}_p(2j)\\
&= \frac{\overline{\chi_3}(2)\cdot\tau^2(\chi_3)}{\tau(\overline{\chi_3})}\\
&= \frac{\overline{\chi_3}(2)\cdot\tau^3(\chi_3)}{p}. \qquad (2.2)
\end{aligned}
$$

From another perspective, we have

$$
\begin{aligned}
\sum_{i=1}^{p}\chi_3(i^2-1) &= \frac{1}{\tau(\overline{\chi_3})}\sum_{j=1}^{p-1}\overline{\chi_3}(j)\sum_{i=1}^{p}\mathrm{e}_p\left(j(i^2-1)\right)\\
&= \frac{1}{\tau(\overline{\chi_3})}\sum_{j=1}^{p-1}\overline{\chi_3}(j)\mathrm{e}_p(-j)\sum_{i=1}^{p}\mathrm{e}_p\left(i^2j\right)\\
&= \frac{1}{\tau(\overline{\chi_3})}\sum_{j=1}^{p-1}\overline{\chi_3}(j)\mathrm{e}_p(-j)\left[1+\sum_{i=1}^{p-1}(1+\chi_2(i))\mathrm{e}_p(ij)\right]\\
&= \frac{1}{\tau(\overline{\chi_3})}\sum_{j=1}^{p-1}\overline{\chi_3}(j)\mathrm{e}_p(-j)\sum_{i=1}^{p-1}\chi_2(i)\mathrm{e}_p(ij)\\
&= \frac{\tau(\chi_2)}{\tau(\overline{\chi_3})}\sum_{j=1}^{p-1}\overline{\chi_3}\chi_2(j)\mathrm{e}_p(-j)\\
&= \frac{\chi_2(-1)\cdot\tau(\chi_2)\cdot\tau(\overline{\chi_3}\chi_2)}{\tau(\overline{\chi_3})}
\end{aligned}
$$

$$= \frac{\chi_2(-1)\cdot\tau(\chi_2)\cdot\tau(\overline{\chi_3}\chi_2)\cdot\tau(\chi_3)}{p}. \tag{2.3}$$

Combining (2.2) and (2.3), we determine the relationship equation between $\tau(\overline{\chi_3}\chi_2)$, $\tau(\chi_3)$ and $\tau(\chi_2)$

$$\tau(\overline{\chi_3}\chi_2) = \frac{\overline{\chi_3}(2)\cdot\tau^2(\chi_3)\cdot\tau(\chi_2)}{p}.$$

This completes the proof. □

Lemma 4. *If p is a prime with $3 \mid \phi(p)$, then*

$$\sum_{\substack{i=1 \\ i+j-s-1\equiv 0 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p} \overline{\chi_3}\left(i^3+j^3-s^3-1\right) = \frac{-\overline{\chi_3}(3)\cdot\tau^3\left(\overline{\chi_3}\right)}{p}.$$

Proof. Using Lemma 3, we have

$$\begin{aligned}
&\sum_{\substack{i=1 \\ i+j-s-1\equiv 0 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p} \overline{\chi_3}\left(i^3+j^3-s^3-1\right)\\
=\ & \sum_{\substack{i=1 \\ i+j\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3+3j^2s+3js^2-1\right)\\
=\ & \chi_3(4)\sum_{\substack{i=1 \\ i+j\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(4i^3+j^3+3j\,(2s+j)^2-4\right)\\
=\ & \frac{\chi_3(4)}{\tau(\chi_3)}\sum_{\substack{i=1 \\ i+j\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{t=1}^{p-1}\chi_3(t)\sum_{s=1}^{p}\mathrm{e}_p\left(t\left(4i^3+j^3+3js^2-4\right)\right)\\
=\ & \frac{\chi_3(4)}{\tau(\chi_3)}\sum_{\substack{i=1 \\ i+j\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{t=1}^{p-1}\chi_3(t)\mathrm{e}_p(t(4i^3+j^3-4))\left[1+\sum_{s=1}^{p-1}(1+\chi_2(s))\mathrm{e}_p\,(3jst)\right]\\
=\ & \frac{\chi_3(4)}{\tau(\chi_3)}\sum_{\substack{i=1 \\ i+j\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{t=1}^{p-1}\chi_3(t)\mathrm{e}_p\left(t(4i^3+j^3-4)\right)\chi_2(3jt)\tau(\chi_2)\\
=\ & \frac{\chi_3(4)\cdot\chi_2(3)\cdot\tau(\chi_2)}{\tau(\chi_3)}\sum_{\substack{i=1 \\ i+j\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\chi_2(j)\tau\left(\chi_3\chi_2\right)\overline{\chi_3}\chi_2\left(4i^3+j^3-4\right)\\
=\ & \frac{\chi_3(4)\cdot\chi_2(3)\cdot\tau(\chi_2)\cdot\tau\left(\chi_3\chi_2\right)}{\tau(\chi_3)}\sum_{i=1}^{p}\chi_2(1-i)\overline{\chi_3}\chi_2\left(3i^3+3i^2-3i-3\right)\\
=\ & \frac{\overline{\chi_3}(6)\cdot\chi_2(-1)\cdot\tau(\chi_2)\cdot\tau\left(\chi_3\chi_2\right)}{\tau(\chi_3)}\sum_{i=1}^{p-1}\overline{\chi_3}\left((i+2)^2 i\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\overline{\chi_3}(6)\cdot\chi_2(-1)\cdot\tau(\chi_2)\cdot\tau\left(\chi_3\chi_2\right)}{\tau(\chi_3)}\sum_{i=1}^{p-1}\overline{\chi_3}(i)\chi_3\left(i+2\right)\\
&= \frac{\overline{\chi_3}(6)\cdot\chi_2(-1)\cdot\tau(\chi_2)\cdot\tau\left(\chi_3\chi_2\right)}{\tau(\chi_3)}\sum_{i=1}^{p-1}\chi_3\left(1+2\cdot\bar{i}\right)\\
&= \frac{\overline{\chi_3}(6)\cdot\chi_2(-1)\cdot\tau(\chi_2)\cdot\tau\left(\chi_3\chi_2\right)}{\tau(\chi_3)}\left(-1+\sum_{i=1}^{p}\chi_3\left(i\right)\right)\\
&= \frac{-\chi_2(-1)\cdot\overline{\chi_3}(6)\cdot\tau(\chi_2)\cdot\tau\left(\overline{\chi_3}\right)\cdot\tau\left(\chi_3\chi_2\right)}{p}\\
&= \frac{-\overline{\chi_3}(3)\cdot\tau^3\left(\overline{\chi_3}\right)}{p}.
\end{aligned}
$$

This completes the proof. □

Lemma 5. *If p is a prime and $3 \mid \phi(p)$, then*

$$
\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3\right)e_p\left(i+j-s\right) = -3p+\overline{\chi_3}(3)\cdot\tau^3(\overline{\chi_3}).
$$

Proof. Note that $\tau(\chi_3)\cdot\tau\left(\overline{\chi_3}\right) = p$ and $\overline{\chi_3}\left(i^3\right) = 1$ with i is an integer relatively prime to p. Therefore we have

$$
\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3\right)\mathrm{e}_p\left(i+j-s\right)\\
=\ &\sum_{i=1}^{p}\sum_{j=1}^{p}\overline{\chi_3}\left(i^3+j^3\right)\mathrm{e}_p\left(i+j\right)+\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}(i^3+j^3-1)\mathrm{e}_p\left(s(i+j-1)\right)\\
&-\sum_{i=1}^{p}\sum_{j=1}^{p}\overline{\chi_3}(i^3+j^3-1)\\
=\ &\sum_{i=1}^{p}\overline{\chi_3}(i^3)\mathrm{e}_p\left(i\right)+\sum_{i=1}^{p}\sum_{j=1}^{p}\overline{\chi_3}\left(i^3+1\right)\mathrm{e}_p\left(j(i+1)\right)-\sum_{i=1}^{p}\overline{\chi_3}(i^3+1)\\
&+p\sum_{\substack{i=1\\ i+j\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\overline{\chi_3}(i^3+j^3-1)-\frac{1}{\tau(\chi_3)}\sum_{t=1}^{p-1}\chi_3(t)\sum_{i=1}^{p}\sum_{j=1}^{p}\mathrm{e}_p\left(t(i^3+j^3-1)\right)\\
=\ &\sum_{i=1}^{p-1}\mathrm{e}_p\left(i\right)+p\sum_{\substack{i=1\\ i+1\equiv 0 \bmod p}}^{p}\overline{\chi_3}\left(i^3+1\right)-1-\sum_{i=1}^{p-1}(1+\chi_3(i)+\overline{\chi_3}(i))\overline{\chi_3}(i+1)\\
&+p\sum_{\substack{i=1\\ i+j\equiv 0 \bmod p}}^{p}\sum_{j=1}^{p}\overline{\chi_3}(i^3+j^3+3j^2+3j)-\frac{1}{\tau(\chi_3)}\sum_{t=1}^{p-1}\chi_3(t)\mathrm{e}_p\left(-t\right)\left(\sum_{i=1}^{p}\mathrm{e}_p\left(it^3\right)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= -1 - \sum_{i=1}^{p}\overline{\chi_3}(i+1) - \sum_{i=1}^{p-1}\chi_3(i)\overline{\chi_3}(i+1) - \sum_{i=1}^{p-1}\overline{\chi_3}(i)\chi_3(i+1) + p\sum_{j=1}^{p}\overline{\chi_3}(3j^2+3j) \\
&\quad -\frac{1}{\tau(\chi_3)}\sum_{t=1}^{p-1}\chi_3(t)\mathrm{e}_p(-t)\left[1+\sum_{i=1}^{p-1}(1+\chi_3(i)+\overline{\chi_3}(i))\mathrm{e}_p(it)\right]^2 \\
&= -1 - \sum_{i=1}^{p-1}\overline{\chi_3}(1+\bar{i}) - \sum_{i=1}^{p-1}\overline{\chi_3}(i^2+i) + p\overline{\chi_3}(3)\sum_{j=1}^{p-1}\overline{\chi_3}(j^2+j) \\
&\quad -\frac{1}{\tau(\chi_3)}\sum_{t=1}^{p-1}\chi_3(t)\mathrm{e}_p(-t)\left(\overline{\chi_3}(t)\tau(\chi_3)+\chi_3(t)\tau(\overline{\chi_3})\right)^2 \\
&= -\frac{1}{\tau(\chi_3)}\left(\tau^2(\chi_3)\sum_{t=1}^{p-1}\overline{\chi_3}(t)\mathrm{e}_p(-t)+\tau^2(\overline{\chi_3})\sum_{t=1}^{p-1}\mathrm{e}_p(-t)+2p\sum_{t=1}^{p-1}\chi_3(t)\mathrm{e}_p(-t)\right) \\
&\quad +\left(-1+p\overline{\chi_3}(3)\right)\frac{1}{\tau(\chi_3)}\sum_{s=1}^{p-1}\chi_3(s)\sum_{i=1}^{p-1}\overline{\chi_3}(i)\,\mathrm{e}_p(s(i+1)) \\
&= -\frac{1}{\tau(\chi_3)}\left(\tau^2(\chi_3)\cdot\tau(\overline{\chi_3})-\tau^2(\overline{\chi_3})+2p\cdot\tau(\chi_3)\right)-\frac{\tau^2(\overline{\chi_3})}{\tau(\chi_3)}+p\cdot\overline{\chi_3}(3)\cdot\frac{\tau^2(\overline{\chi_3})}{\tau(\chi_3)} \\
&= \overline{\chi_3}(3)\cdot\tau^3(\overline{\chi_3})-3p.
\end{aligned}
$$

This proves Lemma 5. □

3. Proofs of main results

Proof of Theorem 1. Recall that $\overline{\chi_3}\left(i^3\right)=1$ with $(i,p)=1$. Hence

$$
\begin{aligned}
&\sum_{u=1}^{p-1}\chi_3(u)\left|\sum_{i=1}^{p}\mathrm{e}_p\left(ui^3+i\right)\right|^4 \\
&= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sum_{u=1}^{p-1}\chi_3(u)\mathrm{e}_p\left(u\left(i^3+j^3-s^3-t^3\right)+i+j-s-t\right) \\
&= \tau(\chi_3)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3-t^3\right)\mathrm{e}_p(i+j-s-t) \\
&= \tau(\chi_3)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p-1}\overline{\chi_3}\left(i^3t^3+j^3t^3-s^3t^3-t^3\right)\mathrm{e}_p(it+jt-st-t) \\
&\quad +\tau(\chi_3)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3\right)\mathrm{e}_p(i+j-s) \\
&= \tau(\chi_3)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3-1\right)\sum_{t=1}^{p}\mathrm{e}_p(t(i+j-s-1))
\end{aligned}
$$

$$
\begin{aligned}
& +\tau(\chi_3)\sum_{i=0}^{p-1}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3\right)\mathrm{e}_p\left(i+j-s\right) \\
& -\tau(\chi_3)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3-1\right) \\
= \;& p\tau(\chi_3)\sum_{\substack{i=1\\ i+j-s\equiv 1 \bmod p}}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3-1\right)-\tau(\chi_3)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3-1\right) \\
& +\tau(\chi_3)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\overline{\chi_3}\left(i^3+j^3-s^3\right)\mathrm{e}_p\left(i+j-s\right).
\end{aligned}
$$

Applying Lemmas 2, 4 and 5 we obtain

$$
\begin{aligned}
& \sum_{u=1}^{p-1}\chi_3(u)\left|\sum_{i=1}^{p}\mathrm{e}_p\left(ui^3+i\right)\right|^4 \\
= \;& -\tau(\chi_3)\cdot\overline{\chi_3}(3)\cdot\tau^3\left(\overline{\chi_3}\right)-\tau(\chi_3)\left(p(\alpha-3)+3\tau^3\left(\overline{\chi_3}\right)\right)+\tau(\chi_3)\left(\overline{\chi_3}(3)\cdot\tau^3(\overline{\chi_3})-3p\right) \\
= \;& -\alpha p\cdot\tau(\chi_3)-3p\cdot\tau^2(\overline{\chi_3}).
\end{aligned}
$$

□

Proof of Theorem 2. Based on Theorem 1 and the identities obtained in [9]

$$
\sum_{u=1}^{p-1}\left|\sum_{i=1}^{p}\mathrm{e}_p\left(ui^3+vi\right)\right|^4=\begin{cases} 2p^3-p^2 & \text{if } \; 3\nmid p-1, \\ 2p^3-7p^2 & \text{if } \; 3\mid p-1. \end{cases}
$$

We have

$$
\begin{aligned}
& \sum_{v=1}^{p-1}\left|\sum_{i=1}^{p}\mathrm{e}_p\left(i^3+vi\right)\right|^4=\sum_{v=1}^{p-1}\left|\sum_{i=1}^{p}\mathrm{e}_p\left((\overline{v}i)^3+i\right)\right|^4 \\
= \;& \sum_{v=1}^{p-1}\left(1+\chi_3(v)+\overline{\chi_3}(v)\right)\left|\sum_{i=1}^{p}\mathrm{e}_p\left(vi^3+i\right)\right|^4 \\
= \;& \sum_{v=1}^{p-1}\left|\sum_{i=1}^{p}\mathrm{e}_p\left(vi^3+i\right)\right|^4+\sum_{v=1}^{p-1}\chi_3(v)\left|\sum_{i=1}^{p}\mathrm{e}_p\left(vi^3+i\right)\right|^4 \\
& +\sum_{v=1}^{p-1}\overline{\chi_3}(v)\left|\sum_{i=1}^{p}\mathrm{e}_p\left(vi^3+i\right)\right|^4 \\
= \;& 2p^3-7p^2-\alpha p\tau(\chi_3)-3p\cdot\tau^2(\overline{\chi_3})-\alpha p\tau(\overline{\chi_3})-3p\cdot\tau^2(\chi_3) \\
= \;& 2p^3-p^2-3p\left(\tau(\chi_3)+\tau(\overline{\chi_3})\right)^2-\alpha p\left(\tau(\chi_3)+\tau(\overline{\chi_3})\right). \qquad (3.1)
\end{aligned}
$$

Now we need to determine the value of the real number $\tau(\chi_3)+\tau(\overline{\chi_3})$ in (3.1). For convenience, write the $A=\tau(\chi_3)+\tau(\overline{\chi_3})$, we construct cubic equation $A^3-3pA-\alpha p=0$ based on (2.1) and $\tau(\chi_3)\cdot\tau(\overline{\chi_3})=p$.

According to Cardans formula (formula of roots of a cubic equation), the three roots of the equation are

$$A_1 = \left[\frac{\alpha p}{2} + \left(\left(\frac{\alpha p}{2}\right)^2 + (-p)^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\frac{\alpha p}{2} - \left(\left(\frac{\alpha p}{2}\right)^2 + (-p)^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}},$$

$$A_2 = \omega\left[\frac{\alpha p}{2} + \left(\left(\frac{\alpha p}{2}\right)^2 + (-p)^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \omega^2\left[\frac{\alpha p}{2} - \left(\left(\frac{\alpha p}{2}\right)^2 + (-p)^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}},$$

$$A_3 = \omega^2\left[\frac{\alpha p}{2} + \left(\left(\frac{\alpha p}{2}\right)^2 + (-p)^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \omega\left[\frac{\alpha p}{2} - \left(\left(\frac{\alpha p}{2}\right)^2 + (-p)^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}},$$

where $\omega = \frac{-1+\sqrt{3}i}{2}$.

It is clear that all A_k ($k = 1, 2$ or 3) are real numbers, So $A = A_1, A_2$ or A_3. Therefore, the proof of theorem is complete. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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