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Research article

A new characterization of Janko simple groups

Zhangjia Han*, **Jiang Hu and Dongyang He**

College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225, China

* **Correspondence:** Email: hzjmm11@163.com.

Abstract: In this paper, we studied the influence of centralizers on the structure of groups, and demonstrated that Janko simple groups can be uniquely determined by two crucial quantitative properties: its even-order components of the group and the set $\pi_{p_m}(G)$. Here, G represents a finite group, $\pi(G)$ is the set of prime factors of the order of G, p_m is the largest element in $\pi(G)$, and $\pi_{p_m}(G)=\{|C_G(x)| \,|\, x \in G \text{ and } |x| = p_m \}$ denotes the set of orders of centralizers of p_m-order elements in G.

Keywords: finite groups; simple groups; order components; centralizers; order
Mathematics Subject Classification: 20D10, 20D20

1. Introduction

The groups mentioned in this paper are all finite groups.

It is well known that the order of a finite group and the orders of its elements are closely related to the structure of the group, as demonstrated by Thompson's famous odd-order theorem. On the other hand, since the completion of the classification of finite simple groups in the 1980s, it has been an important research topic to characterize finite simple groups in terms of their quantity relationships as finite groups, such as using the order of groups and the set of orders of group elements, proposed by Professor W. J. Shi, or the use of order components, proposed by Professor G. Y. Chen, and many other achievements in this area.

In mathematics, a graph is an abstract structure composed of nodes (vertices) and edges. The study of algebraic structures through various graphs defined on algebraic systems has always been an interesting research topic. For example, the prime graph $G(N)$ of a near-ring N is used to investigate the commutativity of prime near-rings (see [1]), while g-noncommuting graph are employed to explore the relationship between finite groups and their subgroups (see [2]). In this article, we will apply the concept of prime graphs of groups G.

Let G be a finite group. K. W. Gruenberg and O. Kegel defined the prime graph $\Gamma(G)$ of a finite group G as follows: the vertex set of G is the set of all prime factors of $|G|$, and two vertices p and q are adjacent if and only if G contains an element of order pq (see [3]). The number of connected components in the prime graph of G is denoted by $t(G)$, and the set of connected components in the prime graph of G is denoted by $T(G) = \{\pi_i(G)|i = 1, 2, \ldots, t(G)\}$. When G is an even-order group, it is stipulated that $2 \in \pi_1(G)$. If $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ are all the connected components of G's prime graph, then $|G| = m_1 m_2 \cdots m_{t(G)}$, where the prime factor set of m_i is π_i for $i = 1, 2, \ldots, t(G)$. The numbers $m_1, m_2, \ldots, m_{t(G)}$ are called the order components of G, and $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ is the set of order components of G (see [4]). For convenience, we denote the even-order component of G as $m_1(G)$. Professor Guiyun Chen gave the order components of all simple groups whose prime graph is disconnected (see [4] Tables 1–4).

For example, the set of prime factors of the alternating groups A_{13} and A_{14} are both $\{2, 3, 5, 7, 11, 13\}$. Therefore, their vertex sets are $\{2, 3, 5, 7, 11, 13\}$. The set of connected components of A_{13} is $T(G)$=$\{\pi_1, \pi_2, \pi_3\}$, where $\pi_1 = \{2, 3, 5, 7\}$, $\pi_2 = 11$, and $\pi_3 = 13$. The order components of A_{13} are $m_1 = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, $m_2 = 11$, and $m_3 = 13$. The set of connected components of A_{14} is $T(G)$=$\{\pi_1, \pi_2\}$, where $\pi_1 = \{2, 3, 5, 7, 11\}$, $\pi_2 = 13$. The order components of A_{14} are $m_1 = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, $m_2 = 13$. Their prime graphs are respectively shown in Figure 1.

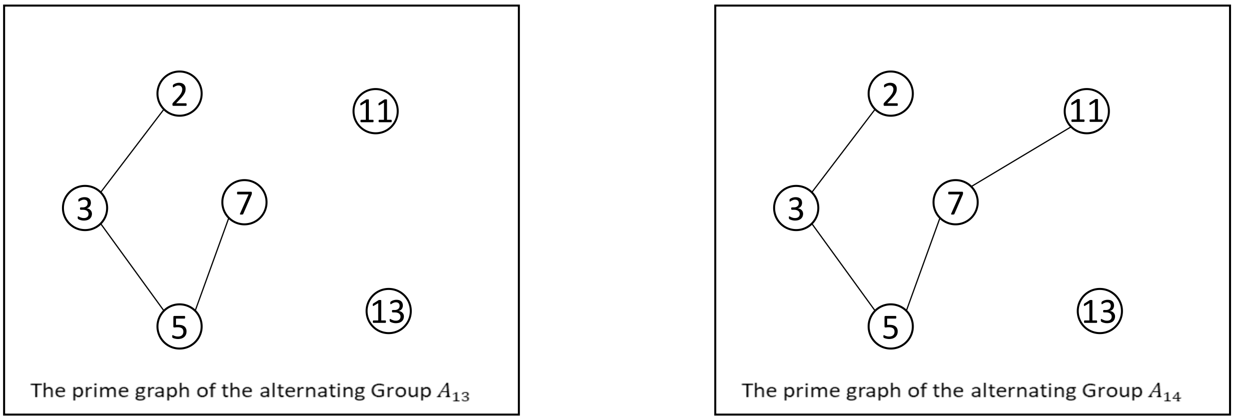

Figure 1. The prime graphs of the alternating groups A_{13} and A_{14}.

Using the concept of order components, Professor G. Y. Chen and other group theorists such as Professor H. G. Shi have studied the following problem:

Let G be a finite group and S be a non-abelian simple group. If $OC(G) = OC(S)$, are G and S isomorphic?

Many group theorists have conducted in-depth studies on this problem, and some of their achievements can be found in [4–15]. From these results, it can be seen that the order components are effective quantitative properties for characterizing simple groups.

In 1962, Professor R. Brauer proposed using centralizers of involutions as labels for finite simple groups at the World Mathematical Congress in Amsterdam. Later, group theorists found that it is sometimes necessary to examine centralizers of elements of odd order in the study of finite simple groups. From this, the properties of centralizers of elements have a significant impact on finite simple groups.

In recent years, important topics in group representation theory such as the McKay conjecture and the Alperin conjecture have been reduced to the cases of simple groups and quasi-simple groups. It is urgent to understand and identify more properties of simple groups. This paper provides a new approach to facilitate better identification of properties of simple groups for group theorists. Compared to known methods, the even-order components and the order of centralizers of the highest-order element in a simple group have the advantages of simplicity, convenience, and wide applicability. In the following, we combine the concepts of order components and centralizers, which has a profound impact on the structure of simple groups, and utilize the quantitative properties of even-order components and centralizers of certain odd-order elements to characterize the Janko simple group.

Main Theorem. *Let G be a finite group, and M be one of the Janko simple groups, namely, J_1, J_2, J_3, or J_4. Then, G is isomorphic to M if and only if:*

(1) $m_1(G) = m_1(M)$;

(2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

Here, the Janko simple groups are four sporadic simple groups discovered by Janko, and they are probably now best constructed by matrix groups over finite fields.

2. Preliminaries

In this paper, we adopt the following conventions: $\pi(G)$ denotes the set of prime factors of the order of G, p_m stands for the largest element of $\pi(G)$, and $\pi_{p_m}(G)$ represents the set of orders of centralizers of p_m-order elements in G. The symbol $|\pi(G)|$ refers to the number of prime factors of the order of G. $Aut(G)$ denotes the automorphism group of G and $Out(G)$ refers to the outer automorphism group of G. Let $|G| = p^{\alpha}m$, where (p, m)=1. Then, a p-subgroup of G which has order p^{α} is called a Sylow p-subgroup of G. For $p_i \in \pi(G)$, S_{p_i} denotes a Sylow p_i-subgroup of G.

If G is a finite group with a subgroup H such that $H \cap H^x$ for all x in $G \setminus H$, then G is called a Frobenius group. Let $N = G \setminus \cup_{x \in G}(H \setminus 1)^x$. Then, $G = HN$, and $H \cap N = 1$. H is called a Frobenius complememt and N the Frobenius kernel. A group G is called a 2-Frobenius group if G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K is a Frobenius group with Frobenius kernel H and G/H is also a Frobenius group with kernel K/H (see [16]). A group G with a solvable series is called a solvable group, where a solvable series is a normal series in which all factor groups are abelian. It is worth mentioning that, in 1963, Professor J. G. Thompson and Professor W. Feit proved the solvability of odd-order groups, from which it can be seen that all simple groups are even-order groups.

All other symbols not explicitly defined are standard and can be found in [17].

The following theorem provides a characterization of the structure of finite groups when $t(G) \geq 2$.

Lemma 2.1. *[3, Corollary] Let G be a finite group with disconnecting prime graph. Then, the structure of G is as follows:*

(1) G is a Frobenius group or a 2-Frobenius group;

(2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent $\pi_1(G)$-group, G/K is a soluble $\pi_1(G)$-group, K/H is a non-abelian simple group, and $|G/K|$ divides $|\mathrm{Out}(K/H)|$.

The following two lemmas respectively provide characterizations of the structure of even-order Frobenius groups and even-order 2-Frobenius groups.

Lemma 2.2. *[16, Theorem 1] Let G be an even-order Frobenius group with Frobenius kernel H and Frobenius complement K. Then,* $t(G) = 2$ *and* $T(G) = \{\pi(H), \pi(K)\}$. *Moreover, the structure of G is one of the following:*

1) If $2 \in \pi(H)$*, then the Sylow subgroups of K are cyclic;*

2) If $2 \in \pi(K)$*, then H is an abelian group. When K is soluble, the odd-order Sylow subgroups of K are cyclic and the Sylow 2-subgroup is either a cyclic group or a generalized quaternion group. When K is insoluble, there exists* $K_0 \leq K$ *such that* $|K : K_0| \leq 2$ *and* $K_0 \simeq Z \times SL(2,5)$*, where* $(|Z|, 30) = 1$ *and the Sylow subgroups of Z are cyclic.*

Lemma 2.3. *[16, Theorem 2] Let G be an even-order 2-Frobenius group. Then,* $t(G) = 2$ *and G has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that* $\pi(K/H) = \pi_2(G)$*,* $\pi(H) \cup \pi(G/K) = \pi_1(G)$*,* $|G/K|$ *divides* $|Aut(K/H)|$*, and both* $|G/K|$ *and* $|K/H|$ *are cyclic groups. In particular,* $|G/K| \leq |K/H|$ *and G is soluble.*

Lemma 2.1 is an important work by J. S. Williams in 1981, which reveals the structure of groups with disconnected prime graphs. The first category consists of Frobenius groups or 2-Frobenius groups, while the second category corresponds to the condition given by Case 2. Lemmas 2.2 and 2.3 are works by G. Y. Chen in 1996, which provide a specific classification for the aforementioned first category. In this paper, we can conveniently apply them to determine whether a group is a Frobenius group or a 2-Frobenius group. These lemmas have been widely applied in related research areas, and readers can refer to the original literature for more information.

3. Proof of Main Theorem

Recall that a simple group which has order $p^a q^b r^c$ is a simple K_3-group, and it is known that all simple K_3-groups are A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$ (see [18]). In the following, we will use those symbols frequently.

Given a subgroup K of a group G, it is obvious that $m_1(K)$ divides $m_1(G)$. In the following proof, we will frequently use this result without further explanation.

Main Theorem. *Let G be a finite group and M be one of the Janko simple groups, namely,* J_1*,* J_2*,* J_3*, or* J_4*. Then, G is isomorphic to M if and only if:*

(1) $m_1(G) = m_1(M)$*;*

(2) $\pi_{p_m}(G) = \pi_{p_m}(M)$*.*

Proof. The necessity of the theorem is evident, so we only need to prove its sufficiency. We will consider four different cases to establish our conclusion.

Case 1. $M \cong J_1(2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$.

In this case, $m_1(G) = m_1(M) = 2^3 \cdot 3 \cdot 5$, and $\pi_{p_m}(M) = \{19\}$. Since $m_1(G) = m_1(M) = 2^3 \cdot 3 \cdot 5$, we have $t(G) \geq 2$, which implies that G has the following structure according to Lemma 2.1:

1) G is a Frobenius group or a 2-Frobenius group;

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5$. Therefore, $|K|\,\big|\,|Aut(S_2)|$. However, since $19\,\big|\,|K|$ and $|Aut(S_2)|\,\big|\,(2^3-1)(2^3-2)(2^3-4)$, we have a contradiction.

2) If $2 \in \pi(K)$, then by the given condition, the Sylow 19-subgroup S_{19} of G is normal in G and has order 19. If we let the Sylow 5-subgroup of G act on S_{19}, we obtain an element of order 95 in G, which contradicts $\pi_{p_m}(M) = \{19\}$. Therefore, G is not a Frobenius group.

G is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^3 \cdot 3 \cdot 5$, we have $19 \in \pi_2(G)$, which means K contains an element of order 19. If we let the 19-order element in K act on the Sylow 2-subgroup of H or the Sylow 3-subgroup of H, we will obtain a contradiction. Therefore, G is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of G is as follows: G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, G/K is a solvable $\pi_1(G)$-group, and K/H is a nonabelian simple group with $|G/K|\,\big|\,|Out(K/H)|$. Since $m_1(G) = m_1(M) = 2^3 \cdot 3 \cdot 5$, it follows that $t(G) \geq 2$, which implies that $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$. Additionally, $19 \in \pi(K/H)$. If H is nontrivial, let us assume $H = S_2 \times S_3 \times S_5$, where $S_i \in Syl_i(H)$ for $i = 2, 3, 5$. Since H is a nilpotent group, we have $S_i \trianglelefteq K$ for $i = 2, 3, 5$. By letting the 19-order element of K act on S_i, we obtain $\pi_{m_p}(M) \neq \{11\}$, which leads to a contradiction. Therefore, we conclude that $H = 1$.

In this way, G has a normal nonabelian simple subgroup K, such that $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5\}$ and $19 \in \pi(K)$. If $|\pi(K)| = 3$, then K is a simple K_3-group. According to [18], no prime factor 19 appears in the order of any simple K_3-group. Therefore, we conclude that $|\pi(K)| \neq 3$. If $|\pi(K)| \geq 4$, then since $m_1(G) = m_1(M) = 2^3 \cdot 3 \cdot 5$ and $\pi_{p_m}(M) = \{19\}$, it follows that $t(K) \geq 2$. By examining Tables 2–4 of [4] and utilizing the condition $\pi_{p_m}(M) = \{19\}$, we can determine that K can only be one of the following groups: $L_2(19)$, $L_3(7)$, $U_3(8)$, J_1, J_3, or HN.

If $K \cong L_2(19)$, then $|K| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$. However, since $m_1(G) = 2^3 \cdot 3 \cdot 5$, $m_1(K) = 2^2 \cdot 5$, we have $6\,\big|\,|G/K|\,\big|\,|Out(L_2(19)| = 2$, which leads to a contradiction. Therefore, K is not isomorphic to $L_2(19)$.

If $K \cong L_3(7)$, then $|K| = 2^5 \cdot 3^2 \cdot 7^3 \cdot 19$. However, in this case, $m_1(K) = 2^5 \cdot 3^2 \cdot 7^3$, which is clearly contradictory to $m_1(G) = 2^3 \cdot 3 \cdot 5$. Therefore, K is not isomorphic to $L_3(7)$. Similarly, K is also not isomorphic to $U_3(8)$, J_3, or HN.

Therefore, we have $K \cong J_1$, which implies that $1 \trianglelefteq J_1 \trianglelefteq G$. In this case, it is clear that $C_G(J_1) = 1$ and $Out(J_1) = 1$. Consequently, we conclude that $G \cong J_1$.

Case 2. $M \cong J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$.

In this case, we have $m_1(G) = m_1(M) = 2^7 \cdot 3^3 \cdot 5^2$ and $\pi_{p_m}(M) = \{7\}$. Since $m_1(G) = m_1(M) = 2^7 \cdot 3^3 \cdot 5^2$, we have $t(G) \geq 2$. Therefore, according to Lemma 2.1, the structure of G is as follows:

1) G is a Frobenius group or a 2-Frobenius group;

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement. In this case, $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is nilpotent, we have $H = S_2 \times S_3 \times S_5$, where

$S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5$. This implies that $|K|\big||Aut(S_5)|$. However, since $7\big||K|$ and $|Aut(S_5)|\big|(5^2-1)(5^2-5)$, we have a contradiction.

2) If $2 \in \pi(K)$, then the Sylow 7-subgroup S_7 of G is normal in G with order 7. By acting the Sylow 5-subgroup of G on S_7, we obtain elements of order 35 in G, which contradicts $\pi_{p_m}(M)$={7}. Hence, G is not a Frobenius group.

G is also not a 2-Frobenius group. Otherwise, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^7 \cdot 3^3 \cdot 5^2$, we have $7 \in \pi_2(G)$, implying that K contains elements of order 7. By acting the 7-order elements of K on either the Sylow 2-subgroup or the Sylow 3-subgroup of H, we obtain a contradiction. Therefore, G is not a 2-Frobenius group.

Therefore, the structure of G is as in Lemma 2.1(2), that is, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, G/K is a solvable $\pi_1(G)$-group, K/H is a non-abelian simple group, and $|G/K|$ divides $|Out(K/H)|$. Since $m_1(G) = m_1(M) = 2^7 \cdot 3^3 \cdot 5^2$, we have $t(G) \geq 2$, implying that $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $7 \in \pi(K/H)$. Since K/H is a nonabelian simple group, we have that 4 divides $|K/H|$. Therefore, the largest possible order of the Sylow 2-subgroup of H is 2^5. If H is non-trivial, without loss of generality, assume $H = S_2 \times S_3 \times S_5$, where $S_i \in Syl_i(H)$ for $i = 2, 3, 5$. Since H is nilpotent, we have $S_i \trianglelefteq K$ for $i = 2, 3, 5$. By acting the 7-order elements of K on S_i, we obtain $\pi_{p_m}(M) \neq \{7\}$, which is a contradiction. Hence, we have $H = 1$.

Now, G have a normal non-abelian simple subgroup K such that $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5\}$ and $7 \in \pi(K)$. If $|\pi(K)| = 3$, then K is a simple K_3-group. According to [18], K can only be one of the following groups: $L_2(7)$, $L_2(8)$, or $U_3(3)$.

If $|\pi(K)| \geq 4$, then due to $m_1(G) = m_1(M) = 2^7 \cdot 3^3 \cdot 5^2$ and $\pi_{p_m}(M) = \{7\}$, we have $t(K) \geq 2$. By examining Tables 2–4 in [4] and using the condition $\pi_{p_m}(M) = \{7\}$, it can be deduced that K can only be one of the following groups: A_7, A_8,A_9,$L_3(4)$, J_2, $U_3(5)$, $S_6(2)$, $O_8^+(2)$, or $U_4(3)$.

If $K \cong L_2(7)$, then $|K| = 2^2 \cdot 3 \cdot 7$. Since $m_1(G) = 2^7 \cdot 3^3 \cdot 5^2$ and $m_1(K) = 2^2 \cdot 3$, we have $2^5{\cdot}3\big||G/K|\big||Out(L_2(7))| = 2$, which is a contradiction. Therefore, K is not isomorphic to $L_2(7)$. Similarly, K is not isomorphic to $L_2(8)$ or $U_3(3)$.

If $K \cong A_7$, then $|K| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. Since $m_1(G) = 2^7 \cdot 3^3 \cdot 5^2$ and $m_1(K) = 2^3 \cdot 3^2$, we have $2^4{\cdot}3\big||G/K|\big||Out(A_7)| = 2$, which is a contradiction. Therefore, K is not isomorphic to A_7. Similarly, K is not isomorphic to A_8, A_9, $L_3(4)$, or $U_3(5)$.

If $K \cong S_6(2)$, then $|K| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$. However, in this case, $m_1(K) = 2^9 \cdot 3^4 \cdot 5$, which contradicts $m_1(G) = 2^7 \cdot 3^3 \cdot 5^2$. Therefore, K is not isomorphic to $S_6(2)$. Similarly, K is not isomorphic to $O_8^+(2)$ or $U_4(3)$.

Therefore, we have $K \cong J_2$, which implies $1 \trianglelefteq J_2 \trianglelefteq G$. It is evident that $C_G(J_2) = 1$ and $|Out(J_2)| = 2$. Hence, we have $G \cong J_2$ or $G \cong Aut(J_2)$. If $G \cong Aut(J_2)$, then it is clear that $m_1(G) > m_1(J_2) = m_1(M)$, which contradicts our assumption. Hence, $G \cong J_2$.

Case 3. $M \cong J_3(2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19)$.

In this case, $m_1(G) = m_1(M) = 2^7 \cdot 3^5 \cdot 5$ and $\pi_{p_m}(M) = \{19\}$. Since $m_1(G) = m_1(M) = 2^7 \cdot 3^5 \cdot 5$, it follows that $t(G) \geq 2$. According to Lemma 2.1, the structure of G is as follows:

1) G is a Frobenius group or a 2-Frobenius group;

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5$. Therefore, $|K|\big||Aut(S_5)|$. However, since $19\big||K|$ and $|Aut(S_5)|\big|5-1 = 4$, we have a contradiction.

2) If $2 \in \pi(K)$, then, by the given condition, the Sylow 19-subgroup S_{19} of G is normal in G and has order 19. If we let the Sylow 5-subgroup of G act on S_{19}, we obtain an element of order 95 in G, which contradicts $\pi_{p_m}(M) = \{19\}$. Therefore, G is not a Frobenius group.

G is also not a 2-Frobenius group. Otherwise, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^7 \cdot 3^5 \cdot 5$, we have $19 \in \pi_2(G)$, which means K contains an element of order 19. If we let the 19-order element in K act on the Sylow 2-subgroup of H or the Sylow 3-subgroup of H, we will obtain a contradiction. Therefore, G is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of G is as follows: G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, G/K is a solvable $\pi_1(G)$-group, and K/H is a nonabelian simple group. Since $m_1(G) = m_1(M) = 2^7 \cdot 3^5 \cdot 5$, we have $t(G) \geq 2$, which implies that $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $19 \in \pi(K/H)$. Similar to Case 1, we can prove that $H = 1$, and thus G has a normal nonabelian simple subgroup K such that $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5\}$, and $19 \in \pi(K)$. If $|\pi(K)| = 3$, then K is a simple K_3-group. According to [18], the orders of all simple K_3-groups do not contain the prime factor 19, which leads to a contradiction. Therefore, $|\pi(K)| \geq 4$. If $|\pi(K)| \geq 4$, then $t(K) \geq 2$ since $m_1(G) = m_1(M) = 2^3 \cdot 3 \cdot 5$ and $\pi_{p_m}(M) = \{19\}$. By examining Tables 2–4 in [4] and using the condition $\pi_{p_m}(M) = \{19\}$, we can conclude that K can only be one of the following groups: $L_2(19)$, $L_3(7)$, $U_3(8)$, J_1, J_3, or HN.

If $K \cong L_2(19)$, then $|K| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$. Since $m_1(G) = 2^7 \cdot 3^5 \cdot 5$ and $m_1(K) = 2^2 \cdot 5$, we have $2^5{\cdot}3^3\big||G/K|\big||Out(L_2(19))| = 2$, which leads to a contradiction. Therefore, K is not isomorphic to $L_2(19)$. Similarly, K is not isomorphic to J_1.

If $K \cong L_3(7)$, then $|K| = 2^5 \cdot 3^2 \cdot 7^3 \cdot 19$. However, in this case $m_1(K) = 2^5 \cdot 3^2 \cdot 7^3$, which contradicts $m_1(G) = 2^7 \cdot 3^5 \cdot 5$. Therefore, K is not isomorphic to $L_3(7)$. Similarly, K is not isomorphic to $U_3(8)$ and HN.

Thus, we have $K \cong J_3$, which implies $1 \trianglelefteq J_3 \trianglelefteq G$. In this case, it is clear that $C_G(J_3) = 1$ and $|Out(J_3)| = 2$. Therefore, we either have $G \cong J_3$ or $G \cong Aut(J_3)$. If $G \cong Aut(J_3)$, then we evidently have that $m_1(G) > m_1(J_3) = m_1(M)$, which contradicts our assumption. Hence, $G \cong J_3$.

Case 4. $M \cong J_4(2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43)$.

In this case, we have $m_1(G) = m_1(M) = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$ and $\pi_{p_m}(M) = \{43\}$. Since $m_1(G) = m_1(M) = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$, we have that $t(G) \geq 2$ according to Lemma 2.1. Thus, the structure of G is as follows:

1) G is a Frobenius group or a 2-Frobenius group;

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7 \times S_{11}$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7, 11$. Therefore, $|K|\big||Aut(S_5)|$. However, since $43\big||K|$ and

$|Aut(S_5)|\|5-1=4$, we have a contradiction.

2) If $2 \in \pi(K)$, then, by the given condition, the Sylow 43-subgroup S_{43} of G is normal in G and has order 43. If we let the Sylow 5-subgroup of G act on S_{43}, we obtain an element of order 215 in G, which contradicts $\pi_{p_m}(M) = \{43\}$. Therefore, G is not a Frobenius group.

G is also not a 2-Frobenius group. Otherwise, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$, we have $43 \in \pi_2(G)$, which means K contains an element of order 43. If we let the 43-order element in K act on the Sylow 2-subgroup of H or the Sylow 3-subgroup of H, we will obtain a contradiction. Therefore, G is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of G is as follows: G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, where H is a nilpotent group, G/K is a solvable $\pi_1(G)$-group, and K/H is a nonabelian simple group. Since $m_1(G) = m_1(M) = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$, we have $t(G) \geq 2$. Therefore, $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $43 \in \pi(K/H)$. Similar to Case 1, we can prove that $H = 1$, implying that G has a normal nonabelian simple subgroup K with $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7, 11\}$ and $43 \in \pi(K)$. If $|\pi(K)| = 3$, then K is a simple K_3-group. According to [18], the order of all simple K_3-groups does not contain the prime factor 43. Hence, $|\pi(K)| \geq 4$. If $|\pi(K)| \geq 4$, then considering that $m_1(G) = m_1(M) = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$ and $\pi_{p_m}(M) = \{43\}$, we have $t(K) \geq 2$. By examining Tables 2 and 4 in [4] and using the condition $\pi_{p_m}(M) = \{43\}$, we conclude that K can only be one of the following groups: $L_2(43)$, $U_3(7)$, or J_4.

If $K \cong L_2(43)$, then $|K| = 2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$. Since $m_1(G) = 2^{21} \cdot 3^3 \cdot 5 \cdot 11^3$ and $m_1(K) = 2^2 \cdot 11$, we have $2^{19}\||G/K|\||Out(L_2(43)| = 2$, which leads to a contradiction. Therefore, K is not isomorphic to $L_2(43)$. Similarly, K is not isomorphic to $U_3(7)$.

Hence, we have $K \cong J_4$, and we have $1 \trianglelefteq J_4 \trianglelefteq G$. It is evident that $C_G(J_4) = 1$ and $Out(J_4) = 1$. Thus, we conclude that $G \cong J_4$. □

4. Conclusions

From the above proof, we can see that the even-order components and the order of centralizers are effective in characterizing Janko finite simple groups, and they have a profound impact on the structure of finite groups. However, due to limitations in length and time, this paper has not verified whether this method is equally effective for more simple groups. Currently, this series of work is progressing in an orderly manner. However, due to the variety of simple groups, it is necessary to obtain more effective results to prove the following conjecture:

Conjecture. *Let G be a finite group, and M a simple group. Then, G is isomorphic to M if and only if:*

(1) $m_1(G) = m_1(M)$;

(2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

Note. In the proof of this conjecture, readers may focus primarily on proving that $\pi_{p_m}(G)$ is a prime number, or deducing from $\pi_{p_m}(G)$ that the group has a corresponding isolated point.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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