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Research article

Effective vague soft environment-based decision-making

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Abstract: The soft idea has a crucial impact on facing uncertainty, but extending soft, fuzzy soft, and vague soft settings is more comprehensive in expressing the problem ambiguity parameters than the soft set. Despite this advancement, the vague soft set falls short in addressing certain decision-making issues. This occurs when we have some external factors that can impact our final decision. These external factors can be represented in the effective parameter set. So, the objective of the current article is to incorporate the new concept of effectiveness into the concept of vague sets. This approach leads the researchers to create a novel and expanded framework for effective decision-making, surpassing any previously introduced methods in applicability. The article also provides an exploration of the types, concepts, and operations of effective vague soft sets, each illustrated with examples. Furthermore, the study delves into the examination of properties like De Morgan's laws, distributive, commutative, absorption, and associative properties for those new sets. Moreover, the framework of effective vague soft sets is used to develop a decision-making methodology. This technique simplifies the process of determining whether a student meets the requirements for a particular level of education or if a patient has a specific disease, among other applications. Additionally, to clarify the proposed algorithm, a detailed case study representing how to classify students toward specializations is examined in detail. Using matrix processes in this example, in addition to $Wolfram\ Mathematica^{®}$, not only renders computations simpler and quicker but also results in more precise optimum effective decisions. In the end, a detailed comparison with existing techniques is performed and summed up in a chart to illustrate the difference between them and the present one.

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1. Introduction

In 1937, Black [4] first introduced the concept of vagueness as follows: The vagueness of a term becomes evident through the creation of "borderline cases", which are individuals for whom it appears equally challenging to either apply or not apply the term. Consequently, a term's vagueness is typically conveyed, to varying degrees, through statements suggesting that there are conceivable situations in which its usage becomes "doubtful" or "ill-defined", situations in which "no one would know how to use it", or situations in which it becomes "impossible" to either affirm or deny its applicability. A proposition is considered vague when there exist potential states of affairs for which it is inherently uncertain whether, when considered by the speaker, they would have been categorized as either permitted or excluded by the proposition. When we say "intrinsically uncertain", we are not referring to the uncertainty resulting from any lack of knowledge on the part of the interpreter, but rather, it stems from the speaker's language habits being indeterminate.

For instance, we can envision an exhibition within an improbable museum dedicated to applied logic, showcasing a collection of "chairs" that differ in quality by the tiniest, barely perceptible increments. At one end of this extensive display, which might encompass thousands of items, you could find an exquisite Chippendale chair, while at the opposite end, there might be a small, unremarkable piece of wood. Any typical observer examining this array encounters significant challenges when attempting to distinctly demarcate what constitutes a chair and what doesn't. The request to make such a determination is considered fundamentally inappropriate. Responses like "chair can not be considered as the type of word that allows for such a sharp division" are commonly provided. If it were, and we were prohibited from using it for any object that deviated most slightly from the ideal term, it would be less valuable to us. This is a practical perspective, but it poses logical difficulties.

Any ordinary set includes items that have a specific characteristic or satisfy a strict condition. This means that if the item satisfies the condition, then it takes membership value equals 1, and then it belongs to the set. Conversely, if this item doesn't satisfy this condition, then it takes membership value equals 0, and then it doesn't belong to the set. However, the challenge occurred when we had some characteristics that their satisfaction couldn't be described specifically by 1 and 0 only because the criterion is relatively different from one circumstance to another or from time to another time. For example, intelligent students can't be described as an ordinary set. To face this challenge, Zadeh [43] developed, in 1965, a novel expansion of the well-defined set theory, known as fuzzy set theory, designed to address uncertainty. Similar to how a crisp set over a universal set X can be naturally defined through its normal characteristic function mapping from the set X to the set $\{0, 1\}$, a fuzzy set over a domain X can also be defined by its membership (characteristic) function that maps from the set X to the interval $[0, 1]$. It's essential to note that this new concept of the fuzzy set, initiated by Zadeh, is entirely devoid of statistical elements. Fuzzy set theory serves as a highly valuable new mathematical method to manage uncertainty. However, the sole value (characteristic function or membership degree) used in this context amalgamates the evidence supporting an element's inclusion and the evidence countering it, all without specifying the relative weight of each. In other words, this single number doesn't provide any information about its accuracy.

So, there was still a challenge which was how to express any vague value. In many situations, one can face problems, where vague expressions must be used like some medical measurements or some evaluations when the degree is ranging or lies between two different values. To address this limitation, in 1993, Gau and Buehrer [20] introduced a novel generalization of both the crisp set theory and the fuzzy set theory known as the vague set theory, building upon Black's initial definition of vagueness. This innovation aimed to address the challenges that occurred when using the fuzzy set theory. Instead of assigning just a single value, they attributed to every element a membership grade in the form of a subinterval within the interval $[0, 1]$. This subinterval effectively captures both the supporting evidence as well as the conflicting evidence. However, it's worth noting that all these aforementioned theories encounter their unique challenges due to limitations in the parametrization tools they employ.

In real-life scenarios, complexity in making judgments predominantly stems from ambiguity and uncertainty. The presence of uncertain data is a fundamental and widespread phenomenon across various critical domains, such as environmental research, sociology, economics, medical science, corporate management, engineering, and numerous others. This uncertainty arises from various sources, including missing data updates, insufficient information, random data variations, limitations of measurement devices, and so forth. Given the substantial volume of uncertain data being collected and stored, coupled with the significance of these applications, there has been a growing interest in recent years in researching effective techniques for representing uncertain data and managing uncertainties. Nonetheless, this endeavor remains challenging. Consequently, numerous complex challenges persist in these fields and others. Conventional mathematical methods are insufficient to address the difficulties arising from uncertainties in such cases.

Decision-making, as a process utilized within a company's managerial echelon, serves the purpose of identifying and selecting choices depending on personal preferences. Each decision's surroundings can be officially characterized as a compilation of data, alternatives, substituting options, and some other values that can be easily accessible when a decision must be made. Given that the effort and time involved in gathering data or exploring alternatives impose limitations on either knowledge or its substitutes, we can frame any reached conclusions within this limited context. Decision-making has gained paramount significance in recent years, primarily due to its close association with achievement and efficiency. Successful individuals attain their work and life objectives using productive adept decision-making techniques. Decision-makers often draw upon their perspectives, values, attitudes, concepts, and ideas to steer their choices.

While various principles can underpin decision-making, the selection of a productive approach that enhances overall performance should be executed with care. It's worth noting that these theories are designed to be an assistant for individuals in improving their decision-making skills in the real world. In recent years, there has been a rapidly growing focus on the current important issue of decision-making in uncertain environments. In the existing body of research and practical applications, numerous specialized mathematical tools have been extensively explored. These tools encompass fuzzy sets [43] and vague sets [20], as well as probability theory, and various other existing mathematical methodologies, all of which serve as valuable means for modeling uncertain data and facilitating the formulation of effective decisions. However, each of these approaches encounters distinct challenges when it comes to dealing with uncertainty. Probability theory, for instance, is a well-established and effective strategy for addressing uncertainty. Nonetheless, it is limited in its applicability to situations involving stochastic processes or events that are entirely governed by chance. One can learn more about decision-making techniques and see more applications using multi-attribute decision-making by referring to [22, 24, 33, 34, 39].

If we had some ordinary, fuzzy, or even vague information, then the lack of the parameterization tool would confuse us when dealing with some problems that require it. So, there was still a critical need for a new extension of the ordinary set that could address this limitation. So, the innovative notion encapsulated within the concept of the theory of the soft sets was initially formalized by Molodtsov in 1999 [31]. One of the motivating factors for the introduction of soft set theory was a contention that arose, suggesting that the inadequacies in the parameterization methods of preceding established frameworks might have been a significant source of challenges and complications. The newly created idea of the soft set theory, introduced through the softness definition framework, represents a novel and practical mathematical tool that is unburdened by the aforementioned issues. As a result, it has been employed for a substantial period to facilitate the management of uncertainties.

Subsequently, in 2002, Maji *et al.* (as documented in [29] and [30]) thoroughly investigated and assessed Molodtsov's soft set novel environment. In addition, they delved into various notions pertaining to soft sets, constructed an extensive conceptual framework for this knowledge, and subsequently incorporated it into a real-world decision-making context. Subsequently, Maji *et al.* [28], in 2001, incorporated the previously-introduced idea of the fuzzy set into the softness term to create the novel idea of fuzzy soft settings. Furthermore, a decision-making approach grounded in the fuzzy soft set theory was developed by Roy and Maji *et al.* [35] to facilitate any needed selection of the optimal item from a range of choices.

In addition, Yang *et al.* [42] established a matrix formulation for the fuzzy soft set theory, relying on the underlying fuzzy soft set theory. Moreover, Çağman *et al.* [6] conducted an extensive exploration of fuzzy soft matrices, as well as performing several algebraic operations and philosophical studies in the wide new field of the fuzzy soft set environments. In that research concerning fuzzy soft matrices, Basu *et al.* [3], as well as Kumar and Kaur *et al.* [25], introduced novel concepts as well as their new related operations. For additional knowledge on the extended fuzzy soft settings and their new associated properties, one could consult [9–15, 17] to access a wealth of theorems, results, and illustrative examples.

Moreover, Alkhazaleh [1] has recently observed a constraint within the fuzzy soft set environment. In this theory, in presented decision-making scenarios, the final resulting decision can just rely usually on conventional parameters, with no regard for external factors. To address this limitation, Alkhazaleh introduced a novel concept, the "effective parameter set", as a means to represent these external variables. Additionally, he devised another new concept which is the "effective fuzzy soft sets", a newly created idea based on the novel concept of effective sets. Furthermore, he specified the operations that applied to the effective fuzzy soft sets environment as well as checked several related properties. Alkhazaleh also demonstrated the usage of the effective fuzzy soft setting at some given decision-making concerns. Ultimately, Alkhazaleh illustrated the utility of this innovative theory in the context of medical diagnosis, providing a hypothetical case study to showcase the methodology.

Notwithstanding this advancement, in practical situations, the mapping can often be too imprecise, rendering the fuzzy soft set concept inadequate for handling it. Thus, there is a need for another broader extension. To meet this need, in 2010, Xu *et al.* [41] introduced the concept of the vague soft set theory and outlined its fundamental properties. The vague soft set theory, in essence, enhances the precision, practicality, and realism of descriptions in the world of objects, creating it as an adaptable instrument, even in specific scenarios. More recently, Wang [40] has contributed a multitude of findings related to the vague soft set environment, examining its linked novel properties and exploring prospective uses. At present, Faried *et al.* [8], as well as Inthumathi and Pavithra [23], have presented the novel concept of vague soft matrices, delving into their general properties, and discussing innovative applications. For further insights into these subjects and numerous other intriguing, interrelated topics, one can refer to [7, 16, 18, 36, 37].

As already indicated in the discussion above, we have defined four different previously introduced types of sets; the fuzzy set, the vague set, as well as the soft set, and the effective set. In addition, there are composite sets that combine the aforementioned, such as the fuzzy soft set, as well as the vague soft set, along with the newly created idea of the effective fuzzy soft set. Nevertheless, there remains a requirement to expand the definition of the novel concept of the effective fuzzy soft set to another wider domain, which is the effective vague soft environment, particularly to address issues characterized by a high degree of vagueness. For instance, this high degree of vagueness occurs when all types of the issue information we have are vague. This means that the problem inputs can include one or more vague soft sets along with a vague effective set. In addition, the resulting sets or the outputs are also vague such as the effective vague soft set or the matrices, etc.

In essence, the motivation for this research stems from the necessity to provide a comprehensive framework that encompasses all the preceding types of sets. This need becomes apparent when we encounter situations involving vague, soft data and the inclusion of effective parameters. The effective vague soft set is ideally suited to fulfill this need by consolidating all the relevant conditions within a unified, generalized definition. In detail, the actual need for this research occurs when we have a decision-making problem that contains some vague soft information along with some external factors that impact the problem. These external factors are very important and should be considered because of their impact on the final decision. These external factors are the effective parameters. For instance, if we have students with some marks, then we can't use only their marks to specify their future track, but also their life skills, etc. can be very important factors in the final decision of their life. The vague soft set theory falls short in addressing this type of decision-making problem. So, we need to introduce the effective vague soft theory.

Consequently, this current article presents the concept of the effective vague soft set, delineating its various categories, properties, operational rules, and practical applications in the realm of educational science. The subsequent sections of this work are structured as the following: Section (2) is dedicated to providing the necessary foundational concepts and definitions. In addition, the aim of section (3) is to delve into the concept of the effective vague soft set, elucidate its various types, and introduce several innovative related concepts. Additionally, section (4) elucidates the operations associated with these sets, such as union and intersection. Moving forward, section (5) consolidates various pertinent properties, including De Morgan's laws, associative properties, commutative properties, distributive laws, and absorption properties. Furthermore, the primary objective of section (6) is to develop an algorithm of decision-making relying on the framework of the effective vague soft set.

This algorithm facilitates the process of determining, for instance, whether a patient is afflicted with a specific ailment or if a student meets the qualifications to enroll in a particular educational program. The procedural steps involved in this algorithm are introduced, and matrices are employed to streamline the computational aspects. The article also presents a detailed example that illustrates how to categorize college students into specialized fields based on their academic performance and life skills. Moreover, the *Wolfram Mathematica*® software is employed for executing both matrix addition and multiplication operations, facilitating the derivation of effective sets and expediting calculations with precision and ease. Additionally, towards the end of section (6), we conduct a comprehensive comparative analysis. This detailed examination serves to underscore the differences between the current methodology and existing approaches. The outcomes of this comparative assessment are also succinctly presented in a chart. In the end, section (7) is designated for presenting concluding important remarks as well as delineating potential future avenues of research. To outline the paper content, Figure 1 shows the structural organization of the paper's content.

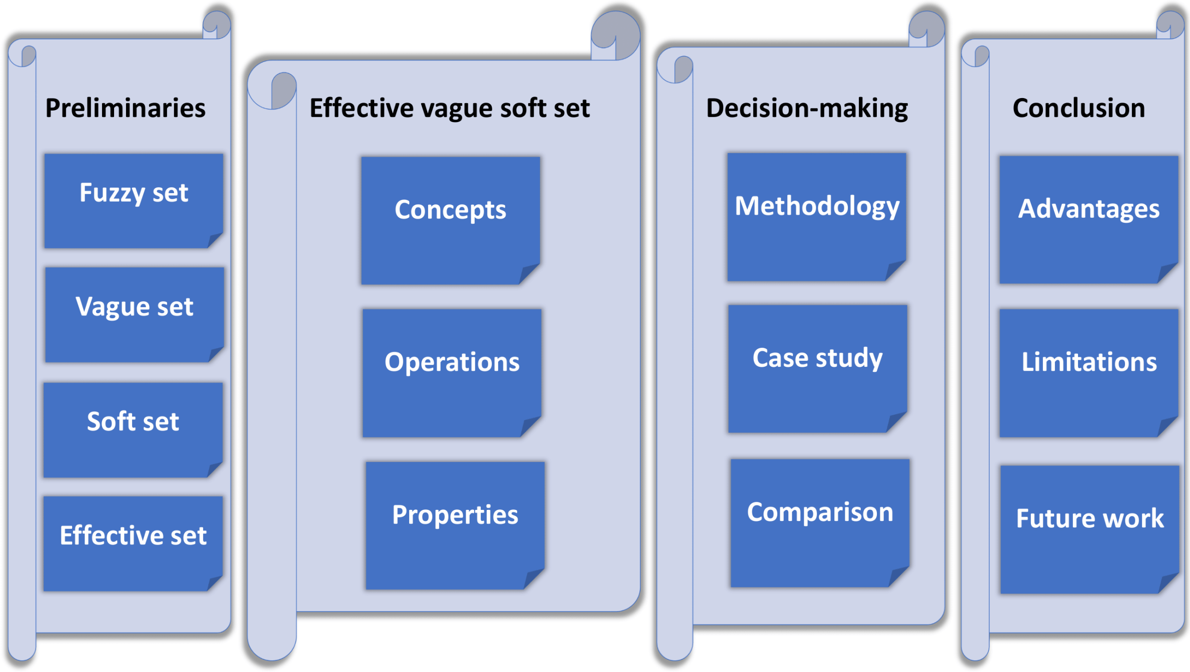


Figure 1. Paper content.

2. Preliminaries

The following part discusses the essential basic definitions that are required in the results that follow. This section contains definitions for the fuzzy set, the vague set, and the soft set, as well as the vague soft set, the effective fuzzy set, and the effective fuzzy soft set. Moreover, the arithmetic operations on closed intervals and some methods to transform vague sets into fuzzy sets are discussed. One can refer to [1, 20, 21, 26, 27, 31, 32, 38, 41, 43] to find more detailed results, explanations and examples about those above concepts.

Definition 2.1. ***(Fuzzy set)*** *[43] Suppose we have an initial universe denoted as Ξ. In such a scenario, we can define a fuzzy class (or set) F over Ξ as a set characterized by a characteristic function $\chi_F : \Xi \to [0,1]$. The term χ_F can also be referred to as the indicator function or the membership function of the fuzzy set F. Additionally, $\chi_F(u)$ is recognized as the degree of membership or the membership grade value of an element ξ in Ξ within the fuzzy set F. In this context, the fuzzy set F over the initial universal set Ξ can be expressed through one of the sequel two forms: $F = \{(\chi_F(\xi)/\xi) : \xi \in \Xi,\ \chi_F(\xi) \in [0,1]\}$, or $F = \{(\xi, \chi_F(\xi)) : \xi \in \Xi,\ \chi_F(\xi) \in [0,1]\}$.*

Definition 2.2. ***(Vague set)*** *[20] Suppose we have the universal set Ξ represented as $\xi_1, \xi_2, ..., \xi_n$. In this context, a vague set V over Ξ is defined by two functions: a truth membership function denoted* *as* τ_V *and a false membership function referred to as* η_V*. The precise membership grade of an element* $\xi \in \Xi$ *(noted as* $\zeta_V(\xi)$*) falls within an interval* $[\tau_V(\xi), 1 - \eta_V(\xi)]$*, which is a subset of the range* $[0, 1]$*. Essentially, this means that* $\zeta_V(\xi)$ *might not be precisely determined, but it is constrained within the bounds of* $\tau_V(\xi) \leq \zeta_V(\xi) \leq 1 - \eta_V(\xi)$*. It's important to note that* $\tau_V(\xi) + \eta_V(\xi) \leq 1$*, and the functions* τ_V *and* η_V *are both defined as mappings from* Ξ *to the interval* $[0, 1]$*. We can represent the vague set V in one of two forms: either as* $V = ([\tau_V(\xi), 1 - \eta_V(\xi)]/\xi) : \xi \in \Xi, ; \tau_V(\xi), \eta_V(\xi) \in [0, 1]$*, or as* $V = (\xi, [\tau_V(\xi), 1 - \eta_V(\xi)]) : \xi \in \Xi, ; \tau_V(\xi), \eta_V(\xi) \in [0, 1]$.

Remark 2.1. *[20] Interval containment happens naturally as follows:* $[a, b] \leq [c, d] \Leftrightarrow a \geq c$ *and* $b \leq d$*.*

Definition 2.3. ***(Arithmetic operations on closed intervals)****([21, 32]) Let* $\star$ *represent any of the four basic arithmetic operations applied to closed intervals: addition "+", subtraction "−", multiplication (or product) "·", (or "×"), and division "/". Consequently,* $[a, b] \star [c, d] = x \star y : a \leq x \leq b, c \leq y \leq d$*. This is a fundamental property that applies to all arithmetic operations on closed intervals, with the exception that* $[a, b]/[c, d]$ *is not defined if* $0 \in [c, d]$*. In other words, the outcome of an arithmetic operation involving closed intervals remains a closed interval. The four basic arithmetic operations on closed intervals are formally defined, using the endpoints of the two intervals, as the following:*

1. *Addition operation:* $[a, b] + [c, d] = [a + c, b + d]$.
2. *Subtraction operation:* $[a, b] - [c, d] = [a - d, b - c]$.
3. *Multiplication operation:* $[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$.
4. *Division operation: Provided that* $0 \notin [c, d]$*, we have:*
   $[a, b]/[c, d] = [a, b] \cdot [1/d, 1/c] = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$.

Remark 2.2. *The multiplication (or product) operation for vague values takes the following form:* $[a, b] \cdot [c, d] = [ac, bd]$ *since all values lie between zero and one. In addition, the formula of the division operation for vague values is as follows:* $[a, b]/[c, d] = [a/d, b/c]$*, provided that* $c, d \neq 0$.

Lemma 2.1. ***(Converting vague sets to fuzzy sets)*** *If* $\mathcal{V}(\Xi)$ *represents the collection of all vague sets across the universe* Ξ*, then, for every* $V \in \mathcal{V}(\Xi)$*, as well as for each* $\xi \in \Xi$ *and its associated vague value* $[\tau_V(\xi), 1 - \eta_V(\xi)]$*, the fuzzy membership function* $\eta_{V^F}(\xi)$ *can be defined. Here,* V^F *denotes the fuzzy set corresponding to the vague set V. This fuzzy membership function is expressed as follows:*

$$\textit{\textbf{Method (1)}} : \eta_{V^F}(\xi) = \frac{1 + \tau_V(\xi) - \eta_V(\xi)}{2}. \tag{2.1}$$

Method (1) ([26]) was created using the median concept. The fuzzy set membership value (the corresponding vague set membership value) can be determined by calculating the corresponding median membership value of the corresponding true and false membership values. In other words, the fuzzy value is defined as the total amount of evidence incorporated in a vague value represented by the median membership value.

$$\textit{\textbf{Method (2)}} : \eta_{V^F}(\xi) = \frac{\tau_V(\xi)}{\tau_V(\xi) + \eta_V(\xi)}. \tag{2.2}$$

Method (2) ([26]) is derived using the defuzzification function. The fuzzy set membership value can be obtained by calculating the related defuzzification value of the relevant true and false membership values.

$$\textbf{\textit{Method (3)}} : \eta_{V^F}(\xi) = \tau_V(\xi) + \frac{1}{2} \times [1 + \frac{\tau_V(\xi) - \eta_V(\xi)}{\tau_V(\xi) + \eta_V(\xi) + 2}][1 - \tau_V(\xi) - \eta_V(\xi)]. \tag{2.3}$$

Method (3) ([27]) investigates the mapping between the elements of imprecise sets and points on a plane. It is discovered that the transition of a vague set into a fuzzy set is a many-to-one mapping relation. It is also discovered to be a generic transformation model for converting fuzzy set membership values to vague set membership values.
For more information on the distinctions between Methods (1), (2), and (3) and what they represent, see [26, 27, 38].

Definition 2.4. ***(Soft set)*** *[31] Consider an initial universe denoted as Ξ, a collection of attributes (or parameters) labeled as Υ, and a subset Λ of Υ. In addition, we have the power set of Ξ constructed from $P(\Xi) = 2^{\Xi}$. In this context, a pair denoted as $(\sqsupset, \Lambda)$, or simply $\sqsupset_\Lambda$, is defined as a soft set over the universal set Ξ, with the understanding that $\sqsupset$ is a mapping symbolized as $\sqsupset : \Lambda \to P(\Xi)$. Furthermore, $\sqsupset_\Lambda$ can be articulated as a collection of ordered pairs: $\sqsupset_\Lambda = (\lambda, \sqsupset_\Lambda(\lambda)) : \lambda \in \Lambda, \sqsupset_\Lambda(\lambda) \in P(\Xi)$. In this context, Λ is referred to as the support of $\sqsupset_\Lambda$, and it's important to note that $\sqsupset_\Lambda(\lambda) \neq \phi$ for any $\lambda \in \Lambda$ and $\sqsupset_\Lambda(\lambda) = \phi$ for any $\lambda \notin \Lambda$. This essentially implies that a soft set $(\sqsupset, \Lambda)$ over Ξ can be viewed as a parameterized collection of subsets of Ξ.*

Definition 2.5. ***(Vague soft set)*** *[41] Let's consider a scenario where Ξ represents the universal set, Υ is the parameter set, and Λ is a subset of Υ. In this context, we define a pair denoted as $(\sqsupset, \Lambda)$, which is expressed as $\sqsupset_\Lambda = (\lambda, \sqsupset_\Lambda(\lambda)) : \lambda \in \Lambda, \sqsupset_\Lambda(\lambda) \in \mathcal{V}(\Xi)$. This combination is referred to as a vague soft set over the universal set Ξ. Here, $\sqsupset$ is a mapping, defined as $\sqsupset : \Lambda \to \mathcal{V}(\Xi)$, and $\mathcal{V}(\Xi)$ represents the power set of vague sets on Ξ, which essentially encompasses the entire family of vague subsets of Ξ following the definition outlined previously in Definition (2.2).*

Definition 2.6. ***(Effective fuzzy set)*** *[1] An effective fuzzy set is characterized as a fuzzy set $\urcorner$ defined over the universal set Δ, with $\urcorner$ represented by the mapping $\urcorner : \Delta \to [0, 1]$. Here, Δ encompasses the collection of all effective attributes or parameters capable of influencing the membership value of each element. It exerts a positive impact on the membership values of the elements once it's applied to them. It's worth noting that, in certain instances, a few membership values stay unaltered even after the application. $\urcorner$ can be expressed using the sequel formulation: $\urcorner = (\delta, \varrho_\neg(\delta)), \delta \in \Delta$, where $\varrho_\neg(\delta)$ signifies the effective membership value for a given $\delta \in \Delta$.*

Definition 2.7. ***(Effective fuzzy soft set)*** *[1] Given an initial universal set Ξ, the collection of all fuzzy subsets of Ξ can be denoted as $\mathfrak{F}(\Xi)$. Let's assume that $\upsilon_i \in \Upsilon$ represents the conventional parameters, along with considering that Δ represents the set of effective parameters, as well as $\urcorner$ represents the effective set over Δ. In this context, we refer to the pair $(\Psi_\neg, \Upsilon)$ as an effective fuzzy soft set over Ξ, with the understanding that the mapping $\Psi : \Delta \to \mathfrak{F}(\Xi)$ is defined according to the sequel expression: $\Psi(\delta_i)\urcorner = (\xi j, \chi_\Psi(\xi_j)\urcorner), \xi j \in \Xi, \delta_i \in \Delta$. The following holds true for all $\delta_k \in \Delta$,*

$$\chi_\Psi(\xi_j, \upsilon_i)_\neg = \begin{cases} \chi_\Psi(\xi_j, \upsilon_i) + \frac{(1-\chi_\Psi(\xi_j,\upsilon_i)) \sum_k \varrho_{\neg_{\xi_j}}(\delta_k)}{|\Delta|}, & if\ \chi_\Psi(\xi_j, \upsilon_i) \in (0, 1), \\ \chi_\Psi(\xi_j), & otherwise. \end{cases} \tag{2.4}$$

In this above context, $|\Delta|$ represents the count of elements within the provided effective parameter set Δ, $\chi_\Psi(\xi_j, \upsilon_i)$ denotes the membership degree value of the item ξ_j concerning the parameter υ_i, as well as $\sum_k \varrho_{\urcorner_{\xi_j}}(\delta_k)$ signifies the total of all effective parameter values associated with ξ_j.

Example 2.1. *Suppose starting with an initial universal set Ξ containing elements ξ_1, ξ_2, and ξ_3, and a parameter set Υ consisting of υ_1, υ_2, and υ_3. Let's consider the fuzzy soft set for the parameter υ_1: $(\Psi, \Upsilon)(\upsilon_1) = \{(\xi_1, 0.3), (\xi_2, 0.7), (\xi_3, 0.5)\}$. To determine the effective membership value for the first element ξ_1, which has a membership value of 0.3 for the first parameter υ_1, we need to consider the given effective set $\urcorner$ for ξ_1: $\urcorner(\xi_1) = \{(\delta_1, 0.8), (\delta_2, 1), (\delta_3, 0), (\delta_4, 0.2)\}$, Here, δ_1, δ_2, δ_3, and δ_4 represent the effective parameters provided. We calculate the effective membership value using Formula 2.4 from Definition (2.7) as follows:*

$$\chi_\Psi(\xi_1, \upsilon_1)_\urcorner = 0.3 + \frac{(1-0.3)(0.8+1+0+0.2)}{4} = 0.3 + \frac{0.7 \times 2}{4} = 0.3 + 0.35 = 0.65.$$

In a similar way, we can compute the effective membership values for the remaining membership values of ξ_1 for the other two parameters, υ_2 and υ_3, and also for the other two elements, ξ_2 and ξ_3. For a more detailed example illustrating this definition, please refer to [1], page 3.

Remark 2.3. *For the sake of simplicity, in place of detailing the entire complex Formula 2.4 stated in Definition 2.7, we may represent the effective membership value $\chi_\Psi(\xi_j, \upsilon_i)\urcorner$ corresponding to the membership value $\chi\Psi(\xi_j, \upsilon_i)$ of a particular element ξ_j for a given parameter υ_i as $\chi_\urcorner$. This abbreviation is especially applicable when we are solely referring to the fuzzy soft set Ψ. Furthermore, we can streamline $\varrho_{\mu_{\xi_j}}$ as ϱ_μ. However, if we are dealing with two or more fuzzy soft sets, such as Ψ_1 and Ψ_2, we should utilize the complete formulas $\chi_{\Psi_1}(\xi_j, \upsilon_i)$ and $\chi_{\Psi_2}(\xi_j, \upsilon_i)$ to distinguish between them, respectively.*

3. Effective vague soft sets

The main purpose of this section is to define the effective vague set and the effective vague soft set, as well as to illustrate them with examples. Additionally, some kinds and several related concepts of effective vague soft sets are conducted.

Definition 3.1. ***(Effective vague set)*** *An effective vague set is characterized as a vague set μ defined over the universal set Δ, with μ being determined by the mapping $\mu : \Delta \to [0, 1]$. It can be affirmed that Δ represents the collection of all pertinent parameters or attributes capable of influencing the truth membership function τ and the false membership function η of each element. This influence manifests positively in the truth and false membership values of the elements subsequent to its application. It is worth noting that in certain scenarios, certain truth and false membership values remain unaltered even after this application. μ can be expressed through the subsequent formula: $\mu = (\delta, [\alpha_\mu(\delta), 1 - \beta_\mu(\delta)]), \delta \in \Delta$, where $\alpha_\mu(\delta)$ and $\beta_\mu(\delta)$ denote the effective truth membership value and the effective false membership value for a given $\delta \in \Delta$, respectively.*

Definition 3.2. ***(Effective vague soft set)*** *Given an initial universal set Ξ, we can represent the collection of all vague subsets on Ξ as $\mathcal{V}(\Xi)$. Let's assume that Υ denotes the parameter set, Δ signifies the effective parameter set, and μ is the effective set defined over Δ. In this context, we refer* *to the pair (Ψ_μ, Υ) as an effective vague soft set over Ξ, and this nomenclature takes into consideration that the mapping $\Psi : \Delta \to \mathcal{V}(\Xi)$ can be defined using the following formula:*

$$\Psi(\delta_i)_\mu = \{(\xi_j, [\tau_\Psi(\xi_j)_\mu, 1 - \eta_\Psi(\xi_j)_\mu], \xi_j \in \Xi, \delta_i \in \Delta\},$$

taking into account that, for each $\delta_k \in \Delta$, the following are satisfied:

$$\tau_\mu = \begin{cases} \tau + \frac{(1-\tau)\sum_k \alpha_\mu(\delta_k)}{|\Delta|}, & if\ \tau \in (0,1), \\ \tau, & otherwise, \end{cases} \tag{3.1}$$

and

$$1 - \eta_\mu = \begin{cases} 1 - \eta + \frac{\eta \sum_k 1-\beta_\mu(\delta_k)}{|\Delta|}, & if\ \eta \in (0,1), \\ \eta, & otherwise, \end{cases} \tag{3.2}$$

regarding that $|\Delta|$ represents the count of elements within the provided Δ, the set of the effective parameters.

Example 3.1. *If we are given an initial universal set Ξ comprising elements ξ_1, ξ_2, and ξ_3, and a parameter set Υ containing elements υ_1, υ_2, and υ_3, then consider the vague soft set associated with the parameter υ_1 as follows:*

$$(\Psi, \Upsilon)(\upsilon_1) = \{(\xi_1, [0.1, 0.3]), (\xi_2, [0.6, 0.7]), (\xi_3, [0.4, 0.5])\}.$$

Next, in order to determine the effective membership value for the initial item ξ_1, given a membership value of 0.3 for the initial parameter υ_1, and the provided effective set μ for ξ_1:

$$\mu(\xi_1) = \{(\delta_1, [0.6, 0.8]), (\delta_2, [0.8, 1]), (\delta_3, [0, 0.1]), (\delta_4, [0.2, 0.3])\},$$

where δ_1, δ_2, δ_3 and δ_4 are the provided effective parameters, we perform the following calculation using Formula 3.1 and Formula 3.2 from Definition (3.2):

$$\begin{aligned} [\tau_\Psi, 1 - \eta_\Psi](\xi_1, \upsilon_1)_\mu &= [0.1, 0.3] + \frac{([1, 1] - [0.1, 0.3])([0.6, 0.8] + [0.8, 1] + [0, 0.1] + [0.2, 0.3])}{4} \\ &= [0.1, 0.3] + \frac{[0.9, 0.7] \times [1.6, 2.2]}{4} \\ &= [0.1, 0.3] + [0.36, 0.385] = [0.46, 0.685]. \end{aligned}$$

Likewise, we can compute the effective membership values for the remaining membership values of ξ_1 concerning the two other parameters, υ_2 and υ_3, and also extend this computation to the other two items, ξ_2 and ξ_3.

Example 3.2. *Consider a universal set of houses denoted as Ξ, consisting of properties like ξ_1, ξ_2, and ξ_3, all of which are being considered for purchase. The set of attributes used to describe these properties includes Υ, where Υ is defined as $\upsilon_i, i = 1, 2, \cdots, 6$. In this context, υ_1 represents proximity to the city center, υ_2 indicates a green surrounding, υ_3 signifies a high cost, υ_4 relates to a large size, υ_5 pertains to being within a stacked apartment complex, and υ_6 refers to a luxurious design. Furthermore, let's assume that $\Delta = \delta_j, j = 1, 2, 3, 4$ represents a set of parameters with an impact on* *the properties, where δ_1 relates to the property having had previous occupants, δ_2 signifies the presence of a broker fee, δ_3 relates to the absence of an elevator, and δ_4 indicates that the property lacks the necessary licensing. According to expert evaluations, the effective vague set for each of the houses in Ξ ($\xi_k, k = 1, 2, 3$) can be described as follows:*

$$\mu(\xi_1) = \{(\delta_1, [0.5, 0.6]), (\delta_2, [0.7, 0.8]), (\delta_3, [0.1, 0.2]), (\delta_4, [0.9, 1])\},$$

$$\mu(\xi_2) = \{(\delta_1, [0.3, 0.4]), (\delta_2, [0.8, 0.9]), (\delta_3, [0.6, 0.7]), (\delta_4, [0.4, 0.5])\},$$

$$\mu(\xi_3) = \{(\delta_1, [0.1, 0.2]), (\delta_2, [0.2, 0.3]), (\delta_3, [0, 0.1]), (\delta_4, [0.3, 0.4])\}.$$

Moreover, the appeal of the mentioned houses in line with the buyer's inclinations can be represented through a vague soft set (Ψ, Υ) within the universal set Ξ in the following manner:

$$\begin{aligned}(\Psi, \Upsilon) = \{&(\upsilon_1, \{(\xi_1, [0.3, 0.4]), (\xi_2, [0.8, 0.9]), (\xi_3, [0.6, 0.7])\}),\\ &(\upsilon_2, \{(\xi_1, [0.5, 0.6]), (\xi_2, [0.1, 0.2]), (\xi_3, [0.9, 1])\}),\\ &(\upsilon_3, \{(\xi_1, [0.7, 0.8]), (\xi_2, [0.4, 0.5]), (\xi_3, [0.2, 0.3])\})\\ &(\upsilon_4, \{(\xi_1, [0.9, 1]), (\xi_2, [0.6, 0.7]), (\xi_3, [0.8, 0.9])\})\\ &(\upsilon_5, \{(\xi_1, [0, 0.1]), (\xi_2, [0.3, 0.4]), (\xi_3, [0.5, 0.6])\})\\ &(\upsilon_6, \{(\xi_1, [0.8, 0.9]), (\xi_2, [0.2, 0.3]), (\xi_3, [0.7, 0.8])\})\}.\end{aligned}$$

Then, by using Formula 3.1 and Formula 3.2 from Definition (3.2), We possess the effective vague soft set (Ψ_μ, Υ) across the universal set Ξ, effectively portraying the appeal of the aforementioned houses as follows:

$$\begin{aligned}(\Psi_\mu, \Upsilon) = \{&(\upsilon_1, \{(\xi_1, [0.685, 0.79]), (\xi_2, [0.905, 0.9625]), (\xi_3, [0.66, 0.775])\}),\\ &(\upsilon_2, \{(\xi_1, [0.775, 0.86]), (\xi_2, [0.5725, 0.7]), (\xi_3, [0.915, 1])\}),\\ &(\upsilon_3, \{(\xi_1, [0.865, 0.93]), (\xi_2, [0.715, 0.8125]), (\xi_3, [0.32, 0.475])\})\\ &(\upsilon_4, \{(\xi_1, [0.955, 1]), (\xi_2, [0.81, 0.8875]), (\xi_3, [0.83, 0.925])\})\\ &(\upsilon_5, \{(\xi_1, [0.55, 0.685]), (\xi_2, [0.6675, 0.775]), (\xi_3, [0.575, 0.7])\})\\ &(\upsilon_6, \{(\xi_1, [0.91, 0.965]), (\xi_2, [0.62, 0.7375]), (\xi_3, [0.745, 0.85])\})\}.\end{aligned} \tag{3.3}$$

The description provided by (Ψ_μ, Υ) can assist the buyer in making an informed decision regarding the most suitable house for their needs and preferences.

Remark 3.1. *The above effective vague soft set (Ψ_μ, Υ) 3.3 from Example (3.2) is easier to handle when represented in a matrix form as follows:*

$$A = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \begin{array}{c} \begin{array}{cccccc} \upsilon_1 & \upsilon_2 & \upsilon_3 & \upsilon_4 & \upsilon_5 & \upsilon_6 \end{array} \\ \begin{pmatrix} [0.68, 0.79] & [0.77, 0.86] & [0.86, 0.93] & [0.95, 1] & [0.55, 0.68] & [0.91, 0.96] \\ [0.9, 0.96] & [0.57, 0.7] & [0.71, 0.81] & [0.81, 0.88] & [0.66, 0.77] & [0.62, 0.73] \\ [0.66, 0.77] & [0.91, 1] & [0.32, 0.47] & [0.83, 0.92] & [0.57, 0.7] & [0.74, 0.85] \end{pmatrix} \end{array},$$

We can call A the effective vague soft matrix that is associated with the effective vague soft set (Ψ_μ, Υ).

Definition 3.3. ***(Complete effective vague soft set)*** *Let's consider an initial universe denoted as* Ξ*, and let's also assume the presence of a set of parameters* Υ*. Therefore, we can refer to any effective vague soft set* (Ψ_μ, Υ) *defined over the initial universe* Ξ *and constructed using an effective set* μ *as an absolute (or complete) vague soft set, represented as* $(C\mu, \Upsilon)$*. This absolute set is defined as follows: for each* $\upsilon \in \Upsilon$*, it holds that* $\Psi\Upsilon(\upsilon)\mu = \mathfrak{BMF}(\Xi)$*, meaning that* $\tau\Psi_\Upsilon(\upsilon)(\xi)\mu = 1$ *and* $\eta\Psi_\Upsilon(\upsilon)(\xi)\mu = 0$ *for all* $\upsilon \in \Upsilon$*. In other words, for each* $\upsilon \in \Upsilon$ *and every* $\xi \in \Xi$*, the absolute set is represented as* $(C\mu, \Upsilon) = (\upsilon, \xi, [1, 1]) : \upsilon \in \Upsilon, \xi \in \Xi$.

Definition 3.4. ***(Null effective vague soft set)*** *Let's consider that* Ξ *serves as the initial universal set, and assuming the existence of a parameter set* Υ*, any effective vague soft set denoted as* (Ψ_μ, Υ) *in this initial universe* Ξ*, constructed with the aid of an effective set* μ*, is termed as an empty (or null) vague soft set, denoted as* (Φ_μ, Υ)*. This empty set is characterized by the following property: for each* $\upsilon \in \Upsilon$*, one can find that* $\Psi_\Upsilon(\upsilon)\mu = \phi$*. In other words,* $\tau\Psi_\Upsilon(\upsilon)(\xi)\mu = 0$ *and* $\eta\Psi_\Upsilon(\upsilon)(\xi)\mu = 1$ *for every* $\upsilon \in \Upsilon$*. This signifies that for each* $\upsilon \in \Upsilon$ *and every* $\xi \in \Xi$*, the empty set is represented as* $(\Phi\mu, \Upsilon) = (\upsilon, \xi, [0, 0]) : \upsilon \in \Upsilon, \xi \in \Xi$.

4. Effective vague soft operations

The primary goal of this section is to present operations on effective vague soft sets. Union, intersection, complement, subset, and many other operations are established. Furthermore, for each operation, an example is provided to demonstrate how this operation can be performed.

Definition 4.1. ***(Union of two effective vague soft sets)*** *Supposing that* Ξ *serves as an initial universal set, letting* Υ_1 *and* Υ_2 *represent two sets of parameters, as well as considering* μ_1 *and* μ_2 *as two sets of effective parameters defined over* Δ*, one can have the operation of the union for two effective vague soft sets denoted as* $(\Psi_{1\mu_1}, \Upsilon_1)$ *and* $(\Psi_{2\mu_2}, \Upsilon_2)$*, over the initial universal set* Ξ *can be represented as a new resulting effective vague soft set, labeled as* $(\Psi^U\mu^U, \Upsilon^U)$*. The effective set* μ^U *is described as* $(\delta, [\alpha^U(\delta), 1 - \beta^U(\delta)]), \delta \in \Delta$*. The effective truth membership union value,* α^U*, and the effective false membership union value,* β^U*, are characterized by* $[\alpha^U, 1-\beta^U] = [\alpha_1, 1-\beta_1]\widetilde{\cup}[\alpha_2, 1-\beta_2]$*, where* α_1*,* α_2*,* β_1*, and* β_2 *represent the effective truth membership values and the effective false membership values associated with* μ_1 *and* μ_2*, respectively. Furthermore, in the case of two vague soft sets, denoted as* (Ψ_1, Υ_1) *and* (Ψ_2, Υ_2)*, we find that* $(\Psi, \Upsilon)^U = (\Psi^U, \Upsilon^U) = (\Psi_1, \Upsilon_1)\widetilde{\cup}(\Psi_2, \Upsilon_2)$*. It's important to note that this operation takes into consideration that* $\Upsilon^U = \Upsilon_1 \cup \Upsilon_2$*. The following formulas can be used to calculate* α^U *and* β^U *for every* $\xi \in \Xi$*:*

$$\alpha_{\mu^U_\xi}(\delta) = \begin{cases} \alpha_{\mu_{1\xi}}(\delta), & if\ \delta \in \mu_1 - \mu_2, \\ \alpha_{\mu_{2\xi}}(\delta), & if\ \delta \in \mu_2 - \mu_1, \\ \max\{\alpha_{\mu_{1\xi}}(\delta), \alpha_{\mu_{2\xi}}(\delta)\}, & if\ \delta \in \mu_1 \cap \mu_2, \end{cases} \tag{4.1}$$

$$\beta_{\mu^U_\xi}(\delta) = \begin{cases} \beta_{\mu_{1\xi}}(\delta), & if\ \delta \in \mu_1 - \mu_2, \\ \beta_{\mu_{2\xi}}(\delta), & if\ \delta \in \mu_2 - \mu_1, \\ \min\{\beta_{\mu_{1\xi}}(\delta), \beta_{\mu_{2\xi}}(\delta)\}, & if\ \delta \in \mu_1 \cap \mu_2, \end{cases} \tag{4.2}$$

for each $\delta \in \Delta$. In addition, for all $\xi \in \Xi$ and for all $\upsilon \in \Upsilon^U$, we can determine the formula that compute $(\Psi^U, \Upsilon^U) = (\Psi_1, \Upsilon_1)\widetilde{\cup}(\Psi_2, \Upsilon_2)$ as follows:

$$(\Psi^U, \Upsilon^U) = \begin{cases} \{(\upsilon, \{\xi, [\tau_{\Psi_1(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi_1(\upsilon)}(\xi)_\mu]\}), \xi \in \Xi\}, \; if \; \upsilon \in \Upsilon_1 - \Upsilon_2, \\ \{(\upsilon, \{\xi, [\tau_{\Psi_2(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi_2(\upsilon)}(\xi)_\mu]\}), \xi \in \Xi\}, \; if \; \upsilon \in \Upsilon_2 - \Upsilon_1, \\ \{(\upsilon, \{\xi, [\max\{\tau_{\Psi_1(\upsilon)}(\xi)_\mu, \tau_{\Psi_2(\upsilon)}(\xi)_\mu\}, \\ \qquad \min\{\eta_{\Psi_1(\upsilon)}(\xi)_\mu, \eta_{\Psi_1(\upsilon)}(\xi)_\mu\}]\}), \xi \in \Xi\}, if \; \upsilon \in \Upsilon_1 \cap \Upsilon_2. \end{cases} \tag{4.3}$$

Example 4.1. *Under assumptions of Example (3.2), one can define two effective sets μ_1 and μ_2 over $\Delta = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, for ξ_1 and ξ_2, as follows:*

$$\mu_1(\xi_1) = \{(\delta_1, [0.4, 0.5]), (\delta_2, [0.2, 0.3]), (\delta_3, [0.7, 0.8]), (\delta_4, [0.6, 0.7])\},$$

$$\mu_1(\xi_2) = \{(\delta_1, [0.5, 0.6]), (\delta_2, [0.1, 0.2]), (\delta_3, [0.9, 1]), (\delta_4, [0.8, 0.9])\},$$

$$\mu_2(\xi_1) = \{(\delta_1, [0.6, 0.7]), (\delta_2, [0.3, 0.4]), (\delta_3, [0.4, 0.5]), (\delta_4, [0.5, 0.6])\},$$

$$\mu_2(\xi_2) = \{(\delta_1, [0.1, 0.2]), (\delta_2, [0, 0.1]), (\delta_3, [1, 1]), (\delta_4, [0, 0])\},$$

respectively, associated with the following two vague soft sets (Ψ_1, Υ_1) and (Ψ_2, Υ_2), over a universal set Ξ:

$$\begin{aligned}(\Psi_1, \Upsilon_1) = \{&(\upsilon_1, \{(\xi_1, [0.7, 0.8]), (\xi_2, [0, 0.1]), (\upsilon_2, \{(\xi_1, [0.9, 1]), (\xi_2, [0.6, 0.7]), \\ &(\upsilon_3, \{(\xi_1, [0.1, 0.2]), (\xi_2, [0, 0])\})\},\end{aligned}$$

$$\begin{aligned}(\Psi_2, \Upsilon_2) = \{&(\upsilon_1, \{(\xi_1, [0.5, 0.6]), (\xi_2, [0.3, 0.4]), (\upsilon_2, \{(\xi_1, [1, 1]), (\xi_2, [0.8, 0.9]), \\ &(\upsilon_3, \{(\xi_1, [0.4, 0.5]), (\xi_2, [0.2, 0.3])\})\}.\end{aligned}$$

Then, one can calculate the union of the two effective sets μ^U, applying Formula 4.1 and Formula 4.2 from Definition (4.1), as follows:

$$\mu^U(\xi_1) = \{(\delta_1, [0.6, 0.7]), (\delta_2, [0.3, 0.4]), (\delta_3, [0.7, 0.8]), (\delta_4, [0.6, 0.7])\},$$

$$\mu^U(\xi_2) = \{(\delta_1, [0.5, 0.6]), (\delta_2, [0.1, 0.2]), (\delta_3, [1, 1]), (\delta_4, [0.8, 0.9])\}.$$

In addition, compute the union of the two vague soft sets (Ψ_1, Υ_1) and (Ψ_2, Υ_2), namely $(\Psi, \Upsilon)^U = (\Psi^U, \Upsilon^U)$, where $\Upsilon^U = \Upsilon_1 \cup \Upsilon_2$, by making use of Formula 4.3 from Definition (4.1), as shown below:

$$\begin{aligned}(\Psi^U, \Upsilon^U) = \{&(\upsilon_1, \{(\xi_1, [0.7, 0.8]), (\xi_2, [0.3, 0.4]), (\upsilon_2, \{(\xi_1, [1, 1]), (\xi_2, [0.8, 0.9]), \\ &(\upsilon_3, \{(\xi_1, [0.4, 0.5]), (\xi_2, [0.2, 0.3])\})\}.\end{aligned}$$

Finally, computing effective union of vague soft sets, namely $(\Psi^U_{\mu^U}, \Upsilon^U)$ using Formula 3.1 and Formula 3.2 from Definition (3.2), results in the following:

$$\begin{aligned}(\Psi^U_{\mu^U}, \Upsilon^U) = \{&(\upsilon_1, \{(\xi_1, [0.865, 0.93]), (\xi_2, [0.72, 0.805]), (\upsilon_2, \{(\xi_1, [1, 1]), (\xi_2, [0.92, 0.9675]), \\ &(\upsilon_3, \{(\xi_1, [0.73, 0.825]), (\xi_2, [0.68, 0.7725])\})\}.\end{aligned}$$

Definition 4.2. ***(Restricted union of two effective vague soft sets)*** *Assuming that Ξ serves as an initial universal set and considering the presence of two sets of parameters, namely, Υ_1 and Υ_2, along with two sets of effective parameters, namely, μ_1 and μ_2 defined over Δ, one can define the operation of the restricted union of two effective vague soft sets $(\Psi_{1\mu_1}, \Upsilon_1)$ and $(\Psi_{2\mu_2}, \Upsilon_2)$ on the given initial universal set Ξ as a new resulting effective vague soft set $(\Psi^{U_R}_{\mu^{U_R}}, \Upsilon^{U_R})$, taking into account that $\mu^{U_R} = \{(\delta, [\alpha^{U_R}(\delta), 1-\beta^{U_R}(\delta)]), \delta \in \Delta\}$. The effective truth membership union value $\alpha^{U_R} : \Delta \to [0,1]$ and the effective false membership union value $\beta^{U_R} : \Delta \to [0,1]$ are then determined by $[\alpha^{U_R}, 1-\beta^{U_R}] = [\alpha_1, 1-\beta_1]\widetilde{\cup_R}[\alpha_2, 1-\beta_2]$, in which α_1, α_2, β_1 and β_2 are, respectively, the effective truth membership values and the effective false membership values related to μ_1 and μ_2. Furthermore, for two vague soft sets (Ψ_1, Υ_1) and (Ψ_2, Υ_2), One can obtain that $(\Psi, \Upsilon)^{U_R} = (\Psi^{U_R}, \Upsilon^{U_R}) = (\Psi_1, \Upsilon_1)\widetilde{\cup_R}(\Psi_2, \Upsilon_2)$, where $\Upsilon^{U_R} = \Upsilon_1 \cap \Upsilon_2 \neq \phi$ and $\mu_1 \cap \mu_2 \neq \phi$. For each $\xi \in \Xi$, the following formulas can be used to calculate α^{U_R} and β^{U_R} as the following: $\alpha_{\mu_\xi^{U_R}}(\delta) = \max\{\alpha_{\mu_{1\xi}}(\delta), \alpha_{\mu_{2\xi}}(\delta)\}$ and $\beta_{\mu_\xi^{U_R}}(\delta) = \min\{\beta_{\mu_{1\xi}}(\delta), \beta_{\mu_{2\xi}}(\delta)\}$, for each $\delta \in \Delta$. Additionally, the formula that compute $(\Psi^{U_R}, \Upsilon^{U_R}) = (\Psi_1, \Upsilon_1)\widetilde{\cup_R}(\Psi_2, \Upsilon_2)$ can be calculated, for each $\upsilon \in \Upsilon^{U_R}$ and $\xi \in \Xi$ as the following: $(\Psi^{U_R}, \Upsilon^{U_R}) = \{(\upsilon, \{\xi, [\max\{\tau_{\Psi_1(\upsilon)}(\xi)_\mu, \tau_{\Psi_2(\upsilon)}(\xi)_\mu\}, \min\{\eta_{\Psi_1(\upsilon)}(\xi)_\mu, \eta_{\Psi_1(\upsilon)}(\xi)_\mu\}]\}), \xi \in \Xi\}$.*

Definition 4.3. ***(Intersection of two effective vague soft sets)*** *Assuming that Ξ serves as an initial universal set, with supposing that Υ_1 and Υ_2 serve as two sets of parameters, and also letting μ_1 and μ_2 represent two sets of effective parameters over Δ, we have the intersection operation of two effective vague soft sets $(\Psi_{1\mu_1}, \Upsilon_1)$ and $(\Psi_{2\mu_2}, \Upsilon_2)$ on an initial universal set Ξ as another resulting effective vague soft set $(\Psi^I_{\mu^I}, \Upsilon^I)$, keeping into account that $\mu^I = \{(\delta, [\alpha^I(\delta), 1-\beta^I(\delta)]), \delta \in \Delta\}$. The effective truth membership intersection value $\alpha^I : \Delta \to [0,1]$ as well as the effective false membership intersection value $\beta^I : \Delta \to [0,1]$ can be determined by $[\alpha^I, 1-\beta^I] = [\alpha_1, 1-\beta_1]\widetilde{\cap}[\alpha_2, 1-\beta_2]$, taking into account that α_1, α_2, β_1 and β_2 represent the effective truth membership values as well as the effective false membership values associated with μ_1 and μ_2, respectively. Moreover, for two vague soft sets (Ψ_1, Υ_1) and (Ψ_2, Υ_2), we have $(\Psi, \Upsilon)^I = (\Psi^I, \Upsilon^I) = (\Psi_1, \Upsilon_1)\widetilde{\cap}(\Psi_2, \Upsilon_2)$, where $\Upsilon^I = \Upsilon_1 \cup \Upsilon_2$. For each $\xi \in \Xi$, the following formulas can be used to calculate α^I as well as β^I, as follows:*

$$\alpha_{\mu^I_\xi}(\delta) = \begin{cases} \alpha_{\mu_{1\xi}}(\delta), & if\ \delta \in \mu_1 - \mu_2, \\ \alpha_{\mu_{2\xi}}(\delta), & if\ \delta \in \mu_2 - \mu_1, \\ \min\{\alpha_{\mu_{1\xi}}(\delta), \alpha_{\mu_{2\xi}}(\delta)\}, & if\ \delta \in \mu_1 \cap \mu_2, \end{cases} \tag{4.4}$$

$$\beta_{\mu^I_\xi}(\delta) = \begin{cases} \beta_{\mu_{1\xi}}(\delta), & if\ \delta \in \mu_1 - \mu_2, \\ \beta_{\mu_{2\xi}}(\delta), & if\ \delta \in \mu_2 - \mu_1, \\ \max\{\beta_{\mu_{1\xi}}(\delta), \beta_{\mu_{2\xi}}(\delta)\}, & if\ \delta \in \mu_1 \cap \mu_2, \end{cases} \tag{4.5}$$

for all $\delta \in \Delta$. Moreover, the formula used to compute $(\Psi^I, \Upsilon^I) = (\Psi_1, \Upsilon_1)\widetilde{\cap}(\Psi_2, \Upsilon_2)$ can be determined, for all $\xi \in \Xi$ and for all $\upsilon \in \Upsilon^I$ as the following:

$$(\Psi^I, \Upsilon^I) = \begin{cases} \{(\upsilon, \{\xi, [\tau_{\Psi_1(\upsilon)}(\xi)_\mu, 1-\eta_{\Psi_1(\upsilon)}(\xi)_\mu]\}), \xi \in \Xi\}, & if\ \upsilon \in \Upsilon_1 - \Upsilon_2, \\ \{(\upsilon, \{\xi, [\tau_{\Psi_2(\upsilon)}(\xi)_\mu, 1-\eta_{\Psi_2(\upsilon)}(\xi)_\mu]\}), \xi \in \Xi\}, & if\ \upsilon \in \Upsilon_2 - \Upsilon_1, \\ \{(\upsilon, \{\xi, [\min\{\tau_{\Psi_1(\upsilon)}(\xi)_\mu, \tau_{\Psi_2(\upsilon)}(\xi)_\mu\}, & \\ \quad \max\{\eta_{\Psi_1(\upsilon)}(\xi)_\mu, \eta_{\Psi_1(\upsilon)}(\xi)_\mu\}]\}), \xi \in \Xi\}, & if\ \upsilon \in \Upsilon_1 \cap \Upsilon_2. \end{cases} \tag{4.6}$$

Example 4.2. *One can compute the intersection of the two effective sets stated in Example (4.1), say $\mu^I = \mu_1 \widetilde{\cap} \mu_2$, by applying Formula 4.4 and Formula 4.5 from Definition (4.3), as the following:*

$$\mu^I(\xi_1) = \{(\delta_1, [0.4, 0.5]), (\delta_2, [0.2, 0.3]), (\delta_3, [0.4, 0.5]), (\delta_4, [0.5, 0.6])\},$$

$$\mu^I(\xi_2) = \{(\delta_1, [0.1, 0.2]), (\delta_2, [0, 0.1]), (\delta_3, [0.9, 1]), (\delta_4, [0, 0])\}.$$

Also, compute the intersection of the two vague soft sets (Ψ_1, Υ_1) and (Ψ_2, Υ_2) stated in Example 4.1, say $(\Psi, \Upsilon)^I = (\Psi^I, \Upsilon^I)$, where $\Upsilon^I = \Upsilon_1 \cup \Upsilon_2$, by making use of Formula 4.6 from Definition (4.3), as shown below:

$$(\Psi^I, \Upsilon^I) = \{(\upsilon_1, \{(\xi_1, [0.5, 0.6]), (\xi_2, [0, 0.1]), (\upsilon_2, \{(\xi_1, [0.9, 1]), (\xi_2, [0.6, 0.7]), (\upsilon_3, \{(\xi_1, [0.1, 0.2]), (\xi_2, [0, 0])\})\},$$

Then, computing effective intersection of vague soft sets $(\Psi^I_{\mu^I}, \Upsilon^I)$ using Formula 3.1 and Formula 3.2 from Definition (3.2), results:

$$(\Psi^I_{\mu^I}, \Upsilon^I) = \{(\upsilon_1, \{(\xi_1, [0.6875, 0.79]), (\xi_2, [0.25, 0.3925]), (\upsilon_2, \{(\xi_1, [0.9375, 1]), (\xi_2, [0.7, 0.7975]), (\upsilon_3, \{(\xi_1, [0.13, 0.2]), (\xi_2, [0, 0])\})\}.$$

Definition 4.4. ***(Restricted intersection of two effective vague soft sets)*** *By letting the set Ξ represent the initial universal set, as well as assuming that Υ_1 and Υ_2 are two sets of parameters, along with considering that μ_1 and μ_2 are two sets of effective parameters over Δ, one can determine the operation of the restricted intersection of two effective vague soft sets $(\Psi_{1\mu_1}, \Upsilon_1)$ and $(\Psi_{2\mu_2}, \Upsilon_2)$ on the initial universal set Ξ as another resulting effective vague soft set $(\Psi^{I_R}_{\mu^{I_R}}, \Upsilon^{I_R})$, in which $\mu^{I_R} = \{(\delta, [\alpha^{I_R}(\delta), 1 - \beta^{I_R}(\delta)]), \delta \in \Delta\}$. The effective truth membership union value $\alpha^{I_R} : \Delta \to [0, 1]$ as well as the effective false membership union value $\beta^{I_R} : \Delta \to [0, 1]$ can be determined by $[\alpha^{I_R}, 1 - \beta^{I_R}] = [\alpha_1, 1 - \beta_1] \widetilde{\cap_R} [\alpha_2, 1 - \beta_2]$, keeping into account that α_1, α_2, β_1 and β_2 are the effective truth membership values as well as the effective false membership values related, respectively, with μ_1 and μ_2. Furthermore, for two vague soft sets (Ψ_1, Υ_1) and (Ψ_2, Υ_2), one has $(\Psi, \Upsilon)^{I_R} = (\Psi^{I_R}, \Upsilon^{I_R}) = (\Psi_1, \Upsilon_1) \widetilde{\cap_R} (\Psi_2, \Upsilon_2)$, where $\Upsilon^{I_R} = \Upsilon_1 \cap \Upsilon_2 \neq \phi$ and $\mu_1 \cap \mu_2 \neq \phi$. One can present the formulas for computing α^{I_R} and β^{I_R}, for each $\xi \in \Xi$, as follows $\alpha_{\mu^{I_R}_{\xi}}(\delta) = \min\{\alpha_{\mu_{1\xi}}(\delta), \alpha_{\mu_{2\xi}}(\delta)\}$ and $\beta_{\mu^{I_R}_{\xi}}(\delta) = \max\{\beta_{\mu_{1\xi}}(\delta), \beta_{\mu_{2\xi}}(\delta)\}$, for each $\delta \in \Delta$. Moreover, the formula that can be used to compute $(\Psi^{I_R}, \Upsilon^{I_R}) = (\Psi_1, \Upsilon_1) \widetilde{\cap_R} (\Psi_2, \Upsilon_2)$ may be given, for each $\upsilon \in \Upsilon^{I_R}$ and $\xi \in \Xi$, by the following: $(\Psi^{I_R}, \Upsilon^{I_R}) = \{(\upsilon, \{\xi, [\min\{\tau_{\Psi_1(\upsilon)}(\xi)_\mu, \tau_{\Psi_2(\upsilon)}(\xi)_\mu\}, \max\{\eta_{\Psi_1(\upsilon)}(\xi)_\mu, \eta_{\Psi_1(\upsilon)}(\xi)_\mu\}]\}), \xi \in \Xi\}$.*

Definition 4.5. ***(Subset of effective vague soft set)*** *Letting Ξ represents an initial universal set, considering that Υ_1 and Υ_2 are two sets of parameters, supposing that μ_1 and μ_2 are two effective sets of parameters over Δ, and assuming that $(\Psi_{1\mu_1}, \Upsilon_1)$ and $(\Psi_{2\mu_2}, \Upsilon_2)$ are two effective vague soft sets on a universal set Ξ, one can call $(\Psi_{1\mu_1}, \Upsilon_1)$ an effective vague soft subset of $(\Psi_{2\mu_2}, \Upsilon_2)$ if, for all $\xi \in \Xi$, the following conditions hold:*

1. *$\mu_1 \subseteq \mu_2$, i.e., $\alpha_{\mu_{1\xi}}(\delta) \leq \alpha_{\mu_{2\xi}}(\delta)$ and $\beta_{\mu_{2\xi}}(\delta) \leq \beta_{\mu_{1\xi}}(\delta)$, for all $\delta \in \Delta$,*
2. *$\Upsilon_1 \subseteq \Upsilon_2$, that is to say that the ordinary inclusion (usual subset) is satisfied, and*
3. *For every $\upsilon \in \Upsilon_1$, $\Psi_1(\upsilon) \subseteq \Psi_2(\upsilon)$, i.e., $\tau_{\Psi_1(\upsilon)}(\xi) \leq \tau_{\Psi_2(\upsilon)}(\xi)$ as well as $\eta_{\Psi_2(\upsilon)}(\xi) \leq \eta_{\Psi_1(\upsilon)}(\xi)$.*

In this context, the following formula can be written $(\Psi_{1\mu_1}, \Upsilon_1)\widetilde{\subseteq}(\Psi_{2\mu_2}, \Upsilon_2)$*. In addition, one can call* $(\Psi_{2\mu_2}, \Upsilon_2)$ *an effective vague soft superset of* $(\Psi_{1\mu_1}, \Upsilon_1)$*, labeled as* $(\Psi_{2\mu_2}, \Upsilon_2)\widetilde{\supseteq}(\Psi_{1\mu_1}, \Upsilon_1)$*.*

Definition 4.6. ***(Equality of two effective vague soft sets)*** *Assuming that* Ξ *represents the initial universal set, with considering that* Υ_1 *and* Υ_2 *are two sets of parameters, and supposing that* μ_1 *and* μ_2 *are two sets of effective parameters over* Δ*, one can call the two effective vague soft sets* $(\Psi_{1\mu_1}, \Upsilon_1)$ *and* $(\Psi_{2\mu_2}, \Upsilon_2)$ *on an initial universe* Ξ *effective vague soft equal provided that each one of them is effective vague soft subset of the other one in the same manner showed in Definition (4.5), i.e.,* $(\Psi_{1\mu_1}, \Upsilon_1)\widetilde{\subseteq}(\Psi_{2\mu_2}, \Upsilon_2)$ *and* $(\Psi_{2\mu_2}, \Upsilon_2)\widetilde{\subseteq}(\Psi_{1\mu_1}, \Upsilon_1)$*.*

Definition 4.7. ***(Complement of effective vague soft set)*** *If we have the set of parameters, namely* Υ*, and the set of effective parameters, namely* Δ*, then we can obtain the operation of the complement of any effective vague soft set* (Ψ_μ, Υ) *on an initial universal set* Ξ *symbolized by* $(\Psi_\mu, \Upsilon)^c = (\Psi^c_{\mu^c}, \Upsilon)$*. One can have* $\mu^c = \{(\delta, [\alpha^c(\delta), 1 - \beta^c(\delta)]), \delta \in \Delta\}$*, keeping into account that* $\alpha^c : \Delta \to [0, 1]$ *and* $\beta^c : \Delta \to [0, 1]$*. The following two formulas can determine* α^c *and* β^c*, respectively, as follows:* $\alpha_{\mu^c_\xi}(\delta) = \beta_{\mu_\xi}(\delta)$ *as well as* $\beta_{\mu^c_\xi}(\delta) = \alpha_{\mu_\xi}(\delta)$*, for every* $\xi \in \Xi$ *and for every* $\delta \in \Delta$*. This means that,* $\mu^c = \{(\delta, [\beta(\delta), 1 - \alpha(\delta)]), \delta \in \Delta\}$*. Moreover, one can determine* $\Psi^c : \Upsilon \to \mathcal{V}(\Xi)$ *by the following two formulas:* $\tau_{\Psi^c_\Upsilon(\upsilon)}(\xi)_{\mu^c} = \eta_{\Psi_\Upsilon(\upsilon)}(\xi)_{\mu^c}$ *as well as* $\eta_{\Psi^c_\Upsilon(\upsilon)}(\xi)_{\mu^c} = \tau_{\Psi_\Upsilon(\upsilon)}(\xi)_{\mu^c}$*, for each* $\xi \in \Xi$ *and for each* $\upsilon \in \Upsilon$*. That is to say that,* $(\Psi_\mu, \Upsilon)^c = \{(\upsilon, \{\xi, [\eta_{\Psi_\Upsilon(\upsilon)}(\xi)_{\mu^c}, 1 - \tau_{\Psi_\Upsilon(\upsilon)}(\xi)_{\mu^c}]\}) : \upsilon \in \Upsilon, \xi \in \Xi\}$*.*

Remark 4.1. *The aforementioned definitions can be expanded to include a family of sets in addition to the scenario of just two sets. The formulas describing those definitions are readily deduced, and an example can be provided for each.*

5. Effective vague soft properties

Many important properties of effective vague soft sets are discussed in this section, including distributive, absorption, commutative, associative, and De Morgan's properties. In this context, the following theorems can be proved by making use of formulas and operations declared in Definitions (3.3), (3.4), (4.1), (4.3), (4.4), (4.5) and (4.7) of section (4).

Theorem 5.1. *Assuming that* Ξ *represents the initial universal set, along with supposing that* Υ *serves as a set of parameters, as well as considering that* (Ψ_μ, Υ) *is an effective vague soft set on the initial universal set* Ξ*, produced by an effective set* μ*, and assuming that* (Φ_μ, Υ) *and* (C_μ, Υ) *are the null effective vague soft set and the absolute effective vague soft set, respectively, on a common initial universal set* Ξ*, we can find the following hold:*

1. $(\Psi_\mu, \Upsilon)\tilde{\cup}(\Psi_\mu, \Upsilon) = (\Psi_\mu, \Upsilon)\tilde{\cap}(\Psi_\mu, \Upsilon) = (\Psi_\mu, \Upsilon)$.
2. $(\Psi_\mu, \Upsilon)\tilde{\cap}(C_\mu, \Upsilon) = (\Psi_\mu, \Upsilon)\tilde{\cup}(\Phi_\mu, \Upsilon) = (\Psi_\mu, \Upsilon)$.
3. $(\Psi_\mu, \Upsilon)\tilde{\cup}(C_\mu, \Upsilon) = (C_\mu, \Upsilon)\tilde{\cup}(\Phi_\mu, \Upsilon) = (C_\mu, \Upsilon)$.
4. $(\Psi_\mu, \Upsilon)\tilde{\cap}(\Phi_\mu, \Upsilon) = (C_\mu, \Upsilon)\tilde{\cap}(\Phi_\mu, \Upsilon) = (\Phi_\mu, \Upsilon)$.

Proof. Our mission is to prove (4). By following the same method, we can prove (1), (2) and (3). For (4) now, we want to prove that $(C_\mu, \Upsilon)\tilde{\cap}(\Phi_\mu, \Upsilon) = (\Phi_\mu, \Upsilon)$. In addition, $(\Psi_\mu, \Upsilon)\tilde{\cap}(\Phi_\mu, \Upsilon) = (\Phi_\mu, \Upsilon)$ can be proved by using the same technique. Depending on Definitions (3.3) and (3.4), we have $(C_\mu, \Upsilon) = \{(\upsilon, \{\xi, [1,1]\}) : \upsilon \in \Upsilon, \xi \in \Xi\}$ as well as $(\Phi_\mu, \Upsilon) = \{(\upsilon, \{\xi, [0,0]\}) : \upsilon \in \Upsilon, \xi \in \Xi\}$, respectively. Let's suppose, for $\Upsilon = \Upsilon \cup \Upsilon = \Upsilon$, that

$$\begin{aligned}(C_\mu, \Upsilon)\tilde{\cap}(\Phi_\mu, \Upsilon) &= (\Psi_\mu, \Upsilon)\\ &= \{(\upsilon, \{\xi, [\tau_{\Psi(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi(\upsilon)}(\xi)_\mu]\}) : \upsilon \in \Upsilon, \xi \in \Xi\}\\ &= \{(\upsilon, \{\xi, [\min\{1,0\}_\mu, \min\{1,0\}_\mu]\}) : \upsilon \in \Upsilon, \xi \in \Xi\}\\ &= \{(\upsilon, \{\xi, [0,0]\}) : \upsilon \in \Upsilon, \xi \in \Xi\} = (\Phi_\mu, \Upsilon).\end{aligned}$$

Therefore, the third case in Definition (4.3) is true depending on satisfying for $\upsilon \in \Upsilon \cap \Upsilon = \Upsilon$. But, for the first case as well as the second case, because of $\upsilon \in \Upsilon - \Upsilon = \phi$, we have no parameters.

Theorem 5.2. *Supposing that Ξ serves as an initial universal set, along with considering that Υ_1 and Υ_2 represent two sets of parameters, as well as assuming that $(\Psi_{1\mu}, \Upsilon_1)$ and $(\Psi_{2\mu}, \Upsilon_2)$ are two effective vague soft sets, for a common effective set μ, on a universal set Ξ, one obtains the absorption properties satisfied as follows:*

1. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}((\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R(\Psi_{2\mu}, \Upsilon_2)) = (\Psi_{1\mu}, \Upsilon_1)$.
2. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R((\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2)) = (\Psi_{1\mu}, \Upsilon_1)$.

Proof. In order to prove (1), suppose first that

$$(\Psi_{1\mu}, \Upsilon_1) = \{(\upsilon, \{\xi, [\tau_{\Psi_1(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi_1(\upsilon)}(\xi)_\mu]\}) : \upsilon \in \Upsilon_1, \xi \in \Xi\},$$

$$(\Psi_{2\mu}, \Upsilon_2) = \{(\upsilon, \{\xi, [\tau_{\Psi_2(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi_2(\upsilon)}(\xi)_\mu]\}) : \upsilon \in \Upsilon_2, \xi \in \Xi\}.$$

According to Definition (4.1), we have to show that (1) is satisfied for all sequel three cases:
(i) Given $\upsilon \in \Upsilon_1 - \Upsilon_2$, therefore, from Definition (4.4), we have:

$$\begin{aligned}(\Psi_{3\mu}, \Upsilon_3) &= (\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R(\Psi_{2\mu}, \Upsilon_2)\\ &= \{(\upsilon, \{\xi, [\tau_{\Psi_3(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi_3(\upsilon)}(\xi)_\mu]\}) : \upsilon \in \Upsilon_1 - \Upsilon_2, \xi \in \Xi\} = \phi.\end{aligned}$$

(ii) If $\upsilon \in \Upsilon_2 - \Upsilon_1$, then we obtain from Definition (4.4) that:

$$\begin{aligned}(\Psi_{3\mu}, \Upsilon_3) &= (\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R(\Psi_{2\mu}, \Upsilon_2)\\ &= \{(\upsilon, \{\xi, [\tau_{\Psi_3(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi_3(\upsilon)}(\xi)_\mu]\}) : \upsilon \in \Upsilon_2 - \Upsilon_1, \xi \in \Xi\} = \phi.\end{aligned}$$

Then, for (i) and (ii), by using (3) from Theorem (5.1), we have:

$$(\Psi_{4\mu}, \Upsilon_4) = (\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{3\mu}, \Upsilon_3) = (\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}\phi = (\Psi_{1\mu}, \Upsilon_1).$$

(iii) When $\upsilon \in \Upsilon_1 \cap \Upsilon_2$, then we obtain from Definition (4.4) that:

$$\begin{aligned}&(\Psi_{3\mu}, \Upsilon_3)\\ &= (\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R(\Psi_{2\mu}, \Upsilon_2)\\ &= \{(\upsilon, \{\xi, [\tau_{\Psi_3(\upsilon)}(\xi)_\mu, 1 - \eta_{\Psi_3(\upsilon)}(\xi)_\mu]\}) : \upsilon \in \Upsilon_1 \cap \Upsilon_2, \xi \in \Xi\}\\ &= \{(\upsilon, \{\xi, [\min\{\tau_{\Psi_1(\upsilon)}(\xi)_\mu, \tau_{\Psi_2(\upsilon)}(\xi)_\mu\}, \max\{\eta_{\Psi_1(\upsilon)}(\xi)_\mu, \eta_{\Psi_2(\upsilon)}(\xi)_\mu\}]\}) : \upsilon \in \Upsilon_1 \cap \Upsilon_2, \xi \in \Xi\}.\end{aligned}$$

Thus, from Definition (4.1), one can obtain:

$$\begin{aligned}(\Psi_{4\mu}, \Upsilon_4) &= (\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{3\mu}, \Upsilon_3)\\ &= \{(\upsilon, \{\xi, [\max\{\eta_{\Psi_{1(\upsilon)}}(\xi)_\mu, \min\{\eta_{\Psi_{1(\upsilon)}}(\xi)_\mu, \eta_{\Psi_{2(\upsilon)}}(\xi)_\mu\}\},\\ &\quad \min\{\eta_{\Psi_{1(\upsilon)}}(\xi)_\mu, \max\{\eta_{\Psi_{1(\upsilon)}}(\xi)_\mu, \eta_{\Psi_{2(\upsilon)}}(\xi)_\mu\}\}]\}) : \upsilon \in \Upsilon_1 \cap \Upsilon_2, \xi \in \Xi\}\\ &= (\Psi_{1\mu}, \Upsilon_1).\end{aligned}$$

In this context, (2) can be proved directly as (1).

Corollary 5.1. *Considering that Ξ is an initial universal set, as well as supposing that Υ_1 and Υ_2 are two sets of parameters, we get, for two effective vague soft sets generated by a common effective set μ on a common initial universal set $(\Psi_{1\mu}, \Upsilon_1)$ and $(\Psi_{2\mu}, \Upsilon_2)$, that:*

$$(\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}((\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R(\Psi_{2\mu}, \Upsilon_2)) = (\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R((\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2)) = (\Psi_{1\mu}, \Upsilon_1).$$

Proof. It can easily be proved by following the same steps of proofing of Theorem (5.2).

Theorem 5.3. *Given that Ξ serves as an initial universal set, along with considering that Υ_1 and Υ_2 represent two sets of parameters, as well as assuming that there is a common effective set μ, related to two effective vague soft sets, namely $(\Psi_{1\mu}, \Upsilon_1)$ and $(\Psi_{2\mu}, \Upsilon_2)$, we have the commutative property is satisfied as the following:*

1. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}(\Psi_{2\mu}, \Upsilon_2) = (\Psi_{2\mu}, \Upsilon_2)\tilde{\cap}(\Psi_{1\mu}, \Upsilon_1)$.
2. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2) = (\Psi_{2\mu}, \Upsilon_2)\tilde{\cup}(\Psi_{1\mu}, \Upsilon_1)$.

Proof. It is easy using Definitions (4.1) and (4.3) similar to Theorem (5.2).

Proposition 5.1. *Assuming that Ξ represents an initial universal set, as well as considering that Υ_1 and Υ_2 serve as two sets of parameters, along with supposing that there is a common effective set μ, related to two effective vague soft sets, namely $(\Psi_{1\mu}, \Upsilon_1)$ and $(\Psi_{2\mu}, \Upsilon_2)$, and if $(\Psi_{1\mu}, \Upsilon_1)\tilde{\subseteq}(\Psi_{2\mu}, \Upsilon_2)$, we have:*

1. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}_R(\Psi_{2\mu}, \Upsilon_2) = (\Psi_{1\mu}, \Upsilon_1)$.
2. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2) = (\Psi_{2\mu}, \Upsilon_2)$.

Proof. It is direct like Theorem (5.2) applying Definitions (4.2) and (4.4).

Theorem 5.4. *Given that Ξ serves as an initial universal set, along with supposing that Υ_1, Υ_2 and Υ_3 represent three sets of parameters, as well as assuming that $(\Psi_{1\mu}, \Upsilon_1)$, $(\Psi_{2\mu}, \Upsilon_2)$ and $(\Psi_{3\mu}, \Upsilon_3)$ are effective vague soft sets, for a common effective set μ, on a common initial universal set Ξ, one obtain the associative laws as well as the distributive laws are satisfied, respectively, as below:*

1. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}((\Psi_{2\mu}, \Upsilon_2)\tilde{\cap}(\Psi_{3\mu}, \Upsilon_3)) = ((\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}(\Psi_{2\mu}, \Upsilon_2))\tilde{\cap}(\Psi_{3\mu}, \Upsilon_3)$.
2. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}((\Psi_{2\mu}, \Upsilon_2)\tilde{\cup}(\Psi_{3\mu}, \Upsilon_3)) = ((\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2))\tilde{\cup}(\Psi_{3\mu}, \Upsilon_3)$.
3. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}((\Psi_{2\mu}, \Upsilon_2)\tilde{\cup}(\Psi_{3\mu}, \Upsilon_3)) = ((\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}(\Psi_{2\mu}, \Upsilon_2))\tilde{\cup}((\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}(\Psi_{3\mu}, \Upsilon_3))$.
4. $(\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}((\Psi_{2\mu}, \Upsilon_2)\tilde{\cap}(\Psi_{3\mu}, \Upsilon_3)) = ((\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2))\tilde{\cap}((\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{3\mu}, \Upsilon_3))$.

Proof. Like Theorem (5.2), using Definitions (4.1) and (4.3), it can be proved.

Theorem 5.5. *The following De Morgan's laws are satisfied for any two effective vague soft sets* $(\Psi_{1\mu}, \Upsilon_1)$ *and* $(\Psi_{2\mu}, \Upsilon_2)$*, for a common effective set* μ*, on a common universal set* Ξ*, where* Υ_1 *and* Υ_2 *represent two sets of parameters:*

1. $((\Psi_{1\mu}, \Upsilon_1)\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2))^c = (\Psi_{1\mu}, \Upsilon_1)^c\tilde{\cap}(\Psi_{2\mu}, \Upsilon_2)^c$.
2. $((\Psi_{1\mu}, \Upsilon_1)\tilde{\cap}(\Psi_{2\mu}, \Upsilon_2))^c = (\Psi_{1\mu}, \Upsilon_1)^c\tilde{\cup}(\Psi_{2\mu}, \Upsilon_2)^c$.

Proof. It can be proved using Theorem (5.2), and Definitions (4.3), (4.1) and (4.7).

6. Decision-making techniques

The purpose of this section is to address a practical issue related to diagnosis, whether it's in the medical or educational domain. We introduce an algorithm for diagnosis that leverages the effective vague soft set. This technique can be implemented using matrix operations and their associated properties to make accurate diagnoses. These diagnoses involve determining which individuals, whether they are students or patients, are suitable for a particular education level or are afflicted by a specific ailment, respectively.
Furthermore, we provide a detailed case study that exemplifies the process of medical diagnosis. We've organized the steps of this method within a framework of matrix operations, simplifying the computational aspects. Additionally, to expedite and enhance the accuracy of tasks such as matrix multiplication and effective membership calculations, we utilize the *Wolfram Mathematica*® program. This choice of tool helps ensure that these computations are not only faster but also more precise and easier to perform.

6.1. Methodology

Let's assume we have a set of n individuals, which could be students, patients, or similar, denoted as $\Pi = \pi_1, \pi_2, ..., \pi_n$. Additionally, there is a set of m exams, symptoms, or similar factors, referred to as $\Upsilon = \upsilon_1, \upsilon_2, ..., \upsilon_m$. These factors relate to a well-defined set of k levels, diseases, or similar conditions, designated as $\Xi = \xi_1, \xi_2, ..., \xi_k$. Furthermore, $\Delta = \delta_1, \delta_2, ..., \delta_r$ can be considered as a set of r effective attributes or parameters specifically designed for any issue. In this context, the effective set μ is established based on the input and responses from the individuals (students, patients, etc.).
Moreover, the vague soft set (Ψ, Υ) is constructed through a process involving questioning individuals and subjecting them to analyses or tests done by experts. Additionally, based on expert documentation, the description of the levels or diseases and, respectively, their related exams or symptoms is created approximately by the vague soft set $(\sqsupset, \Upsilon)$. With these assumptions in place, the algorithm proceeds to determine which individuals are succeeding at what level of education or diagnosed with what disease. In this context, the first step involves computing the effective vague soft set (Ψ_μ, Υ) using Formula 3.1 and Formula 3.2 from Definition (3.2). This is done by combining the given vague soft set (Ψ, Υ) along with the provided effective set μ for the individuals. Subsequently, as a second step, we need to extract the matrix corresponding to the effective vague soft set, which consists of the interval values of membership of the relevant items, denoted as $\widetilde{\Psi_\mu}$. As a similar step, extracting the matrix corresponding to the given vague soft set, referred to as $\widetilde{\sqsupset}$ represents the third step. The fourth step

involves multiplying $\widetilde{\Psi_\mu}$ by $\widetilde{\Xi}$ (or by the transpose of $\widetilde{\Xi}$ if the problem conditions require it), resulting in the diagnosis matrix $\widetilde{D}$. This matrix serves as a link between the individuals (students, patients, etc.) and the levels, diseases, etc.
Finally, the fifth step is to determine the diagnosis for each individual from the diagnosis matrix $\widetilde{D}$. The best diagnosis for each individual is identified to be the maximum value in their respective row. In addition, Figure 2, for simplicity, provides a concise visual representation of the algorithm's steps in the form of a basic flowchart.

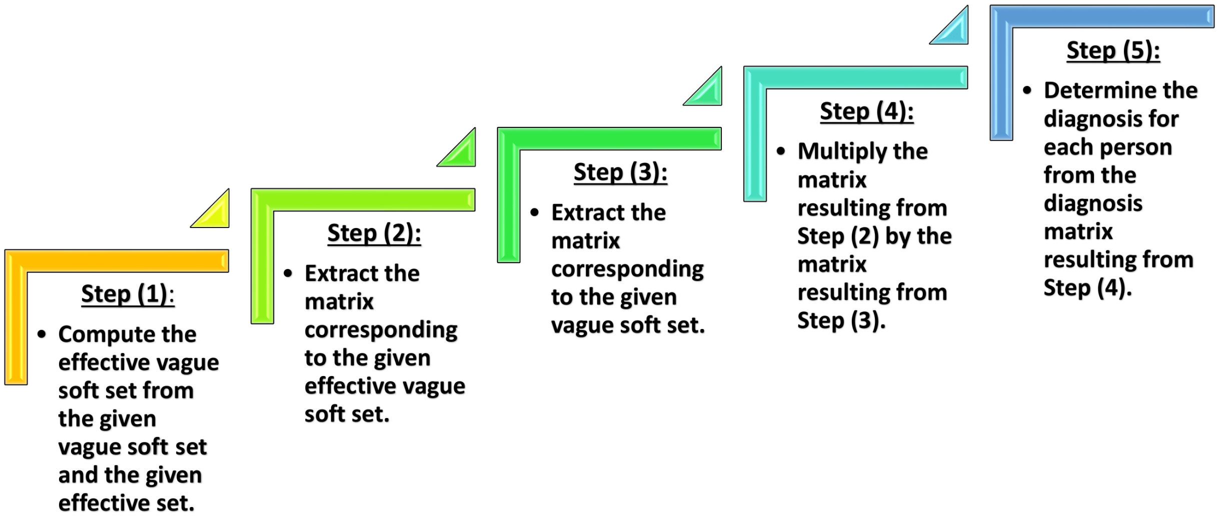


Figure 2. The proposed algorithm's steps.

6.2. Case study

The following part discusses an applied example to classify students into specializations (specialized sections). In this example, the aforementioned algorithmic steps are used to choose the most appropriate specialization at the next level for every student at a certain level. These students already have some grades in the subjects they have studied at the current level in college. In addition, we consider that every student may have some additional life skills that can make him/her qualified to study a certain specialization at the next level. This consideration makes the decision-making process more effective. Additionally, we utilize the *Wolfram Mathematica*® program to carry out any calculations required in each step, such as matrix multiplication and effective value computations.

Example 6.1. *Assume that we have a group of five students from the Architecture Department in the Faculty of Engineering representing the universal set* $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$. *They have passed the second year and they will be in the third year. Each student has to study subjects of only one specialized branch (section) from three offered sections that represent another universal set* $\Xi = \{\xi_1, \xi_2, \xi_3\}$, *where* ξ_1 *= Public Buildings,* ξ_2 *= Housing,* ξ_3 *= Urban Design according to their subject grades in the second year as well as their life skills. Let the five vital subjects they have studied in the second year representing the set of attributes be as follows:* $\Upsilon = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$, *where* υ_1 *= Theory of structures,* υ_2 *= Architectural Design,* υ_3 *= Execution Design,* υ_4 *= Technical systems in buildings, and* υ_5 *= History of Islamic Architecture. The following vague soft set* (Ψ, Υ) *represents second year* *subject grades of the students:*

$$\begin{aligned}(\Psi,\Upsilon)=\{&(\upsilon_1,\{(\pi_1,[0.9,1]),(\pi_2,[0.6,0.7]),(\pi_3,[0.5,0.6]),(\pi_4,[0.8,0.9]),(\pi_5,[0.7,0.8])\}),\\&(\upsilon_2,\{(\pi_1,[0.5,0.6]),(\pi_2,[0.8,0.9]),(\pi_3,[0.7,0.8]),(\pi_4,[1,1]),(\pi_5,[0.9,1])\}),\\&(\upsilon_3,\{(\pi_1,[1,1]),(\pi_2,[0.5,0.6]),(\pi_3,[0.8,0.9]),(\pi_4,[1,1]),(\pi_5,[0.9,1])\})\\&(\upsilon_4,\{(\pi_1,[0.6,0.7]),(\pi_2,[0.8,0.9]),(\pi_3,[0.5,0.6]),(\pi_4,[0.7,0.8]),(\pi_5,[1,1])\})\\&(\upsilon_5,\{(\pi_1,[0.7,0.8]),(\pi_2,[0.5,0.6]),(\pi_3,[0.8,0.9]),(\pi_4,[0.9,1]),(\pi_5,[0.5,0.6])\})\}.\end{aligned}$$

In addition, one can have, depending on expert documentation, another vague soft set $(\sqsupset,\Upsilon)$ that shows a preliminary explanation of the relation between the three sections (specialized branches) and the pre-studied subjects as follows:

$$\begin{aligned}(\sqsupset,\Upsilon)=\{&(\upsilon_1,\{(\xi_1,[0.5,0.6]),(\xi_2,[0.3,0.4]),(\xi_3,[0.1,0.2])\}),\\&(\upsilon_2,\{(\xi_1,[0.2,0.3]),(\xi_2,[0.7,0.8]),(\xi_3,[0.4,0.5])\}),\\&(\upsilon_3,\{(\xi_1,[0.8,0.9]),(\xi_2,[0.2,0.3]),(\xi_3,[0.6,0.7])\}),\\&(\upsilon_4,\{(\xi_1,[0.6,0.7]),(\xi_2,[0.4,0.5]),(\xi_3,[0.2,0.3])\}),\\&(\upsilon_5,\{(\xi_1,[0.4,0.5]),(\xi_2,[0.2,0.3]),(\xi_3,[0.8,0.9])\})\}.\end{aligned}$$

Moreover, suppose that $\Delta=\{\delta_1,\delta_2,\delta_3,\delta_4\}$ is a set of effective parameters, where δ_1 = the student has flexibility skill, δ_2 = the student has problem solving skills, δ_3 = the student has good communication skills, and δ_4 = the student has creativity. After talking to the students and subjecting them to some tests by experts, the effective set μ over Δ can be created, for $\pi_i, i=1,2,3,4,5$, as the following:

$$\mu(\pi_1)=\{(\delta_1,[0.3,0.4]),(\delta_2,[0.6,0.7]),(\delta_3,[0.2,0.3]),(\delta_4,[0.8,0.9])\},$$

$$\mu(\pi_2)=\{(\delta_1,[0.5,0.6]),(\delta_2,[0.4,0.5]),(\delta_3,[0.1,0.2]),(\delta_4,[0.7,0.8])\},$$

$$\mu(\pi_3)=\{(\delta_1,[0.8,0.9]),(\delta_2,[0.5,0.6]),(\delta_3,[0.4,0.5]),(\delta_4,[0.2,0.3])\},$$

$$\mu(\pi_4)=\{(\delta_1,[0.4,0.5]),(\delta_2,[0.2,0.3]),(\delta_3,[0.6,0.7]),(\delta_4,[0.3,0.4])\},$$

$$\mu(\pi_5)=\{(\delta_1,[0.7,0.8]),(\delta_2,[0.1,0.2]),(\delta_3,[0.8,0.9]),(\delta_4,[0.4,0.5])\}.$$

What is the most appropriate section for each student?

Solution. $Step(1)$: Compute the effective vague soft set (Ψ_μ,Υ), that determines the cases of the above students, as a first step, depending on calculations extracted from both Formula 3.1, as well as Formula 3.2 from Definition (3.2), as follows:

$$\begin{aligned}(\Psi_\mu,\Upsilon)=\{&(\upsilon_1,\{(\pi_1,[0.9475,1]),(\pi_2,[0.77,0.8575]),(\pi_3,[0.7375,0.83]),(\pi_4,[0.875,0.9475]),\\&(\pi_5,[0.85,0.92])\}),(\upsilon_2,\{(\pi_1,[0.7375,0.83]),(\pi_2,[0.885,0.9525]),(\pi_3,[0.8425,0.915]),\\&(\pi_4,[1,1]),(\pi_5,[0.95,1])\}),(\upsilon_3,\{(\pi_1,[1,1]),(\pi_2,[0.7125,0.81]),(\pi_3,[0.895,0.9575]),\\&(\pi_4,[1,1]),(\pi_5,[0.95,1])\}),(\upsilon_4,\{(\pi_1,[0.79,0.8725]),(\pi_2,[0.885,0.9525]),\\&(\pi_3,[0.7375,0.83]),(\pi_4,[0.8125,0.895]),(\pi_5,[1,1])\}),(\upsilon_5,\{(\pi_1,[0.8425,0.915]),\\&(\pi_2,[0.7125,0.81]),(\pi_3,[0.895,0.9575]),(\pi_4,[0.9375,1]),(\pi_5,[0.75,0.84])\})\}.\end{aligned}$$

$Step(2)$: Extract the matrix $\widetilde{\Psi_\mu}$ representing the students' grades corresponding to the effective vague soft set (Ψ_μ,Υ) as follows:

$$\widetilde{\Psi_\mu} = \begin{array}{c} \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \upsilon_1 & \upsilon_2 & \upsilon_3 & \upsilon_4 & \upsilon_5 \end{array} \\ \begin{pmatrix} [0.9475, 1] & [0.7375, 0.83] & [1, 1] & [0.79, 0.8725] & [0.8425, 0.915] \\ [0.77, 0.8575] & [0.885, 0.9525] & [0.7125, 0.81] & [0.885, 0.9525] & [0.7125, 0.81] \\ [0.7375, 0.83] & [0.8425, 0.915] & [0.895, 0.9575] & [0.7375, 0.83] & [0.895, 0.9575] \\ [0.875, 0.9475] & [1, 1] & [1, 1] & [0.8125, 0.895] & [0.9375, 1] \\ [0.85, 0.92] & [0.95, 1] & [0.95, 1] & [1, 1] & [0.75, 0.84] \end{pmatrix} \end{array}.$$

$Step(3)$: In addition, extract the matrix $\widetilde{\sqsupset}$ representing the section-subject relations from the membership values of the vague soft set $(\sqsupset, \Upsilon)$ as follows:

$$\widetilde{\sqsupset} = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \begin{array}{c} \begin{array}{ccccc} \upsilon_1 & \upsilon_2 & \upsilon_3 & \upsilon_4 & \upsilon_5 \end{array} \\ \begin{pmatrix} [0.5, 0.6] & [0.2, 0.3] & [0.8, 0.9] & [0.6, 0.7] & [0.4, 0.5] \\ [0.3, 0.4] & [0.7, 0.8] & [0.2, 0.3] & [0.4, 0.5] & [0.2, 0.3] \\ [0.1, 0.2] & [0.4, 0.5] & [0.6, 0.7] & [0.2, 0.3] & [0.8, 0.9] \end{pmatrix} \end{array}.$$

$Step(4)$: To obtain the student-section matrix (the diagnosis matrix), we take the transpose for $\widetilde{\sqsupset}$, then find the product $\widetilde{\Psi_\mu} \times \widetilde{\sqsupset}^T$ as the following:

$$\widetilde{D} = \widetilde{\Psi_\mu} \times \widetilde{\sqsupset}^T = \begin{array}{c} \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{array} \begin{array}{c} \begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array} \\ \begin{pmatrix} [2.23225, 2.81725] & [1.485, 2.07475] & [1.82175, 2.40025] \\ [1.948, 2.601] & [1.4895, 2.06725] & [1.6055, 2.2295] \\ [2.05375, 2.694] & [1.464, 2.0535] & [1.81125, 2.4045] \\ [2.3, 2.895] & [1.675, 2.2265] & [2, 2.558] \\ [2.275, 2.872] & [1.66, 2.22] & [1.835, 2.44] \end{pmatrix} \end{array}.$$

$Step(5)$: Finally, to arrive at a final diagnosis from the above resulting final diagnosis matrix $\widetilde{D}$, it is evident from Definition 2.1 that the first value, in each row, is the maximum value. This means that the values $[2.23225, 2.81725]$, $[1.948, 2.601]$, $[2.05375, 2.694]$, $[2.3, 2.895]$ and $[2.275, 2.872]$ are the maximum values for the students π_1, π_2, π_3, π_4 and π_5, respectively, corresponding to the first section ξ_1.

Then, we conclude that this group of students (π_1, π_2, π_3, π_4 and π_5) is expected to be qualified to study the subjects of the first section ξ_1, which is "Public Buildings" section. If more than one student

is expected to be qualified to study the subjects of exactly the same section, just like received in the example at hand, one can determine which student is more qualified to study the subjects of this section than the others.
The final resulting diagnosis matrix $\widetilde{D}$ above indicates the following order of students (considered as alternatives): $\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$. This order can be obtained by ordering the values of the first column (the column of the first section) from the greatest to the lowest value.
This shows that we must follow that priority if we need less than five students in this section. For other students, we can take the second maximum value in each row in the diagnosis matrix $\widetilde{D}$. For example, if there is no need for the last student (by priority) which is the second student π_2 in the "Public Buildings" section, then he/she can study in the third section ξ_3, which is "Urban Design" section. This is because $[1.6055, 2.2295] > [1.4895, 2.06725]$ according to Definition 2.1.
Additionally, the ranking of the students, according to priority, for the second section ξ_2 (Housing section) is $\pi_4 > \pi_5 > \pi_2 > \pi_1 > \pi_3$. Moreover, the students in this third section ξ_3 (Urban Design section) are arranged as follows, with priority: $\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$.

6.3. Comparative analysis

To compare decision-making in the effective vague soft set environment with earlier models or environments, a comparative analysis is carried out. We address the identical Example (6.1) using the previously established variations. The following is an overview of the comparative analysis's findings:

1. If we take the final determination under the fuzzy soft set, supplied by Yang *et al.* [42] employing algorithm stages of development, the results are shown below. The final diagnosis matrix D appears in the following manner:

$$D = \begin{array}{c} \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{array} \begin{array}{c} \begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array} \\ \begin{pmatrix} 2.14 & 1.33 & 1.73 \\ 1.74 & 1.36 & 1.34 \\ 1.8 & 1.31 & 1.7 \\ 2.31 & 1.71 & 2.03 \\ 2.27 & 1.6 & 1.69 \end{pmatrix} \end{array}.$$

So, from the final determined diagnosis matrix D, the greatest number for every student is the first value in each row, which is 2.14, 1.74, 1.8,2.31 and 2.27, for students π_1, π_2, π_3, π_4, and π_5, respectively. Then, We infer that this set of students ($\pi_1, \pi_2, \pi_3, \pi_4$, and π_5) are thought to be ready to study the subjects of the first section ξ_1, which is "Public Buildings" section.
Furthermore, the order of the students in the first section ξ_1, according to priority, is $\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$. Furthermore, for the second section ξ_2 (Housing section), the students are ranked as follows in order of priority: $\pi_4 > \pi_5 > \pi_2 > \pi_1 > \pi_3$. Additionally, the students are grouped with priority in this third section ξ_3 (Urban Design section) in the following order: $\pi_4 > \pi_1 > \pi_3 > \pi_5 > \pi_2$.

2. When the conclusion is made under the vague soft set described by Faried *et al.* [8] applying the procedure steps, we get the following findings. The final diagnosis matrix D is provided by:

$$D = \begin{array}{c} \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{array} \begin{array}{c} \begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array} \\ \begin{pmatrix} [1.99, 2.57] & [1.2, 1.77] & [1.57, 2.13] \\ [1.54, 2.16] & [1.26, 1.81] & [1.24, 1.82] \\ [1.65, 2.28] & [1.16, 1.72] & [1.55, 2.14] \\ [2.18, 2.8] & [1.6, 2.16] & [1.94, 2.52] \\ [2.05, 2.68] & [1.52, 2.1] & [1.57, 2.2] \end{pmatrix} \end{array}.$$

Then, based on the final diagnosis matrix D shown above, it can be determined that the first value in each row represents the maximum value for each of the students; π_1, π_2, π_3, π_4, and π_5 as follows: $[1.99, 2.57]$, $[1.54, 2.16]$, $[1.65, 2.28]$, $[2.18, 2.8]$ and $[2.05, 2.68]$, respectively. We, then, conclude that the students in this group (π_1, π_2, π_3, π_4, and π_5) are supposed to be able to study the topics covered in the first section ξ_1, which is the section on "Public Buildings".
The students in the first section ξ_1 are then arranged as follows, with priority: $\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$. Further, the students are prioritized as follows for the second section ξ_2 (Housing section): $\pi_4 > \pi_5 > \pi_2 > \pi_1 > \pi_3$. In addition, in this third section ξ_3 (the Urban Design section), the students are prioritized according to the following order: $\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$.

3. The outcomes are as follows if the final choice is reached utilizing the process steps and the effective fuzzy soft set, as proposed by Alkhazaleh [1]. The final diagnosis matrix D is obtained as follows:

$$D = \begin{array}{c} \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{array} \begin{array}{c} \begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array} \\ \begin{pmatrix} 2.36 & 1.59 & 1.95 \\ 2.13 & 1.58 & 1.72 \\ 2.2 & 1.59 & 1.93 \\ 2.41 & 1.76 & 2.08 \\ 2.43 & 1.73 & 1.94 \end{pmatrix} \end{array}.$$

Therefore, we have the largest value for each student, the first value, from this final diagnosis matrix D, which is respectively as 2.36, 2.13, 2.2, 2.41, and 2.43 for students π_1, π_2, π_3, π_4, and π_5. Subsequently, we deduce that this group of students ($\pi_1, \pi_2, \pi_3, \pi_4$, and π_5) is deemed prepared to study the topics addressed in the first section ξ_1, titled "Public Buildings".
Furthermore, the ranking of the students in the first section ξ_1 is $\pi_5 > \pi_4 > \pi_1 > \pi_3 > \pi_2$. In

addition, for the second section ξ_2 (Housing section), the students are ranked according to the following order: $\pi_4 > \pi_5 > \pi_1 = \pi_3 > \pi_2$. Moreover, the students are ranked in the following order in this third section ξ_3, which is the Urban Design section: $\pi_4 > \pi_1 > \pi_5 > \pi_3 > \pi_2$.

We conclude that there is an agreement from the above methods that the section nominated first for each student is the first section, but the priorities vary according to the needs of the sections for a certain number of students, and then we move to the next priority for each student according to the order of alternatives resulting from the different methods used.

We have for Example (6.1), Tables 1 and 2 summarize, respectively, the diagnosis values and the ranking order of the students for the first section; Public Buildings Section (ξ_1), using the four different models. In addition, Tables 3 and 4, respectively, describe the diagnosis values and the student ranking order for the second section; Housing section (ξ_2), using the four alternative models. Furthermore, Tables 5 and 6, which utilize the four different models, detail the diagnosis values and the student ranking order, respectively, for the third section; Urban Design section (ξ_3).

Table 1. Diagnosis values for the first section using different models on Example (6.1).

Models	Yang *et al.* [42]	Faried *et al.* [8]	Alkhazaleh [1]	Proposed model
π_1	2.14	[1.99,2.57]	2.36	[2.23225,2.81725]
π_2	1.74	[1.54,2.16]	2.13	[1.948,2.601]
π_3	1.8	[1.65,2.28]	2.2	[2.05375,2.694]
π_4	2.31	[2.18,2.8]	2.41	[2.3,2.895]
π_5	2.27	[2.05,2.68]	2.43	[2.275,2.872]

Table 2. Students' order for the first section using different models on Example (6.1).

Models	Public Buildings Section (ξ_1)
Yang *et al.* [42]	$\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$
Faried *et al.* [8]	$\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$
Alkhazaleh [1]	$\pi_5 > \pi_4 > \pi_1 > \pi_3 > \pi_2$
Proposed model	$\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$

Table 3. Diagnosis values for the second section using different models on Example (6.1).

Models	Yang *et al.* [42]	Faried *et al.* [8]	Alkhazaleh [1]	Proposed model
π_1	1.33	[1.2,1.77]	1.59	[1.485,2.07475]
π_2	1.36	[1.26,1.81]	1.58	[1.4895,2.06725]
π_3	1.31	[1.16,1.72]	1.59	[1.464,2.0535]
π_4	1.71	[1.6,2.16]	1.76	[1.675,2.2265]
π_5	1.6	[1.52,2.1]	1.73	[1.66,2.22]

Table 4. Students' order for the second section using different models on Example (6.1).

Models	Housing Section (ξ_2)
Yang *et al.* [42]	$\pi_4 > \pi_5 > \pi_2 > \pi_1 > \pi_3$
Faried *et al.* [8]	$\pi_4 > \pi_5 > \pi_2 > \pi_1 > \pi_3$
Alkhazaleh [1]	$\pi_4 > \pi_5 > \pi_1 = \pi_3 > \pi_2$
Proposed model	$\pi_4 > \pi_5 > \pi_2 > \pi_1 > \pi_3$

Table 5. Diagnosis values for the third section using different models on Example (6.1).

Models	Yang *et al.* [42]	Faried *et al.* [8]	Alkhazaleh [1]	Proposed model
π_1	1.73	[1.57,2.13]	1.95	[1.82175,2.40025]
π_2	1.34	[1.24,1.82]	1.72	[1.6055,2.2295]
π_3	1.7	[1.55,2.14]	1.93	[1.81125,2.4045]
π_4	2.03	[1.94,2.52]	2.08	[2,2.558]
π_5	1.69	[1.57,2.2]	1.94	[1.835,2.44]

Table 6. Students' order for the third section using different models on Example (6.1).

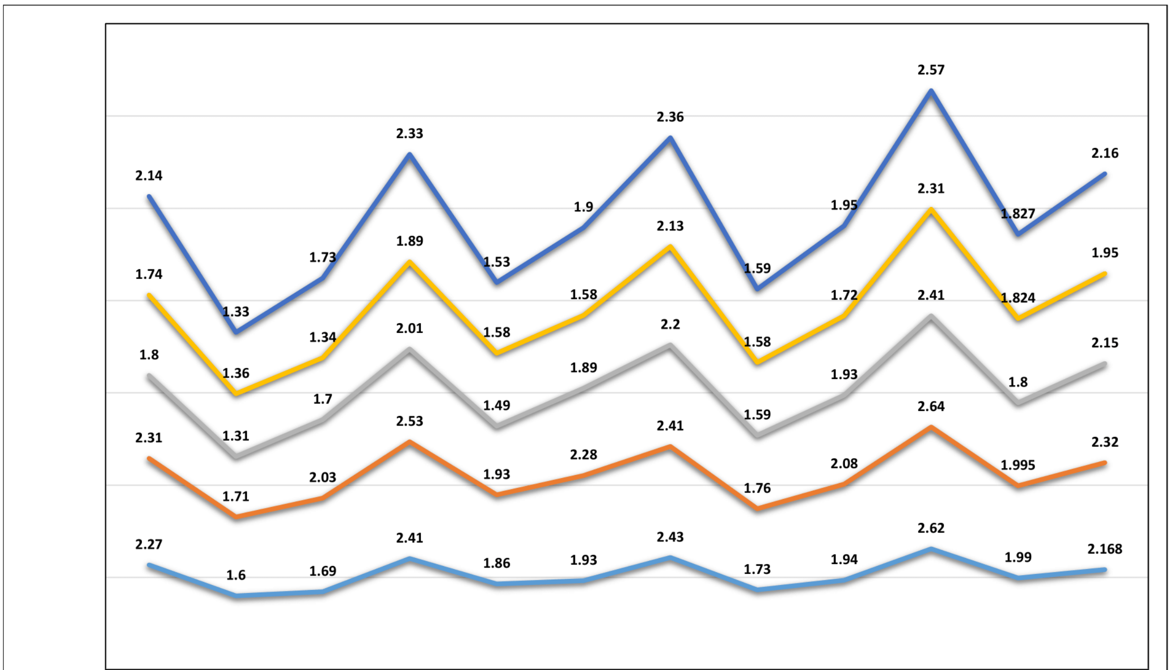
Models	Urban Design Section (ξ_3)
Yang *et al.* [42]	$\pi_4 > \pi_1 > \pi_3 > \pi_5 > \pi_2$
Faried *et al.* [8]	$\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$
Alkhazaleh [1]	$\pi_4 > \pi_1 > \pi_5 > \pi_3 > \pi_2$
Proposed model	$\pi_4 > \pi_5 > \pi_1 > \pi_3 > \pi_2$

Moreover, conversely to the above, one can also determine the priority order of sections for each student. This order can be obtained also from the diagnosis matrix by ordering the values of the row corresponding to each student from the greatest to the lowest value. That is to say that the best ranking of the three sections; Public Buildings Section (ξ_1), Housing Section (ξ_2), and Urban Design Section (ξ_3) as alternatives for each student can be found in Table 7.

Table 7. Sections' order for each student using different models on Example (6.1).

Models	Yang *et al.* [42]	Faried *et al.* [8]	Alkhazaleh [1]	Proposed model
π_1	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$
π_2	$\xi_1 > \xi_2 > \xi_3$	$\xi_1 > \xi_2 > \xi_3$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$
π_3	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$
π_4	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$
π_5	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$	$\xi_1 > \xi_3 > \xi_2$

Finally, Figure 3 summarizes the diagnosis values of Table 1, Table 3, and Table 5 in a chart (after converting the vague values to fuzzy values to be comparable with others on the chart). It is clear from the above tables and Figure 3 that the proposed method is more general and accurate than other methods. For instance, when applying the proposed model, if the first section needs only two students, then the fourth and the fifth students will join it. In addition, if the second section needs only one student, then he/she will be the second student. Moreover, if the third section needs two students, they will be the first and the third students. When applying Alkhazaleh model, there is a contradiction, which is that the first and the third students have the same value and then the same order for the second section students' order, but the second section in this assumed case needs only one student. For Yang and Faried, it is clear that they neglected the external factors (effective parameters), which can impact the decision on many other issues when the scores vary.



	xi_1	xi_2	xi_3	xi_1	xi_2	xi_3	xi_1	xi_2	xi_3	xi_1	xi_2	xi_3
	Yang et al. [2011]			Faried et al. [2019]			Alkhazaleh [2022]			Proposed model		
pi_1	2.14	1.33	1.73	2.33	1.53	1.9	2.36	1.59	1.95	2.57	1.827	2.16
pi_2	1.74	1.36	1.34	1.89	1.58	1.58	2.13	1.58	1.72	2.31	1.824	1.95
pi_3	1.8	1.31	1.7	2.01	1.49	1.89	2.2	1.59	1.93	2.41	1.8	2.15
pi_4	2.31	1.71	2.03	2.53	1.93	2.28	2.41	1.76	2.08	2.64	1.995	2.32
pi_5	2.27	1.6	1.69	2.41	1.86	1.93	2.43	1.73	1.94	2.62	1.99	2.168

Figure 3. Different models' comparative results.

7. Final conclusions and upcoming studies

The present research aims to develop an innovative hybrid expansion extending the standard crisp set concept, referred to as the effective vague soft set. It encompasses a discussion of the various types and newly introduced essential concepts and operations. Additionally, it explores De Morgan's laws, as well as distributive laws. In addition, it presents absorption properties, and commutative properties, along with associative properties. The article additionally presents a decision-making strategy that relies on effective vague soft sets. A comprehensive example illustrates the classification of college students into specializations based on their subject grades and life skills, demonstrating the practical application of the proposed technique. To streamline the decision-making process and enhance clarity, matrix representations of the technique steps replace set extensions. To ensure accurate and efficient results, the article utilizes $Wolfram\ Mathematica^{®}$ to perform matrix multiplications and other computations throughout. This methodology streamlines the process of determining whether a patient is suffering from a particular disease or whether a student is eligible for a specific education level. Furthermore, the article includes a comparison with other existing algorithms to highlight the advantages of the proposed algorithm.

One of the key benefits of the suggested framework is its generality, encompassing and generalizing previous models such as fuzzy soft sets, effective fuzzy soft sets, and vague soft sets. This means that the fuzzy soft set, effective fuzzy soft set, and vague soft set are all special cases of the effective vague soft set. For instance, when all effective parameter values for the effective vague soft set are all zeros, then the effective set has no impact on the vague soft set values (can be considered as an empty set). Then, this effective vague soft set coincides with the vague soft set as a special case. Moreover, this vague soft set can be reduced to a fuzzy soft set, if all membership interval values are reduced to single values (by using the methods that transform the vague values into fuzzy values). Furthermore, again for the effective vague soft set, if all membership interval values are reduced to single values, then the effective vague soft set coincides with the effective fuzzy soft set.

However, it's important to acknowledge that, the suggested approach, like any technique or framework, may have inherent drawbacks or limitations. Specifically, it may face challenges when dealing with a substantial number of attributes or items, resulting in a high volume of computations. To address this limitation, mathematical programs like $Wolfram\ Mathematica^{®}$ or $MATLAB^{®}$ can be employed to efficiently process large amounts of data. Furthermore, the article acknowledges that the effective vague soft model may not be suitable for addressing scenarios involving bipolarity or a multi-attribute environment with vague soft data, which are common in real-world situations. As a direction for future work, the authors may consider extending their ideas to encompass effective bipolar-valued vague soft sets, or effective multi-vague soft sets, as well as effective bipolar-valued multi-vague soft sets.

Use of AI tools declaration

The authors affirm that they did not employ Artificial Intelligence (AI) tools in the development of this article.

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Conflict of interest

The authors affirm the absence of any conflict of interest.

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