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Research article

Hermite-Hadamard and Ostrowski type inequalities via α-exponential type convex functions with applications

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Abstract: This paper introduced and investigated a new form of convex mapping known as α-exponential type convexity. We presented several algebraic properties associated with this newly introduced convexity. Additionally, we established novel adaptations of well-known inequalities, including the Hermite-Hadamard and Ostrowski-type inequalities, specifically for this convex function. We also derived special cases of these newly established results. Furthermore, we provided new estimations for the trapezoidal formula, demonstrating practical applications of this research.

Keywords: Hermite Hadamard inequality; exponential type convexity; Ostrowski inequality; convex function; Holder's inequality
Mathematics Subject Classification: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction and preliminaries

The study of convex functions has become increasingly significant due to their versatile nature. Recently, this concept has been extended and generalized in different directions. For more details, see [1–8].

These days the investigation on convexity theory is considered as a unique symbol in the study of the theoretical conduct of mathematical inequalities. As of late, a few articles have been published with a special reference to integral inequalities for convex functions. Specifically, much consideration has been given to the theoretical investigations of inequalities on various kinds of convex functions; for example, s-type convex functions, Harmonic convex functions, strongly quasi convex function, (p, h)-convex functions, tgs-convex functions, Exponential type convex functions, GA-convex functions, MT-convex functions, Exponential s-type convex functions and so on. Many researchers have worked on the above mentioned convexities in different directions with some innovative applications. One intriguing feature of these different forms of convex functions is that each definition can be seen as a generalization of the other under certain specific conditions. For more details, see [9–18].

Motivated by ongoing developments and studies in this subject, it has been revealed that there is one particular type of convexity known as exponential convexity, and lots of researchers are now trying to enhance it. Dragomir [19] and Antczak [20] presented the concept of exponential type convexity, and Awan [21] investigated another class of exponential convex function. More recently, Mahir Kadakal and Iscan [22] introduced another meaning of exponential-type convexity.

The main purpose of the article is to introduce the notion of an α-exponential type convex function and derive the variants of the classical Hermite-Hadamard and Ostrowski type inequalities by use of the class of α-exponential type convex functions. We also discuss several new special cases for the obtained results, which show that our obtained results are generalizations and extensions of some previously known results.

Researchers have shown a great interest in big data analysis, deep learning and information theory, utilizing the concept of exponential convex functions. As a result, we anticipate that the introduction of the concept of α-exponential convex functions could capture the attention of these scientists, leading to further advancements in the fields of deep learning, data analysis and information theory. Moreover, many mathematicians have done studies in q-calculus analysis; the interested reader can see [23–26].

Integral inequalities are commonly satisfied by convex functions, including the well-known Hermite-Hadamard inequality. The Hermite-Hadamard inequality for a convex function $\Phi : \Im \to \mathbb{R}$ on an interval $\Im$ is

$$\Phi\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \Phi(\mu) d\mu \leq \frac{\Phi(\mu_1) + \Phi(\mu_2)}{2}.$$

This inequality holds for all $\mu_1, \mu_2 \in \Im$ with $\mu_1 < \mu_2$. Some refinements and generalizations of the H-H inequality have been obtained by [27] and the references therein.

Let a differentiable function $\Phi : \Im \subseteq \mathbb{R} \longrightarrow \Re$ be defined on the interior of $\Im$ along with $\mu_1, \mu_2 \in \Im^o$, where $\mu_1 < \mu_2$ and also $\Phi \in L[\mu_1, \mu_2]$. If $|\Phi'(z)| \leq K$ for all $z \in [\mu_1, \mu_2]$ then the subsequent inequality satisfies,

$$\left|\Phi(z) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \Phi(\mu) d\mu\right| \leq \kappa(\mu_2 - \mu_1)\left[\frac{1}{4} + \frac{\left(z - \frac{\mu_1+\mu_2}{2}\right)^2}{(\mu_2 - \mu_1)^2}\right].$$

The above inequality is a well-known Ostowski inequality. For more details, see [28–31]. Here, we recall some known concepts. The exponential convex functions are defined as follows.

Definition 1.1. *[21] A function $\Phi : \Im \subseteq \mathbb{R} \to \Re$ is referred to as an exponential convex function if*

$$\Phi(\upsilon\mu_1 + (1 - \upsilon)\mu_2) \leq \upsilon \frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (1 - \upsilon) \frac{\Phi(\mu_2)}{e^{\alpha\mu_2}} \tag{1.1}$$

satisfied $\forall\ \mu_1, \mu_2 \in \Im$, $\alpha \in \mathbb{R}$ and $\upsilon \in [0, 1]$.

Definition 1.2. *[22] A function $\Phi : \Im \subseteq \mathbb{R} \to \Re$ is said to be a convex function of exponential type if*

$$\Phi(\upsilon\mu_1 + (1 - \upsilon)\mu_2) \leq (e^{\upsilon} - 1)\Phi(\mu_1) + (e^{1-\upsilon} - 1)\Phi(\mu_2) \tag{1.2}$$

holds for all $\mu_1, \mu_2 \in \Im$, $0 \leq \upsilon \leq 1$.

The present paper is structured in the following way: In section two, we explore the concept of an α-exponential type convex function and give some of its algebraic properties. In section three, we derive the Hermite-Hadamard inequality for an α-exponential type convex function. In section four, we establish an Ostrowski type inequality for an α-exponential type convex function. Additionally, in section five, we provide new estimations for the trapezoidal formula as practical applications. Finally, in the next section, the conclusion is presented.

2. α-exponential type convex function and its properties:

Now, we introduce an α-exponential type convex function and give some of its algebraic properties for the newly defined class of function.

Definition 2.1. *A function $\Phi : \Im \subseteq \mathbb{R} \to \Re$ is said to be an α-exponential type convex function if*

$$\Phi(\upsilon\mu_1 + (1-\upsilon)\mu_2) \leq (e^{\upsilon} - 1)\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}} \tag{2.1}$$

holds true for all $\alpha \in \Re$, $\mu_1, \mu_2 \in \Im$ and $\upsilon \in [0, 1]$.

Remark 1. *By employing $\alpha = 0$ in the above inequality* (2.1)*, exponential type convexity, which was investigated by Iscan in [22], is obtained.*

We study specific relationships between the class of exponential convex functions and other forms of convex functions.

Lemma 2.1. *The subsequent inequalities hold*

$$e^{\upsilon} - 1 \geq \upsilon, \qquad e^{1-\upsilon} - 1 \geq 1 - \upsilon \tag{2.2}$$

for $\upsilon \in [0, 1]$.

Proof. The proof follows directly by expanding the exponential series. □

Proposition 1. *Every exponential convex function is an α-exponential type convex function.*

Proof. By Lemma 2.1, since $\upsilon \leq e^{\upsilon} - 1$ and $1 - \upsilon \leq e^{1-\upsilon} - 1$ for all $\upsilon \in [0, 1]$ and $\alpha \in \mathbb{R}$, we obtain

$$\Phi(\upsilon\mu_1 + (1-\upsilon)\mu_2) \leq \upsilon\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (1-\upsilon)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}} \leq (e^{\upsilon} - 1)\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}.$$

□

Theorem 1. *Let $\Phi, \Psi : [\mu_1, \mu_2] \longrightarrow \Re$ be an α-exponential type convex function, then*
(i) $\Psi + \Phi$ would be an α-exponential type convex function.
(ii) If $\kappa \geq 0$, $\kappa\Phi$ is an α-exponential type convex function.

Proof. (i) Let Φ be an α-exponential type convex function

$$\begin{aligned}
(\Phi + \Psi)(\upsilon\mu_1 + (1-\upsilon)\mu_2) &= \Phi(\upsilon\mu_1 + (1-\upsilon)\mu_2) + \Psi(\upsilon\mu_1 + (1-\upsilon)\mu_2) \\
&\leq (e^{\upsilon} - 1)\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\mu} - 1)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}} + (e^{\upsilon} - 1)\frac{\Psi(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\Psi(\mu_2)}{e^{\alpha\mu_2}} \\
&= (e^{\upsilon} - 1)\left[\frac{\Phi(\mu_1) + \Psi(\mu_1)}{e^{\alpha\mu_1}}\right] + (e^{1-\upsilon} - 1)\left[\frac{\Phi(\mu_2) + \Psi(\mu_2)}{e^{\alpha\mu_2}}\right] \\
&= (e^{\upsilon} - 1)\frac{\Big(\Phi + \Psi\Big)(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\Big(\Phi + \Psi\Big)(\mu_2)}{e^{\alpha\mu_2}}.
\end{aligned}$$

(ii) Let Φ be an α-exponential type convex function and $\kappa \in \Re$ $(\kappa \geq 0)$, so

$$(\kappa\Phi)(\upsilon\mu_1 + (1-\upsilon)\mu_2) \leq \kappa\left[(e^{\upsilon} - 1)\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}\right]$$
$$= (e^{\upsilon} - 1)\frac{\kappa\Phi(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\kappa\Phi(\mu_2)}{e^{\alpha\mu_2}}$$
$$= (e^{\upsilon} - 1)\frac{\big(\kappa\Phi\big)(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\big(\kappa\Phi\big)(\mu_2)}{e^{\alpha\mu_2}}.$$

□

Theorem 2. *Let $\Phi : \Im \longrightarrow \mathbb{J}$ be an exponential type convex and $\Psi : \mathbb{J} \longrightarrow \Re$ be an α-exponential type convex function and nondecreasing, then $\Psi o \Phi : \Im \longrightarrow \Re$ be an α-exponential type convex function.*

Proof. Let $\mu_1, \mu_2 \in \Im$ with $0 \leq \upsilon \leq 1$, for $\alpha \in \Re$ and we get

$$(\Psi o\Phi)(\upsilon\mu_1 + (1-\upsilon)\mu_2) = \Psi(\Phi(\upsilon\mu_1 + (1-\upsilon)\mu_2))$$
$$\leq \Psi\left(\upsilon\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (1-\upsilon)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}\right)$$
$$\leq (e^{\mu} - 1)\Psi\left(\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}}\right) + (e^{1-\upsilon} - 1)\Psi\left(\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}\right)$$
$$= (e^{\upsilon} - 1)\frac{(\Psi o\Phi)(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{(\Psi o\Phi)(\mu_2)}{e^{\alpha\mu_2}}.$$

□

Theorem 3. *If $\Phi : [\mu_1, \mu_2] \longrightarrow \Re$ is an α-exponential type convex function , then Φ will be bounded on the closed interval $[\mu_1, \mu_2]$.*

Proof. Suppose that $\varkappa = max\left\{\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}}, \frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}\right\}$ and $\check{x} \in [\mu_1, \mu_2]$ is any arbitrary point. Also, consider $\exists\, 0 \leq \upsilon \leq 1$ such that $\check{x} = \upsilon\mu_1 + (1-\upsilon)\mu_2$. Thus, since $e^{\upsilon} \leq e$ and $e^{1-\upsilon} \leq e$, for $0 \leq \upsilon \leq 1$, we have

$$\Phi(\check{x}) = \Phi(\upsilon\mu_1 + (1-\upsilon)\mu_2),$$
$$\leq (e^{\upsilon} - 1)\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + (e^{1-\upsilon} - 1)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}$$
$$\leq (e^{\upsilon} - 1)\varkappa + (e^{1-\upsilon} - 1)\varkappa$$
$$\leq (e^{\upsilon} + e^{1-\upsilon} - 2)\varkappa$$
$$\leq (e + e - 2)\varkappa$$
$$\leq 2(e - 1)\varkappa = \mathfrak{B}.$$

We established that Φ is bounded above by the real number $\mathfrak{B}$. Similarly, we can show that Φ is bounded below. □

3. Hermite-Hadamard inequality for α-exponential type convex function

The primary objective of this section is to introduce H-H type inequalities applicable to α-exponential type convex functions.

Theorem 4. *Let $\Phi : [\mu_1, \mu_2] \longrightarrow \Re$ be an α-exponential type convex function. If $\mu_1 < \mu_2$ and $\Phi \in L[\mu_1, \mu_2]$, then subsequent H-H inequality holds:*

$$\frac{1}{2(e^{\frac{1}{2}}-1)}\Phi(\frac{\mu_1+\mu_2}{2}) \le \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\Phi(x)}{e^{\alpha x}}dx \le A(\upsilon)\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}} + B(\upsilon)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}, \tag{3.1}$$

where

$$A(\upsilon) = \int_0^1 \frac{(e^{\upsilon}-1)}{e^{\alpha(\upsilon\mu_1+(1-\upsilon)\mu_2)}}d\upsilon, \qquad B(\upsilon) = \int_0^1 \frac{(e^{1-\upsilon}-1)}{e^{\alpha(\upsilon\mu_1+(1-\upsilon)\mu_2)}}d\upsilon.$$

Proof. Since

$$\Phi\Big(\frac{\mu_1+\mu_2}{2}\Big) = \Phi\Big(\frac{(\upsilon\mu_1+(1-\upsilon)\mu_2)+(\upsilon\mu_2+(1-\upsilon)\mu_1)}{2}\Big),$$

assume that

$$\mu_1 = \upsilon\mu_1 + (1-\upsilon)\mu_2, \qquad \mu_2 = \upsilon\mu_2 + (1-\upsilon)\mu_1,$$

$$\Phi\Big(\frac{\mu_1+\mu_2}{2}\Big) = \Phi\Big(\frac{1}{2}(\upsilon\mu_1+(1-\upsilon)\mu_2) + \frac{1}{2}(\upsilon\mu_2+(1-\upsilon)\mu_1)\Big). \tag{3.2}$$

By definition of an α-exponential type convex function to Eq (3.2), we get

$$\Phi\Big(\frac{\mu_1+\mu_2}{2}\Big) \le (e^{\frac{1}{2}}-1)\frac{\Phi(\upsilon\mu_1+(1-\upsilon)\mu_2)}{e^{\alpha(\upsilon\mu_1+(1-\upsilon)\mu_2)}} + (e^{\frac{1}{2}}-1)\frac{\Phi(\upsilon\mu_2+(1-\upsilon)\mu_1)}{e^{\alpha(\upsilon\mu_2+(1-\upsilon)\mu_1)}}. \tag{3.3}$$

Integrating above Eq (3.3) w.r.t $\upsilon \in [0,1]$ and using the change of variable, we have

$$\frac{1}{2(e^{\frac{1}{2}}-1)}\Phi\Big(\frac{\mu_1+\mu_2}{2}\Big) \le \Big[\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\Phi(u)}{e^{\alpha u}}du\Big]. \tag{3.4}$$

By Definition 2.1,

$$\frac{\Phi(\upsilon\mu_1+(1-\upsilon)\mu_2)}{e^{\alpha(\upsilon\mu_1+(1-\upsilon)\mu_2)}} \le \frac{(e^{\upsilon}-1)\frac{\Phi(\mu_1)}{e^{\alpha\mu_1}}}{e^{\alpha(\upsilon\mu_1)+(1-\upsilon)\mu_2}} + \frac{(e^{1-\upsilon}-1)\frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}}{e^{\alpha(\upsilon\mu_1)+(1-\upsilon)\mu_2}}. \tag{3.5}$$

Integrating (3.5) w.r.t $\upsilon \in [0,1]$, we obtained

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\Phi(u)}{e^{\alpha u}}du \le \frac{\Phi(\mu_1)}{e^{\alpha\mu_1}}\int_0^1\frac{(e^{\upsilon}-1)}{e^{\alpha(\upsilon\mu_1+(1-\upsilon)\mu_2)}}d\upsilon + \frac{\Phi(\mu_2)}{e^{\alpha\mu_2}}\int_0^1\frac{(e^{1-\upsilon}-1)}{e^{\alpha(\upsilon\mu_1+(1-\upsilon)\mu_2)}}d\upsilon. \tag{3.6}$$

From Eqs (3.4) and (3.6), we obtain (3.1). □

3.1. Some new inequalities for α-exponential type convex function

The objective of this section is to investigate various estimates that enhance the H-H inequality for functions in which the first derivative in absolute value at certain power is an α exponential type convex. Dragomir and Agarwal employed the subsequent lemma in their work [32].

Lemma 3.1. *Let $\Phi : \Im \subseteq \Re \longrightarrow \Re$ be a differentiable mapping on $\Im^o$. Consider $\mu_1, \mu_2 \in \Im^o$ with $\mu_1 < \mu_2$. If $\Phi' \in L[\mu_1, \mu_2]$, then the following identity holds:*

$$\frac{\Phi(\mu_1) + \Phi(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \Phi(x)dx = \frac{\mu_2 - \mu_1}{2} \int_0^1 (1 - 2\upsilon)\Phi'(\upsilon\mu_1 + (1 - \upsilon)\mu_2)d\upsilon. \tag{3.7}$$

Theorem 5. *Let a differentiable function $\Phi : \Im \longrightarrow \Re$ be defined on the interior of $\Im$ along with $\mu_1, \mu_2 \in \Im^o$, where $\mu_1 < \mu_2$ and also $\Phi' \in L[\mu_1, \mu_2]$. If on $[\mu_1, \mu_2]$, $|\Phi'|$ is an α-exponential type convex function, then the subsequent inequality satisfied for $0 \le \upsilon \le 1$:*

$$\left|\frac{\Phi(\mu_1) + \Phi(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \le \frac{\mu_2 - \mu_1}{2} \left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right)\left[\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right| + \left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|\right]. \tag{3.8}$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}\left|\frac{\Phi(\mu_1) + \Phi(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \Phi(x)dx\right| &= \frac{\mu_2 - \mu_1}{2}\left|\int_0^1 (1 - 2\upsilon)\Phi'(\upsilon\mu_1 + (1 - \upsilon)\mu_2)d\upsilon\right| \\ &\le \frac{\mu_2 - \mu_1}{2} \int_0^1 \left|1 - 2\upsilon\right|\left|\Phi'(\upsilon\mu_1 + (1 - \upsilon)\mu_2)\right|d\upsilon.\end{aligned}$$

Using an α-exponential type convexity of Φ', we get

$$\begin{aligned}&\left|\frac{\Phi(\mu_1) + \Phi(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \\ &\le \frac{\mu_2 - \mu_1}{2} \int_0^1 |1 - 2\upsilon|\left[(e^{\upsilon} - 1)\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right| + \left(e^{1-\upsilon} - 1\right)\left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|\right] d\upsilon \\ &= \frac{\mu_2 - \mu_1}{2}\left[\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right| \int_0^1 |(1 - 2\upsilon)|\,(e^{\upsilon} - 1)\,d\upsilon + \left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right| \int_0^1 |(1 - 2\upsilon)|(e^{1-\upsilon} - 1)d\upsilon\right] \\ &= \frac{\mu_2 - \mu_1}{2}\left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right)\left[\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right| + \left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|\right].\end{aligned} \tag{3.9}$$

Since

$$\int_0^1 \left|1 - 2\upsilon\right|(e^{\upsilon} - 1)\,d\upsilon = \int_0^1 \left|1 - 2\upsilon\right|\left(e^{1-\upsilon} - 1\right)d\upsilon = 4e^{\frac{1}{2}} - e - \frac{7}{2}. \tag{3.10}$$

by substituting equality (3.10) in (3.9), we get inequality (3.8). □

Remark 2. *(i) By letting $\alpha = 0$, we obtain Theorem 4.1 in [22].*

Theorem 6. *Let a differentiable function $\Phi : \Im \longrightarrow \Re$ be defined on the interior of $\Im$ along with $\mu_1, \mu_2 \in \Im^o$ where $\mu_1 < \mu_2$. Additionally, suppose that the derivative Φ' is integrable on the interval*

$[\mu_1, \mu_2]$. *If on* $[\mu_1, \mu_2]$ *the function* $|\Phi'|^q$ *exhibits an* α*-exponential type convexity, then the subsequent inequality satisfied for* $0 \leq \upsilon \leq 1$*:*

$$\left|\frac{\Phi(\mu_1)+\Phi(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \leq \frac{\mu_2-\mu_1}{2}(e-2)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right|^q+\left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|^q\right]^{\frac{1}{q}}, \quad (3.11)$$

where $p^{-1}+q^{-1}=1$.

Proof. From Lemma 3.1, we have

$$\begin{aligned}&\left|\frac{\Phi(\mu_1)+\Phi(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right|\\&=\frac{\mu_2-\mu_1}{2}\left|\int_0^1(1-2\upsilon)\Phi'(\upsilon\mu_1+(1-\upsilon)\mu_2)d\upsilon\right|\\&\leq\frac{\mu_2-\mu_1}{2}\int_0^1\left|1-2\upsilon\right|\left|\Phi'(\upsilon\mu_1+(1-\upsilon)\mu_2)\right|d\upsilon.\end{aligned}$$

Applying Holder's integral inequality, we find

$$\begin{aligned}&\frac{\mu_2-\mu_1}{2}\int_0^1\left|1-2\upsilon\right|\left|\Phi'(\upsilon\mu_1+(1-\upsilon)\mu_2)\right|d\upsilon\\&\leq\frac{\mu_2-\mu_1}{2}\left(\int_0^1\left|1-2\upsilon\right|^pd\upsilon\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'(\upsilon\mu_1+(1-\upsilon)\mu_2)\right|^qd\upsilon\right)^{\frac{1}{q}}.\end{aligned} \quad (3.12)$$

Since $|\Phi|^q$ is an α-exponential type convex function, we get

$$\begin{aligned}&\int_0^1\left|\Phi'(\upsilon\mu_1+(1-\upsilon)\mu_2)\right|^qd\upsilon\\&\leq\int_0^1\left[(e^\upsilon-1)\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right|^q+\left(e^{1-\upsilon}-1\right)\left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|^q\right]d\upsilon\\&=(e-2)\left[\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right|^q+\left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|^q\right].\end{aligned} \quad (3.13)$$

Since

$$\int_0^1(e^\upsilon-1)\,d\upsilon=\int_0^1\left(e^{1-\upsilon}-1\right)d\upsilon=e-2. \quad (3.14)$$

$$\int_0^1\left|1-2\upsilon\right|^pd\upsilon=\frac{1}{p+1}, \quad (3.15)$$

using (3.13)–(3.15) in (3.12), we get (3.11).

Remark 3. *(i) By letting* $\alpha=0$*, we obtain Theorem 4.2 in* [22].

□

Theorem 7. *Let a differentiable function $\Phi : \Im \longrightarrow \Re$ be defined on the interior of $\Im$ along with $\mu_1, \mu_2 \in \Im^o$, where $\mu_1 < \mu_2$ and $q > 1$ and also $\Phi' \in L[\mu_1, \mu_2]$. If on $[\mu_1, \mu_2]$ $|\Phi'|^q$ is a convex function of an α-exponential type then the subsequent inequality satisfied for $0 \leq \upsilon \leq 1$:*

$$\left|\frac{\Phi(\mu_1)+\Phi(\mu_2)}{2} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \leq \frac{\mu_2-\mu_1}{2^{2-\frac{1}{q}}}\left[\left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right)\right]^{\frac{1}{q}}\left[\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right|^q + \left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|^q\right]. \quad (3.16)$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&\left|\frac{\Phi(\mu_1)+\Phi(\mu_2)}{2} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \\
&= \frac{\mu_2-\mu_1}{2}\left|\int_0^1 (1-2\upsilon)\Phi'(\upsilon\mu_1 + (1-\upsilon)\mu_2)d\upsilon\right| \\
&\leq \frac{\mu_2-\mu_1}{2}\int_0^1 \left|1-2\upsilon\right|\left|\Phi'(\upsilon\mu_1 + (1-\upsilon)\mu_2)\right|d\upsilon.
\end{aligned}$$

Applying the power mean inequality, we find

$$\begin{aligned}
&\frac{\mu_2-\mu_1}{2}\int_0^1 \left|1-2\upsilon\right|\left|\Phi'(\upsilon\mu_1 + (1-\upsilon)\mu_2)\right|d\upsilon \\
&\leq \frac{\mu_2-\mu_1}{2}\left(\int_0^1 \left|1-2\upsilon\right|d\upsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|1-2\upsilon\right|\left|\Phi'(\upsilon\mu_1 + (1-\upsilon)\mu_2)\right|^q d\upsilon\right)^{\frac{1}{q}}.
\end{aligned} \quad (3.17)$$

Since $|\Phi|^q$ is an α-exponential type convex function, we get

$$\begin{aligned}
&\int_0^1 \left|1-2\upsilon\right|\left|\Phi'(\upsilon\mu_1 + (1-\upsilon)\mu_2)\right|^q d\upsilon \\
&\leq \int_0^1 \left|1-2\upsilon\right|\left[(e^{\upsilon}-1)\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right|^q + \left(e^{1-\upsilon}-1\right)\left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|^q\right]d\upsilon \\
&= \left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right)\left[\left|\frac{\Phi'(\mu_1)}{e^{\alpha\mu_1}}\right|^q + \left|\frac{\Phi'(\mu_2)}{e^{\alpha\mu_2}}\right|^q\right].
\end{aligned} \quad (3.18)$$

Since

$$\int_0^1 \left|1-2\upsilon\right|d\upsilon = \frac{1}{2}, \quad (3.19)$$

by substituting inequality (3.18) and equality (3.19) in (3.17), we get inequality (3.16). □

Remark 4. *(i) By letting $\alpha = 0$, we obtain Theorem 4.3 in* [22].

4. Refinements of Ostrowski type inequality for an α-exponential type convex functions

Here, we introduced several improvements to the Ostrowski type inequality applicable to differentiable α-exponential type convex functions. Cerone and Dragomir employed the subsequent lemma in their work [33].

Lemma 4.1. *Let a differentiable function $\Phi : \Im \subseteq \Re \longrightarrow \Re$ be defined on the interior of $\Im$. Take $\mu_1, \mu_2 \in \Im^o$ where $\mu_1 < \mu_2$. If $\Phi' \in L[\mu_1, \mu_2]$, then the subsequent identity satisfied:*

$$\begin{aligned}&\Phi(z) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx \\ &= \frac{(z-\mu_1)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\Phi'(\upsilon z + (1-\upsilon)\mu_1)d\upsilon - \frac{(\mu_2-z)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\Phi'(\upsilon z + (1-\upsilon)\mu_2)d\upsilon,\end{aligned} \tag{4.1}$$

for all $z \in [\mu_1, \mu_2]$.

Theorem 8. *Let a differentiable function $\Phi : \Im \longrightarrow \Re$ be defined on the interior of $\Im$. Take $\mu_1, \mu_2 \in \Im^o$ where $\mu_1 < \mu_2$. Also, assume that $\Phi' \in L[\mu_1, \mu_2]$. If on the interval $[\mu_1, \mu_2]$ the absolute value of the derivative $|\Phi'|$ is an α-exponential type convex function and satisfies $|\Phi'| \leq K$ for all $z \in [\mu_1, \mu_2]$, then the subsequent inequality satisfied for $0 \leq \upsilon \leq 1$:*

$$\left|\Phi(z) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \leq \frac{K(e-2)}{\mu_2-\mu_1}\Big[(z-\mu_1)^2 + (\mu_2 - z)^2\Big], \tag{4.2}$$

for each $z \in [\mu_1, \mu_2]$.

Proof. Using Lemma 4.1, since $|\Phi'|$ is an α-exponential type convex function and $|\Phi'| \leq K$,

$$\begin{aligned}&\left|\Phi(z) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \\ &\leq \frac{(z-\mu_1)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\Big|\Phi'(\upsilon z + (1-\upsilon)\mu_1)\Big|d\upsilon + \frac{(\mu_2-z)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\Big|\Phi'(\upsilon z + (1-\upsilon)\mu_2)\Big|d\upsilon \\ &\leq \frac{(z-\mu_1)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\Big\{(e^\upsilon - 1)\frac{|\Phi'(z)|}{e^{\alpha z}} + (e^{1-\upsilon}-1)\frac{|\Phi'(\mu_1)|}{e^{\alpha\mu_1}}\Big\}d\upsilon \\ &+ \frac{(\mu_2-z)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\Big\{(e^\upsilon - 1)\frac{|\Phi'(z)|}{e^{\alpha z}} + (e^{1-\upsilon}-1)\frac{|\Phi'(\mu_2)|}{e^{\alpha\mu_2}}\Big\}d\upsilon \\ &\leq \frac{(z-\mu_1)^2}{\mu_2-\mu_1}\left[\frac{|\Phi'(z)|}{e^{\alpha z}}\int_0^1 \upsilon(e^\upsilon-1)d\upsilon + \frac{|\Phi'(\mu_1)|}{e^{\alpha\mu_1}}\int_0^1 \upsilon(e^{1-\upsilon}-1)d\upsilon\right] \\ &+ \frac{(\mu_2-z)^2}{\mu_2-\mu_1}\left[\frac{|\Phi'(z)|}{e^{\alpha z}}\int_0^1 \upsilon(e^\upsilon-1)d\upsilon + \frac{|\Phi'(\mu_2)|}{e^{\alpha\mu_2}}\int_0^1 \upsilon(e^{1-\upsilon}-1)d\upsilon\right] \\ &\leq \frac{K(z-\mu_1)^2}{\mu_2-\mu_1}\left\{\frac{1}{2} + e - \frac{5}{2}\right\} + \frac{K(\mu_2-z)^2}{\mu_2-\mu_1}\left\{\frac{1}{2} + e - \frac{5}{2}\right\} \\ &\leq \frac{K(z-\mu_1)^2}{\mu_2-\mu_1}\Big[e-2\Big] + \frac{K(\mu_2-z)^2}{\mu_2-\mu_1}\Big[e-2\Big] \\ &\leq \frac{K(e-2)}{\mu_2-\mu_1}\Big[(z-\mu_1)^2 + (\mu_2-z)^2\Big].\end{aligned}$$

□

Corollary 4.1. *(1) By assuming $z = \frac{\mu_1+\mu_2}{2}$ in Theorem 8 yields the subsequent midpoint inequality:*

$$\left|\Phi(\frac{\mu_1+\mu_2}{2}) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \leq \frac{K(\mu_2-\mu_1)}{2}\Big[e-2\Big]. \tag{4.3}$$

(2) By assuming $z = \mu_1$ in Theorem 8 yields the subsequent inequality:

$$\left|\Phi(\mu_1) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \leq K(\mu_2 - \mu_1)\Big[e - 2\Big]. \tag{4.4}$$

(3) By assuming $z = \mu_2$ in Theorem 8 yields the subsequent inequality:

$$\left|\Phi(\mu_2) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \leq K(\mu_2 - \mu_1)\Big[e - 2\Big]. \tag{4.5}$$

Theorem 9. *Suppose a mapping $\Phi : \Im \longrightarrow \Re$, which is differentiable on $\Im^o$. Take $\mu_1, \mu_2 \in \Im^o$ with $\mu_1 < \mu_2$. Additionally, suppose $\Phi' \in L[\mu_1, \mu_2]$ and consider $q > 1$ such that $1 - \frac{1}{p} = q^{-1}$. If on the interval $[\mu_1, \mu_2]$ $|\Phi'|$ is an α-exponential type convex function and $|\Phi'| \leq K$ for all $z \in [\mu_1, \mu_2]$, then the subsequent inequality holds true for $0 \leq \upsilon \leq 1$:*

$$\begin{aligned}&\left|\Phi(z) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \\ &\leq \frac{2^{\frac{1}{q}}K}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(z - \mu_1)^2\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_1}}\right)^{\frac{1}{q}} + (\mu_2 - z)^2\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_2}}\right)^{\frac{1}{q}}\right],\end{aligned} \tag{4.6}$$

for each $z \in [\mu_1, \mu_2]$.

Proof. Utilizing both well-known Holder's inequality and Lemma 4.1 given that $|\Phi'|^q$ is an α-exponential type convex function and $|\Phi'(z)|^q \leq K$, we deduce:

$$\begin{aligned}&\left|\Phi(z) - \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \Phi(x)dx\right| \\
&\leq \frac{(z-\mu_1)^2}{\mu_2 - \mu_1}\int_0^1 \upsilon\Big|\Phi'(\upsilon z + (1-\upsilon)\mu_1)\Big|d\upsilon + \frac{(\mu_2 - z)^2}{\mu_2 - \mu_1}\int_0^1 \upsilon\Big|\Phi'(\upsilon z + (1-\upsilon)\mu_2)\Big|d\upsilon \\
&\leq \frac{(z-\mu_1)^2}{\mu_2 - \mu_1}\left(\int_0^1 \upsilon d\upsilon\right)^{\frac{1}{p}}\left(\int_0^1 \Big|\Phi'(\upsilon z + (1-\upsilon)\mu_1)\Big|d\upsilon\right)^{\frac{1}{q}} \\
&+ \frac{(\mu_2 - z)^2}{\mu_2 - \mu_1}\left(\int_0^1 \upsilon d\upsilon\right)^{\frac{1}{p}}\left(\int_0^1 \Big|\Phi'(\upsilon z + (1-\upsilon)\mu_2)\Big|d\upsilon\right)^{\frac{1}{q}} \\
&\leq \frac{(z-\mu_1)^2}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 (e^{\upsilon} - 1)\frac{|\Phi'(z)|^q}{e^{\alpha z}}d\upsilon + \int_0^1 \left(e^{1-\upsilon} - 1\right)\frac{|\Phi'(\mu_1)|^q}{e^{\alpha\mu_1}}d\upsilon\right)^{\frac{1}{q}} \\
&+ \frac{(\mu_2 - z)^2}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 (e^{\upsilon} - 1)\frac{|\Phi'(z)|^q}{e^{\alpha z}}d\upsilon + \int_0^1 \left(e^{1-\upsilon} - 1\right)\frac{|\Phi'(\mu_2)|^q}{e^{\alpha\mu_2}}d\upsilon\right)^{\frac{1}{q}} \\
&\leq \frac{(2K^q)^{\frac{1}{q}}(z-\mu_1)^2}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_1}}\right)^{\frac{1}{q}}\right] + \frac{(2K^q)^{\frac{1}{q}}(\mu_2 - z)^2}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_2}}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{2^{\frac{1}{q}}K(z-\mu_1)^2}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_1}}\right)^{\frac{1}{q}}\right] + \frac{2^{\frac{1}{q}}K(\mu_2 - z)^2}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_2}}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{2^{\frac{1}{q}}K}{\mu_2 - \mu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(z-\mu_1)^2\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_1}}\right)^{\frac{1}{q}} + (\mu_2 - z)^2\left(\frac{(e-2)}{e^{\alpha z}} + \frac{(e-2)}{e^{\alpha\mu_2}}\right)^{\frac{1}{q}}\right].\end{aligned}$$

□

Corollary 4.2. *(1) By assuming* $z = \frac{\mu_1+\mu_2}{2}$ *in Theorem 9 yields the subsequent midpoint inequality:*

$$\left|\Phi(\frac{\mu_1+\mu_2}{2}) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \leq 2^{\frac{1}{q}-1}K(\mu_2-\mu_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(e-2\right)^{\frac{1}{q}}. \tag{4.7}$$

(2) By assuming $z = \mu_1$ *in Theorem 9 yields the subsequent inequality:*

$$\left|\Phi(\mu_1) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \leq 2^{\frac{1}{q}}K(\mu_2-\mu_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(e-2\right)^{\frac{1}{q}}. \tag{4.8}$$

(3) By assuming $z = \mu_2$ *in Theorem 9 yields the subsequent inequality:*

$$\left|\Phi(\mu_2) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \leq 2^{\frac{1}{q}}K(\mu_2-\mu_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(e-2\right)^{\frac{1}{q}}. \tag{4.9}$$

Theorem 10. *Let a differentiable function* $\Phi : \Im \longrightarrow \Re$ *be defined on the interior of* $\Im$ *along with* $\mu_1, \mu_2 \in \Im^o$*, where* $\mu_1 < \mu_2$ *and also* $\Phi' \in L[\mu_1, \mu_2]$*. If on* $[\mu_1, \mu_2]$ $|\Phi'|$ *is an* α*-exponential type convex function and* $|\Phi'| \leq K$ *for all* $z \in [\mu_1, \mu_2]$ *then the subsequent inequality satisfied for* $0 \leq \upsilon \leq 1$*:*

$$\begin{aligned}
&\left|\Phi(z) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \\
&\leq \frac{K}{(\mu_2-\mu_1)2^{1-\frac{1}{q}}}\left[(z-\mu_1)^2\left(\left(\frac{1}{2e^{\alpha z}}\right)+\left(\frac{2e-5}{2e^{\alpha\mu_1}}\right)\right)^{\frac{1}{q}} + (\mu_2-z)^2\left(\left(\frac{1}{2e^{\alpha z}}\right)+\left(\frac{2e-5}{2e^{\alpha\mu_2}}\right)\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{4.10}$$

for each $z \in [\mu_1, \mu_2]$.

Proof. Employing from both Lemma 4.1 and the power mean inequality and considering that $|\Phi'|^q$ is an α- exponential type convex function while $|\Phi(z)| \leq K$, we arrive at the following result:

$$\begin{aligned}
&\left|\Phi(z) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right| \\
&\leq \frac{(z-\mu_1)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\left|\Phi'(\upsilon z+(1-\upsilon)\mu_1)\right|d\upsilon + \frac{(\mu_2-z)^2}{\mu_2-\mu_1}\int_0^1 \upsilon\left|\Phi'(\upsilon z+(1-\upsilon)\mu_2)\right|d\upsilon \\
&\leq \frac{(z-\mu_1)^2}{\mu_2-\mu_1}\left(\int_0^1 \upsilon d\upsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left|\Phi'(\upsilon z+(1-\upsilon)\mu_1)\right|d\upsilon\right)^{\frac{1}{q}} \\
&+ \frac{(\mu_2-z)^2}{\mu_2-\mu_1}\left(\int_0^1 \upsilon d\upsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left|\Phi'(\upsilon z+(1-\upsilon)\mu_2)\right|d\upsilon\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \frac{(z-\mu_1)^2}{\mu_2-\mu_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left(e^{\upsilon}-1\right)\frac{|\Phi'(z)|^q}{e^{\alpha z}}d\upsilon+\int_0^1 \upsilon\left(e^{1-\upsilon}-1\right)\frac{|\Phi'(\mu_1)|^q}{e^{\alpha\mu_1}}d\upsilon\right)^{\frac{1}{q}}$$
$$+\frac{(\mu_2-z)^2}{\mu_2-\mu_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 \upsilon\left(e^{\upsilon}-1\right)\frac{|\Phi'(z)|^q}{e^{\alpha z}}d\upsilon+\int_0^1 \upsilon\left(e^{1-\upsilon}-1\right)\frac{|\Phi'(\mu_2)|^q}{e^{\alpha\mu_2}}d\upsilon\right)^{\frac{1}{q}}$$
$$\leq \frac{K(z-\mu_1)^2}{\mu_2-\mu_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \frac{\upsilon\left(e^{\upsilon}-1\right)}{e^{\alpha z}}d\upsilon+\int_0^1 \frac{\upsilon\left(e^{1-\upsilon}-1\right)}{e^{\alpha\mu_1}}d\upsilon\right)^{\frac{1}{q}}\right]$$
$$+\frac{K(\mu_2-z)^2}{\mu_2-\mu_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 \frac{\upsilon\left(e^{\upsilon}-1\right)}{e^{\alpha z}}d\upsilon+\int_0^1 \frac{\upsilon\left(e^{1-\upsilon}-1\right)}{e^{\alpha\mu_2}}d\upsilon\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{K(z-\mu_1)^2}{\mu_2-\mu_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{1}{2e^{\alpha z}}\right)+\left(\frac{2e-5}{2e^{\alpha\mu_1}}\right)\right)^{\frac{1}{q}}\right]+\frac{K(\mu_2-z)^2}{\mu_2-\mu_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{1}{2e^{\alpha z}}\right)+\left(\frac{2e-5}{2e^{\alpha\mu_2}}\right)\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{K}{(\mu_2-\mu_1)2^{1-\frac{1}{q}}}\left[(z-\mu_1)^2\left(\left(\frac{1}{2e^{\alpha z}}\right)+\left(\frac{2e-5}{2e^{\alpha\mu_1}}\right)\right)^{\frac{1}{q}}+(\mu_2-z)^2\left(\left(\frac{1}{2e^{\alpha z}}\right)+\left(\frac{2e-5}{2e^{\alpha\mu_2}}\right)\right)^{\frac{1}{q}}\right].$$

□

Corollary 4.3. *(1) By assuming $z=\frac{\mu_1+\mu_2}{2}$ in Theorem 10 yields the subsequent midpoint inequality:*

$$\left|\Phi(\frac{\mu_1+\mu_2}{2})-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right|\leq \frac{2^{\frac{1}{q}}K(\mu_2-\mu_1)}{4}\left[e-2\right]^{\frac{1}{q}}. \tag{4.11}$$

(2) If we choose $z=\mu_1$ in Theorem 10 it yields the subsequent inequality:

$$\left|\Phi(\mu_1)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right|\leq \frac{2^{\frac{1}{q}}K(\mu_2-\mu_1)}{2}\left[e-2\right]^{\frac{1}{q}}. \tag{4.12}$$

(3) If we choose $z=\mu_2$ in Theorem 10 it yields the subsequent inequality:

$$\left|\Phi(\mu_2)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Phi(x)dx\right|\leq \frac{2^{\frac{1}{q}}K(\mu_2-\mu_1)}{2}\left[e-2\right]^{\frac{1}{q}}. \tag{4.13}$$

5. Applications

Assuming that d represents a partition of the interval $[\mu_1,\mu_2]$ such that $d:\mu_1=w_0<w_1<\cdots<w_{m-1}<w_m=\mu_2$, the trapezoidal formula can be expressed as follows:

$$T(\Phi,d)=\sum_{n=0}^{m-1}\frac{\Phi(w_n)+\Phi(w_{n+1})}{2}(w_{n+1}-w_n).$$

It has been clear that if $\Phi:[\mu_1,\mu_2]\to\mathbb{R}$ is twice differentiable on the open interval (μ_1,μ_2) and $M=\max_{w\in(\mu_1,\mu_2)}|\Phi(w)|<\infty$, then

$$\int_{\mu_1}^{\mu_2}\Phi(w)dw=T(\Phi,d)+R(\Phi,d). \tag{5.1}$$

The remainder term $R(\Phi, d)$ satisfies the inequality

$$|R(\Phi, d)| \leq \frac{M}{12} \sum_{n=0}^{m-1} (w_{n+1} - w_n)^3. \tag{5.2}$$

If either the second derivative of Φ does not exist or is unbounded, then (5.1) cannot be used. However, Dragomir and Wang [34–36] demonstrated that $R(\Phi, d)$ can be calculated using only the first derivative, which can have several practical applications.

Proposition 2. *Suppose $\Phi : \Im \subseteq \mathbb{R}^0 \to \mathbb{R}$ is a differentiable function defined on $\Im^\circ$. Let $\mu_1, \mu_2 \in \Im$ with $\mu_1 < \mu_2$. If the absolute value of Φ is an α-exponentially convex on the interval $[\mu_1, \mu_2]$, then for any partition d of the interval $[\mu_1, \mu_2]$, the following holds in Eq* (5.1)*:*

$$\begin{aligned} \left| R(\Phi, d) \right| &\leq \frac{1}{2} \sum_{n=0}^{m-1} (\mu_{n+1} - \mu_n)^2 \left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right) \left[\left| \frac{\Phi'(\mu_n)}{e^{\alpha \mu_n}} \right| + \left| \frac{\Phi'(\mu_{n+1})}{e^{\alpha \mu_{n+1}}} \right| \right] \\ &\leq Max \left[\left| \frac{\Phi'(\mu_n)}{e^{\alpha \mu_n}} \right|, \left| \frac{\Phi'(\mu_{n+1})}{e^{\alpha \mu_{n+1}}} \right| \right] \left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right) \sum_{n=0}^{m-1} (\mu_{n+1} - \mu_n)^2 . \end{aligned} \tag{5.3}$$

Proof. Applying Theorem 5 on the sub interval $[\mu_n, \mu_{n+1}]$ (n = 0, 1, . . . , m– 1) of the partition d, we obtain

$$\begin{aligned} &\mu_1 = \mu_n, \qquad \qquad \mu_2 = \mu_{n+1}, \\ &\left| \frac{\Phi(\mu_n) + \Phi(\mu_{n+1})}{2} - \frac{1}{\mu_{n+1} - \mu_n} \int_{\mu_n}^{\mu_{n+1}} \Phi(x) dx \right| \\ &\leq \frac{\mu_{n+1} - \mu_n}{2} \left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right) \left[\left| \frac{\Phi'(\mu_n)}{e^{\alpha \mu_n}} \right| + \left| \frac{\Phi'(\mu_{n+1})}{e^{\alpha \mu_{n+1}}} \right| \right]. \end{aligned} \tag{5.4}$$

By summing over the range of n from zero to m-1, we get

$$\begin{aligned} &\left| T(\Phi, d) - \int_{\mu_1}^{\mu_2} \Phi(\mu) d\mu \right| \\ &\leq \frac{1}{2} \sum_{n=0}^{m-1} (\mu_{n+1} - \mu_n)^2 \left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right) \left[\left| \frac{\Phi'(\mu_n)}{e^{\alpha \mu_n}} \right| + \left| \frac{\Phi'(\mu_{n+1})}{e^{\alpha \mu_{n+1}}} \right| \right] \\ &\leq Max \left[\left| \frac{\Phi'(\mu_1)}{e^{\alpha \mu_1}} \right|, \left| \frac{\Phi'(\mu_2)}{e^{\alpha \mu_2}} \right| \right] \left(4e^{\frac{1}{2}} - e - \frac{7}{2}\right) \sum_{n=0}^{m-1} (\mu_{n+1} - \mu_n)^2 . \end{aligned} \tag{5.5}$$

□

Proposition 3. *Let a differentiable function $\Phi : \Im \subseteq \mathbb{R}_0 \to \mathbb{R}$ be defined on the interior of $\Im$, along with $\mu_1, \mu_2 \in \Im$ where $\mu_1 < \mu_2$. Assuming that $|\Phi|^q$ is an α-exponentially convex function on the interval $[\mu_1, \mu_2]$, and given that $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then within the context of* (5.1)*, for any partition d of the interval $[\mu_1, \mu_2]$, it follows that:*

$$\begin{aligned} \left| R(\Phi, d) \right| &\leq \frac{(e-2)^{\frac{1}{q}}}{2} \sum_{n=0}^{m-1} (\mu_{n+1} - \mu_n)^2 \left[\left| \frac{\Phi'(\mu_n)}{e^{\alpha \mu_n}} \right| + \left| \frac{\Phi'(\mu_{n+1})}{e^{\alpha \mu_{n+1}}} \right| \right]^{\frac{1}{q}} \\ &\leq Max \left[\left| \frac{\Phi'(\mu_n)}{e^{\alpha \mu_n}} \right|, \left| \frac{\Phi'(\mu_{n+1})}{e^{\alpha \mu_{n+1}}} \right| \right] \frac{(e-2)^{\frac{1}{q}}}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \sum_{n=0}^{m-1} (\mu_{n+1} - \mu_n)^2 . \end{aligned} \tag{5.6}$$

Proof. By employing Theorem 6 and employing analogous reasoning as presented in Proposition 2, we obtained the desired result. □

6. Conclusions

This paper focused on examining the notion of α-exponential type convex functions, which appears to be as an extension of the traditional exponential type convex functions. The study included establishing the Hermite-Hadamard inequality for α-exponential type convex functions. Moreover, novel Ostrowski type inequalities were derived specifically for α-exponential type convex functions. The research also explored applications derived from these findings. As far as our understanding goes, these outcomes are original contributions that have not been previously documented in existing literatures. The concept of α-exponential convexity typically applies to functions with specific domains and mathematical forms. Functions with complex or irregular domains may not exhibit exponential convexity. The upcoming researchers can establish similar inequalities by using different types of convexities in their future works. Additionally, it will be an interesting problem to prove similar inequalities for the functions of two variables.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interests.

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