AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(4): 9494–9507.
DOI: 10.3934/math.2024463
Received: 05 January 2024
Revised: 05 March 2024
Accepted: 06 March 2024
Published: 07 March 2024

Research article

A new design method to global asymptotic stabilization of strict-feedforward nonlinear systems with state and input delays

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Abstract: This paper studies the global asymptotic stabilization problem of strict-feedforward nonlinear systems with state and input delays. We will first transform the considered system into an equivalent system by constructing the novel parameter-dependent state feedback controller and introducing the appropriate coordinate transformation. After that, the global asymptotic stability of the closed system is proved by giving the proper Lyapunov-Krasovskii functional and using the stability criterion of time-delay system.

Keywords: strict-feedforward nonlinear systems; delays in state and input; global asymptotic stabilization; state feedback; Lyapunov-Krasovskii functional method
Mathematics Subject Classification: 34D20

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1. Introduction

Feedforward systems (namely upper-triangular systems) are a class of important nonlinear systems. A mass of physical devices, such as the planar vertical takeoff and landing aircraft [1], the ball-beam with a friction term and the translational oscillator with a rotational actuator system [2], the cart-pendulum system [3, 4], can be described by equations with the upper-triangular structure. Moreover, feedforward nonlinear systems cannot be linearized, which results that it is more hard to researchers to find appropriate control method. Based on this, the research of feedforward nonlinear systems has attracted considerable attention, see [5–9] and the references therein.

On the other hand, time-delay systems constitute basic mathematical models of real phenomena and time delays are often encountered in multifarious engineering systems. Hence, the research of control problem for time-delay systems is one of the most interesting and significant problems, and the Lyapunov-Krasovskii method and Lyapunov-Razumikhin method are two powerful tools in the stability analysis and controller design for time-delay systems [10, 11]. There are many results focused on time-delay systems. [12–15] considered the one-order feedforward nonlinear systems, [16–18] considered high-order feedforward nonlinear systems. However, these results only considered state time delay or input delay only appearing in the nonlinearities. Input delay is often unavoidable in practice and often generate instability due to sensors, information processing or transport [19]. [20] considered adaptive dynamic high-gain scaling based output-feedback control of nonlinear feedforward systems with time delays in input and state. [21–23] considered the stabilization of feedforward nonlinear systems with linear growth condition. [24] designed stabilizing controllers for high order feedforward nonlinear systems with input delay. [25] studied memoryless linear feedback control for a class of upper-triangular systems with large delays in state and input. [26] considered global stabilization by memoryless feedback for nonlinear systems with a small input delay and large state delays. [27] developed homogeneous output feedback design for time-delay nonlinear integrators and beyond.

It is worth noting that the most mentioned above conclusions on feedforward time-delay systems take advantage of the homogeneous domination approach and it is useful to handle the special structure of feedforward nonlinear systems (see Remark 1 for the detailed discussion). However, this method does not work well for strict-feedforward nonlinear systems with state and input delays. The purpose of this paper is to find a useful method. The main contributions are:

(i) By applying the stability criterion on time-delay system, a novel parameter-dependent state feedback controller is proposed to guarantee the global asymptotic stabilization of strict-feedforward nonlinear systems with time delay in state and input.

(ii) The parameter-dependent state feedback controller is very simple and flexible, because this controller only depends on a positive parameter. And the design process and computing effort are greatly reduced.

(iii) Due to the appearance of time-delay in control input, the L-K functionals in the existing papers are no longer applicable, a difficult work is how to find an appropriate L-K functional. And how to deal with the terms related to nonlinear function and controller is another difficulty.

This paper is organized as follows. Section 2 gives some preliminaries. The main results is given in Section 3. Section 4 presents the extended results. Two numerical examples are given in Section 5. Section 6 concludes this paper.

2. Preliminaries

Some notions and lemmas are to be used throughout this paper.

Notations. $|\cdot|$ is the Euclidean norm of a vector and $||\cdot||$ stands for the Frobenius norm of a matrix. A function $f: R^n \to R$ is C if it is continuous and is C^1 if it is continuously differential. $\mathcal{K}$ denotes the set of all functions: $R^+ \to R^+$ that are continuous, strictly increasing and vanishing at zero. $\mathcal{K}_\infty$ denotes the set of all functions that are of class $\mathcal{K}$ and unbounded.

Lemma 2.1: [28] *Consider system*

$$\dot{x} = f(t, x(t+\theta)), \tag{2.1}$$

where $\theta \in [-d, 0]$, $x(t) \in R^n$ and $f: R \times C \to R^n$ with $f(t, 0) \equiv 0$. Suppose that $f: R \times C \to R^n$ given in (2.1), maps every $R\times$ (bounded set in C) into a bounded set in R^n, and that $u, v, w: R^+ \to R^+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s > 0$, and

$u(0) = v(0) = 0$. *If there exists a continuous differentiable functional* $V : R \times C \to R$ *such that*

$$u(\|x(0)\|) \leq V(t, x) \leq v\left(\sup_{-d\leq\theta\leq 0} |x(t+\theta)|\right)$$

and

$$\dot{V}(t, x) \leq -w(\|x(0)\|),$$

then the trivial solution of (2.1) is uniformly stable. If $w(s) > 0$ *for* $s > 0$*, then it is uniformly asymptotically stable. In addition, if* $\lim_{s\to\infty} u(s) = \infty$*, then it is globally uniformly asymptotically stable.*

Lemma 2.2: [29]. *For any given vectors* y, z *and constant* $a > 0$*, there are real numbers* $\mu > 1$ *and* $\nu > 1$ *satisfying* $(\mu - 1)(\nu - 1) = 1$ *such that*

$$y^\top z \leq \frac{a^\mu}{\mu}|y|^\mu + \frac{1}{\nu a^\nu}|z|^\nu.$$

Lemma 2.3: [30]. *For any function* $f(t) \in C([-\tau, \infty) : R^+)$ *and positive integer* $p, \tau \in R^+$*, then*

$$\left(\int_{t-\tau}^{t} f(\sigma)d\sigma\right)^p \leq \tau^{p-1}\int_{t-\tau}^{t} f^p(\sigma)d\sigma.$$

3. Main result

In this paper, we consider the following strict-feedforward nonlinear systems with state and input delays described by

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + f_1(x_2(t), \cdots, x_n(t), x_2(t-\tau_2), \cdots, x_n(t-\tau_n)),\\
\dot{x}_2(t) &= x_3(t) + f_2(x_3(t), \cdots, x_n(t), x_3(t-\tau_3), \cdots, x_n(t-\tau_n)),\\
&\vdots\\
\dot{x}_n(t) &= u(t-\tau_1),
\end{aligned} \tag{3.1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^\top \in R^n$ and $u(t) \in R$ are system state and control input, respectively. For $i = 1, \cdots, n$, $\tau_i > 0$ is time-invariant delay, $\bar{\tau} = \max\{\tau_1, \tau_2, \cdots, \tau_n\}$, $x_i(t - \tau_i)$ and $u(t - \tau_1)$ are time-delayed systems state and time-delayed control input. Nonlinear functions f_i, $i = 1, \cdots, n-1$, are continuous.

This paper aims to construct a novel parameter-dependent state feedback controller of system (3.1) such that the eqailibrium at the origin of the closed-loop system is global asymptotically stable. In order to achieve this purpose, the following assumption is needed.

Assumption 3.1: *There is a known positive constant c such that*

$$|f_i(\cdot)| \leq c\sum_{j=i+1}^{n}(|x_j(t)| + |x_j(t-\tau_j)|), \quad i = 1, \cdots, n-1.$$

Remark 3.1: As discussed in feedforward nonlinear systems [14, 21–23], the linear growth condition in Assumption 1 is a general condition for dealing with the nonlinearity $f_i(\cdot)$ in feedforward systems.

It is not hard to see from Assumption 3.1 that the nonlinear term $f_i(\cdot)$ in this paper contains system states $x_{i+1}, \cdots, x_n$ rather than $x_{i+2}, \cdots, x_n$ in [14, 21–23]. Hence, system (3.1) can be viewed as a class of strict-feedfroward nonlinear systems. □

It is obvious that system (3.1) can be rewritten as

$$\dot{x} = \mathcal{A}x + \mathcal{B}u(t-\tau_1) + \mathcal{F}, \tag{3.2}$$

where

$$\mathcal{A} = \begin{bmatrix} 0_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \mathcal{F} = \begin{bmatrix} f_1(\cdot) \\ \vdots \\ f_{n-1}(\cdot) \\ 0 \end{bmatrix}.$$

The main result of this paper is stated in the following theorem.

Theorem 3.1: *If Assumption 3.1 holds for system (3.1) and there exists a positive parameter λ such that $\bar{\Phi}(\lambda) - \tilde{\Phi}(\lambda) - 2\bar{\tau}\hat{\Phi}(\lambda) > 0$, then the equilibrium at the origin of closed-loop system is global asymptotically stable by adopting the parameter-dependent state feedback controller*

$$u = \left[\frac{\Theta_1}{\lambda^n}, \frac{\Theta_2}{\lambda^{n-1}}, \cdots, \frac{\Theta_n}{\lambda}\right] x =: \Theta(\lambda)x, \tag{3.3}$$

where $\bar{\Phi}(\lambda)$, $\tilde{\Phi}(\lambda)$ and $\hat{\Phi}(\lambda)$ are defined in (3.14) and $\bar{\tau}$ is defined in system (3.1), $[\Theta_1, \cdots, \Theta_n] = \bar{\Theta}$ satisfying $\mathcal{A}_{\bar{\Theta}} =: \mathcal{A} + \mathcal{B}\bar{\Theta}$ is Hurwitz.

Proof. The proof procedure of Theorem 3.1 can be divided into two parts.

Part I: Introduce the coordinate transformations:

$$\xi = \begin{bmatrix} x_1 \\ \lambda x_2 \\ \vdots \\ \lambda^{n-1}x_n \end{bmatrix} =: \Gamma(\lambda)x, \tag{3.4}$$

where $\xi = [\xi_1, \xi_2, \cdots, \xi_n]^\top$ and $\Gamma(\lambda) = \mathrm{diag}[1, \lambda, \cdots, \lambda^{n-1}]$.

Meanwhile, according to (3.2)-(3.4) and using the fact of $\lambda\Gamma(\lambda)(\mathcal{A} + \mathcal{B}\Theta(\lambda)) = \mathcal{A}_{\bar{\Theta}}\Gamma(\lambda)$, the closed-loop system is transformed into

$$\dot{\xi} = \lambda^{-1}\mathcal{A}_{\bar{\Theta}}\xi + \Gamma(\lambda)\mathcal{B}(u(t-\tau_1) - u(t)) + \Gamma(\lambda)\mathcal{F}. \tag{3.5}$$

Choose the candidate Lyapunov function

$$V(\xi) = \frac{\delta_1}{2}\xi^\top P\xi, \tag{3.6}$$

where δ_1 is a positive constant and P is a symmetric positive definite matrix that satisfies $P\mathcal{A}_{\bar{\Theta}} + \mathcal{A}_{\bar{\Theta}}^\top P = -I$. Applying (3.5) and (3.6), one has

$$\dot{V} \leq -\delta_1\lambda^{-1}|\xi|^2 + 2\delta_1(\xi^\top P)(\Gamma(\lambda)\mathcal{F}) + 2\delta_1(\xi^\top P)\Gamma(\lambda)\mathcal{B}(u(t-\tau_1) - u(t)). \tag{3.7}$$

Let us consider the last two terms on the right-hand side.

By Assumption 3.1 and (3.4), one has

$$\begin{aligned}|\Gamma(\lambda)\mathcal{F}| &\leq c\sum_{i=1}^{n-1}\lambda^{i-1}(\sum_{l=i+1}^{n}|x_i|+\sum_{l=i+1}^{n}|x_l(t-\tau_l)|)\\ &= c\sum_{i=2}^{n}\sum_{k=0}^{i-2}\lambda^k(|x_i|+|x_i(t-\tau_i)|)\\ &= c\sum_{i=2}^{n}\sum_{k=1}^{i-1}\frac{1}{\lambda^k}(|\xi_i|+|\xi_i(t-\tau_i)|)\\ &\leq c\sum_{i=1}^{n-1}\frac{1}{\lambda^i}\sum_{i=2}^{n}(|\xi_i|+|\xi_i(t-\tau_i)|)\\ &\leq c\sqrt{n-1}\sum_{i=1}^{n-1}\frac{1}{\lambda^i}|\xi|+c\sum_{i=1}^{n-1}\frac{1}{\lambda^i}\sum_{j=2}^{n}|\xi_j(t-\tau_j)|. \end{aligned} \tag{3.8}$$

Then, by Lemma 2.2 and (3.8), one can obtain

$$\begin{aligned}2\delta_1(\xi^\top P)(\Gamma(\lambda)\mathcal{F}) &\leq \sqrt{n-1}\Phi_1(\lambda)|\xi|^2+\Phi_1(\lambda)\sum_{j=2}^{n}|\xi||\xi_j(t-\tau_j)|\\ &\leq \left(\frac{c_1^2(n-1)}{2}+\sqrt{n-1}\right)\Phi_1(\lambda)|\xi|^2+\frac{\Phi_1(\lambda)}{2c_1^2}\sum_{j=2}^{n}|\xi_j(t-\tau_j)|^2, \end{aligned} \tag{3.9}$$

where $\Phi_1(\lambda) = 2c\delta_1\|P\|\sum_{i=1}^{n-1}\frac{1}{\lambda^i}$ and c_1 is a positive real number.

By using Lemma 2.2, Assumption 3.1 and (3.3), one leads to

$$\begin{aligned}&2\delta_1(\xi^\top P)\Gamma(\lambda)\mathcal{B}(u(t-\tau_1)-u(t))\\ &\leq 2\frac{\|\Gamma(\lambda)\|}{\lambda^n}\sum_{i=1}^{n}\Theta_i\delta_1\|P\||\xi|\sum_{j=1}^{n}|\xi_j(t-\tau_1)-\xi_j(t)|\\ &\leq \Phi_2(\lambda)\left(c_2^2|\xi|^2+\frac{1}{c_2^2}\Big(\sum_{j=1}^{n}(\xi_j(t-\tau_1)-\xi_j(t))\Big)^2\right), \end{aligned} \tag{3.10}$$

where $\Phi_2(\lambda) = \frac{\|\Gamma(\lambda)\|}{\lambda^n}\delta_1\|P\|\sum_{i=1}^{n}|\Theta_i|$ and c_2 is a positive real number. When $i = \cdots, n-1$,

$$\begin{aligned}|\xi_j(t-\tau_1)-\xi_j(t)| &\leq \left|\int_{t-\tau_1}^{t}\dot{\xi}_j(\sigma)\mathrm{d}\sigma\right|\\ &\leq \int_{t-\tau_1}^{t}|\lambda^{-1}\xi_{j+1}+\lambda^{j-1}f_j|\mathrm{d}\sigma\\ &\leq \Phi_{3,j}(\lambda)\int_{t-\tau_1}^{t}\sum_{j=2}^{n}(|\xi_j(\sigma)|+|\xi_j(\sigma-\tau_j)|)\mathrm{d}\sigma, \end{aligned} \tag{3.11}$$

where $\Phi_{3,j}(\lambda) = (c+1)(\frac{1}{\lambda}+\sum_{l=j+1}^{n}\frac{1}{\lambda^{l-j}})$.

When $i = n$,

$$\begin{aligned} |\xi_n(t-\tau_1) - \xi_n(t)| &\le \left| \int_{t-\tau_1}^{t} \dot{\xi}_n(\sigma) \mathrm{d}\sigma \right| \\ &\le \frac{1}{\lambda} \int_{t-\tau_1}^{t} \sum_{j=1}^{n} |\Theta_j \xi_j(\sigma - \tau_1)| \mathrm{d}\sigma \\ &\le \Phi_4(\lambda) \int_{t-2\bar{\tau}}^{t} \Big(\sum_{j=1}^{n} |\xi_j(\sigma)| \Big) \mathrm{d}\sigma, \end{aligned} \tag{3.12}$$

where $\Phi_4(\lambda) = \frac{1}{\lambda} \sum_{j=1}^{n} |\Theta_j|$. With the help of Lemma 2.3, (3.10)-(3.12), one can obtain

$$\begin{aligned} & 2\delta_1 (\xi^\top P) \Gamma(\lambda) \mathcal{B}(u(t-\tau_1) - u(t)) \\ & \le \Phi_2(\lambda) \left(c_2^2 |\xi|^2 + \frac{2\bar{\tau}((\sum_{j=1}^{n-1} \Phi_{3,j}(\lambda))^2 + \Phi_4(\lambda)^2)}{c_2^2} \int_{t-2\bar{\tau}}^{t} \sum_{i=2}^{n} |\xi_i(\sigma)|^2 \mathrm{d}\sigma \right). \end{aligned} \tag{3.13}$$

Combining (3.9) and (3.13), one leads to

$$\begin{aligned} \dot{V} \le\ & -\left\{ \delta_1 \lambda^{-1} - \left(\frac{c_1^2(n-1)}{2} + \sqrt{n-1} \right) \Phi_1(\lambda) - c_2^2 \Phi_2(\lambda) \right\} |\xi|^2 \\ & + \frac{1}{2c_1^2} \Phi_1(\lambda) \sum_{j=1}^{n} |\xi_j(t-\tau_j)|^2 \\ & + \frac{1}{c_2^2} \Phi_2(\lambda) \Big(2\bar{\tau} \Big((\sum_{j=1}^{n-1} \Phi_{3,j}(\lambda))^2 + \Phi_4(\lambda)^2 \Big) \Big) \int_{t-2\bar{\tau}}^{t} \sum_{j=1}^{n} \xi_j^2(\sigma) \mathrm{d}\sigma \\ =:\ & -\bar{\Phi}(\lambda) |\xi|^2 + \tilde{\Phi}(\lambda) \sum_{j=1}^{n} |\xi_j(t-\tau_j)|^2 + \hat{\Phi}(\lambda) \int_{t-2\bar{\tau}}^{t} \sum_{j=1}^{n} |\xi_j(\sigma)|^2 \mathrm{d}\sigma. \end{aligned} \tag{3.14}$$

Construct the following L-K functional

$$\bar{V} = V + \tilde{\Phi}(\lambda) \sum_{i=1}^{n} \int_{t-\tau_j}^{t} |\xi_j(\sigma)|^2 \mathrm{d}\sigma + \hat{\Phi}(\lambda) \int_{t-2\bar{\tau}}^{t} \int_{\mu}^{t} \sum_{j=1}^{n} |\xi_j(\sigma)|^2 \mathrm{d}\sigma \mathrm{d}\mu, \tag{3.15}$$

then

$$\dot{\bar{V}} \le -(\bar{\Phi}(\lambda) - \tilde{\Phi}(\lambda) - 2\bar{\tau}\hat{\Phi}(\lambda)) |\xi|^2 =: -\phi(\lambda)|\xi|^2. \tag{3.16}$$

Taking $\gamma(s) = \phi(\lambda)s^2$ and applying $\bar{\Phi}(\lambda) - \tilde{\Phi}(\lambda) - 2\bar{\tau}\Phi(\lambda) > 0$, then $\gamma(s)$ is a $\mathcal{K}$ function and (3.16) is changed into

$$\dot{\bar{V}} \le -\gamma(|\xi|). \tag{3.17}$$

Part II: Next, we verify that $\bar{V}$ satisfies the first condition of Lemma 2.1.
On the basis of $|\xi| \le \sup_{-2\bar{\tau} \le \theta \le 0} |\xi(\theta + t)|$ and (3.6), one has

$$\pi_1(|\xi|) \le V \le \pi_{21}(\sup_{-2\bar{\tau} \le \theta \le 0} |\xi(\theta + t)|), \tag{3.18}$$

where $\pi_{21}(s) = \frac{\delta_1}{2}\lambda^2_{min}(P)s^2$ and $\pi_{21}(s) = \frac{\delta_1}{2}\lambda^2_{max}(P)s^2$ are class $\mathcal{K}_\infty$ functions.

$$\begin{aligned}
\tilde{\Phi}(\lambda)\sum_{i=2}^{n}\int_{t-\tau_j}^{t}|\xi_j(\sigma)|^2\mathrm{d}\sigma &\le \tilde{\Phi}(\lambda)\sum_{i=2}^{n}\int_{-\bar{\tau}}^{0}|\xi_j(\theta+t)|^2\mathrm{d}\theta \\
&\le \tilde{\Phi}(\lambda)\bar{\tau}\sup_{-\bar{\tau}\le\theta\le 0}|\xi(\theta+t)|^2 \\
&\le \tilde{\Phi}(\lambda)\bar{\tau}(\sup_{-2\bar{\tau}\le\theta\le 0}|\xi(\theta+t)|)^2 \\
&=: \pi_{22}(\sup_{-2\bar{\tau}\le\theta\le 0}|\xi(\theta+t)|),
\end{aligned} \tag{3.19}$$

where $\pi_{22}(s) = \tilde{\Phi}(\lambda)\bar{\tau}s^2$ is a class $\mathcal{K}_\infty$ function.

$$\begin{aligned}
&\hat{\Phi}(\lambda)\int_{t-2\bar{\tau}}^{t}\int_{\mu}^{t}\sum_{j=2}^{n}|\xi_j(\sigma)|^2\mathrm{d}\sigma\mathrm{d}\mu \\
&\le 2\hat{\Phi}(\lambda)\bar{\tau}\int_{t-2\bar{\tau}}^{t}\sum_{j=2}^{n}|\xi_j(\sigma)|^2\mathrm{d}\sigma \\
&\le 2\hat{\Phi}(\lambda)\bar{\tau}\int_{-2\bar{\tau}}^{0}\sum_{j=2}^{n}|\xi_j(s+\theta)|^2\mathrm{d}(s+\theta) \\
&\le 2\hat{\Phi}(\lambda)\bar{\tau}(\sup_{-2\bar{\tau}\le\theta\le 0}|\xi(\theta+t)|)^2 \\
&=: \pi_{23}(\sup_{-2\bar{\tau}\le\theta\le 0}|\xi(\theta+t)|),
\end{aligned} \tag{3.20}$$

where $\pi_{23}(s) = \hat{\Phi}(\lambda)\bar{\tau}s^2$ is a class $\mathcal{K}_\infty$ function. Choosing $\pi_2 = \pi_{21} + \pi_{22} + \pi_{23}$, one has

$$\pi_1(|\xi|) \le \bar{V} \le \pi_2(\sup_{-2\bar{\tau}\le\theta\le 0}|\xi(\theta+t)|). \tag{3.21}$$

According to (3.17), (3.21) and Lemma 2.1, one concludes that the equilibrium at the origin of closed-loop system is global asymptotic stabilization.

Remark 3.2: In the existing atricles, backstepping method is a usually method to design state feedback controller. However, this method requires calculation step-by-step. For n-order systems, the nonlinear term must be calculated at each step. From the above calculation process, it can be seen that the form of controller u has been given. We only need to calculate the last two terms of (3.7). And then the appropriate parameter λ can be selected according to (3.16). □

4. Extension

As a matter of fact, the design scheme of section 3 can also be generalized to a class of strict-feedforward stochastic nonlinear systems with multiple time-variant delays in the following form

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + \tilde{f}_1(x_2(t),\cdots,x_n(t),x_2(t-\tau_2(t)),\cdots,x_n(t-\tau_n(t))), \\
\dot{x}_2(t) &= x_3(t) + \tilde{f}_2(x_3(t),\cdots,x_n(t),x_3(t-\tau_3(t)),\cdots,x_n(t-\tau_n(t))), \\
&\vdots
\end{aligned}$$

$$\dot{x}_n(t) = u(t - \tau_1(t)), \tag{4.1}$$

where $\tau_j(t) : R^+ \to [0, \tau^*]$ is time-variant delay, $\tau^* > 0, j = 1, \cdots, n$. Nonlinear functions $\tilde{f}_i$, $i = 1, \cdots, n-1$, are continuous.

To obtain the stability theorem of system (4.1), we need the following assumptions.

Assumption 4.1: *There is a known positive constant $\bar{c}$ such that*

$$|f_j| \leq \bar{c} \sum_{j=i+1}^{n} (|x_j(t)| + |x_j(t - \tau_j(t))|), \quad j = 1, \cdots, n-1.$$

Assumption 4.2: *For $j = 1, \cdots, n$, there is a known constant β such that $\dot{\tau}_j(t) \leq \beta < 1$.*

Similar to (3.2), system (4.1) can be rewritten as

$$\dot{x} = \mathcal{A}x + \mathcal{B}u(t - \tau_1(t)) + \tilde{\mathcal{F}}, \tag{4.2}$$

where

$$\tilde{\mathcal{F}} = \begin{bmatrix} \tilde{f}_1(\cdot) \\ \vdots \\ \tilde{f}_{n-1}(\cdot) \\ 0 \end{bmatrix}.$$

Then the extended result is summarized in the following theorem.

Theorem 4.1: *If Assumptions 4.1 and 4.2 hold for systems (4.1) and there exists a positive parameter λ such that $\bar{\Psi}(\lambda) - \frac{\tilde{\Psi}(\lambda)}{1-\beta} - 2\tau^* \hat{\Psi}(\lambda) > 0$, then the equilibrium at the origin of closed-loop system is global asymptotically stable by adopting the parameter-dependent state feedback controller (3.3), where $\bar{\Psi}(\lambda)$, $\tilde{\Psi}(\lambda)$ and $\hat{\Phi}(\lambda)$ are defined in (4.8).*

Proof. We will prove it by two parts as well as Theorem 4.1.

Part I: Using (3.4)-(3.5) and (4.2), the closed-loop system is transformed into

$$\dot{\xi} = \lambda^{-1} \mathcal{A}_{\bar{\Theta}} \xi + +\Gamma(\lambda)\mathcal{B}(u(t - \tau_1(t)) - u(t)) + \Gamma(\lambda)\tilde{\mathcal{F}}. \tag{4.3}$$

Choose the candidate Lyapunov function

$$V_1(\xi) = \frac{\delta_2}{2} \xi^\top Q \xi, \tag{4.4}$$

where δ_2 is a positive constant and Q is a symmetric positive definite matrix that satisfies $Q\mathcal{A}_{\bar{\Theta}} + \mathcal{A}_{\bar{\Theta}}^\top Q = -I$. Applying (3.5) and (3.6), one has

$$\dot{V}_1 \leq -\delta_2 \lambda^{-1} |\xi|^2 + 2\delta_2(\xi^\top Q)(\Gamma(\lambda)\tilde{\mathcal{F}}) + 2\delta_2(\xi^\top Q)\Gamma(\lambda)\mathcal{B}(u(t - \tau_1(t)) - u(t)). \tag{4.5}$$

Similar to inequality (3.9), according to Assumption 4.1, one has

$$2\delta_2(\xi^\top P)(\Gamma(\lambda)\tilde{\mathcal{F}}) \quad \leq \quad \sqrt{n-1}\Psi_1(\lambda)|\xi|^2 + \Psi_1(\lambda) \sum_{j=2}^{n} |\xi||\xi_j(t - \tau_j)|$$

$$\leq \left(\frac{c_3^2(n-1)}{2} + \sqrt{n-1}\right)\Psi_1(\lambda)|\xi|^2 + \frac{1}{2c_3^2}\Psi_1(\lambda)\sum_{j=1}^{n}|\xi_j(t-\tau_j(t))|^2, \tag{4.6}$$

where $\Psi_1(\lambda) = 2\delta_2\|Q\|\sum_{k=1}^{n-1}\frac{1}{\lambda^k}$ and c_3 is a positive real number.

Similar to the proof of (3.13), using Assumptions 4.1,4.2 and (3.4), one leads to

$$\begin{aligned}&2\delta_2(\xi^\top P)\Gamma(\lambda)\mathcal{B}(u(t-\tau_1(t)) - u(t))\\ &\leq \Psi_2(\lambda)\left(c_4^2|\xi|^2 + \frac{2\tau^*\left((\sum_{j=1}^{n-1}\Psi_{3,j})^2 + \Psi_4(\lambda)^2\right)}{c_4^2}\int_{t-2\tau^*}^{t}\sum_{i=1}^{n}|\xi_i(\sigma)|^2\mathrm{d}\sigma\right),\end{aligned} \tag{4.7}$$

where $\Psi_2(\lambda) = \frac{1}{\lambda^n}\delta_2\|P\|\sum_{j=1}^{n}|\Theta_i|$, $\Psi_{3,j}(\lambda) = (\bar{c}+1)(\frac{1}{\lambda} + \sum_{l=j+1}^{n}\frac{1}{\lambda^{l-j}})$, $\Psi_4(\lambda) = \frac{1}{\lambda^n}\sum_{j=1}^{n}|\Theta_i|$ and c_4 is a positive real number. Combining (4.6) and (4.7), one leads to

$$\begin{aligned}\dot{V}_1 \leq\; & -\left\{\delta_2\lambda^{-1} - \left(\frac{c_3^2(n-1)}{2} + \sqrt{n-1}\right)\Psi_1(\lambda) - c_4^2\psi_2(\lambda)\right\}|\xi|^2\\ & + \frac{1}{2c_3^2}\Psi_1(\lambda)\sum_{j=1}^{n}|\xi_j(t-\tau_j)|^2\\ & + \frac{2\tau^*}{c_4^2}\Psi_2(\lambda)\Big((\sum_{j=1}^{n-1}\Psi_{3,j})^2 + \Psi_4(\lambda)^2\Big)\int_{t-2\tau^*}^{t}\sum_{j=1}^{n}\xi_j^2(\sigma)\mathrm{d}\sigma\\ =:\; & -\bar{\Psi}(\lambda)|\xi|^2 + \tilde{\Psi}(\lambda)\sum_{j=1}^{n}|\xi_j(t-\tau_j)|^2 + \hat{\Psi}(\lambda)\int_{t-2\tau^*}^{t}\sum_{j=1}^{n}|\xi_j(\sigma)|^2\mathrm{d}\sigma.\end{aligned} \tag{4.8}$$

Introduce the following L-K functional

$$\bar{V}_1 = V_1 + \frac{\tilde{\Psi}(\lambda)}{1-\beta}\sum_{i=2}^{n}\int_{t-\tau_j(t)}^{t}|\xi_j(\sigma)|^2\mathrm{d}\sigma + \hat{\Psi}(\lambda)\int_{t-2\tau^*}^{t}\int_{\mu}^{t}\sum_{j=2}^{n}|\xi_j(\sigma)|^2\mathrm{d}\sigma\mathrm{d}\mu, \tag{4.9}$$

then

$$\dot{\bar{V}}_1 \leq -\left(\bar{\Psi}(\lambda) - \frac{\tilde{\Psi}(\lambda)}{1-\beta} - 2\tau^*\hat{\Psi}(\lambda)\right)|\xi|^2 =: -\psi(\lambda)|\xi|^2. \tag{4.10}$$

Taking $\tilde{\gamma}(s) = \psi(\lambda)s^2$ and applying $\bar{\Psi}(\lambda) - \frac{\tilde{\Psi}(\lambda)}{1-\beta} - 2\tau^*\Psi(\lambda) > 0$, then $\tilde{\gamma}(s)$ is a class $\mathcal{K}$ function and (3.16) is changed into

$$\dot{\bar{V}}_1 \leq -\tilde{\gamma}(|\xi|). \tag{4.11}$$

Part II: Next, we verify that $\bar{V}_1$ satisfies the first condition of Lemma 2.1.

On the basis of $|\xi| \leq \sup_{-2\tau^*\leq\theta\leq 0}|\xi(\theta+t)|$ and (3.6), one has

$$\pi_3(|\xi|) \leq V_1 \leq \pi_{41}(\sup_{-2\tau^*\leq\theta\leq 0}|\xi(\theta+t)|), \tag{4.12}$$

where $\pi_{31}(s) = \frac{\delta_2}{2}\lambda_{min}^2(Q)s^2$ and $\pi_{41}(s) = \frac{\delta_2}{2}\lambda_{max}^2(Q)s^2$ are class $\mathcal{K}_\infty$ functions.

$$\frac{\tilde{\Psi}(\lambda)}{1-\beta}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t}|\xi_j(\sigma)|^2\mathrm{d}\sigma \leq \pi_{42}(\sup_{-2\tau^*\leq\theta\leq 0}|\xi(\theta+t)|), \tag{4.13}$$

where $\pi_{42}(s) = \frac{\tilde{\Psi}(\lambda)}{1-\beta}$ is a class $\mathcal{K}_\infty$ function.

$$\hat{\Psi}(\lambda)\int_{t-2\tau^*}^{t}\int_{\mu}^{t}\sum_{j=1}^{n}|\xi_j(\sigma)|^2\mathrm{d}\sigma\mathrm{d}\mu \leq \pi_{43}(\sup_{-2\tau^*\leq\theta\leq 0}|\xi(\theta+t)|), \tag{4.14}$$

where $\pi_{23}(s) = \tilde{\Psi}(\lambda)\bar{\tau}s^2$ is a class $\mathcal{K}_\infty$ function. Choosing $\pi_4 = \pi_{41} + \pi_{42} + \pi_{43}$, one has

$$\pi_3(|\xi|) \leq \bar{V}_1 \leq \pi_4(\sup_{-2\tau^*\leq\theta\leq 0}|\xi(\theta+t)|). \tag{4.15}$$

According to (4.12), (4.15) and Lemma 2.1, one concludes that the equilibrium at the origin of closed-loop system is global asymptotic stabilization.

5. Simulation example

For the sake of verifying the effectiveness of the proposed controller, we consider the following numerical example.

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + \frac{1}{10}x_2(t-2) + \frac{1}{3}\ln(1+x_3^2(t)),\\ \dot{x}_2(t) &= x_3(t) + \frac{1}{5}\sin(x_3(t-1))),\\ \dot{x}_3(t) &= u(t-1).\end{aligned} \tag{5.1}$$

With the help of $|\sin x| \leq |x|$, $\ln(1+x^2) \leq |x|$, Assumption 3.1 is held with $c = \frac{1}{3}$.

It is obvious that Assumption 3.1 holds. By system (5.1), $\mathcal{A} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0\\ 0\\ 1\end{bmatrix}$. Choosing $\Theta_1 = -\frac{3}{8}, \Theta_2 = -\frac{11}{8}, \Theta_3 = -\frac{5}{2}$, one has $\mathcal{A}_{\bar{\Theta}} = \mathcal{A} + \mathcal{B}\bar{\Theta} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{3}{8} & -\frac{11}{8} & -\frac{5}{2}\end{bmatrix}$. According to $P\mathcal{A}_{\bar{\Theta}} + \mathcal{A}_{\bar{\Theta}}^\top P = -I$, one obtains $P = \begin{bmatrix} 3.0298 & 3.8869 & 1.3333\\ 3.8869 & 8.6726 & 3.1905\\ 1.3333 & 3.1905 & 1.4762\end{bmatrix}$. Constructing the parameter-dependent controller

$$u = -\frac{3}{8\lambda^3}x_1 - \frac{11}{8\lambda^2}x_2 - \frac{5}{2\lambda}x_3. \tag{5.2}$$

Define $\xi = \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} = \begin{bmatrix} x_1\\ \lambda x_2\\ \lambda^2 x_3\end{bmatrix}$, consider the L-K functional $\bar{V} = \frac{1}{2}\xi^\top P\xi + 0.258(\frac{1}{\lambda} + \frac{2}{\lambda^2} + \frac{2}{\lambda^3})\sum_{i=1}^{3}\int_{t-\tau_i}^{t}|\xi_i(\sigma)|^2\mathrm{d}\sigma + 0.0446(\frac{1}{\lambda} + \frac{2}{\lambda^2} + \frac{4}{\lambda^3})\sum_{i=1}^{3}\int_{t-4}^{t}\int_{t}^{\mu}|\xi_i(\sigma)|^2\mathrm{d}\sigma\mathrm{d}\mu$. According to the design procedure, one can deduce $\bar{\Phi}(\lambda) = \frac{0.258}{\lambda} - 0.0516(\frac{1}{\lambda} + \frac{1}{\lambda^2}) - \frac{0.067\sqrt{1+\lambda^2+\lambda^4}}{\lambda^3}$, $\tilde{\Phi}(\lambda) = 0.021(\frac{1}{\lambda} + \frac{1}{\lambda^2})$, $\hat{\Phi}(\lambda) = \frac{0.0016\sqrt{1+\lambda^2+\lambda^4}}{\lambda^3}(\frac{48}{\lambda^3} + \frac{4.16}{\lambda^2})$. Then $\bar{\Phi}(\lambda) - \tilde{\Phi}(\lambda) - 2\bar{\tau}\hat{\Phi}(\lambda) > 0$ by taking $\lambda = 2$. Therefore, the condition of Theorem 3.1 is satisfied.

In the simulation, we take the initial data $x_1(0) = 1, x_2 = -0.5, x_3 = 0.5$, Figure 1 demonstrates the effectiveness of the controller.

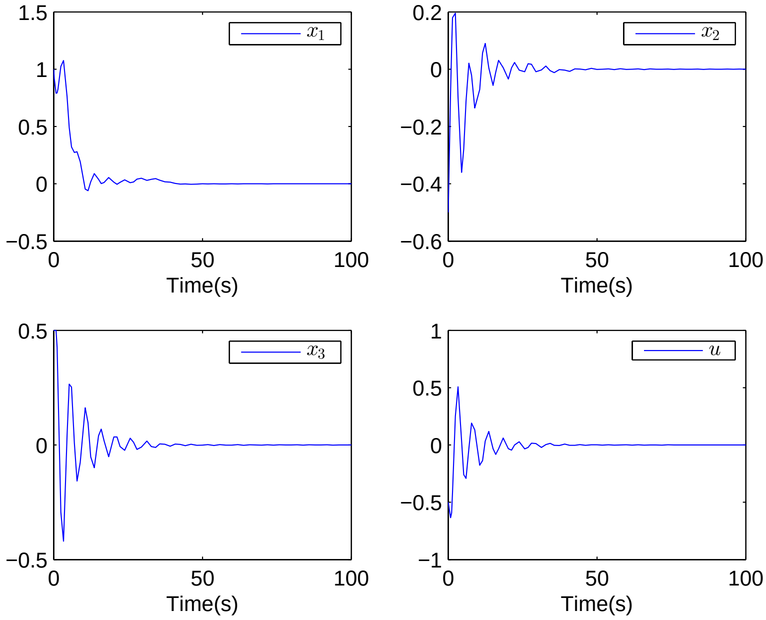

Figure 1. The responses of the closed-loop systems (5.1) and (5.2).

6. Conclusions

By introducing the Lyapunov-Krasoviskii functional and applying the stability criterion on time-delay system, a novel parameter-dependent state feedback controller is proposed to guarantee the global asymptotic stabilization of strict-feedforward nonlinear systems with time delays in state and input.

Some problems are still remained, e.g., 1) How to solve the problem of output feedback stabilization of the nonlinear strict-feedforward systems? 2) For the stochastic nonlinear systems with state and input delays, can we design the parameter-dependent state feedback controller?

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China [grant number 62103175].

Conflict of interest

All the authors declare no conflict of interest.

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