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Research article

The true twin classes-based investigation for connected local dimensions of connected graphs

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Abstract: Let G be a connected graph of order n. The representation of a vertex v of G with respect to an ordered set $W = \{w_1, w_2, ..., w_k\}$ is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$, where $d(v, w_i)$ represents the distance between vertices v and w_i for $1 \leq i \leq k$. An ordered set W is called a connected local resolving set of G if distinct adjacent vertices have distinct representations with respect to W, and the subgraph $\langle W \rangle$ induced by W is connected. A connected local resolving set of G of minimum cardinality is a connected local basis of G, and this cardinality is the connected local dimension $\text{cld}(G)$ of G. Two vertices u and v of G are true twins if $N[u] = N[v]$. In this paper, we establish a fundamental property of a connected local basis of a connected graph G. We analyze the connected local dimension of a connected graph without a singleton true twin class and explore cases involving singleton true twin classes. Our investigation reveals that a graph of order n contains at most two non-singleton true twin classes when $\text{cld}(G) = n - 2$. Essentially, our work contributes to the characterization of graphs with a connected local dimension of $n - 2$.

Keywords: representation; connected local resolving set; connected local dimension; true twin graph
Mathematics Subject Classification: 05C69

1. Introduction

For vertices u and v in a connected graph G, the distance $d(u, v)$ between u and v is the length of the shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices in G. The representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If the representations of any two distinct vertices in G with respect to W are distinct, then W is called a resolving set of G. A minimal cardinality resolving set is referred to as a minimum resolving set or a basis of G, and this cardinality is referred to as the dimension of G, which is denoted by dim(G).

The concept of a resolving set of a connected graph G was introduced by Slater in [16]. The usefulness of the concept was mentioned in [4–6]. Similar concepts were also discovered independently; see [3, 9]. The connected graphs of order n with dimension $n-2$ and $n-3$ were characterized in [2, 17], respectively. The concept of the resolving set lies within the theme of irregularity of graphs; see [1]. Further studies and applications of resolving sets were presented in [7, 10, 11, 13].

Some interesting developments in the concept of resolving sets are locality and connectivity. For any two adjacent vertices u and v of G, if $r(u|W) \neq r(v|W)$, then W is called a local resolving set of G. A minimum cardinality local resolving set is called a minimum local resolving set or a local basis of G, and this cardinality is said to be the local dimension ld(G) of G. For connectivity, a resolving set W of G is called a connected resolving set of G if the induced subgraph $\langle W \rangle$ is connected. The minimum cardinality of a connected resolving set of G is the connected dimension cd(G) of G, and a resolving set of G having this cardinality is called a minimum connected resolving set or a connected basis of G. To illustrate these concepts, consider the graph G of Figure 1. For an ordered set $W_1 = \{u, z\}$, the representations of vertices of G with respect to W_1 are

$$r(u|W_1) = (0,2), \quad r(v|W_1) = (1,2), \quad r(w|W_1) = (1,1),$$
$$r(x|W_1) = (2,1), \quad r(y|W_1) = (2,1), \quad r(z|W_1) = (2,0).$$

Hence, W_1 is a local resolving set of G since any two adjacent vertices of G have distinct representations with respect to W_1. However, W_1 is not a resolving set. Since G contains no local resolving set of cardinality 1, it follows that W_1 is a local basis of G, and so ld(G) = 2. For an ordered set $W_2 = \{u, x\}$, the representations of vertices of G with respect to W_2 are

$$r(u|W_2) = (0,2), \quad r(v|W_2) = (1,2), \quad r(w|W_2) = (1,1),$$
$$r(x|W_2) = (2,0), \quad r(y|W_2) = (2,2), \quad r(z|W_2) = (2,1).$$

We can see that W_2 is a resolving set of G. However, since $\langle W_2 \rangle$ is not connected, it follows that W_2 is not a connected resolving set of G. The idea of a local resolving set was introduced by Okamoto and others in [14]. They characterized all nontrivial connected graphs of order n with local dimensions 1, $n-1$, and $n-2$. The concept of a connected resolving set has been described in [15], and the term connected resolving number has been used to denote what we have referred to as the connected dimension.

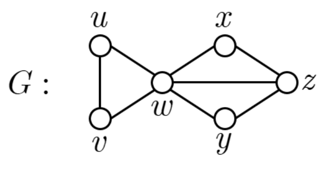

Figure 1. A connected graph G.

The two developments mentioned above lead us to study a local resolving set W of a connected graph G with the property that the induced subgraph $\langle W \rangle$ is connected in G. An ordered set W of vertices of a connected graph G is said to be a connected local resolving set of G if W is a local resolving set of G and the induced subgraph $\langle W \rangle$ of G is connected. A minimal cardinality connected local resolving set of G is called a minimum connected local resolving set or a connected local basis of G. The cardinality of a connected local basis of G is the connected local dimension, denoted by $\text{cld}(G)$.

Consider the graph G in Figure 1. Observe that $W_1 = \{u, z\}$ is a local resolving set, but it is not a connected local resolving set. For an ordered set $W_3 = \{u, w, z\}$, the representations of vertices in G with respect to W_3 are

$$r(u|W_3) = (0, 1, 2), \quad r(v|W_3) = (1, 1, 2), \quad r(w|W_3) = (1, 0, 1),$$
$$r(x|W_3) = (2, 1, 1), \quad r(y|W_3) = (2, 1, 1), \quad r(z|W_3) = (2, 1, 0).$$

Since the representations of two adjacent vertices are distinct, and $\langle W_3 \rangle = P_3$ is connected, it follows that W_3 is a connected local resolving set of G. Through a case-by-case analysis, it can be shown that W_3 is also a connected local basis of G, and thus $\text{cld}(G) = 3$. Connected local resolving sets were further studied in [8, 12].

Note that every connected local resolving set of G is a local resolving set of G, but the converse is not true in general. Furthermore, every connected resolving set of G is a connected local resolving set of G. Nevertheless, not every connected local resolving set of G is necessarily a connected resolving set of G. Therefore, we have arrived at the following:

$$1 \leq \text{ld}(G) \leq \text{cld}(G) \leq \text{cd}(G) \leq n - 1. \tag{1.1}$$

In fact, a characterization of local metric dimensions 1, $n-2$, and $n-1$ in a nontrivial connected graph of order n was established in [14]. Additionally, all connected graphs G of order $n \geq 2$ with $\text{cd}(G) = 1$, $n - 1$ were characterized in [15].

For every ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of a connected graph G, the only vertex of G whose representation with respect to W contains 0 in its i^{th} coordinate is w_i. Therefore, the vertices of W necessarily have distinct representations with respect to W. Furthermore, the representations of vertices of G that do not belong to W have coordinates, all of which are positive. Indeed, to determine whether an ordered set W is a connected local resolving set of G, we only need to verify that any two adjacent vertices in $V(G) - W$ have distinct representations with respect to W, and $\langle W \rangle$ is connected.

2. Connected local dimension with prescribed true twin classes

First, we present a principal property of a connected local basis of a connected graph G. We then recall that a vertex v of a connected graph G is a cut-vertex of G if $G - v$ is not connected. Furthermore, a set U of vertices of G is called a vertex-cut if $G - U$ is not connected.

Proposition 2.1. *Let W be a connected local basis of a connected graph G. Then, every vertex v of W satisfies at least one of the following conditions*:

(i) *$\langle W - \{v\} \rangle$ is not connected, or*

(ii) *there are two adjacent vertices x and y in $V(G) - (W - \{v\})$ such that $d(x, w) = d(y, w)$ for all vertices $w \in W - \{v\}$.*

Proof. Let v be a vertex of a connected local basis of a connected graph G. If v is a cut-vertex of $\langle W \rangle$, then (i) holds. Assume that v is not a cut-vertex of $\langle W \rangle$. Then, v does not satisfy (i). Hence, $\langle W - \{v\} \rangle$ is connected. Since W is a connected local basis of G, it follows that $\langle W - \{v\} \rangle$ is not a local resolving set of G. In other words, there exist two adjacent vertices x and y in G such that $r(x|W-\{v\}) = r(y|W-\{v\})$. This implies that $d(x, w) = d(y, w)$ for all $w \in W - \{v\}$. □

The open neighborhood, or simply the neighborhood, of a vertex u of a connected graph G is defined as the set of all vertices that are adjacent to u, which is denoted by $N(u) = \{v \in V(G) \mid uv \in E(G)\}$. The closed neighborhood $N[u]$ of u is defined as $N(u) \cup \{u\}$. Two vertices u and v of G are true twins if $N[u] = N[v]$. Observe that the true twin relation is an equivalence relation on $V(G)$, and as such, this relation partitions $V(G)$ into equivalence classes, which are called the true twin equivalence classes or simply the true twin classes on $V(G)$. Observe that if G contains l distinct true twin classes $U_1, U_2, ..., U_l$, then every local resolving set of G must contain at least $|U_i|-1$ vertices from U_i for each integer i with $1 \leq i \leq l$. This observation was presented in [14] as follows.

Proposition 2.2. *[14] Let G be a connected graph having l true twin classes $U_1, U_2, ..., U_l$. Then, every local resolving set of G must contain every vertex, except at most one, in each true twin class U_i, where $1 \leq i \leq l$. Moreover,* $\mathrm{ld}(G) \geq \sum_{i=1}^{l} |U_i| - l$.

The following result which appeared in [14] will be useful to us.

Theorem 2.1. *[14] If G is a nontrivial connected graph of order n with l true twin classes, none of which is a singleton set, then* $\mathrm{ld}(G) = n - l$.

The following theorem provides the connected local dimension of a connected graph that does not have a singleton true twin class.

Theorem 2.2. *If G is a connected graph of order n with l true twin classes, none of which is a singleton set, then* $\mathrm{cld}(G) = n - l$.

Proof. By Theorem 2.1, it follows that $\mathrm{ld}(G) = n - l$. Consequently, by (1.1), $\mathrm{cld}(G) \geq n - l$. Next, we show that there exists a connected local resolving set of G having cardinality $n - l$. In order to do this, let W be a local basis of G. By Proposition 2.2 and Theorem 2.1, $W = V(G) - \{u_1, u_2, ..., u_l\}$, where $u_1, u_2, ..., u_l$ belong to distinct true twin classes, resulting in $|W| = n-l$. We claim that $\langle W \rangle$ is connected. Let x and y represent two distinct vertices of W. Since G is connected, it follows that there is an $x-y$ path P in G. If $V(P) \subseteq W$, then x and y are connected in $\langle W \rangle$. We therefore assume that $V(P) \nsubseteq W$. Then, $V(P)$ contains u_i for some integer i with $1 \leq i \leq l$. Since G contains only non-singleton true twin classes, there is a vertex v_i such that v_i and u_i belong to the same true twin class. We construct an $x-y$ path Q from P by replacing u_i with v_i. If $V(Q) \subseteq W$, then x and y are connected in $\langle W \rangle$. If this is not the case, we continue the above procedure until finally arriving at x and y are connected in $\langle W \rangle$. Consequently, W is a connected local resolving set of G, that is, $\mathrm{cld}(G) \leq n-l$. Thus, $\mathrm{cld}(G) = n-l$. □

If a connected graph G contains some singleton true twin classes, then vertices in these true twin classes may or may not be in a connected local resolving set of G. Next, we investigate the connected local dimension of G having some singleton true twin classes. To do that, we first establish a definition. Let G be a connected graph containing at least two true twin classes. For two distinct true twin classes U and V of G, define the true twin distance $d(U, V)$ between U and V by $d(U, V) = d(u, v)$, where $u \in U$ and $v \in V$. Observe that $d(U, V) \geq 1$. Next, we present a useful lemma.

Lemma 2.1. *Let G be a connected graph having l true twin classes, and $d(U, V) = l - 1$ for some true twin classes U and V of G. Then, for each $u \in U$ and $v \in V$, every $u - v$ geodesic contains exactly one vertex from each true twin class. Furthermore, every $u - v$ path contains at least one vertex from each true twin class.*

Proof. Let U and V be distinct true twin classes of G with $d(U, V) = l - 1$, and let $u \in U$ and $v \in V$. Consider a $u - v$ geodesic $P = (u = u_1, u_2, ..., u_l = v)$ in G. Suppose that P contains two vertices u_i and u_j from the same true twin class for some integer i, j with $1 \leq i < j \leq l$. If $u_j \neq u_l$, then deleting the vertices $u_{i+1}, u_{i+2}, ..., u_j$ from P yields the $u - v$ path $(u = u_1, u_2, ..., u_i, u_{j+1}, ..., u_l = v)$ with length less than $l - 1$, which is impossible. If $u_j = u_l$, then deleting the vertices $u_i, u_{i+1}, ..., u_{j-1}$ from P yields the $u - v$ path $(u = u_1, u_2, ..., u_{i-1}, u_j = u_l = v)$ with length less than $l - 1$, which is also impossible. Thus, no two vertices of P belong to the same true twin class. Since P contains l vertices, it follows that P contains exactly one vertex from each true twin class.

Next, let $P' = (u = u'_1, u'_2, ..., u'_k = v)$ be a $u - v$ path of length $k - 1 \geq l - 1$. Assume that there is a true twin class U' of G such that every vertex in U' does not lie on P'. If P' contains two vertices u'_i and u'_j from the same true twin class for some integer i, j with $1 \leq i < j \leq k$, then, as in the case of P, we delete the vertices $u'_{i+1}, u'_{i+2}, ..., u'_j$ or $u'_i, u'_{i+1}, ..., u'_{j-1}$ from P', arriving at a $u - v$ path with length less than $k - 1$. If there are two vertices of this $u - v$ path belonging to the same true twin class, we continue the procedure until arriving at a $u - v$ path, Q', such that no two of its vertices belong to the same true twin class. Since Q' contains no vertices of U', the length of Q' is less than $l - 1$, which is a contradiction. Hence, P' contains at least one vertex from each true twin class. □

We are now prepared to present the connected local dimension of a connected graph containing some singleton true twin classes.

Theorem 2.3. *Let G be a connected graph having l true twin classes, and $d(U, V) = l - 1$ for some non-singleton true twin classes U and V of G. If there are p singleton true twin classes of G, then* $\text{cld}(G) = n - l + p$.

Proof. Let p be the number of singleton true twin classes in G. Then, $1 \leq p \leq l - 2$. Let $U_1, U_2, ..., U_l$ be true twin classes of G, where $|U_i| \geq 2$ for $1 \leq i \leq l - p$ and $|U_i| = 1$ for $l - p + 1 \leq i \leq l$, and let $u_i \in U_i$ for $1 \leq i \leq l$. First, we show that $W = V(G) - \{u_1, u_2, ..., u_{l-p}\}$ is a connected local resolving set of G. Let u_i and u_j be adjacent vertices in $V(G) - W$, where $1 \leq i < j \leq l - p$. As u_i and u_j belong to distinct true twin equivalence classes, there exists a vertex $v \in W$ that is adjacent to either u_i or u_j, but not both, say u_i. Consequently, $d(u_i, v) = 1 < 2 = d(u_j, v)$, implying that W is a local resolving set of G. We now claim that $\langle W \rangle$ is connected. Let x and y be vertices of W. Since G is connected, it follows that there is an $x - y$ path P in G. If P contains no u_i for $1 \leq i \leq l - p$, then $\langle W \rangle$ is connected. If P contains some vertices u_i for $1 \leq i \leq l - p$, then an $x - y$ path Q is obtained from P by replacing each u_i by v_i, where v_i is a vertex of U_i for $1 \leq i \leq l - p$. Thus, $\langle W \rangle$ is connected, and so W is a connected local resolving set of G. Therefore, $\text{cld}(G) \leq n - l + p$. To demonstrate $\text{cld}(G) \geq n - l + p$, let W' be a connected local resolving set of G. Since $d(U, V) = l - 1$ for some non-singleton true twin classes U and V of G, there exists a $u - v$ path P' of length $l - 1$, where $u \in U$ and $v \in V$. By Lemma 2.1, P' contains exactly one vertex from each true twin class. Consequently, W' must contain $u_i \in U_i$ for $l - p + 1 \leq i \leq l$. Since $|U_i| \geq 2$ for $1 \leq i \leq l - p$, W' must include at least $|U_i| - 1$ vertices from U_i for $1 \leq i \leq l - p$. Therefore, $|W'| \geq n - l + p$, that is, $\text{cld}(G) \geq n - l + p$. Hence, $\text{cld}(G) = n - l + p$. □

3. The characterization of connected graphs with connected local dimension $n - 2$

Consider a connected graph G with l distinct true twin classes denoted as $U_1, U_2, ..., U_l$. The true twin graph tG of G is defined as a graph having a vertex set $\{U_1, U_2, ..., U_l\}$. In tG, two distinct vertices U_i and U_j are adjacent if and only if $d(U_i, U_j) = 1$ (in G), where $1 \leq i < j \leq l$. Actually, if each of the true twin classes of G consists of a single vertex, then $tG = G$.

For example, the connected graph G given in Figure 2(a) has eight true twin classes $U_1 = \{u_1\}$, $U_2 = \{u_2, u_{10}\}$, $U_3 = \{u_3\}$, $U_4 = \{u_4\}$, $U_5 = \{u_5\}$, $U_6 = \{u_6, u_7\}$, $U_7 = \{u_8\}$, and $U_8 = \{u_9\}$. Then, the true twin graph tG has the vertex set $\{U_1, U_2, ..., U_8\}$, and this true twin graph is shown in Figure 2(b).

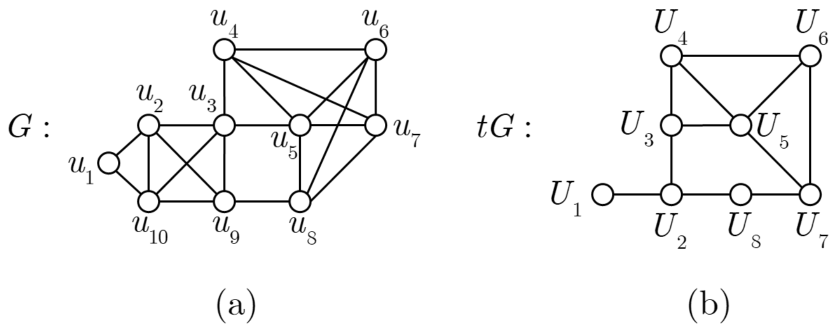

Figure 2. The connected graph G and its true twin graph tG.

Let u and v be vertices of a connected graph G belonging to distinct true twin classes. Then, $N[u] \neq N[v]$, and so there is a vertex w of G that is adjacent to either u or v, but not both. This concept leads to the following useful result.

Lemma 3.1. *Let x, y, and z be vertices belonging to distinct true twin classes of a connected graph G. Assume that $G - \{x, y, z\}$ is connected. If*

(i) $\langle\{x, y, z\}\rangle = K_3$,

(ii) $\langle\{x, y, z\}\rangle = (x, y, z)$*, a path of order* 3*, where x and z belong to non-singleton true twin classes,*

(iii) $\langle\{x, y, z\}\rangle = K_2 \cup K_1$*, or*

(iv) $\langle\{x, y, z\}\rangle = \overline{K_3}$,

then $V(G) - \{x, y, z\}$ is a connected local resolving set of G.

Proof. Let $W = V(G) - \{x, y, z\}$. Since $G - \{x, y, z\}$ is connected, it remains to prove that W is a local resolving set of G.

(i) Assume that $\langle\{x, y, z\}\rangle = K_3$. For any distinct $u, v \in \{x, y, z\}$, since u and v belong to distinct true twin classes, there exists a vertex w of W that is adjacent to either u or v, but not both. Consequently, $r(u|W) \neq r(v|W)$, and hence W is a local resolving set of G.

(ii) Assume that $\langle\{x, y, z\}\rangle = P_3 = (x, y, z)$, where x and z belong to non-singleton true twin classes. Then, there are two vertices x' and z' such that both x and x' belong to the same true twin class and both z and z' belong to the same true twin class. Since $d(x, z') = 2 > 1 = d(y, z')$ and $d(z, x') = 2 > 1 = d(y, x')$, $r(x|W) \neq r(y|W)$ and $r(z|W) \neq r(y|W)$, respectively. Therefore, W is a local resolving set of G.

(iii) Assume that $\langle\{x, y, z\}\rangle = K_2 \cup K_1$. Without loss of generality, let $V(K_2) = \{x, y\}$, and $V(K_1) = \{z\}$. Since x and y belong to distinct true twin classes, there is a vertex w of W such that w is adjacent to either x or y, but not both. Therefore, $r(x|W) \neq r(y|W)$, implying that W is a local resolving set of G.

(iv) Assume that $\langle\{x, y, z\}\rangle = \overline{K_3}$. Since $\{x, y, z\}$ is independent, it follows that W is a local resolving set of G. □

As we mentioned earlier, every connected local resolving set of a connected graph G must contain at least $|U| - 1$ vertices from U, where U is a true twin class of G. This implies that a connected graph G of order n contains at most two non-singleton true twin classes if $\text{cld}(G) = n - 2$, as we present next.

Theorem 3.1. *Let G be a connected graph of order n. If* $\text{cld}(G) = n - 2$*, then G contains at most two non-singleton true twin classes.*

Proof. Suppose, to the contrary, that there are three non-singleton true twin classes denoted as U_1, U_2, and U_3. For $1 \leq i \leq 3$, let $u_i \in U_i$. Observe that $G - \{u_1, u_2, u_3\}$ is connected. There are four possibilities of each induced subgraph $\langle\{u_1, u_2, u_3\}\rangle$ of G: $\langle\{u_1, u_2, u_3\}\rangle = K_3$, $\langle\{u_1, u_2, u_3\}\rangle = P_3$, $\langle\{u_1, u_2, u_3\}\rangle = K_2 \cup K_1$, or $\langle\{u_1, u_2, u_3\}\rangle = \overline{K_3}$. That $V(G) - \{u_1, u_2, u_3\}$ is a connected local resolving set of G is an immediate consequence of Lemma 3.1. Therefore, $\text{cld}(G) \leq n - 3$, which contradicts the fact that $\text{cld}(G) = n - 2$. □

Theorem 3.1 gives a necessary condition for a connected graph G of order n with $\text{cld}(G) = n - 2$. However, a connected graph G of order n containing at most two non-singleton true twin classes is not a sufficient condition for a graph G having $\text{cld}(G) = n - 2$. For example, when $n \geq 4$, a path P_n contains no non-singleton true twin class, but $\text{cld}(P_n) = 1 \neq n - 2$. Furthermore, Theorem 3.1 provides an important point for investigating a connected graph G of order n with the connected local dimension $n - 2$. To characterize all such graphs G, it suffices to consider connected graphs containing at most two non-singleton true twin classes. We first present the characterization of connected graphs G of order n that do not contain non-singleton true twin classes satisfying $\text{cld}(G) = n - 2$.

Theorem 3.2. *Let G be a connected graph of order n containing no non-singleton true twin class. Then,* $\text{cld}(G) = n - 2$ *if and only if* $tG = P_3$.

Proof. If $tG = P_3$, then $G = P_3$ since G contains only singleton true twin classes. It can be shown that $\text{cld}(P_3) = 1$. To verify the converse, assume that $\text{cld}(G) = n - 2$. For $n = 3$, only the graph $G = P_3$ has the desired property. For $n = 4$, all connected graphs of order 4 having only singleton true twin classes are P_4, $K_{1,3}$, and C_4. It is routine to verify that all of them have connected local dimensions of 1. This implies that there is no connected graph of order 4 with the connected local dimension 2. We therefore assume that $n \geq 5$. Then, there are three vertices x, y, and z of G such that $G - \{x, y, z\}$ is connected. Let $W = \{x, y, z\}$. If $\langle W\rangle = K_3$, $\langle W\rangle = K_2 \cup K_1$, or $\langle W\rangle = \overline{K_3}$, then $V(G) - W$ is a connected local resolving set of G by Lemma 3.1(i), (iii), and (iv), respectively. Therefore, $\text{cld}(G) \leq n - 3$, which contradicts the fact that $\text{cld}(G) = n - 2$. Assume that $\langle W\rangle = P_3 = (x, y, z)$. Since $\text{cld}(G) = n - 2$, it follows that $V(G) - W$ is not a local resolving set of G. Thus, we may assume, without loss of generality, that $r(x|W) = r(y|W)$. Then, $N[x] = N[y] - \{z\}$. Let $G' = G - \{x, y\}$. We consider two cases.

Case 1. z is adjacent to some vertex in G'.

Since G' is connected, it follows that there is a vertex $u \neq z$ in G' such that $G' - u$ is connected. Thus, $G - \{x, y, u\}$ is connected. We observe that the induced subgraph $\langle\{x, y, u\}\rangle$ of G is either K_3 or

$K_2 \cup K_1$. Nevertheless, $\langle\{x, y, u\}\rangle$ is a connected local resolving set of G by Lemma 3.1(i) and (iii), respectively. Therefore, $\text{cld}(G) \leq n-3$, establishing a contradiction.

Case 2. z is not adjacent to every vertex in G'.

Since G is connected and $N[x] = N[y] - \{z\}$, it follows that there is a vertex in $G' - z$ that is adjacent to both x and y, so $G - \{x, z\}$ remains connected. Thus, there is a vertex $v \neq y$ in $G - \{x, z\}$ such that $G - \{x, z, v\}$ is connected. We now obtain that the induced subgraph $\langle\{x, z, v\}\rangle$ of G is either $K_2 \cup K_1$ or $\overline{K_3}$. However, $\langle\{x, z, v\}\rangle$ is a connected local resolving set of G by Lemma 3.1(iii) and (iv), respectively. Thus, $\text{cld}(G) \leq n-3$, which is impossible.

Hence, for $n \geq 5$, there is no connected graph G of order n containing only singleton true twin classes such that $\text{cld}(G) = n-2$. This implies that $G = tG = P_3$. □

Next, we will identify all connected graphs G of order n containing exactly one non-singleton true twin class such that $\text{cld}(G) = n-2$. To do this, we first introduce some key notation. The eccentricity $e(u)$ of a vertex u in a connected graph G is the distance between u and a vertex farthest from u in G. The following lemma is useful.

Lemma 3.2. *Let G be a connected graph of order n containing exactly one non-singleton true twin class U. If* $\text{cld}(G) = n-2$, *then $e(u) \leq 2$, where $u \in U$.*

Proof. Assume, to the contrary, that $e(u) \geq 3$. Then, there is a vertex v of G such that $d(u, v) = k = e(u) \geq 3$. Let $(u = u_0, u_1, u_2, ..., u_k = v)$ be a $u - v$ geodesic in G. For the set $V(G) - \{u, u_{k-1}, u_k\}$, if $G - \{u, u_{k-1}, u_k\}$ is connected, then $V(G) - \{u, u_{k-1}, u_k\}$ is a connected local resolving set of G, and so $\text{cld}(G) \leq n-3$, contradicting the fact that $\text{cld}(G) = n-2$. Assume that $G - \{u, u_{k-1}, u_k\}$ is not connected. In other words, $\{u, u_{k-1}, u_k\}$ is a vertex-cut. Since u belongs to a non-singleton true twin class, $\{u_{k-1}, u_k\}$ is also a vertex-cut. Let G_1 be a component of $G - \{u_{k-1}, u_k\}$ that contains u. We claim that every vertex $x \in V(G) - (V(G_1) \cup \{u_{k-1}, u_k\})$ is adjacent to u_{k-1} in G. Suppose, contrary to our claim, that such a vertex x is not adjacent to u_{k-1} in G. Consequently, there exists a $u - x$ geodesic in G containing $u_1, u_2, ..., u_{k-1}, u_k$. This implies that G contains a $u - x$ path of length at least $k+1$, which contradicts the fact that $e(u) = k$. Hence, every vertex $x \in V(G) - (V(G_1) \cup \{u_{k-1}, u_k\})$ is adjacent to u_{k-1}. Therefore, $G - \{u, x, u_k\}$ is connected. Since the induced subgraph $\langle\{u, x, u_k\}\rangle$ of G is either $K_2 \cup K_1$ or $\overline{K_3}$, it follows by Lemma 3.1(iii) and (iv) that $V(G) - \{u, x, u_k\}$ is a connected local resolving set of G, that is, $\text{cld}(G) \leq n-3$, which is impossible. Thus, $e(u) \leq 2$. □

Theorem 3.3. *Let G be a connected graph of order n containing exactly one non-singleton true twin class. Then,* $\text{cld}(G) = n-2$ *if and only if $tG = P_3$.*

Proof. Assume that the true twin graph tG of G is the path $P_3 = (X, Y, Z)$. Since G contains three true twin classes, it follows by Proposition 2.2 that $\text{ld}(G) \geq n-3$, and so $\text{cld}(G) \geq n-3$. If either X or Z is a non-singleton true twin class of G, say X, then X is a connected local resolving set of G, that is, $\text{cld}(G) \leq n-2$. Indeed, G contains no connected local resolving set of cardinality $n-3$. Otherwise, a connected local resolving set W of G consists of $|X| - 1$ vertices of X. This implies that there are two vertices x and y, where $x \in X - W$ and $y \in Y$ with $r(x|W) = r(y|W)$, which is impossible. If Y is a non-singleton true twin class of G, then Y is a connected local resolving set of G, and so $\text{cld}(G) \leq n-2$. Similarly, G contains no connected resolving set of cardinality $n-3$. Therefore, $\text{cld}(G) = n-2$. Conversely, assume that $\text{cld}(G) = n-2$. Let U be the non-singleton true twin class of G. For a vertex u in U, we have that $e(u) \leq 2$ by Lemma 3.2. We therefore consider two cases according to the eccentricity of u.

Case 1. $e(u) = 1$.

Since G is not complete, and U is the only non-singleton true twin class of G, it follows that there are at least two vertices x and y that do not belong to U. Since $e(u) = 1$, it follows that u is adjacent to both x and y. We claim that $|V(G) - U| = 2$. Suppose, contrary to our claim, that there are at least three vertices of $V(G) - U$. Then, there are two adjacent vertices in $G - U$. Otherwise, there is a set S of three independent vertices in $V(G) - U$ such that $G - S$ is connected, and $\langle S \rangle = \overline{K_3}$. By Lemma 3.1(iv), S is a connected local resolving set of G, that is, $\mathrm{cld}(G) \leq n-3$, a contradiction. Let x and y be two adjacent vertices in $G - U$. Since $G - \{u, x, y\}$ is connected, and $\langle \{u, x, y\} \rangle = K_3$, it follows by Lemma 3.1(i) that $\{u, x, y\}$ is a connected local resolving set of G, and so $\mathrm{cld}(G) \leq n - 3$, which is impossible. Thus, as claimed, $|V(G) - U| = 2$. Since the two vertices in $V(G) - U$ are not true twins, it follows that $|V(tG)| = 3$, and so $tG = P_3$.

Case 2. $e(u) = 2$.

Then, there is a vertex $v \notin U$ with $d(u, v) = 2$. Let (u, x, v) be a $u - v$ geodesic in G. We first prove the following claim.

Claim. Every vertex in $V(G) - (U \cup \{x, v\})$ must be adjacent to u.

Suppose, contrary to our claim, that there is a vertex $z \in V(G) - (U \cup \{x, v\})$ that is not adjacent to u. Then, $d(u, z) = 2$, that is, $G - \{v, z\}$ is connected, so is $G - \{u, v, z\}$. Since the induced subgraph $\langle \{u, v, z\} \rangle$ is either $K_2 \cup K_1$ or $\overline{K_3}$, it follows by Lemma 3.1(iii) and (iv) that $V(G) - \{u, v, z\}$ is a connected local resolving set of G, which is impossible. Thus, every vertex in $V(G) - (U \cup \{x, v\})$ must be adjacent to u.

Next, we show that $V(G) - U = \{x, v\}$. Assume, to the contrary, that there is a vertex $z \in V(G) - (U \cup \{x, v\})$. By the claim, we obtain that z is adjacent to u. If x is not adjacent to z, then $V(G) - \{u, v, z\}$ is a connected local resolving set of G. This is a contradiction. Thus, x and z are adjacent. Assume that z is not adjacent to v. Since $\langle \{u, v, z\} \rangle = K_2 \cup K_1$, it follows by Lemma 3.1(iii) that $V(G) - \{u, v, z\}$ is a connected local resolving set of G, producing a contradiction. Therefore, z is adjacent to v. Since $\langle \{v, x, z\} \rangle = K_3$, it follows by Lemma 3.1(i) that $V(G) - \{v, x, z\}$ is a connected local resolving set of G. This is a contradiction. Hence, $V(G) - U = \{x, v\}$. Since x and v are not true twins, it follows that $|V(tG)| = 3$, and hence $tG = P_3$. □

Last, we investigate all connected graphs G containing two non-singleton true twin classes such that $\mathrm{cld}(G) = n - 2$.

Theorem 3.4. *Let G be a connected graph of order n containing exactly two non-singleton true twin classes. Then,* $\mathrm{cld}(G) = n - 2$ *if and only if*

$$tG = \begin{cases} P_3, & \text{if } d(U, V) = 1, \\ P_{k+1}, & \text{if } d(U, V) = k \geq 2, \end{cases}$$

where U and V are two distinct non-singleton true twin classes of G.

Proof. For $k \geq 2$, if $tG = P_{k+1}$, then G has $k + 1$ true twin classes. Since U and V are non-singleton true twin classes and $d(U, V) = k$, it follows by Theorem 2.3 that $\mathrm{cld}(G) = n - 2$. For $d(U, V) = 1$, if $tG = P_3$, then G has three true twin classes. Without loss of generality, consider $tG = (U, V, \{x\})$ and let $u \in U$ and $v \in V$. Since G contains no connected local resolving set of cardinality $n - 3$, it follows that $\mathrm{cld}(G) \geq n - 2$. It can be shown that $V(G) - \{u, v\}$ is a connected local resolving set of G. This implies that $\mathrm{cld}(G) = n - 2$. We now verify the converse. Assume that $\mathrm{cld}(G) = n - 2$. Let $u \in U$ and $v \in V$. We consider two cases.

Case 1. $d(U, V) = 1$.

Since u and v are not true twins, it follows that there is a vertex $x \in V(G) - (U \cup V)$ such that x is adjacent to every vertex in either U or V, but not both, say V. We claim that G have only three true twin classes U, V, and $\{x\}$. Suppose, contrary to our claim, that G contains another true twin class. If $e(v) = 1$, then there is a vertex $y \in V(G) - (U \cup V \cup \{x\})$ that is adjacent to v. Since every vertex in G is adjacent to v, it follows that $G - y$ is connected, and so is $G - \{u, v, y\}$. Since u, v, and y are not true twins, it follows that $V(G) - \{u, v, y\}$ is a connected local resolving set of G and so $\mathrm{cld}(G) \leq n - 3$, contradicting the fact that $\mathrm{cld}(G) = n - 2$. We may assume that $e(v) \geq 2$. Then, there is a vertex $z \in V(G) - (U \cup V \cup \{x\})$ with $d(v, z) = e(v) \geq 2$. Notice that $G - \{u, v, z\}$ is connected. Thus, $V(G) - \{u, v, z\}$ is a connected local resolving set of G, and so $\mathrm{cld}(G) \leq n - 3$, which is impossible. Hence, G has only three true twin classes U, V, and $\{x\}$, that is, $tG = P_3$.

Case 2. $d(U, V) = k \geq 2$.

Let $P = (u = u_0, u_1, ..., u_k = v)$ be a $u - v$ geodesic of G. Then, every internal vertex of P belongs to a singleton true twin class. We claim that $V(G) - (U \cup V) = \{u_1, u_2, ..., u_{k-1}\}$. Suppose, contrary to our claim, that there is a vertex x in $V(G) - (U \cup V \cup V(P))$. We consider two subcases.

Subcase 2.1. Neither u nor v is adjacent to x.

If x is not a cut-vertex of G, then $G - \{u, v, x\}$ is connected. Since $\langle\{u, v, x\}\rangle = \overline{K_3}$, it follows by Lemma 3.1(iv) that $V(G) - \{u, v, x\}$ is a connected local resolving set of G, producing a contradiction. We may assume that x is a cut-vertex of G. Then, there are at least two components G_1 and G_2 of $G - x$. Suppose that G_1 contains U and V. Thus, there is a vertex x' of G_2 such that $G - x'$ is connected, that is, $G - \{u, v, x'\}$ is also connected. Since $\langle\{u, v, x'\}\rangle = \overline{K_3}$, it follows that $V(G) - \{u, v, x'\}$ is a connected local resolving set of G, which is impossible.

Subcase 2.2. Either u or v is adjacent to x, say u.

Similarly, if x is not a cut-vertex of G, then $G - \{u, v, x\}$ is connected. Since $\langle\{u, v, x\}\rangle$ is $K_2 \cup K_1$ or P_3, it follows by Lemma 3.1(ii) and (iii), respectively, that $V(G) - \{u, v, x\}$ is a connected local resolving set of G. This is a contradiction. We therefore assume that x is a cut-vertex of G. Thus, there is a vertex x' in a component of $G - x$ not containing U and V such that $G - \{u, v, x'\}$ is connected. Observe that $\langle\{u, v, x'\}\rangle = \overline{K_3}$. Consequently, $V(G) - \{u, v, x'\}$ is a connected local resolving set of G by Lemma 3.1(iv). This is also a contradiction.

Hence, as claimed, $V(G) - (U \cup V) = \{u_1, u_2, ..., u_{k-1}\}$, and so $tG = P_{k+1}$. □

All connected graphs of order n with connected local dimension $n - 2$ are characterized by Theorems 3.2–3.4. The following result is a consequence of these theorems.

Corollary 3.1. *Let G be a connected graph of order n. Then,* $\mathrm{cld}(G) = n - 2$ *if and only if one of the following holds:*

(i) $tG = P_3$, *and G contains at most two non-singleton true twin classes.*

(ii) $tG = P_{k+1}$, *and G contains exactly two non-singleton true twin classes U and V, with* $d(U, V) = k \geq 3$.

Some examples of graphs with connected local dimension $n - 2$ are shown in Figure 3. Vertices in the same non-singleton true twin class in each graph are enclosed by a dashed circle. The true twin graphs of the graphs G_1 and G_2 are P_3, as seen Figure 3(a) and (b). In Figure 3(c), the true twin graph of the graph G_3 is P_5, and the distance between the two non-singleton true twin classes of G_3 is 4.

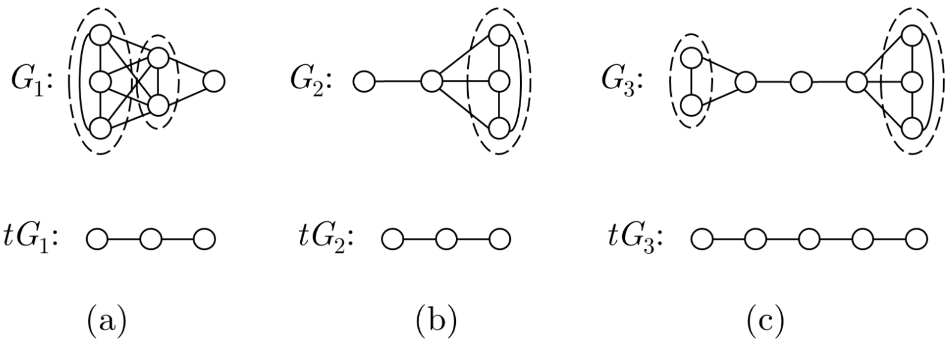

Figure 3. Graphs with connected local dimension $n-2$.

4. Conclusions

In this paper, we have established a principal property of a connected local basis of a connected graph G. In our analysis, we determined that for a connected graph G of order n with l true twin classes, none of which is a singleton set, the connected local dimension is given by $\mathrm{cld}(G) = n - l$. Extending our investigation to involve a connected graph G with l true twin classes and $d(U, V) = l - 1$ for some non-singleton true twin classes U and V of G, and if there are p singleton true twin classes in G, then $\mathrm{cld}(G) = n - l + p$. We demonstrated that, in a connected graph G of order n with a connected local dimension $\mathrm{cld}(G) = n-2$, there exist at most two non-singleton true twin classes. Ultimately, our research significantly contributes to the characterization of graphs with a connected local dimension of $n - 2$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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