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Research article

Combinatorial identities concerning trigonometric functions and Fibonacci/Lucas numbers

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Abstract: In this work, by means of the generating function method and the De Moivre's formula, we derive some interesting combinatorial identities concerning trigonometric functions and Fibonacci/Lucas numbers. One of them confirms the formula proposed recently by Svinin (2022).

Keywords: Fibonacci numbers; Lucas numbers; binomial coefficients; trigonometric functions
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1. Introduction and motivation

The Fibonacci and Lucas numbers are defined by the following recurrence relations [5, 6]

$$\begin{cases} F_0 = 0, F_1 = 1, \\ F_n = F_{n-1} + F_{n-2}, n \geq 2 \end{cases} \quad \text{and} \quad \begin{cases} L_0 = 2, L_1 = 1, \\ L_n = L_{n-1} + L_{n-2}, n \geq 2 \end{cases}$$

with the explicit formulae of Binet forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. It is interesting that they can be expressed by binomial coefficients, which can be found in Koshy [8, Eqs 12.1 and 13.5] and [1]

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} = F_n \quad \text{and} \quad \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} = L_n \quad \text{with} \quad n > 0.$$

In fact, there are two general binomial identities, which can be found in Carlitz [2, Page 23], Comtet [3, §4.9] and [5]

$$\sum_{0\le k<n/2}(-1)^k\binom{n-k-1}{k}(uv)^k(u+v)^{n-2k-1}=\frac{u^n-v^n}{u-v},$$
$$\sum_{0\le k<n/2}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(uv)^k(u+v)^{n-2k}=u^n+v^n.$$

When specifying $u=\alpha$ and $v=\beta$, they reduce to the formulae mentioned above.

Recently, Svinin [10] proposed a problem, where including demands to show that

$$\sum_{k=0}^{\lfloor\frac{n-1}{3}\rfloor}\frac{1}{2k+1}\binom{n-k-1}{2k}\left(\frac{4}{27}\right)^k=\frac{4^n-1}{3^{n-1}(2n+1)}, \tag{1.1}$$

which can be rewritten as

$$\sum_{k=0}^{\lfloor\frac{n-1}{3}\rfloor}\frac{2n+1}{n-k}\binom{n-k}{2k+1}\left(\frac{4}{27}\right)^k=\frac{4^n-1}{3^{n-1}}.$$

When making attempts to resolve (1.1), we find that a series of similar identities, concerning trigonometric function or Fibonacci/Lucas numbers, can be established. In the process of proving some of these formulae, we will use the De Moivre's formula

$$(\cos\theta+\mathbf{i}\sin\theta)^n=\cos n\theta+\mathbf{i}\sin n\theta,$$

by which we can get the following identities

$$(\cos\theta+\mathbf{i}\sin\theta)^n+(\cos\theta-\mathbf{i}\sin\theta)^n=2\cos(n\theta), \tag{1.2}$$
$$(\cos\theta+\mathbf{i}\sin\theta)^n-(\cos\theta-\mathbf{i}\sin\theta)^n=2\mathbf{i}\sin(n\theta), \tag{1.3}$$

where $\mathbf{i}$ denotes the imaginary unit.

In the following sections, we will evaluate, by means of generating functions and binomial linear relations, the following binomial sums

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{\tau n+\delta}{n-\upsilon k}\binom{n-\upsilon k}{\tau k+\delta}x^k,\quad\text{with}\quad \tau,\upsilon,\delta\in\{0,1,2\}.$$

Specifically, in the next section, we shall consider the sums

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}x^k,$$

where $\delta\in\{0,1\}$. Then in Section 3, we will examine the binomial sums

$$\sum_{k=0}^{\lfloor\frac{n}{3}\rfloor}\frac{n}{n-2k}\binom{n-2k}{k}x^k.$$

Finally in Section 4, the paper will end with two formulae, one of which is equivalent to (1.1).

For convenience, throughout the paper we shall make use of the notations: for a real number x, the symbol $\lfloor x \rfloor$ denotes the greatest integer $\le x$. When n is a natural number, the symbol $i \equiv_n j$ stands for "i is congruent to j modulo n". For a formal power series $f(x)$, its coefficient of x^n denote by $[x^n]f(x)$. Considering two formal power series $f(x)$ and $g(x)$, the following two equalities are well known [4, 9]

$$[x^n]xf(x) = [x^{n-1}]f(x), \tag{1.4}$$
$$[x^n](\alpha f(x) + \beta g(x)) = \alpha[x^n]f(x) + \beta[x^n]g(x), \tag{1.5}$$

which will be used frequently in the proofs in subsequent sections.

2. Case $\upsilon = 1$ and $\tau = 2$

Lemma 1. *The generating function of the sequence* $A_n(x) = \sum\limits_{k=0}^{n} \binom{n-k+\delta}{2k+\delta} x^k$ *is*

$$\mathcal{A}(z, x, \delta) = \sum_{n=0}^{\infty} A_n(x) z^n = \frac{1}{(1-z)^{\delta+1} - xz^3(1-z)^{\delta-1}}. \tag{2.1}$$

Proof. By exchanging the summation order, we have

$$\mathcal{A}(z, x, \delta) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n-k+\delta}{2k+\delta} x^k z^n = \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} \binom{n-k+\delta}{2k+\delta} z^n.$$

Making the replacement $n \to n + k - \delta$, the last equation becomes

$$\mathcal{A}(z, x, \delta) = \sum_{k=0}^{\infty} x^k z^{k-\delta} \sum_{n=0}^{\infty} \binom{n}{2k+\delta} z^n = \sum_{k=0}^{\infty} x^k z^{k-\delta} \sum_{n=2k+\delta}^{\infty} \binom{n}{2k+\delta} z^n.$$

Keeping in mind the formula [11, Eq 5.57]

$$\sum_{k=0}^{\infty} \binom{k}{n} z^k = \frac{z^n}{(1-z)^{n+1}},$$

we get the generating function

$$\mathcal{A}(z, x, \delta) = \sum_{k=0}^{\infty} \frac{x^k z^{3k}}{(1-z)^{2k+\delta+1}} = \frac{1}{(1-z)^{\delta+1} - xz^3(1-z)^{\delta-1}}.$$

□

2.1. Four combinatorial identities concerning trigonometric functions

Firstly, we establish four combinatorial identities concerning trigonometric functions.

Theorem 2 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor \frac{n-\delta}{3} \rfloor} \frac{2n+\delta}{n-k} \binom{n-k}{2k+\delta} 2^k = \begin{cases} 2^n + 2\cos\frac{n\pi}{2}, & \delta = 0; \\ 2^n + \sqrt{2}\sin\frac{(2n-1)\pi}{4}, & \delta = 1. \end{cases} \tag{2.2}$$

Proof. Letting $\delta = 1$ and $x = 2$ in (2.1), we have

$$\mathcal{A}(z, 2, 1) = \frac{1}{(1-z)^2 - 2z^3} = \frac{1}{(1-2z)(1+\mathbf{i}z)(1-\mathbf{i}z)}.$$

According to partial fraction decomposition, it can be decomposed into

$$\mathcal{A}(z, 2, 1) = \frac{1}{(1-2z)(1+\mathbf{i}z)(1-\mathbf{i}z)} = \frac{A}{1-2z} + \frac{B}{1+\mathbf{i}z} + \frac{C}{1-\mathbf{i}z},$$

where A, B and C are three parameters to be determined. Multiplying both sides of the equation by $1 - 2z$ and taking the limit at $z \to \frac{1}{2}$ yields

$$A = \lim_{z\to\frac{1}{2}} \frac{1-2z}{(1-2z)(1+\mathbf{i}z)(1-\mathbf{i}z)} = \lim_{z\to\frac{1}{2}} \frac{1}{(1+\mathbf{i}z)(1-\mathbf{i}z)} = \frac{4}{5}.$$

Similarly, multiplying both sides of the equation by $1 + \mathbf{i}z$ and taking the limit at $z \to \mathbf{i}$ yields

$$B = \lim_{z\to\mathbf{i}} \frac{1+\mathbf{i}z}{(1-2z)(1+\mathbf{i}z)(1-\mathbf{i}z)} = \lim_{z\to\mathbf{i}} \frac{1}{(1-2z)(1-\mathbf{i}z)} = \frac{2\mathbf{i}+1}{10}.$$

Multiplying both sides of the equation by $1 - \mathbf{i}z$ and taking the limit at $z \to -\mathbf{i}$ yields

$$C = \lim_{z\to-\mathbf{i}} \frac{1-\mathbf{i}z}{(1-2z)(1+\mathbf{i}z)(1-\mathbf{i}z)} = \lim_{z\to-\mathbf{i}} \frac{1}{(1-2z)(1+\mathbf{i}z)} = -\frac{2\mathbf{i}-1}{10}.$$

Therefore, we can decompose the rational fraction

$$\mathcal{A}(z, 2, 1) = \frac{1}{(1-z)^2 - 2z^3} = \frac{4}{5}\frac{1}{1-2z} + \frac{2\mathbf{i}+1}{10}\frac{1}{1+\mathbf{i}z} - \frac{2\mathbf{i}-1}{10}\frac{1}{1-\mathbf{i}z}.$$

Keeping in mind (1.4) and the fact that $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$, we can determine the coefficient of z^n

$$[z^n]\frac{1}{(1-z)^2 - 2z^3} = \frac{2^{n+2}}{5} + \frac{1}{10}\{[\mathbf{i}^n + (-\mathbf{i})^n] - 2\mathbf{i}[\mathbf{i}^n - (-\mathbf{i})^n]\}.$$

By means of the relations (1.2) and (1.3), we can get the formulae

$$\mathbf{i}^n + (-\mathbf{i})^n = 2\cos\frac{n\pi}{2} \quad \text{and} \quad \mathbf{i}^n - (-\mathbf{i})^n = 2\mathbf{i}\sin\frac{n\pi}{2},$$

which results in the following binomial summation formula

$$\sum_{k=0}^{n} \binom{n-k}{2k+1} 2^k = \frac{2^{n+1} + \sin\frac{n\pi}{2} - 2\cos\frac{n\pi}{2}}{5}. \tag{2.3}$$

Noting that when $\delta = 0$ and $x = 2$

$$\mathcal{A}(z, 2, 0) = (1-z)\mathcal{A}(z, 2, 1),$$

we have

$$[z^n]\mathcal{A}(z,2,0) = [z^n]\mathcal{A}(z,2,1) - [z^{n-1}]\mathcal{A}(z,2,1),$$

which yields another combinatorial identity below.

$$\sum_{k=0}^{n} \binom{n-k}{2k} 2^k = \frac{2^{n+1} + 3\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}}{5}. \tag{2.4}$$

Then by means of the binomial relation below

$$\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta} = 3\binom{n-k}{2k+\delta} - \binom{n-k-1}{2k+\delta}, \tag{2.5}$$

and after a little manipulation, we can derive, from (2.4) and (2.3), the identity in **Theorem** 2. □

Theorem 3 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor} \frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}(-4)^k = \begin{cases} 2^{n+1}\cos(n\theta_1) + (-1)^n, & \delta = 0; \\ 2^{n+\frac{1}{2}}\sin(n\theta_1+\theta_2) - \frac{(-1)^n}{2}, & \delta = 1, \end{cases} \tag{2.6}$$

where $\theta_1 = \arctan\frac{\sqrt{7}}{3}$ *and* $\theta_2 = \arctan\frac{\sqrt{7}}{7}$.

Proof. Letting $\delta = 1$ and $x = -4$ in (2.1), we have

$$(1-z)^2 + 4z^3 = 4(z+1)\left(z - \frac{3+\sqrt{7}\mathbf{i}}{8}\right)\left(z - \frac{3-\sqrt{7}\mathbf{i}}{8}\right).$$

By using partial fractional decomposition, we get the generating function

$$\begin{aligned}\mathcal{A}(z,-4,1) &= \frac{1}{(1-z)^2+4z^3} \\ &= \frac{1}{8}\frac{1}{1+z} + \frac{49+13\sqrt{7}\mathbf{i}}{112}\frac{1}{1-\frac{3-\sqrt{7}\mathbf{i}}{2}z} + \frac{49-13\sqrt{7}\mathbf{i}}{112}\frac{1}{1-\frac{3+\sqrt{7}\mathbf{i}}{2}z}.\end{aligned}$$

Computing the coefficient of z^n, we obtain

$$\begin{aligned}[z^n]\mathcal{A}(z,-4,1) =& \frac{(-1)^n}{8} + \frac{1}{112\cdot 2^n}\Big\{49[(3+\sqrt{7}\mathbf{i})^n + (3-\sqrt{7}\mathbf{i})^n] \\ &- 13\sqrt{7}\mathbf{i}[(3+\sqrt{7}\mathbf{i})^n - (3-\sqrt{7}\mathbf{i})^n]\Big\}.\end{aligned}$$

By means of the relations (1.2) and (1.3), we have

$$\left(3+\sqrt{7}\mathbf{i}\right)^n + \left(3-\sqrt{7}\mathbf{i}\right)^n = 2\cdot 4^n\cos\left(n\arctan\frac{\sqrt{7}}{3}\right),$$

$$\left(3+\sqrt{7}\mathbf{i}\right)^n - \left(3-\sqrt{7}\mathbf{i}\right)^n = 2\cdot 4^n\mathbf{i}\sin\left(n\arctan\frac{\sqrt{7}}{3}\right),$$

we can establish the following combinatorial identity:

$$\sum_{k=0}^{n}\binom{n-k}{2k+1}(-4)^k=\frac{2^n}{56}\left\{7\cos\left(n\arctan\frac{\sqrt{7}}{3}\right)+11\sqrt{7}\sin\left(n\arctan\frac{\sqrt{7}}{3}\right)\right\}-\frac{(-1)^n}{8}. \tag{2.7}$$

Keeping in mind the relations

$$\mathcal{A}(z,-4,0)=(1-z)\mathcal{A}(z,-4,1)$$

and

$$[z^n]\mathcal{A}(z,-4,0)=[z^n]\mathcal{A}(z,-4,1)-[z^{n-1}]\mathcal{A}(z,-4,1),$$

we can derive the identity below:

$$\sum_{k=0}^{n}\binom{n-k}{2k}(-4)^k=\frac{(-1)^n}{4}+\frac{2^n}{28}\left\{21\cos\left(n\arctan\frac{\sqrt{7}}{3}\right)+\sqrt{7}\sin\left(n\arctan\frac{\sqrt{7}}{3}\right)\right\}. \tag{2.8}$$

By employing the relation (2.5), we can get, from (2.8) and (2.7), the identity in **Theorem** 3. □

Theorem 4 ($n\in\mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}(-18)^k=\begin{cases}2\cdot 3^n\cos(n\theta_1)+(-2)^n, & \delta=0;\\ 3^{n-1}\sqrt{6}\sin(n\theta_1+\theta_2)-\frac{(-2)^n}{3}, & \delta=1,\end{cases} \tag{2.9}$$

where $\theta_1=\arctan\frac{\sqrt{5}}{2}$ *and* $\theta_2=\arctan\frac{\sqrt{5}}{5}$.

Proof. By setting $t=1$ and $x=-18$ in (2.1), we have

$$(1-z)^2+18z^3=18\left(z+\frac{1}{2}\right)\left(z-\frac{2+\sqrt{5}\mathbf{i}}{9}\right)\left(z-\frac{2-\sqrt{5}\mathbf{i}}{9}\right).$$

By means of the partial fractional decomposition, we can get the generating function

$$\mathcal{A}(z,-18,1)=\frac{4}{21}\frac{1}{1+2z}+\frac{85+16\sqrt{5}\mathbf{i}}{210}\frac{1}{1-\left(2-\sqrt{5}\mathbf{i}\right)z}+\frac{85-16\sqrt{5}\mathbf{i}}{210}\frac{1}{1-\left(2+\sqrt{5}\mathbf{i}\right)z},$$

and determine the coefficient of z^n

$$\begin{aligned}[z^n]\mathcal{A}(z,-18,1)=&\frac{4(-2)^n}{21}+\frac{3^n}{210}\left\{85\left[\left(\frac{2}{3}+\frac{\sqrt{5}}{3}\mathbf{i}\right)^n+\left(\frac{2}{3}-\frac{\sqrt{5}}{3}\mathbf{i}\right)^n\right]\right.\\&\left.-16\sqrt{5}\mathbf{i}\left[\left(\frac{2}{3}+\frac{\sqrt{5}}{3}\mathbf{i}\right)^n-\left(\frac{2}{3}-\frac{\sqrt{5}}{3}\mathbf{i}\right)^n\right]\right\}.\end{aligned}$$

Then we can derive, by utilizing the relations

$$\left(\frac{2}{3}+\frac{\sqrt{5}}{3}\mathbf{i}\right)^n+\left(\frac{2}{3}-\frac{\sqrt{5}}{3}\mathbf{i}\right)^n=2\cos\left(n\arctan\frac{\sqrt{5}}{2}\right),$$
$$\left(\frac{2}{3}+\frac{\sqrt{5}}{3}\mathbf{i}\right)^n-\left(\frac{2}{3}-\frac{\sqrt{5}}{3}\mathbf{i}\right)^n=2\mathbf{i}\sin\left(n\arctan\frac{\sqrt{5}}{2}\right),$$

the following summation formula:

$$\sum_{k=0}^{n}\binom{n-k}{2k+1}(-18)^k=\frac{4(-2)^{n-1}}{21}+\frac{3^n}{105}\chi(n),\tag{2.10}$$

where

$$\chi(n)=10\cos\left(n\arctan\frac{\sqrt{5}}{2}\right)+13\sqrt{5}\sin\left(n\arctan\frac{\sqrt{5}}{2}\right).$$

According to the relation

$$\mathcal{A}(z,-18,0)=(1-z)\mathcal{A}(z,-18,1),$$

we have

$$[z^n]\mathcal{A}(z,-18,0)=[z^n]\mathcal{A}(z,-18,1)-[z^{n-1}]\mathcal{A}(z,-18,1),$$

which leads to the following binomial sum:

$$\sum_{k=0}^{n}\binom{n-k}{2k}(-18)^k=\frac{2(-2)^n}{7}+\frac{3^n}{35}\left\{25\cos\left(n\arctan\frac{\sqrt{5}}{2}\right)+\sqrt{5}\sin\left(n\arctan\frac{\sqrt{5}}{2}\right)\right\}.\tag{2.11}$$

By means of the binomial relation (2.5), we can get, from (2.11) and (2.10), the summation formula stated in **Theorem** 4. □

Theorem 5 ($n\in\mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}36^k=\begin{cases}4^n+2(-3)^n\cos(n\theta_1), & \delta=0;\\ \frac{4^n}{3}+(-3)^{n-1}\sin(n\theta_1+\theta_2), & \delta=1,\end{cases}\tag{2.12}$$

where $\theta_1=\arctan 2\sqrt{2}$ *and* $\theta_2=\arctan\frac{\sqrt{2}}{2}$.

Proof. Letting $\delta=1$ and $x=36$ in (2.1), we have

$$(1-z)^2-36z^3=-36\left(z-\frac{1}{4}\right)\left(z-\frac{2\sqrt{2}\mathbf{i}-1}{9}\right)\left(z+\frac{2\sqrt{2}\mathbf{i}+1}{9}\right),$$

which results in, by the partial fractional decomposition, the generating function

$$\begin{aligned}\mathcal{A}(z,36,1)=\frac{16}{33}\frac{1}{1-4z}&+\frac{68+19\sqrt{2}\mathbf{i}}{264}\frac{1}{1+\left(1+2\sqrt{2}\mathbf{i}\right)z}\\&+\frac{68-19\sqrt{2}\mathbf{i}}{264}\frac{1}{1+\left(1-2\sqrt{2}\mathbf{i}\right)z}.\end{aligned}$$

Computing it's coefficient of z^n, we have

$$\begin{aligned}[z^n]\mathcal{A}(z,36,1)=&\frac{4^{n+2}}{33}+\frac{(-1)^n}{264}\Big\{68\Big[\left(1+2\sqrt{2}\mathbf{i}\right)^n+\left(1-2\sqrt{2}\mathbf{i}\right)^n\Big]\\&+19\sqrt{2}\mathbf{i}\Big[\left(1+2\sqrt{2}\mathbf{i}\right)^n-\left(1-2\sqrt{2}\mathbf{i}\right)^n\Big]\Big\}.\end{aligned}$$

Combining it with the relations

$$\left(1+2\sqrt{2}\mathbf{i}\right)^n+\left(1-2\sqrt{2}\mathbf{i}\right)^n=2\cdot 3^n\cos\left(n\arctan 2\sqrt{2}\right),$$
$$\left(1+2\sqrt{2}\mathbf{i}\right)^n-\left(1-2\sqrt{2}\mathbf{i}\right)^n=2\cdot 3^n\mathbf{i}\sin\left(n\arctan 2\sqrt{2}\right),$$

we can obtain the following identity:

$$\sum_{k=0}^{n}\binom{n-k}{2k+1}36^k=\frac{4^{n+1}}{33}-\frac{(-3)^n}{132}\eta(n), \tag{2.13}$$

where

$$\eta(n)=16\cos\left(n\arctan 2\sqrt{2}\right)+13\sqrt{2}\sin\left(n\arctan 2\sqrt{2}\right).$$

According to the relation between the generating functions of $\delta=1$ and $\delta=0$

$$\mathcal{A}(z,36,0)=(1-z)\mathcal{A}(z,36,1),$$

we have

$$[z^n]\mathcal{A}(z,36,0)=[z^n]\mathcal{A}(z,36,1)-[z^{n-1}]\mathcal{A}(z,36,1),$$

which yields the summation formula below:

$$\sum_{k=0}^{n}\binom{n-k}{2k}36^k=\frac{4^{n+1}}{11}+\frac{(-3)^n}{22}\left\{14\cos\left(n\arctan 2\sqrt{2}\right)-\sqrt{2}\sin\left(n\arctan 2\sqrt{2}\right)\right\}. \tag{2.14}$$

Now, employing the binomial relation (2.5), we can get, from (2.14) and (2.13), the identity stated in **Theorem** 5. □

2.2. Three identities concerning Fibonacci and Lucas numbers

In this section, three identities concerning Fibonacci and Lucas numbers are derived.

For $\lambda\neq\{0,\pm1,-2,-\frac{1}{2}\}$, letting $\delta=1$ and $x=x(\lambda)=\frac{(\lambda^2+\lambda)^2}{(\lambda^2+\lambda+1)^3}$ in (2.1), we have

$$(1-z)^2-xz^3=-\frac{(\lambda^2+\lambda)^2}{(\lambda^2+\lambda+1)^3}(z-(\lambda^2+\lambda+1))\left(z-\frac{\lambda^2+\lambda+1}{\lambda^2}\right)\left(z-\frac{\lambda^2+\lambda+1}{(\lambda+1)^2}\right).$$

By means of the partial fractional decomposition, we can get the generating function

$$\mathcal{A}(z,x(\lambda),1)=\frac{1}{(\lambda^2-1)(\lambda^2+2\lambda)}\frac{1}{1-\frac{z}{\lambda^2+\lambda+1}}-\frac{\lambda^4}{(\lambda^2-1)(2\lambda+1)}\frac{1}{1-\frac{\lambda^2}{\lambda^2+\lambda+1}z}$$
$$+\frac{(\lambda+1)^4}{(\lambda^2+\lambda)(2\lambda+1)}\frac{1}{1-\frac{(\lambda+1)^2}{\lambda^2+\lambda+1}z}.$$

Evaluating the coefficient of z^n, we get

$$[z^n]\mathcal{A}(z,x(\lambda),1)=\left\{\frac{1}{(\lambda^2-1)(\lambda^2+2\lambda)}-\frac{\lambda^{2n+4}}{(\lambda^2-1)(2\lambda+1)}+\frac{(\lambda+1)^{2n+4}}{(\lambda^2+2\lambda)(2\lambda+1)}\right\}$$

$$\times \frac{1}{(\lambda^2+\lambda+1)^n}$$
$$= \frac{2\lambda+1-\lambda^{2n+4}(\lambda^2+2\lambda)+(\lambda+1)^{2n+4}(\lambda^2-1)}{(\lambda^2-1)(\lambda^2+2\lambda)(2\lambda+1)(\lambda^2+\lambda+1)^n},$$

which results in the following identity:

$$\sum_{k=0}^{n}\binom{n-k}{2k+1}\left(\frac{(\lambda^2+\lambda)^2}{(\lambda^2+\lambda+1)^3}\right)^k = \frac{2\lambda+1-\lambda^{2n+2}(\lambda^2+2\lambda)+(\lambda+1)^{2n+2}(\lambda^2-1)}{(\lambda^2-1)(\lambda^2+2\lambda)(2\lambda+1)(\lambda^2+\lambda+1)^{n-1}}. \tag{2.15}$$

Theorem 6 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}\left(\frac{5}{64}\right)^k = \begin{cases}\frac{5^n+L_{2n}}{4^n}, & \delta=0;\\ \frac{5^n-F_{2n+1}}{4^{n-1}}, & \delta=1.\end{cases} \tag{2.16}$$

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}\left(\frac{9}{512}\right)^k = \begin{cases}\frac{9^n+L_{4n}}{8^n}, & \delta=0;\\ \frac{3^{2n+1}-L_{4n+2}}{3\cdot 8^{n-1}}, & \delta=1.\end{cases} \tag{2.17}$$

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}\left(\frac{5}{216}\right)^k = \begin{cases}\frac{5^n+L_{4n}}{6^n}, & \delta=0;\\ \frac{F_{4n+2}-5^n}{6^{n-1}}, & \delta=1.\end{cases} \tag{2.18}$$

Proof. Letting $\lambda=\alpha^2$, $\lambda=\alpha^4$ and $\lambda=-\alpha^4$ in (2.15), we can get the following three combinatorial identities:

$$\boxed{\lambda=\alpha^2} \qquad \sum_{k=0}^{n}\binom{n-k}{2k+1}\left(\frac{5}{64}\right)^k = \frac{5^{n+1}-F_{2n}-F_{2n+5}}{11\cdot 4^{n-1}},$$

$$\boxed{\lambda=\alpha^4} \qquad \sum_{k=0}^{n}\binom{n-k}{2k+1}\left(\frac{9}{512}\right)^k = \frac{3^{2n+3}-2F_{4n+4}-F_{4n+8}}{57\cdot 8^{n-1}},$$

$$\boxed{\lambda=-\alpha^4} \qquad \sum_{k=0}^{n}\binom{n-k}{2k+1}\left(\frac{5}{216}\right)^k = \frac{F_{4n+5}-5^{n+1}}{9\cdot 6^{n-1}}.$$

Observing that the generating function

$$\mathcal{A}(z,x(\lambda),0) = (1-z)\mathcal{A}(z,x(\lambda),1),$$

we can establish another three identities below about Fibonacci and Lucas numbers:

$$\sum_{k=0}^{n}\binom{n-k}{2k}\left(\frac{5}{64}\right)^k = \frac{5^{n+1}+10F_{2n}+3L_{2n}}{11\cdot 4^n},$$

$$\sum_{k=0}^{n}\binom{n-k}{2k}\left(\frac{9}{512}\right)^k = \frac{2\cdot 3^{2n+2}+9F_{4n+4}-L_{4n+4}}{38\cdot 8^n},$$

$$\sum_{k=0}^{n}\binom{n-k}{2k}\left(\frac{5}{216}\right)^k = \frac{5^{n+1}+L_{4n+3}}{9\cdot 6^n}.$$

Applying the binomial relation (2.5) to the above six identities, we can recover the three summation formulae stated in **Theorem** 6 □.

3. Case $\upsilon = 2$ and $\tau = 1$

Lemma 7. *The generating function of the sequence* $B_n(x) = \sum_{k=0}^{n} \binom{n-2k}{k} x^k$ *is*

$$\mathcal{B}(z,x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n-2k}{k} x^k z^n = \frac{1}{1-z-xz^3}. \tag{3.1}$$

Proof. Exchanging the summation order, we can evaluate

$$\mathcal{B}(z,x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n-2k}{k} x^k z^n = \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} \binom{n-2k}{k} z^n.$$

By making the replacement $n \to n + 2k$, the last equation becomes

$$\mathcal{B}(z,x) = \sum_{k=0}^{\infty} x^k z^{2k} \sum_{n=0}^{\infty} \binom{n}{k} z^n.$$

By means of the formula

$$\sum_{k=0}^{\infty} \binom{k}{n} z^k = \frac{z^n}{(1-z)^{n+1}},$$

we get the generating function

$$\mathcal{B}(z,x) = \sum_{k=0}^{\infty} \frac{x^k z^{3k}}{(1-z)^{k+1}} = \frac{1}{1-z-xz^3}. \qquad \square$$

3.1. Three formulae concerning trigonometric functions

Analogously, in this section we establish another three formulae concerning trigonometric functions.

Theorem 8 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{n}{n-2k} \binom{n-2k}{k} (-2)^k = 2^{\frac{n+2}{2}} \cos \frac{n\pi}{4} + (-1)^n. \tag{3.2}$$

Proof. Letting $x = -2$ in Lemma 7, we have the generating function

$$\mathcal{B}(z,-2) = \frac{1}{1-z+2z^3} = \frac{1}{5}\frac{1}{1+z} + \frac{2-\mathbf{i}}{5}\frac{1}{1-(1+\mathbf{i})z} + \frac{2+\mathbf{i}}{5}\frac{1}{1-(1-\mathbf{i})z}.$$

By extracting the coefficient of z^n, we get

$$\begin{aligned}[z^n]\mathcal{B}(z,-2) = &= \frac{(-1)^n}{5} + \frac{2-\mathbf{i}}{5}(1+\mathbf{i})^n + \frac{2+\mathbf{i}}{5}(1-\mathbf{i})^n \\ &= \frac{(-1)^n}{5} + \frac{1}{5}\Big\{2\big[(1+\mathbf{i})^n + (1-\mathbf{i})^n\big] + \mathbf{i}\big[(1-\mathbf{i})^n - (1+\mathbf{i})^n\big]\Big\}.\end{aligned}$$

By means of the identities [7, Eqs. 1.90 and 1.96]

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}(-1)^k = \frac{(1+\mathbf{i})^n + (1-\mathbf{i})^n}{2} = 2^{\frac{n}{2}} \cos\frac{n\pi}{4},$$
$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1}(-1)^k = \frac{(1+\mathbf{i})^n - (1-\mathbf{i})^n}{2\mathbf{i}} = 2^{\frac{n}{2}} \sin\frac{n\pi}{4},$$

we obtain the following identity

$$\sum_{k=0}^{n} \binom{n-2k}{k}(-2)^k = \frac{1}{5}\left\{2^{\frac{n+2}{2}}\left(2\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right) + (-1)^n\right\}.$$

Then the desired identity follows by using the binomial relation (3.7). □

Theorem 9 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{n}{n-2k}\binom{n-2k}{k}18^k = 2\left(-\sqrt{6}\right)^n \cos\left(n \arctan\sqrt{5}\right) + 3^n. \tag{3.3}$$

Proof. Letting $x = 18$ in Lemma 7, we have the generating function

$$\begin{aligned}\mathcal{B}(z, 18) &= \frac{1}{1 - z - 18z^3}\\ &= \frac{3}{7}\frac{1}{1-3z} + \frac{10+\sqrt{5}\mathbf{i}}{35}\frac{1}{1+\left(\sqrt{5}\mathbf{i}+1\right)z} + \frac{10-\sqrt{5}\mathbf{i}}{35}\frac{1}{1-\left(\sqrt{5}\mathbf{i}-1\right)z}.\end{aligned}$$

Evaluating the coefficient of z^n, we get

$$\begin{aligned}[z^n]\mathcal{B}(z, 18) = \frac{3^{n+1}}{7} + \frac{(-1)^n}{35}\Big\{&10\Big[\left(1+\sqrt{5}\mathbf{i}\right)^n + \left(1-\sqrt{5}\mathbf{i}\right)^n\Big]\\ &+ \sqrt{5}\mathbf{i}\Big[\left(1+\sqrt{5}\mathbf{i}\right)^n - \left(1-\sqrt{5}\mathbf{i}\right)^n\Big]\Big\}.\end{aligned}$$

Keeping in mind that

$$(\cos\theta + \mathbf{i}\sin\theta)^n + (\cos\theta - \mathbf{i}\sin\theta)^n = 2\cos(n\theta),$$
$$(\cos\theta + \mathbf{i}\sin\theta)^n - (\cos\theta - \mathbf{i}\sin\theta)^n = 2\mathbf{i}\sin(n\theta),$$

we have

$$\begin{aligned}\left(1+\sqrt{5}\mathbf{i}\right)^n + \left(1-\sqrt{5}\mathbf{i}\right)^n &= 6^{\frac{n}{2}}\left\{\left(\frac{1}{\sqrt{6}} + \frac{\sqrt{5}}{\sqrt{6}}\mathbf{i}\right)^n + \left(\frac{1}{\sqrt{6}} - \frac{\sqrt{5}}{\sqrt{6}}\mathbf{i}\right)^n\right\}\\ &= 6^{\frac{n}{2}}\left\{(\cos t + \mathbf{i}\sin t)^n + (\cos t - \mathbf{i}\sin t)^n\right\}\\ &= 2\cdot 6^{\frac{n}{2}}\cos(nt),\end{aligned}$$

and

$$\left(1+\sqrt{5}\mathbf{i}\right)^n-\left(1-\sqrt{5}\mathbf{i}\right)^n=6^{\frac{n}{2}}\left\{\left(\frac{1}{\sqrt{6}}+\frac{\sqrt{5}}{\sqrt{6}}\mathbf{i}\right)^n-\left(\frac{1}{\sqrt{6}}-\frac{\sqrt{5}}{\sqrt{6}}\mathbf{i}\right)^n\right\}$$
$$=6^{\frac{n}{2}}\{(\cos t+\mathbf{i}\sin t)^n-(\cos t-\mathbf{i}\sin t)^n\}$$
$$=2\cdot 6^{\frac{n}{2}}\mathbf{i}\sin(nt),$$

where $\tan t=\sqrt{5}$, which yield the identity

$$\sum_{k=0}^{n}\binom{n-2k}{k}18^k=\frac{3^{n+1}}{7}+\frac{2(-\sqrt{6})^n}{35}\left\{10\cos\left(n\arctan\sqrt{5}\right)-\sqrt{5}\sin\left(n\arctan\sqrt{5}\right)\right\}.$$

The desired identity follows by means of (3.7). □

Theorem 10 ($n\in\mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n}{3}\rfloor}\frac{n}{n-2k}\binom{n-2k}{k}(-36)^k=2\left(2\sqrt{3}\right)^n\cos\left(n\arctan\sqrt{2}\right)+(-3)^n. \tag{3.4}$$

Proof. Letting $x=-36$ in Lemma 7, we have the generating function

$$\mathcal{B}(z,-36)=\frac{1}{36z^3-z+1}$$
$$=\frac{3}{11}\frac{1}{1+3z}+\frac{8+\sqrt{2}\mathbf{i}}{22}\frac{1}{1-2\left(1-\sqrt{2}\mathbf{i}\right)z}+\frac{8-\sqrt{2}\mathbf{i}}{22}\frac{1}{1-2\left(1+\sqrt{2}\mathbf{i}\right)z}.$$

and its coefficient of z^n

$$[z^n]\mathcal{B}(z,-36)=\frac{3(-3)^n}{11}+\frac{2^n}{22}\left\{8\left[\left(1-\sqrt{2}\mathbf{i}\right)^n+\left(1+\sqrt{2}\mathbf{i}\right)^n\right]\right.$$
$$\left.+\sqrt{2}\mathbf{i}\left[\left(1-\sqrt{2}\mathbf{i}\right)^n-\left(1+\sqrt{2}\mathbf{i}\right)^n\right]\right\}.$$

By means of the following results

$$\left(1-\sqrt{2}\mathbf{i}\right)^n+\left(1+\sqrt{2}\mathbf{i}\right)^n=2\cdot 3^{\frac{n}{2}}\cos\left(n\arctan\sqrt{2}\right),$$

and

$$\left(1-\sqrt{2}\mathbf{i}\right)^n-\left(1+\sqrt{2}\mathbf{i}\right)^n=2\cdot 3^{\frac{n}{2}}\mathbf{i}\sin\left(n\arctan\sqrt{2}\right),$$

we can get the identity

$$\sum_{k=0}^{n}\binom{n-2k}{k}(-36)^k=\frac{3(-3)^n}{11}+\frac{(2\sqrt{3})^n}{11}\left\{8\cos\left(n\arctan\sqrt{2}\right)+\sqrt{2}\sin\left(n\arctan\sqrt{2}\right)\right\}.$$

Then the proof completes by the aid of (3.7). □

3.2. Four identities concerning Fibonacci and Lucas numbers

For $x = x(\lambda) = -\frac{(\lambda^2+\lambda)^2}{(\lambda^2+\lambda+1)^3}$, $\lambda \neq \{0, \pm 1, -2, -\frac{1}{2}\}$, we have

$$1 - z - xz^3 = \frac{(\lambda^2+\lambda)^2}{(\lambda^2+\lambda+1)^3}\Big(z + \frac{\lambda^2+\lambda+1}{\lambda}\Big)\Big(z - \frac{\lambda^2+\lambda+1}{\lambda+1}\Big)\Big(z - \frac{\lambda^2+\lambda+1}{\lambda^2+\lambda}\Big).$$

By means of the partial fractional decomposition, we can get the generating function

$$\begin{aligned}\mathcal{B}(z, x(\lambda)) = & \frac{\lambda}{(\lambda+2)(2\lambda+1)}\frac{1}{1+\frac{\lambda}{\lambda^2+\lambda+1}z} - \frac{\lambda+1}{(2\lambda+1)(\lambda-1)}\frac{1}{1-\frac{\lambda+1}{\lambda^2+\lambda+1}z}\\ & + \frac{\lambda(\lambda+1)}{(\lambda+2)(\lambda-1)}\frac{1}{1-\frac{\lambda(\lambda+1)}{\lambda^2+\lambda+1}z}.\end{aligned}$$

Evaluating the coefficient of z^n, we get

$$[z^n]\mathcal{B}(z, x(\lambda)) = \frac{1}{(\lambda^2+\lambda+1)^n}\Big\{\frac{(-1)^n\lambda^{n+1}}{(\lambda+2)(2\lambda+1)} - \frac{(\lambda+1)^{n+1}}{(2\lambda+1)(\lambda-1)} + \frac{(\lambda^2+\lambda)^{n+1}}{(\lambda+2)(\lambda-1)}\Big\},$$

which results in the following identity:

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n-2k}{k}\Big(\frac{(\lambda^2+\lambda)^2}{(\lambda^2+\lambda+1)^3}\Big)^k\\ &= \frac{1}{(\lambda^2+\lambda+1)^n}\Big\{\frac{(-1)^n\lambda^{n+1}}{(\lambda+2)(2\lambda+1)} - \frac{(\lambda+1)^{n+1}}{(2\lambda+1)(\lambda-1)} + \frac{(\lambda^2+\lambda)^{n+1}}{(\lambda+2)(\lambda-1)}\Big\}.\end{aligned} \tag{3.5}$$

Theorem 11 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{n}{n-2k}\binom{n-2k}{k}\Big(-\frac{1}{8}\Big)^k = \frac{L_n+1}{2^n}. \tag{3.6}$$

Proof. Let $\lambda = \alpha$ in (3.5), we have $x(\lambda) = -\frac{1}{8}$ and

$$\sum_{k=0}^{n}\binom{n-2k}{k}\Big(-\frac{1}{8}\Big)^k = \frac{F_{n+3}-1}{2^n}.$$

By means of the relation

$$\frac{n}{n-2k}\binom{n-2k}{k} = 3\binom{n-2k}{k} - 2\binom{n-2k-1}{k}, \tag{3.7}$$

we have

$$\begin{aligned}\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{n}{n-2k}\binom{n-2k}{k}\Big(-\frac{1}{8}\Big)^k &= 3\sum_{k=0}^{n}\binom{n-2k}{k}\Big(-\frac{1}{8}\Big)^k - 2\sum_{k=0}^{n-1}\binom{n-2k-1}{k}\Big(-\frac{1}{8}\Big)^k\\ &= \frac{3F_{n+3}-3}{2^n} - \frac{2F_{n+2}-2}{2^{n-1}} = \frac{3F_{n+3}-4F_{n+2}+1}{2^n}.\end{aligned}$$

Then the proof follows by using

$$3F_{n+3} - 4F_{n+2} = L_n.$$

□

Theorem 12 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n}{3}\rfloor} \frac{n}{n-2k}\binom{n-2k}{k}\Big(-\frac{5}{64}\Big)^k = \begin{cases} \frac{5^{\frac{n}{2}}L_n+1}{4^n}, & n \equiv_2 0; \\ \frac{5^{\frac{n+1}{2}}F_n-1}{4^n}, & n \equiv_2 1. \end{cases} \tag{3.8}$$

Proof. Let $\lambda = \alpha^2$ in (3.5), we have $x(\lambda) = -\frac{5}{64}$ and

$$\sum_{k=0}^{n}\binom{n-2k}{k}\Big(-\frac{5}{64}\Big)^k = \begin{cases} \frac{1}{11\cdot 4^n}\Big\{1+5^{\frac{n+2}{2}}(L_n+4F_n)\Big\}, & n \equiv_2 0; \\ \frac{1}{11\cdot 4^n}\Big\{5^{\frac{n+1}{2}}(4L_n+5F_n)-1\Big\}, & n \equiv_2 1. \end{cases}$$

Then the proof follows by using the relation (3.7). □

Theorem 13 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n}{3}\rfloor} \frac{n}{n-2k}\binom{n-2k}{k}\Big(-\frac{5}{216}\Big)^k = \begin{cases} \frac{5^{\frac{n}{2}}L_{2n}+1}{6^n}, & n \equiv_2 0; \\ \frac{5^{\frac{n+1}{2}}F_{2n}+1}{6^n}, & n \equiv_2 1. \end{cases} \tag{3.9}$$

Proof. Let $\lambda = -\alpha^4$ in (3.5), we have $x(\lambda) = -\frac{5}{216}$ and

$$\sum_{k=0}^{n}\binom{n-2k}{k}\Big(-\frac{5}{216}\Big)^k = \begin{cases} \frac{1}{9\cdot 6^n}\Big\{5^{\frac{n+2}{2}}F_{2n+3}-1\Big\}, & n \equiv_2 0; \\ \frac{1}{9\cdot 6^n}\Big\{5^{\frac{n+1}{2}}L_{2n+3}-1\Big\}, & n \equiv_2 1. \end{cases}$$

By means of the relation (3.7), we can complete the proof. □

Theorem 14 ($n \in \mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n}{3}\rfloor} \frac{n}{n-2k}\binom{n-2k}{k}\Big(-\frac{9}{512}\Big)^k = \frac{3^nL_{2n}+(-1)^n}{8^n}. \tag{3.10}$$

Proof. Let $\lambda = \alpha^4$ in (3.5), we have $x(\lambda) = -\frac{9}{512}$ and

$$\sum_{k=0}^{n}\binom{n-2k}{k}\Big(-\frac{9}{512}\Big)^k = \frac{3^{n+1}(8F_{2n}+3L_{2n})+(-1)^n}{19\cdot 8^n}.$$

By using the binomial relation (3.7), we can get the identity stated in the theorem. □

4. Concluding comments

There should be more interesting identities by choosing appropriate parameters λ or x in Lemmas 2.1 and 7. For instance, by setting $\delta = 1$ and $x = \frac{4}{27}$ in (2.1), we have the generating function

$$\mathcal{A}\Big(z, \frac{4}{27}, 1\Big) = -\frac{27}{4z^3-27z^2+54z-27} = \frac{16}{9}\frac{1}{1-\frac{4}{3}z} + \frac{4}{27}\frac{z}{\big(1-\frac{z}{3}\big)^2} - \frac{7}{9}\frac{1}{\big(1-\frac{z}{3}\big)^2}.$$

Extracting the coefficient of z^{n-1}, we can get the following identity:

$$\sum_{k=0}^{n}\binom{n-k}{2k+1}\Big(\frac{4}{27}\Big)^k=\frac{4^{n+1}-3n-4}{3^{n+1}}. \tag{4.1}$$

In view of (1.4) and (1.5), as well as the relation

$$\mathcal{A}\Big(z,\frac{4}{27},0\Big)=(1-z)\mathcal{A}\Big(z,\frac{4}{27},1\Big),$$

we have

$$[z^n]\mathcal{A}\Big(z,\frac{4}{27},0\Big)=[z^n]\mathcal{A}\Big(z,\frac{4}{27},1\Big)-[z^{n-1}]\mathcal{A}\Big(z,\frac{4}{27},1\Big),$$

and derive immediately another summation formula below:

$$\sum_{k=0}^{n}\binom{n-k}{2k}\Big(\frac{4}{27}\Big)^k=\frac{4^{n+1}+6n+5}{3^{n+2}}. \tag{4.2}$$

By means of the binomial relation (2.5), we can get, from (4.2) and (4.1), the theorem below.

Theorem 15 ($n\in\mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n-\delta}{3}\rfloor}\frac{2n+\delta}{n-k}\binom{n-k}{2k+\delta}\Big(\frac{4}{27}\Big)^k=\begin{cases}\frac{4^n+2}{3^n}, & \delta=0;\\ \frac{4^n-1}{3^{n-1}}, & \delta=1.\end{cases} \tag{4.3}$$

In fact, the case of $\delta=1$ is equivalent to the identity (1.1) anticipated in the introduction because of the binomial relation

$$\frac{1}{n-k}\binom{n-k}{2k+1}=\frac{1}{2k+1}\binom{n-k-1}{2k}.$$

Analogously, by letting $x=-\frac{4}{27}$ in Lemma 7, we have the generating function

$$\mathcal{B}\Big(z,-\frac{4}{27}\Big)=\frac{1}{9+3z}-\frac{4}{3}\frac{z}{(2z-3)^2}+\frac{8}{(2z-3)^2},$$

which yields, by evaluating the coefficient of z^n, the identity

$$\sum_{k=0}^{n}\binom{n-2k}{k}\Big(-\frac{4}{27}\Big)^k=\frac{2^{n+1}(3n+4)+(-1)^n}{3^{n+2}}.$$

Then using the binomial relation (3.7), we can get the following theorem.

Theorem 16 ($n\in\mathbb{N}$)**.**

$$\sum_{k=0}^{\lfloor\frac{n}{3}\rfloor}\frac{n}{n-2k}\binom{n-2k}{k}\Big(-\frac{4}{27}\Big)^k=\frac{2^{n+1}+(-1)^n}{3^n}. \tag{4.4}$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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