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Research article

N-dimension for dynamic generalized inequalities of Hölder and Minkowski type on diamond alpha time scales

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Abstract: Expanding on our research, this paper introduced novel generalizations of Hölder's and Minkowski's dynamic inequalities on diamond alpha time scales. Specifically, as particular instances of our findings, we replicated the discrete inequalities established when $\mathbb{T} = \mathbb{N}$. Furthermore, our investigation extended to the continuous case with $\mathbb{T} = \mathbb{R}$, revealing additional inequalities that are both new and valuable for readers seeking a comprehensive understanding of the topic.

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1. Introduction

In 1889, Hölder [1] proved that

$$\sum_{k=1}^{n} x_k y_k \leq \left(\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n}$ are positive sequences, p and q are two positive numbers, such that

$$1/p + 1/q = 1.$$

The inequality reverses if $p\,q < 0$. The integral form of (1.1) is

$$\int_{\eta}^{\epsilon} \lambda(\tau)\omega(\tau)d\tau \leq \left[\int_{\eta}^{\epsilon} \lambda^{\gamma}(\tau)d\tau\right]^{\frac{1}{\gamma}} \left[\int_{\eta}^{\epsilon} \omega^{\nu}(\tau)d\tau\right]^{\frac{1}{\nu}}, \tag{1.2}$$

where $\eta, \epsilon \in \mathbb{R}, \gamma > 1,\ 1/\gamma + 1/\nu = 1$, and $\lambda, \omega \in C([a, b], \mathbb{R})$. If $0 < \gamma < 1$, then (1.2) is reversed.

The subsequent inequality, widely recognized as Minkowski's inequality, asserts that, for $p \geq 1$, if φ, ϖ are nonnegative continuous functions on $[\eta, \epsilon]$ such that

$$0 < \int_{\eta}^{\epsilon} \varphi^{p}(\tau)d\tau < \infty \quad \text{and} \quad 0 < \int_{\eta}^{\epsilon} \varpi^{p}(\tau)d\tau < \infty,$$

then

$$\left(\int_{\eta}^{\epsilon} (\varphi(\tau) + \varpi(\tau))^{p}\, d\tau\right)^{\frac{1}{p}} \leq \left(\int_{\eta}^{\epsilon} \varphi^{p}(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_{\eta}^{\epsilon} \varpi^{p}(\tau)d\tau\right)^{\frac{1}{p}}.$$

Sulaiman [2] introduced the following result pertaining to the reverse Minkowski's inequality: for any $\varphi, \varpi > 0$, if $p \geq 1$ and

$$1 < m \leq \frac{\varphi(\vartheta)}{\varpi(\vartheta)} \leq M,\ \vartheta \in [\eta, \epsilon],$$

then

$$\begin{aligned}\frac{M+1}{(M-1)}\left(\int_{\eta}^{\epsilon} (\varphi(\vartheta) - \varpi(\vartheta))^{p}\, d\vartheta\right)^{\frac{1}{p}} &\leq \left(\int_{\eta}^{\epsilon} \varphi^{p}(\vartheta)d\vartheta\right)^{\frac{1}{p}} + \left(\int_{\eta}^{\epsilon} \varpi^{p}(\vartheta)d\vartheta\right)^{\frac{1}{p}} \\ &\leq \left(\frac{m+1}{m-1}\right)\left(\int_{\eta}^{\epsilon} (\varphi(\vartheta) - \varpi(\vartheta))^{p}\, d\vartheta\right)^{\frac{1}{p}}.\end{aligned} \tag{1.3}$$

Sroysang [3] proved that if $p \geq 1$ and $\varphi, \varpi > 0$ with

$$0 < c < m \leq \frac{\varphi(\vartheta)}{\varpi(\vartheta)} \leq M, \quad \vartheta \in [\eta, \epsilon],$$

then

$$\begin{aligned}\frac{M+1}{(M-c)}\left(\int_{\eta}^{\epsilon} (\varphi(\vartheta) - c\varpi(\vartheta))^{p}\, d\vartheta\right)^{\frac{1}{p}} &\leq \left(\int_{\eta}^{\epsilon} \varphi^{p}(\vartheta)d\vartheta\right)^{\frac{1}{p}} + \left(\int_{\eta}^{\epsilon} \varpi^{p}(\vartheta)d\vartheta\right)^{\frac{1}{p}} \\ &\leq \left(\frac{m+1}{m-c}\right)\left(\int_{\eta}^{\epsilon} (\varphi(\vartheta) - c\varpi(\vartheta))^{p}\, d\vartheta\right)^{\frac{1}{p}}.\end{aligned}$$

In 2023, Kirmaci [4] proved that for nonnegative sequences $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n}$, if $p > 1$, $s \geq 1$,

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

then

$$\sum_{k=1}^{n} (x_k y_k)^{\frac{1}{s}} \leq \left(\sum_{k=1}^{n} x_k^p \right)^{\frac{1}{sp}} \left(\sum_{k=1}^{n} y_k^q \right)^{\frac{1}{sq}}, \tag{1.4}$$

and if $s \geq 1$, $sp < 0$ and

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

then (1.4) is reversed. Also, the author [4] generalized Minkowski's inequality and showed that for nonnegative sequences $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$, if $m \in \mathbb{N}$, $p > 1$,

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

then

$$\left(\sum_{k=1}^{n} (x_k + y_k)^p \right)^{\frac{1}{mp}} \leq \left(\sum_{k=1}^{n} x_k^p \right)^{\frac{1}{mp}} + \left(\sum_{k=1}^{n} y_k^p \right)^{\frac{1}{mp}}. \tag{1.5}$$

Hilger was a trailblazer in integrating continuous and discrete analysis, introducing calculus on time scales in his doctoral thesis, later published in his influential work [5]. The exploration of dynamic inequalities on time scales has garnered significant interest in academic literature. Agarwal et al.'s book [6] focuses on essential dynamic inequalities on time scales, including Young's inequality, Jensen's inequality, Hölder's inequality, Minkowski's inequality, and more. For further insights into dynamic inequalities on time scales, refer to the relevant research papers [7–20]. For the solutions of some differential equations, see, for instance, [21–27] and the references therein.

Expanding upon this trend, the present paper aims to broaden the scope of the inequalities (1.4) and (1.5) on diamond alpha time scales. We will establish the generalized form of Hölder's inequality and Minkowski's inequality for the forward jump operator and the backward jump operator in the discrete calculus. In addition, we prove these inequalities in diamond $-\alpha$ calculus, where for $\alpha = 1$, we can get the inequalities for the forward operator, and when $\alpha = 0$, we get the inequalities for the backward jump operator. Also, we can get the special cases of our results in the discrete and continuous calculus.

The paper is organized as follows: In Section 2, we present some definitions, theorems, and lemmas on time scales, which are needed to get our main results. In Section 3, we state and prove new dynamic inequalities and present their special cases in different (continuous, discrete) calculi.

2. Preliminaries and basic lemmas

In 2001, Bohner and Peterson [28] defined the time scale $\mathbb{T}$ as an arbitrary, nonempty, closed subset of the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by

$$\sigma(\xi) := \inf\{s \in \mathbb{T} : s > \xi\}$$

and

$$\rho(\xi) := \sup\{s \in \mathbb{T} : s < \xi\},$$

respectively. The graininess function μ for a time scale $\mathbb{T}$ is defined by

$$\mu(\xi) := \sigma(\xi) - \xi \geq 0,$$

and for any function φ: $\mathbb{T} \to \mathbb{R}$, the notation $\varphi^{\sigma}(\xi)$ and $\varphi^{\rho}(\xi)$ denote $\varphi(\sigma(\xi))$ and $\varphi(\rho(\xi))$, respectively. We define the time scale interval $[\eta, \epsilon]_{\mathbb{T}}$ by

$$[\eta, \epsilon]_{\mathbb{T}} := [\eta, \epsilon] \cap \mathbb{T}.$$

Definition 2.1. (The Delta derivative [28]) Assume that φ: $\mathbb{T} \to \mathbb{R}$ is a function and let $\xi \in \mathbb{T}$. We define $\varphi^{\Delta}(\xi)$ to be the number, provided it exists, as follows: for any $\epsilon > 0$, there is a neighborhood U of ξ,

$$U = (\xi - \delta, \xi + \delta) \cap \mathbb{T}$$

for some $\delta > 0$, such that

$$|\varphi(\sigma(\xi)) - \varphi(s) - \varphi^{\Delta}(\xi)(\sigma(\xi) - s)| \leq \epsilon|\sigma(\xi) - s|, \quad \forall s \in U, \; s \neq \sigma(\xi).$$

In this case, we say $\varphi^{\Delta}(\xi)$ is the delta or Hilger derivative of φ at ξ.

Definition 2.2. (The nabla derivative [29]) A function λ: $\mathbb{T} \to \mathbb{R}$ is said to be ∇-differentiable at $\xi \in \mathbb{T}$, if ψ is defined in a neighborhood U of ξ and there exists a unique real number $\psi^{\nabla}(\xi)$, called the nabla derivative of ψ at ξ, such that for each $\epsilon > 0$, there exists a neighborhood N of ξ with $N \subseteq U$ and

$$\left|\psi(\rho(\xi)) - \psi(s) - \psi^{\nabla}(\xi)[\rho(\xi) - s]\right| \leq \epsilon \left|\rho(\xi) - s\right|,$$

for all $s \in N$.

Definition 2.3. ([6]) Let $\mathbb{T}$ be a time scale and $\Xi(\xi)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ sense. For $\xi \in \mathbb{T}$, we define the diamond $-\alpha$ derivative $\Xi^{\diamond_\alpha}(\xi)$ by

$$\Xi^{\diamond_\alpha}(\xi) = \alpha\Xi^{\Delta}(\xi) + (1-\alpha)\Xi^{\nabla}(\xi), \quad 0 \leq \alpha \leq 1.$$

Thus, Ξ is diamond $-\alpha$ differentiable if, and only if, Ξ is Δ and ∇ differentiable.

For $\alpha = 1$, we get that

$$\Xi^{\diamond_\alpha}(\xi) = \alpha\Xi^{\Delta}(\xi),$$

and for $\alpha = 0$, we have that

$$\Xi^{\diamond_\alpha}(\xi) = \Xi^{\nabla}(\xi).$$

Theorem 2.1. *([6]) Let Ξ, Ω: $\mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $\xi \in \mathbb{T}$, then*

(1) $\Xi + \Omega$: $\mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $\xi \in \mathbb{T}$, with

$$(\Xi + \Omega)^{\diamond_\alpha}(\xi) = \Xi^{\diamond_\alpha}(\xi) + \Omega^{\diamond_\alpha}(\xi).$$

(2) $\Xi.\Omega$: $\mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $\xi \in \mathbb{T}$, with

$$(\Xi.\Omega)^{\diamond_\alpha}(\xi) = \Xi^{\diamond_\alpha}(\xi)\Omega(\xi) + \alpha\Xi^{\sigma}(\xi)\Omega^{\Delta}(\xi) + (1-\alpha)\Xi^{\rho}(\xi)\Omega^{\nabla}(\xi).$$

(3) For $\Omega(\xi)\Omega^{\sigma}(\xi)\Omega^{\rho}(\xi) \neq 0$, Ξ/Ω: $\mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $\xi \in \mathbb{T}$, with

$$\left(\frac{\Xi}{\Omega}\right)^{\Diamond_\alpha}(\xi) = \frac{\Xi^{\Diamond_\alpha}(\xi)\Omega^{\sigma}(\xi)\Omega^{\rho}(\xi) - \alpha\Xi^{\sigma}(\xi)\Omega^{\rho}(\xi)\Omega^{\Delta}(\xi) - (1-\alpha)\Xi^{\rho}(\xi)\Omega^{\sigma}(\xi)\Omega^{\nabla}(\xi)}{\Omega(\xi)\Omega^{\sigma}(\xi)\Omega^{\rho}(\xi)}.$$

Theorem 2.2. *([6]) Let Ξ, Ω: $\mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $\xi \in \mathbb{T}$, then the following holds*

(1) $(\Xi)^{\Diamond_\alpha\Delta}(\xi) = \alpha\Xi^{\Delta\Delta}(\xi) + (1-\alpha)\Xi^{\nabla\Delta}(\xi)$,

(2) $(\Xi)^{\Diamond_\alpha\nabla}(\xi) = \alpha\Xi^{\Delta\nabla}(\xi) + (1-\alpha)\Xi^{\nabla\nabla}(\xi)$,

(3) $(\Xi)^{\Delta\Diamond_\alpha}(\xi) = \alpha\Xi^{\Delta\Delta}(\xi) + (1-\alpha)\Xi^{\Delta\nabla}(\xi) \neq (\Xi)^{\Diamond_\alpha\Delta}(\xi)$,

(4) $(\Xi)^{\nabla\Diamond_\alpha}(\xi) = \alpha\Xi^{\nabla\Delta}(\xi) + (1-\alpha)\Xi^{\nabla\nabla}(\xi) \neq (\Xi)^{\Diamond_\alpha\nabla}(\xi)$,

(5) $(\Xi)^{\Diamond_\alpha\Diamond_\alpha}(\xi) = \alpha^2\Xi^{\Delta\Delta}(\xi) + \alpha(1-\alpha)[\Xi^{\Delta\nabla}(\xi) + \Xi^{\nabla\Delta}(\xi)]$.

Theorem 2.3. *([6]) Let $a, \xi \in \mathbb{T}$ and h: $\mathbb{T} \to \mathbb{R}$, then the diamond$-\alpha$ integral from a to ξ of h is defined by*

$$\int_a^t h(s)\Diamond_\alpha s = \alpha\int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

It is known that

$$\left(\int_a^\xi h(s)\Delta s\right)^{\Delta} = h(\xi)$$

and

$$\left(\int_a^\xi h(s)\nabla s\right)^{\nabla} = h(\xi),$$

but in general

$$\left(\int_a^\xi h(s)\Diamond_\alpha s\right)^{\Diamond_\alpha} \neq h(\xi), \quad \text{for} \quad \xi \in \mathbb{T}.$$

Example 2.1. [30] Let $\mathbb{T} = \{0,\ 1,\ 2\}$, $a = 0$, and $h(\xi) = \xi^2$ for $\xi \in \mathbb{T}$. This gives

$$\left.\left(\int_a^\xi h(s)\Diamond_\alpha s\right)^{\Diamond_\alpha}\right|_{\xi=1} = 1 + 2\alpha(1-\alpha),$$

so that the equality above holds only when $\Diamond_\alpha = \Delta$ or $\Diamond_\alpha = \nabla$.

Theorem 2.4. *([6]) Let $a, b, \xi \in \mathbb{T}$, β, γ, $c \in \mathbb{R}$, and Ξ and Ω be continuous functions on $[a, b] \cup \mathbb{T}$, then the following properties hold.*

(1) $\int_a^b [\gamma\Xi(\xi) + \beta\Omega(\xi)]\Diamond_\alpha\xi = \gamma\int_a^b \Xi(\xi)\Diamond_\alpha\xi + \beta\int_a^b \Omega(\xi)\Diamond_\alpha\xi$,

(2) $\int_a^b c\Xi(\xi)\Diamond_\alpha\xi = c\int_a^b \Xi(\xi)\Diamond_\alpha\xi$,

(3) $\int_a^b \Xi(\xi)\Diamond_\alpha\xi = -\int_b^a \Xi(\xi)\Diamond_\alpha\xi$,

(4) $\int_a^c \Xi(\xi)\Diamond_\alpha\xi = \int_a^b \Xi(\xi)\Diamond_\alpha\xi + \int_b^c \Xi(\xi)\Diamond_\alpha\xi$.

Theorem 2.5. *([6]) Let $\mathbb{T}$ be a time scale $a, b \in \mathbb{T}$ with $a < b$. Assume that Ξ and Ω are continuous functions on $[a, b]_{\mathbb{T}}$,*

(1) If $\Xi(\xi) \geq 0$ for all $\xi \in [a, b]_{\mathbb{T}}$, then $\int_a^b \Xi(\xi)\Diamond_\alpha \xi \geq 0$;

(2) If $\Xi(\xi) \leq \Omega(\xi)$ for all $\xi \in [a, b]_{\mathbb{T}}$, then $\int_a^b \Xi(\xi)\Diamond_\alpha \xi \leq \int_a^b \Omega(\xi)\Diamond_\alpha \xi$;

(3) If $\Xi(\xi) \geq 0$ for all $\xi \in [a, b]_{\mathbb{T}}$, then $\Xi(\xi) = 0$ if, and only if, $\int_a^b \Xi(\xi)\Diamond_\alpha \xi = 0$.

Example 2.2. ([6]) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b \Xi(\xi)\Diamond_\alpha \xi = \int_a^b \Xi(\xi) d\xi, \text{ for } a, b \in \mathbb{R},$$

and if $\mathbb{T} = \mathbb{N}$ and $m < n$, then we obtain

$$\int_m^n \Xi(\xi)\Diamond_\alpha \xi = \sum_{i=m}^{n-1} [\alpha \Xi(i) + (1 - \alpha)\Xi(i + 1)], \quad \text{for } m, n \in \mathbb{N}. \tag{2.1}$$

Lemma 2.1. *(Hölder's inequality [6]) If $\eta, \epsilon \in \mathbb{T}$, $0 \leq \alpha \leq 1$, and $\lambda, \omega \in C([\eta, \epsilon]_{\mathbb{T}}, \mathbb{R}^+)$, then*

$$\int_\eta^\epsilon \lambda(\tau)\omega(\tau)\Diamond_\alpha \tau \leq \left[\int_\eta^\epsilon \lambda^\gamma(\tau)\Diamond_\alpha \tau\right]^{\frac{1}{\gamma}} \left[\int_\eta^\epsilon \omega^\nu(\tau)\Diamond_\alpha \tau\right]^{\frac{1}{\nu}}, \tag{2.2}$$

where $\gamma > 1$ and $1/\gamma + 1/\nu = 1$. The inequality (2.2) is reversed for $0 < \gamma < 1$ or $\gamma < 0$.

Lemma 2.2. *(Minkowski's inequality [6]) Let $\eta, \epsilon \in \mathbb{T}$, $\epsilon > \eta$, $0 \leq \alpha \leq 1$, $p > 1$, and $\Xi, \Omega \in C([\eta, \epsilon]_{\mathbb{T}}, \mathbb{R}^+)$, then*

$$\left(\int_\eta^\epsilon (\Xi(\tau) + \Omega(\tau))^p \Diamond_\alpha \tau\right)^{\frac{1}{p}} \leq \left(\int_\eta^\epsilon \Xi^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{p}} + \left(\int_\eta^\epsilon \Omega^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{p}}. \tag{2.3}$$

Lemma 2.3. *(Generalized Young inequality [4]) If $a, b \geq 0$, $sp, sq > 1$ with*

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

then

$$\frac{a^{sp}}{sp} + \frac{b^{sq}}{sq} \geq ab. \tag{2.4}$$

Lemma 2.4. *(See [31]) If $x, y > 0$ and $s > 1$, then*

$$(x^s + y^s)^{\frac{1}{s}} \leq x + y, \tag{2.5}$$

and if $0 < s < 1$, then

$$(x^s + y^s)^{\frac{1}{s}} \geq x + y. \tag{2.6}$$

3. Main results

In the manuscript, we will operate under the assumption that the considered integrals are presumed to exist. Also, we denote $\Diamond_\alpha \tau = \Diamond_\alpha \tau_1, \cdots, \Diamond_\alpha \tau_N$ and $\varpi(\tau) = \varpi(\tau_1, \cdots, \tau_N)$.

Theorem 3.1. *Assume that η_i, $\epsilon_i \in \mathbb{T}$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, $0 \le \alpha \le 1$, p, $s > 0$, $ps > 1$ such that*

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and ϖ, Θ: $\mathbb{T}^N \to \mathbb{R}^+$ are continuous functions, then

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{\frac{1}{s}}(\tau)\, \Theta^{\frac{1}{s}}(\tau) \Diamond_\alpha \tau \le \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\tau) \Diamond_\alpha \tau \right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\tau) \Diamond_\alpha \tau \right)^{\frac{1}{sq}}. \tag{3.1}$$

Proof. Applying (2.4) with

$$a = \varpi^{\frac{1}{s}}(\xi) \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\tau) \Diamond_\alpha \tau \right)^{\frac{-1}{sp}}$$

and

$$b = \Theta^{\frac{1}{s}}(\xi) \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\tau) \Diamond_\alpha \tau \right)^{\frac{-1}{sq}},$$

we get

$$\begin{aligned} &\frac{\varpi^p(\xi)}{sp \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\tau) \Diamond_\alpha \tau} + \frac{\Theta^q(\xi)}{sq \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\tau) \Diamond_\alpha \tau} \\ &\ge \varpi^{\frac{1}{s}}(\xi)\, \Theta^{\frac{1}{s}}(\xi) \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\tau) \Diamond_\alpha \tau \right)^{\frac{-1}{sp}} \times \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\tau) \Diamond_\alpha \tau \right)^{\frac{-1}{sq}}. \end{aligned} \tag{3.2}$$

Integrating (3.2) over ξ_i from η_i to ϵ_i, $i = 1, 2, \cdots, N$, we observe that

$$\begin{aligned} &\frac{\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\xi) \Diamond_\alpha \xi}{sp \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\tau) \Diamond_\alpha \tau} + \frac{\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\xi) \Diamond_\alpha \xi}{sq \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\tau) \Diamond_\alpha \tau} \\ &\ge \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\tau) \Diamond_\alpha \tau \right)^{\frac{-1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\tau) \Diamond_\alpha \tau \right)^{\frac{-1}{sq}} \times \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{\frac{1}{s}}(\xi)\, \Theta^{\frac{1}{s}}(\xi) \Diamond_\alpha \xi, \end{aligned}$$

and then (note that $\frac{1}{sp} + \frac{1}{sq} = 1$)

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{\frac{1}{s}}(\tau)\, \Theta^{\frac{1}{s}}(\tau) \Diamond_\alpha \tau \le \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^p(\tau) \Diamond_\alpha \tau \right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^q(\tau) \Diamond_\alpha \tau \right)^{\frac{1}{sq}},$$

which is (3.1). □

Remark 3.1. If $s = 1$ and $N = 1$, we get the dynamic diamond alpha Hölder's inequality (2.2).

Remark 3.2. If $\mathbb{T} = \mathbb{R}$, η_i, $\epsilon_i \in \mathbb{R}$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, p, $s > 0$, $ps > 1$ such that

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and ϖ, Θ: $\mathbb{R}^N \to \mathbb{R}^+$ are continuous functions, then

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{\frac{1}{s}}(\tau)\,\Theta^{\frac{1}{s}}(\tau)\,\mathbf{d}\tau \leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{p}(\tau)\,\mathbf{d}\tau\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^{q}(\tau)\,\mathbf{d}\tau\right)^{\frac{1}{sq}},$$

where $\mathbf{d}\tau = d\tau_1, \cdots, d\tau_N$.

Remark 3.3. If $\mathbb{T} = \mathbb{N}$, $N = 1$, η, $\epsilon \in \mathbb{N}$, $\epsilon > \eta$, $0 \leq \alpha \leq 1$, p, $s > 0$, $ps > 1$ such that

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and ϖ, Θ are positive sequences, then

$$\begin{aligned}
&\sum_{\tau=\eta}^{\epsilon-1} \left[\alpha\varpi^{\frac{1}{s}}(\tau)\,\Theta^{\frac{1}{s}}(\tau) + (1-\alpha)\,\varpi^{\frac{1}{s}}(\tau+1)\,\Theta^{\frac{1}{s}}(\tau+1)\right] \\
&\leq \left(\sum_{\tau=\eta}^{\epsilon-1} \alpha\varpi^{p}(\tau) + (1-\alpha)\,\varpi^{p}(\tau+1)\right)^{\frac{1}{sp}} \times \left(\sum_{\tau=\eta}^{\epsilon-1} \alpha\Theta^{q}(\tau) + (1-\alpha)\,\Theta^{q}(\tau+1)\right)^{\frac{1}{sq}}.
\end{aligned}$$

Example 3.1. If $\mathbb{T} = \mathbb{R}$, $\epsilon \in \mathbb{R}$, $N = 1$, $\eta = 0$, $sp = 3$, $sq =$, $\varpi(\tau) = \tau^s$, and $\Theta(\tau) = \tau^s$, then

$$\int_{\eta}^{\epsilon} \varpi^{\frac{1}{s}}(\tau)\,\Theta^{\frac{1}{s}}(\tau)\,d\tau < \left(\int_{\eta}^{\epsilon} \varpi^{p}(\tau)\,d\tau\right)^{\frac{1}{sp}} \left(\int_{\eta}^{\epsilon} \Theta^{q}(\tau)\,d\tau\right)^{\frac{1}{sq}}.$$

Proof. Note that

$$\int_{\eta}^{\epsilon} \varpi^{\frac{1}{s}}(\tau)\,\Theta^{\frac{1}{s}}(\tau)\,d\tau = \int_{0}^{\epsilon} \tau^2 d\tau = \left.\frac{\tau^3}{3}\right|_0^{\epsilon} = \frac{\epsilon^3}{3}. \tag{3.3}$$

Using the above assumptions, we observe that

$$\begin{aligned}
\int_{\eta}^{\epsilon} \varpi^{p}(\tau)\,d\tau &= \int_{0}^{\epsilon} \tau^{sp} d\tau \\
&= \left.\frac{\tau^{sp+1}}{sp+1}\right|_0^{\epsilon} \\
&= \frac{\epsilon^{sp+1}}{sp+1},
\end{aligned}$$

then

$$\begin{aligned}
\left(\int_{\eta}^{\epsilon} \varpi^{p}(\tau)\,d\tau\right)^{\frac{1}{sp}} &= \left(\frac{\epsilon^{sp+1}}{sp+1}\right)^{\frac{1}{sp}} \\
&= \frac{\epsilon^{1+\frac{1}{sp}}}{(sp+1)^{\frac{1}{sp}}}.
\end{aligned} \tag{3.4}$$

Also, we can get

$$\int_{\eta}^{\epsilon} \Theta^{q}(\tau)\, d\tau = \int_{0}^{\epsilon} \tau^{sq}(\tau)\, d\tau = \left.\frac{\tau^{sq+1}}{sq+1}\right|_{0}^{\epsilon} = \frac{\epsilon^{sq+1}}{sq+1},$$

and so

$$\left(\int_{\eta}^{\epsilon} \Theta^{q}(\tau)\, d\tau\right)^{\frac{1}{sq}} = \left(\frac{\epsilon^{sq+1}}{sq+1}\right)^{\frac{1}{sq}} = \frac{\epsilon^{1+\frac{1}{sq}}}{(sq+1)^{\frac{1}{sq}}}. \tag{3.5}$$

From (3.4) and (3.5) (note $\frac{1}{sp}+\frac{1}{sq}=1$), we have that

$$\left(\int_{\eta}^{\epsilon} \varpi^{p}(\tau)\, d\tau\right)^{\frac{1}{sp}} \left(\int_{\eta}^{\epsilon} \Theta^{q}(\tau)\, d\tau\right)^{\frac{1}{sq}} = \frac{\epsilon^{1+\frac{1}{sp}}}{(sp+1)^{\frac{1}{sp}}} \frac{\epsilon^{1+\frac{1}{sq}}}{(sq+1)^{\frac{1}{sq}}} = \frac{\epsilon^{3}}{(sp+1)^{\frac{1}{sp}}(sq+1)^{\frac{1}{sq}}}. \tag{3.6}$$

Since $sp = 3$ and $sq = 3/2$,

$$(sp+1)^{\frac{1}{sp}}(sq+1)^{\frac{1}{sq}} = 4^{\frac{1}{3}}\left(\frac{5}{2}\right)^{\frac{2}{3}} = 4^{\frac{1}{3}}\left(\frac{25}{4}\right)^{\frac{1}{3}} = (25)^{\frac{1}{3}} = 2.9240177 < 3,$$

then

$$\frac{\epsilon^{3}}{(sp+1)^{\frac{1}{sp}}(sq+1)^{\frac{1}{sq}}} > \frac{\epsilon^{3}}{3}. \tag{3.7}$$

From (3.3), (3.6) and (3.7), we see that

$$\int_{\eta}^{\epsilon} \varpi^{\frac{1}{s}}(\tau)\, \Theta^{\frac{1}{s}}(\tau)\, d\tau < \left(\int_{\eta}^{\epsilon} \varpi^{p}(\tau)\, d\tau\right)^{\frac{1}{sp}} \left(\int_{\eta}^{\epsilon} \Theta^{q}(\tau)\, d\tau\right)^{\frac{1}{sq}}.$$

The proof is complete. □

Corollary 3.1. *If* $\eta_i,\ \epsilon_i \in \mathbb{T}$, $\alpha = 1$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, $p,\ s > 0$, $ps > 1$ *such that*

$$\frac{1}{sp} + \frac{1}{sq} = 1$$

and ϖ, Θ*:* $\mathbb{T}^N \to \mathbb{R}^+$*, then*

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{\frac{1}{s}}(\tau)\,\Theta^{\frac{1}{s}}(\tau)\,\Delta\tau_1\cdots\Delta\tau_N$$
$$\leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{p}(\tau)\,\Delta\tau_1\cdots\Delta\tau_N\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^{q}(\tau)\,\Delta\tau_1\cdots\Delta\tau_N\right)^{\frac{1}{sq}}. \tag{3.8}$$

Corollary 3.2. *If* $\eta_i,\ \epsilon_i \in \mathbb{T}$, $\alpha = 0$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, $p,\ s > 0$, $ps > 1$ *such that*

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and ϖ, Θ*:* $\mathbb{T}^N \to \mathbb{R}^+$*, then*

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{\frac{1}{s}}(\tau)\,\Theta^{\frac{1}{s}}(\tau)\,\nabla\tau_1\cdots\nabla\tau_N$$
$$\leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{p}(\tau)\,\nabla\tau_1\cdots\nabla\tau_N\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^{q}(\tau)\,\nabla\tau_1\cdots\nabla\tau_N\right)^{\frac{1}{sq}}. \tag{3.9}$$

Theorem 3.2. *Assume that* $\eta_i,\ \epsilon_i \in \mathbb{T}$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, $0 \leq \alpha \leq 1$, $p,\ s > 0$, $ps > 1$ *such that*

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and Ξ, Ω*:* $\mathbb{T}^N \to \mathbb{R}^+$ *are continuous functions. If* $0 < m \leq \Xi/\Omega \leq M < \infty$*, then*

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{ps}}(\tau)\,\Omega^{\frac{1}{qs}}(\tau)\,\Diamond_\alpha\tau \leq \frac{M^{\frac{1}{p^2s^2}}}{m^{\frac{1}{q^2s^2}}} \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{qs}}(\tau)\,\Omega^{\frac{1}{ps}}(\tau)\,\Diamond_\alpha\tau. \tag{3.10}$$

Proof. Applying (3.1) with $\varpi(\tau) = \Xi^{\frac{1}{p}}(\tau)$ and $\Theta(\tau) = \Omega^{\frac{1}{q}}(\tau)$, we see that

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{ps}}(\tau)\,\Omega^{\frac{1}{qs}}(\tau)\,\Diamond_\alpha\tau \leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Omega(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sq}}. \tag{3.11}$$

Since

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

then (3.11) becomes

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{ps}}(\tau)\,\Omega^{\frac{1}{qs}}(\tau)\,\Diamond_\alpha\tau \tag{3.12}$$
$$\leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{ps}}(\tau)\,\Xi^{\frac{1}{qs}}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sp}} \times \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Omega^{\frac{1}{ps}}(\tau)\,\Omega^{\frac{1}{qs}}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sq}}.$$

Since

$$\Xi^{\frac{1}{ps}}(\tau) \leq M^{\frac{1}{ps}}\Omega^{\frac{1}{ps}}(\tau)$$

and
$$\Omega^{\frac{1}{qs}}(\tau) \leq m^{\frac{-1}{qs}} \Xi^{\frac{1}{qs}}(\tau),$$
we have from (3.12) that
$$\begin{aligned}
&\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{ps}}(\tau) \Omega^{\frac{1}{qs}}(\tau) \Diamond_\alpha \tau \\
&\leq \frac{M^{\frac{1}{p^2s^2}}}{m^{\frac{1}{q^2s^2}}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{qs}}(\tau) \Omega^{\frac{1}{ps}}(\tau) \Diamond_\alpha \tau \right)^{\frac{1}{sp}} \times \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{qs}}(\tau) \Omega^{\frac{1}{ps}}(\tau) \Diamond_\alpha \tau \right)^{\frac{1}{sq}} \\
&= \frac{M^{\frac{1}{p^2s^2}}}{m^{\frac{1}{q^2s^2}}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{qs}}(\tau) \Omega^{\frac{1}{ps}}(\tau) \Diamond_\alpha \tau \right)^{\frac{1}{sp}+\frac{1}{sq}},
\end{aligned}$$
then we have for
$$\frac{1}{sp} + \frac{1}{sq} = 1,$$
that
$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{ps}}(\tau) \Omega^{\frac{1}{qs}}(\tau) \Diamond_\alpha \tau \leq \frac{M^{\frac{1}{p^2s^2}}}{m^{\frac{1}{q^2s^2}}} \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{qs}}(\tau) \Omega^{\frac{1}{ps}}(\tau) \Diamond_\alpha \tau,$$
which is (3.10). □

Remark 3.4. Take $\mathbb{T} = \mathbb{R}$, η_i, $\epsilon_i \in \mathbb{R}$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, p, $s > 0$, $ps > 1$ such that
$$\frac{1}{sp} + \frac{1}{sq} = 1$$
and Ξ, Ω: $\mathbb{R}^N \to \mathbb{R}^+$ are continuous functions. If $0 < m \leq \Xi/\Omega \leq M < \infty$, then
$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{ps}}(\tau) \Omega^{\frac{1}{qs}}(\tau) \mathbf{d}\tau \leq \frac{M^{\frac{1}{p^2s^2}}}{m^{\frac{1}{q^2s^2}}} \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{qs}}(\tau) \Omega^{\frac{1}{ps}}(\tau) \mathbf{d}\tau.$$

Remark 3.5. Take $\mathbb{T} = \mathbb{N}$, $N = 1$, η, $\epsilon \in \mathbb{N}$, $\epsilon > \eta$, $0 \leq \alpha \leq 1$, p, $s > 0$, $ps > 1$ such that
$$\frac{1}{sp} + \frac{1}{sq} = 1,$$
and Ξ, Ω are positive sequences. If $0 < m \leq \Xi/\Omega \leq M < \infty$, then
$$\begin{aligned}
&\sum_{\tau=\eta}^{\epsilon-1} \left[\alpha \left(\Xi^{\frac{1}{ps}}(\tau) \Omega^{\frac{1}{qs}}(\tau) \right) + (1-\alpha) \left(\Xi^{\frac{1}{ps}}(\tau+1) \Omega^{\frac{1}{qs}}(\tau+1) \right) \right] \\
&\leq \frac{M^{\frac{1}{p^2s^2}}}{m^{\frac{1}{q^2s^2}}} \sum_{\tau=\eta}^{\epsilon-1} \left[\alpha \left(\Xi^{\frac{1}{qs}}(\tau) \Omega^{\frac{1}{ps}}(\tau) \right) + (1-\alpha) \left(\Xi^{\frac{1}{qs}}(\tau+1) \Omega^{\frac{1}{ps}}(\tau+1) \right) \right].
\end{aligned}$$

Theorem 3.3. *Assume that η_i, $\epsilon_i \in \mathbb{T}$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, $0 \leq \alpha \leq 1$, $sp < 0$ such that*
$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and ϕ, λ: $\mathbb{T}^N \to \mathbb{R}^+$ are continuous functions, then

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{\frac{1}{s}}(\tau)\,\lambda^{\frac{1}{s}}(\tau)\,\Diamond_\alpha\tau \geq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{p}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \lambda^{q}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sq}}. \tag{3.13}$$

Proof. Since $sp < 0$, by applying (3.1) with indices

$$sP = \frac{-sp}{sq} = 1 - sp > 1, \quad sQ = \frac{1}{sq},$$

(note that $\frac{1}{sP} + \frac{1}{sQ} = 1$), $\varpi(\tau) = \phi^{-sq}(\tau)$ and $\Theta(\tau) = \phi^{sq}(\tau)\,\lambda^{sq}(\tau)$, then

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{\frac{1}{s}}(\tau)\,\Theta^{\frac{1}{s}}(\tau)\,\Diamond_\alpha\tau \leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \varpi^{P}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sP}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Theta^{Q}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sQ}},$$

and substituting with values $\varpi(\tau)$ and $\Theta(\tau)$, the last inequality gives us

$$\begin{aligned} &\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{-q}(\tau)\left[\phi^{q}(\tau)\,\lambda^{q}(\tau)\right]\Diamond_\alpha\tau \\ &\leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \left[\phi^{-sq}(\tau)\right]^{\frac{-p}{sq}}\Diamond_\alpha\tau\right)^{\frac{-sq}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \left[\phi^{sq}(\tau)\,\lambda^{sq}(\tau)\right]^{\frac{1}{s^2q}}\Diamond_\alpha\tau\right)^{sq}. \end{aligned}$$

Thus

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \lambda^{q}(\tau)\,\Diamond_\alpha\tau \leq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{p}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{-q}{p}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{\frac{1}{s}}(\tau)\,\lambda^{\frac{1}{s}}(\tau)\,\Diamond_\alpha\tau\right)^{sq},$$

then we have for $sp < 0$ and $sq = sp/(sp-1) > 0$ that

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{\frac{1}{s}}(\tau)\,\lambda^{\frac{1}{s}}(\tau)\,\Diamond_\alpha\tau \geq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{p}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \lambda^{q}(\tau)\,\Diamond_\alpha\tau\right)^{\frac{1}{sq}},$$

which is (3.13). □

Remark 3.6. If $\mathbb{T} = \mathbb{R}$, η_i, $\epsilon_i \in \mathbb{R}$, $\epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, $0 \leq \alpha \leq 1$, $sp < 0$ such that

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and ϕ, λ: $\mathbb{R}^N \to \mathbb{R}^+$are continuous functions, then

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{\frac{1}{s}}(\tau)\,\lambda^{\frac{1}{s}}(\tau)\,\mathbf{d}\tau \geq \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \phi^{p}(\tau)\,\mathbf{d}\tau\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \lambda^{q}(\tau)\,\mathbf{d}\tau\right)^{\frac{1}{sq}}.$$

Remark 3.7. If $\mathbb{T} = \mathbb{N}$, $N = 1, \eta$, $\epsilon \in \mathbb{N}$, $\epsilon > \eta$, $0 \leq \alpha \leq 1$, $sp < 0$, such that

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and ϕ, λ are positive sequences, then

$$\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha\left(\phi^{\frac{1}{s}}(\tau)\lambda^{\frac{1}{s}}(\tau)\right)+(1-\alpha)\left(\phi^{\frac{1}{s}}(\tau+1)\lambda^{\frac{1}{s}}(\tau+1)\right)\right]$$
$$\geq\left(\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha\phi^{p}(\tau)+(1-\alpha)\phi^{p}(\tau+1)\right]\right)^{\frac{1}{sp}}\left(\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha\lambda^{q}(\tau)+(1-\alpha)\lambda^{q}(\tau+1)\right]\right)^{\frac{1}{sq}}.$$

Theorem 3.4. *Assume that $\eta_i,\ \epsilon_i\in\mathbb{T}$, $\epsilon_i>\eta_i$, $i=1,2,\cdots,N$, $0\leq\alpha\leq1$, $r>u>t>0$, and Ξ, Ω: $\mathbb{T}^N\to\mathbb{R}^+$are continuous functions, then*

$$\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{u}(\tau)\Diamond_\alpha\tau\right)^{r-t}\leq\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{t}(\tau)\Diamond_\alpha\tau\right)^{r-u}\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{r}(\tau)\Diamond_\alpha\tau\right)^{u-t}. \quad (3.14)$$

Proof. Applying (3.1) with

$$p=\frac{r-t}{r-u},\quad q=\frac{r-t}{u-t},$$

(note that $s=1$),

$$\varpi^{p}(\tau)=\Xi(\tau)\Omega^{t}(\tau)\quad\text{and}\quad\Theta^{q}(\tau)=\Xi(\tau)\Omega^{r}(\tau),$$

we see that

$$\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\left[\Xi(\tau)\Omega^{t}(\tau)\right]^{\frac{r-u}{r-t}}\left[\Xi(\tau)\Omega^{r}(\tau)\right]^{\frac{u-t}{r-t}}\Diamond_\alpha\tau$$
$$\leq\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{t}(\tau)\Diamond_\alpha\tau\right)^{\frac{r-u}{r-t}}\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{r}(\tau)\Diamond_\alpha\tau\right)^{\frac{u-t}{r-t}},$$

and then

$$\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{u}(\tau)\Diamond_\alpha\tau\right)^{r-t}\leq\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{t}(\tau)\Diamond_\alpha\tau\right)^{r-u}\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{r}(\tau)\Diamond_\alpha\tau\right)^{u-t},$$

which is (3.14). □

Remark 3.8. If $\mathbb{T}=\mathbb{R}$, $\eta_i,\ \epsilon_i\in\mathbb{T},\ \epsilon_i>\eta_i, i=1,2,\cdots,N$, $r>u>t>0$ and Ξ, Ω: $\mathbb{R}^N\to\mathbb{R}^+$ are continuous functions, then

$$\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{u}(\tau)\mathbf{d}\tau\right)^{r-t}\leq\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{t}(\tau)\mathbf{d}\tau\right)^{r-u}\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi(\tau)\Omega^{r}(\tau)\mathbf{d}\tau\right)^{u-t}.$$

Remark 3.9. If $\mathbb{T}=\mathbb{N}$, $N=1,\eta,\ \epsilon\in\mathbb{N},\ \epsilon>\eta$, $0\leq\alpha\leq1$, $r>u>t>0$ and Ξ, Ω are positive sequences, then

$$\left(\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha\left(\Xi(\tau)\Omega^{u}(\tau)\right)+(1-\alpha)\left(\Xi(\tau+1)\Omega^{u}(\tau+1)\right)\right]\right)^{r-t}$$
$$\leq\left(\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha\left(\Xi(\tau)\Omega^{t}(\tau)\right)+(1-\alpha)\left(\Xi(\tau+1)\Omega^{t}(\tau+1)\right)\right]\right)^{r-u}$$
$$\times\left(\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha\left(\Xi(\tau)\Omega^{r}(\tau)\right)+(1-\alpha)\left(\Xi(\tau+1)\Omega^{r}(\tau+1)\right)\right]\right)^{u-t}.$$

Theorem 3.5. *Assume that* $\eta_i,\ \epsilon_i \in \mathbb{T}, \epsilon_i > \eta_i$, $i = 1, 2, \cdots, N$, $0 \le \alpha \le 1$, $s \ge 1$, $sp > 1$ *such that*

$$\frac{1}{sp} + \frac{1}{sq} = 1,$$

and Ξ, Ω*:* $\mathbb{T}^N \to \mathbb{R}^+$ *are continuous functions, then*

$$\left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} (\Xi(\tau) + \Omega(\tau))^p \Diamond_\alpha \tau\right)^{\frac{1}{sp}} \le \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{sp}} + \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Omega^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{sp}}. \quad (3.15)$$

Proof. Note that

$$(\Xi(\tau) + \Omega(\tau))^p = (\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}} (\Xi(\tau) + \Omega(\tau))^{p-\frac{1}{s}}. \quad (3.16)$$

Using the inequality (2.5) with replacing $x,\ y$ by $\Xi^{\frac{1}{s}}(\tau)$ and $\Omega^{\frac{1}{s}}(\tau)$, respectively, we have

$$(\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}} \le \Xi^{\frac{1}{s}}(\tau) + \Omega^{\frac{1}{s}}(\tau),$$

therefore, (3.16) becomes

$$(\Xi(\tau) + \Omega(\tau))^p \le \Xi^{\frac{1}{s}}(\tau) (\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}(sp-1)} + \Omega^{\frac{1}{s}}(\tau) (\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}(sp-1)},$$

then

$$\begin{aligned} \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} (\Xi(\tau) + \Omega(\tau))^p \Diamond_\alpha \tau \le & \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{s}}(\tau) (\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}(sp-1)} \Diamond_\alpha \tau \\ & + \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Omega^{\frac{1}{s}}(\tau) (\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}(sp-1)} \Diamond_\alpha \tau. \end{aligned} \quad (3.17)$$

Applying (3.1) on the two terms of the righthand side of (3.17) with indices $sp > 1$ and $(sp)^* = sp/(sp-1)$ (note $\frac{1}{sp} + \frac{1}{(sp)^*} = 1$), we obtain

$$\begin{aligned} & \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^{\frac{1}{s}}(\tau) (\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}(sp-1)} \Diamond_\alpha \tau \\ & \le \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} (\Xi(\tau) + \Omega(\tau))^p \Diamond_\alpha \tau\right)^{\frac{sp-1}{sp}} \end{aligned} \quad (3.18)$$

and

$$\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Omega^{\frac{1}{s}}(\tau) (\Xi(\tau) + \Omega(\tau))^{\frac{1}{s}(sp-1)} \Diamond_\alpha \tau \le \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Omega^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{sp}} \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} (\Xi(\tau) + \Omega(\tau))^p \Diamond_\alpha \tau\right)^{\frac{sp-1}{sp}}. \quad (3.19)$$

Adding (3.18) and (3.19) and substituting into (3.17), we see that

$$\begin{aligned} \int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} (\Xi(\tau) + \Omega(\tau))^p \Diamond_\alpha \tau \le & \left[\left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Xi^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{sp}} + \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} \Omega^p(\tau) \Diamond_\alpha \tau\right)^{\frac{1}{sp}}\right] \\ & \times \left(\int_{\eta_N}^{\epsilon_N} \cdots \int_{\eta_1}^{\epsilon_1} (\Xi(\tau) + \Omega(\tau))^p \Diamond_\alpha \tau\right)^{\frac{sp-1}{sp}}, \end{aligned}$$

thus,

$$\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\diamond_\alpha\tau\right)^{\frac{1}{sp}}\le\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^p(\tau)\diamond_\alpha\tau\right)^{\frac{1}{sp}}+\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^p(\tau)\diamond_\alpha\tau\right)^{\frac{1}{sp}},$$

which is (3.15). □

Remark 3.10. If $s = 1$ and $N = 1$, then we get the dynamic Minkowski inequality (2.3).

In the following, we establish the reversed form of inequality (3.15).

Theorem 3.6. *Assume that η_i, $\epsilon_i\in\mathbb{T}$, $\epsilon_i>\eta_i$, $i=1,2,\cdots,N$, $0\le\alpha\le1$, $p<0$, $0<s<1$ and Ξ,Ω: $\mathbb{T}^N\to\mathbb{R}^+$ are continuous functions, then*

$$\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\diamond_\alpha\tau\right)^{\frac{1}{sp}}\ge\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^p(\tau)\diamond_\alpha\tau\right)^{\frac{1}{sp}}+\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^p(\tau)\diamond_\alpha\tau\right)^{\frac{1}{sp}}. \quad (3.20)$$

Proof. Applying (2.6) with replacing x, y by $\Xi^{\frac{1}{s}}(\tau)$ and $\Omega^{\frac{1}{s}}(\tau)$, respectively, we have that

$$(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}}\ge\Xi^{\frac{1}{s}}(\tau)+\Omega^{\frac{1}{s}}(\tau). \quad (3.21)$$

Since

$$(\Xi(\tau)+\Omega(\tau))^p=(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}}(\Xi(\tau)+\Omega(\tau))^{p-\frac{1}{s}},$$

by using (3.21), we get

$$(\Xi(\tau)+\Omega(\tau))^p\ge\Xi^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)}+\Omega^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)},$$

then

$$\begin{aligned}\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\diamond_\alpha\tau\ge&\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)}\diamond_\alpha\tau\\&+\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)}\diamond_\alpha\tau.\end{aligned} \quad (3.22)$$

Applying (3.13) on $\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)}\diamond_\alpha\tau$, with

$$\phi(\tau)=\Xi(\tau),\quad\lambda(\tau)=(\Xi(\tau)+\Omega(\tau))^{sp-1},$$

and the indices $sp<0$, $(sp)^*=sp/(sp-1)$ (note $\frac{1}{sp}+\frac{1}{(sp)^*}=1$), we see that

$$\begin{aligned}&\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)}\diamond_\alpha\tau\\&\ge\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^p(\tau)\diamond_\alpha\tau\right)^{\frac{1}{sp}}\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\diamond_\alpha\tau\right)^{\frac{sp-1}{sp}}.\end{aligned} \quad (3.23)$$

Again by applying (3.13) on $\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)}\Diamond_\alpha\tau$ with

$$\phi(\tau)=\Omega(\tau),\quad \lambda(\tau)=(\Xi(\tau)+\Omega(\tau))^{sp-1},$$

and the indices $sp<0,\ (sp)^*=sp/(sp-1)$ (note $\frac{1}{sp}+\frac{1}{(sp)^*}=1$), we have that

$$\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^{\frac{1}{s}}(\tau)(\Xi(\tau)+\Omega(\tau))^{\frac{1}{s}(sp-1)}\Diamond_\alpha\tau\geq\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^p(\tau)\Diamond_\alpha\tau\right)^{\frac{1}{sp}}\times\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\Diamond_\alpha\tau\right)^{\frac{sp-1}{sp}}. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.22), we see that

$$\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\Diamond_\alpha\tau\geq\left[\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^p(\tau)\Diamond_\alpha\tau\right)^{\frac{1}{sp}}+\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^p(\tau)\Diamond_\alpha\tau\right)^{\frac{1}{sp}}\right]\times\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\Diamond_\alpha\tau\right)^{\frac{sp-1}{sp}},$$

then

$$\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\Diamond_\alpha\tau\right)^{\frac{1}{sp}}\geq\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^p(\tau)\Diamond_\alpha\tau\right)^{\frac{1}{sp}}+\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^p(\tau)\Diamond_\alpha\tau\right)^{\frac{1}{sp}},$$

which is (3.20). □

Remark 3.11. If $\mathbb{T}=\mathbb{R}$, $\eta_i,\ \epsilon_i\in\mathbb{T}$, $\epsilon_i>\eta_i$, $i=1,2,\cdots,N$, $p<0$, $0<s<1$ and Ξ, Ω: $\mathbb{R}^N\to\mathbb{R}^+$ are continuous functions, then

$$\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}(\Xi(\tau)+\Omega(\tau))^p\,\mathbf{d}\tau\right)^{\frac{1}{sp}}\geq\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Xi^p(\tau)\,\mathbf{d}\tau\right)^{\frac{1}{sp}}+\left(\int_{\eta_N}^{\epsilon_N}\cdots\int_{\eta_1}^{\epsilon_1}\Omega^p(\tau)\,\mathbf{d}\tau\right)^{\frac{1}{sp}}.$$

Remark 3.12. If $\mathbb{T}=\mathbb{N}$, $N=1,\eta,\ \epsilon\in\mathbb{N},\ \epsilon>\eta$, $0\leq\alpha\leq1$, $p<0$, $0<s<1$, and Ξ, Ω are positive sequences, then

$$\left(\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha(\Xi(\tau)+\Omega(\tau))^p+(1-\alpha)(\Xi(\tau+1)+\Omega(\tau+1))^p\right]\right)^{\frac{1}{sp}}$$
$$\geq\left(\sum_{\tau=\eta}^{\epsilon-1}\left[\alpha\Xi^p(\tau)+(1-\alpha)\Xi^p(\tau+1)\right]\right)^{\frac{1}{sp}}+\left(\sum_{\tau=\eta}^{\epsilon-1}\alpha\Omega^p(\tau)+(1-\alpha)\Omega^p(\tau+1)\right)^{\frac{1}{sp}}.$$

4. Conclusions and future work

In this paper, we present novel generalizations of Hölder's and Minkowski's dynamic inequalities on diamond alpha time scales. These inequalities give us the inequalities on delta calculus when $\alpha=1$

and the inequalities on nabla calculus when $\alpha = 0$. Also, we introduced some of the continuous and discrete inequalities as special cases of our results. In addition, we added an example in our results to indicate the work.

In the future, we will establish some new generalizations of Hölder's and Minkowski's dynamic inequalities on conformable delta fractional time scales. Also, we will prove some new reversed versions of Hölder's and Minkowski's dynamic inequalities on time scales.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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