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Research article

Pinning clustering component synchronization of nonlinearly coupled complex dynamical networks

Jie Liu[1,2] **and Jian-Ping Sun**[1,2,*]

[1] College of Electrical and Information Engineering, Lanzhou University of Technology, Lanzhou 730050, China
[2] Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, China

* **Correspondence:** Email: jpsun@lut.edu.cn.

Abstract: In this paper, the clustering component synchronization of nonlinearly coupled complex dynamical networks with nonidentical nodes was investigated. By applying feedback injections to those nodes who have connections with other clusters, some criteria for achieving clustering component synchronization were obtained. A numerical simulation was also included to verify the correctness of the results obtained.

Keywords: complex dynamical network; clustering component synchronization; pinning control; Lyapunov stability theory; nonlinearly coupling
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1. Introduction

Synchronization, similarly consensus, as a typical collective behavior in complex networks and systems, has been extensively studied in the past decades due to its potential applications in various fields, such as neural networks, biology, and secure communication and information processing [1–9]. Many kinds of synchronization, including complete synchronization [10, 11], lag synchronization [12, 13], cluster synchronization [14, 15], generalized synchronization [16, 17], etc., have been investigated. Among them, cluster synchronization, as a particular synchronization phenomenon, requires that synchronization occurs in each cluster, but there is no synchronization among the different clusters. Cluster synchronization has attracted increasing attention recently since it is considered to be more significant in biological science and communication engineering [18–26]. For example, in 2016, under the event-based mechanism, Li et al. [18] proposed a new event-triggered sampled-data transmission strategy, where only local and event-triggering states were utilized to update the broadcasting state of each agent, to realize cluster synchronization of the coupled neural networks. In 2019, Yang et al. [21]

investigated the cluster lag synchronization of the delayed heterogeneous complex dynamical networks that involved both the transmission delay in communication channels and the time-varying delays in self-dynamics simultaneously. In 2022, Li et al. [23] studied the cluster synchronization for a class of complex dynamical networks with parameters mismatched to the selected cluster pattern and proposed an impulsive control strategy with multiple control gains.

Generally speaking, synchronous trajectories are closely related to the topological structure of the network and the self-dynamics of the isolated node, as well as the strength of the couplings among the nodes. Therefore, it is almost impossible for a complex network to synchronize to the trajectory that we desired. In this case, some controllers should be designed and applied to tame the network to approach the synchronization trajectory that we desired. Pinning control, as a feasible and effective strategy, has been proposed and widely studied; see [27–37]. For instance, in [28], without assuming symmetry, irreducibility, or linearity of the couplings, Chen et al. proved that a single controller can pin a coupled complex network to a homogenous solution. In [29], Wang et al. considered the cluster synchronization of some dynamical networks with community structure and nonidentical nodes and with identical local dynamics for all individual nodes in each community by using feedback control schemes. In 2018, Liu and Chen [32] studied the finite-time and fixed-time cluster synchronization problem for complex networks, designed some simple distributed protocols with or without pinning control, and proved the effectiveness. For relevant works, one can refer to [38–46].

The aforementioned synchronization refers to the convergence on all components of a node's state variables. However, in some cases, we only need to focus on the convergence on some components (rather than all components) of a node's state variables. In [47] and [48], Li et al. gave the definitions of partial component synchronization and clustering component synchronization, and obtained some sufficient conditions on partial component synchronization and clustering component synchronization for a class of chaotic dynamical networks, respectively.

It is necessary to point out that most of the existing works focus on linearly coupled networks, that is to say, the inner coupling is linear. However, in some networks, such as neural and metabolic networks, the coupling configurations are oscillate continuously between two fixed states, which means that the inner coupling is nonlinear. At the same time, the individual nodes exhibit different dynamic behaviors according to their functions. This means that these networks are formed by nonidentical nodes. In this paper, we investigate the clustering component synchronization of nonlinearly coupled complex dynamical networks with nonidentical nodes. By applying controllers to those selected nodes and making mild assumptions, we obtain some sufficient conditions for achieving clustering component synchronization. The novelty of this paper is that the synchronization discussed in this paper refers to the synchronization of any k specified components of a node's state variables of the network rather than all or the first k components. The difficulty of the exploration method is how to construct an effective Lyapunov function to investigate the synchronization behavior of any k specified components of a node's state variables of the network. Compared with the previous works, the advantage of the proposed research method is that it can solve the synchronization problem where the specified components of the node's state variables are synchronized while the rest of the components of the node's state variables may not be synchronized. The main contributions of this paper are listed as follows:

(i) The complex dynamical network discussed in this paper is formed by nonidentical nodes and the inner coupling of the network is nonlinear, which is more realistic since the individual nodes often tend to exhibit various dynamics due to the influence of their functions or other factors, such as external perturbation, and the observed data is usually a nonlinear function of the state variable rather than itself.

(ii) In this paper, the clustering component synchronization of the complex dynamical network is studied, which not only has certain practical significance, but also has the characteristics of less control difficulty and lower control cost. Compared with [48], the clustering component synchronization discussed in this paper is the synchronization of any k specified components of a node's state variables instead of the first k components.

The rest of this paper is organized as follows: In Section 2, we recall some notations, definitions, and lemmas. Our main results are established in Section 3. In Section 4, a numerical simulation is provided to verify the correctness of our theoretical results. The paper is concluded in Section 5.

2. Preliminaries

Throughout this paper, we use the following notations:
R^n denotes the n-dimensional Euclidean space and $\|\cdot\|$ stands for its Euclidean norm;
$R^{n\times n}$ denotes the set of all $n\times n$ real matrices;
I_n denotes the $n\times n$ identity matrix;
$\mathrm{diag}(d_1, d_2, \cdots, d_n)$ denotes the diagonal matrix whose diagonal entries are d_1 to d_n;
the superscript "T" stands for the transpose of a matrix;
for symmetric matrix P, $P < 0$ means that P is negative definite; $\lambda_{\max}(P)$ denotes the maximum eigenvalue of P;
the symbol $\otimes$ denotes the Kronecker product.

Consider the s-dimensional nonautonomous system

$$\dot{x} = f(t, x),\ t \in R^+ = [0, +\infty), \tag{2.1}$$

where $x = (y^T, z^T)^T$, $y = (x_1, \cdots, x_l)^T$ and $z = (x_{l+1}, \cdots, x_s)^T$, and $f \in C[R^+ \times R^s, R^s]$ and $f(t, 0) \equiv 0$ for $t \in R^+$.

Assume that the existence and uniqueness of solutions to the system (2.1) subject to $x(t_0) = x_0$, as well as their dependence on initial values, are guaranteed.

Definition 2.1. *([48,49]) The trivial solution of the system* (2.1) *is said to be stable with respect to the variable y if* $\forall\, \varepsilon > 0$, $\forall\, t_0 \in R^+$, $\exists\, \delta(\varepsilon, t_0) > 0$, $\forall\, x_0 \in S_\delta \triangleq \{x : \|x\| \le \delta\}$*, such that*

$$\|y(t, t_0, x_0)\| < \varepsilon$$

for $t \ge t_0$.

Definition 2.2. *([48,49]) The trivial solution of the system* (2.1) *is said to be attractive with respect to the variable y if* $\forall\, t_0 \in R^+$, $\exists\, \delta(t_0) > 0$, $\forall\, x_0 \in S_\delta = \{x : \|x\| \le \delta\}$, $\forall\, \varepsilon > 0$, $\exists\, T(\varepsilon, t_0, x_0) > 0$*, such that*

$$\|y(t, t_0, x_0)\| < \varepsilon$$

for $t \ge t_0 + T$. Furthermore, S_δ is called the region of attraction with respect to the variable y.

Definition 2.3. *([48, 49]) The trivial solution of the system* (2.1) *is said to be asymptotically stable with respect to the variable y if it is both stable and attractive with respect to the variable y.*

Definition 2.4. *([49]) A function ψ is said to belong to the K class function, denoted by $\psi \in K$, if $\psi : R^+ \to R^+$ is continuous and strictly monotone increasing and $\psi(0) = 0$.*

Definition 2.5. *([50]) Let $h : R \to R$ be a function. If there exist constants $\beta \geq \alpha > 0$ such that for any $v_1, v_2 \in R$ with $v_1 \neq v_2$, the inequality*

$$\alpha \leq \frac{h(v_1) - h(v_2)}{v_1 - v_2} \leq \beta$$

holds, then h is said to belong to $UNI(\alpha,\beta)$, denoted by $h \in UNI(\alpha,\beta)$.

Lemma 2.1. *([48, 49]) Let ϕ, ψ, $\alpha \in K$. If there is a Lyapunov function $V : R^+ \times R^s \to R^+$ with $V(t, 0) = 0$ for $t \in R^+$, such that*

$$\phi(\|y\|) \leq V(t, x) \leq \psi(\|y\|) \text{ for } (t, x) \in (R^+, R^s), \tag{2.2}$$

and its derivative along the trajectories of (2.1) meets

$$\left.\frac{dV}{dt}\right|_{(2.1)} \leq -\alpha(\|y(t)\|), \; t \in R^+, \tag{2.3}$$

then the trivial solution of the system (2.1) is asymptotically stable with respect to the variable y.

Lemma 2.2. *Let $B = (b_{ij}) \in R^{m\times m}$ and $C = (c_{ij}) \in R^{n\times n}$, then for any permutation $\mu_1, \mu_2, \cdots, \mu_n$ of $1, 2, \cdots, n$, there exist orthogonal matrices $P \in R^{mn\times mn}$ and $Q \in R^{n\times n}$, such that the equality*

$$P(B \otimes C)P^T = (QCQ^T) \otimes B \tag{2.4}$$

holds.

Proof. For any permutation $\mu_1, \mu_2, \cdots, \mu_n$ of $1, 2, \cdots, n$, we define

$$P = (\xi^T_{\mu_1}, \xi^T_{n+\mu_1}, \cdots, \xi^T_{(m-1)n+\mu_1}, \xi^T_{\mu_2}, \xi^T_{n+\mu_2}, \cdots, \xi^T_{(m-1)n+\mu_2}, \cdots, \xi^T_{\mu_n}, \xi^T_{n+\mu_n}, \cdots, \xi^T_{(m-1)n+\mu_n})^T,$$

and

$$Q = (\epsilon^T_{\mu_1}, \epsilon^T_{\mu_2}, \cdots, \epsilon^T_{\mu_n})^T,$$

where ξ_k ($k = 1, 2, \cdots, mn$) is an mn-dimensional row vector whose kth element is 1 and all the other elements are 0, while ϵ_l ($l = 1, 2, \cdots, n$) is an n-dimensional row vector whose lth element is 1 and all the other elements are 0. It is obvious that P and Q are orthogonal matrices, and (2.4) is true after direct calculation. □

Lemma 2.3. *([2]) For any $x, y \in R^n$ and positive definite matrix $Q \in R^{n\times n}$,*

$$x^T y \leq \frac{1}{2}x^T Qx + \frac{1}{2}y^T Q^{-1} y.$$

3. Main results

Consider a complex network consisting of m nonidentical nodes, which can be divided into r ($2 \leq r < m$) disjoint nonempty clusters due to a nodes' behavior or other properties. Without loss of generality, let the partition be $\{U_1, U_2, \cdots, U_r\}$, where

$$U_1 = \{1, 2, \cdots, q_1\},\ U_2 = \{q_1 + 1, q_1 + 2, \cdots, q_2\}, \cdots, U_r = \{q_{r-1} + 1, q_{r-1} + 2, \cdots, m\}. \tag{3.1}$$

Thus, the network can be described as

$$\dot{x}_i(t) = f_{\varphi_i}(x_i(t)) + c \sum_{j=1, j\neq i}^{m} a_{ij}\Big(g(x_j(t)) - g(x_i(t))\Big),\ t \in R^+,\ i = 1, 2, \cdots, m, \tag{3.2}$$

where $x_i(t) = \big(x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)\big)^T \in R^n$ is the state variable of the node i at time t; $f_{\varphi_i} : R^n \to R^n$ is a nonlinear function that describes the local dynamic of the nodes in the φ_ith cluster and $f_{\varphi_i} \neq f_{\varphi_j}$ for $\varphi_i \neq \varphi_j$, where φ_i is defined as follows: If $i \in U_l$, then $\varphi_i = l$; $c > 0$ denotes the coupling strength; $g : R^n \to R^n$ is the nonlinear coupling function, which is defined by $g(v) = \big(g_1(v_1), g_2(v_2), \cdots, g_n(v_n)\big)^T$ for $v = (v_1, v_2, \cdots, v_n)^T \in R^n$; and a_{ij} is defined as follows: If there is a connection between node i and node j ($i \neq j$), then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$. Let $a_{ii} = -\sum\limits_{j=1, j\neq i}^{m} a_{ij}$, then $A := (a_{ij}) \in R^{m\times m}$ is called the coupling configuration matrix, which represents the topological structure of the network, and (3.2) can be rewritten as

$$\dot{x}_i(t) = f_{\varphi_i}(x_i(t)) + c \sum_{j=1}^{m} a_{ij} g(x_j(t)),\ t \in R^+,\ i = 1, 2, \cdots, m. \tag{3.3}$$

In what follows, we consider the pinning controlled network

$$\dot{x}_i(t) = f_{\varphi_i}(x_i(t)) + c \sum_{j=1}^{m} a_{ij} g(x_j(t)) + u_i(t),\ t \in R^+,\ i = 1, 2, \cdots, m, \tag{3.4}$$

where $u_i(t)$ is the controller to be designed.

Select any k ($1 \leq k \leq n$) components of a node's state variables as the components, which are required to be synchronized. Denote these k components as $p_1, p_2, \cdots, p_k$ and the remaining components as $p_{k+1}, p_{k+2}, \cdots, p_n$. Let $e_i(t) = x_i(t) - s_{\varphi_i}(t)$, where $s_{\varphi_i}(t)$ is a solution of an isolated node in the φ_ith cluster, i.e., $\dot{s}_{\varphi_i}(t) = f_{\varphi_i}(s_{\varphi_i}(t))$, $t \in R^+$, $i = 1, 2, \cdots, m$. Define

$$\hat{e}_{p_l}(t) = \big(e_{1p_l}(t), e_{2p_l}(t), \cdots, e_{mp_l}(t)\big)^T,\ t \in R^+,\ l = 1, 2, \cdots, n.$$

First, we give the definition of clustering component synchronization of the pinning controlled network (3.4) with respect to the specified components $p_1, p_2, \cdots, p_k$.

Definition 3.1. *If* $\lim\limits_{t\to+\infty} \sum\limits_{l=1}^{k} \|\hat{e}_{p_l}(t)\| = 0$*, then the pinning controlled network (3.4) is said to achieve clustering component synchronization with respect to the specified components* $p_1, p_2, \cdots, p_k$.

In order to make the pinning controlled network (3.4) realize clustering component synchronization with respect to the specified components $p_1, p_2, \cdots, p_k$, we design the pinning controller

$$u_i(t) = -cd_i\Big(g(x_i(t)) - g(s_{\varphi_i}(t))\Big) - c\sum_{j=1}^{m} a_{ij}g(s_{\varphi_j}(t)),\ t \in R^+,\ i \in \widetilde{U}_{\varphi_i}, \tag{3.5}$$

where $d_i > 0$ is the feedback control gain and $\widetilde{U}_{\varphi_i}$ denotes the set of all nodes in the φ_ith cluster, which has connections with other clusters. Since A is a zero-row-sum matrix, we have

$$\sum_{j=1}^{m} a_{ij}g(s_{\varphi_j}(t)) = 0,\ t \in R^+,\ i \in U_{\varphi_i} \setminus \widetilde{U}_{\varphi_i}.$$

So, if we let $d_i = 0$ for $i \in U_{\varphi_i} \setminus \widetilde{U}_{\varphi_i}$, then (3.5) can be rewritten as

$$u_i(t) = -cd_i\Big(g(x_i(t)) - g(s_{\varphi_i}(t))\Big) - c\sum_{j=1}^{m} a_{ij}g(s_{\varphi_j}(t)),\ t \in R^+,\ i = 1, 2, \cdots, m. \tag{3.6}$$

Now, we list the following assumptions that will be used later.

(A1) There exists a constant $\omega > 0$ such that for any $\eta, \zeta \in R^n$, the inequality

$$(\eta - \zeta)^T M(f_{\varphi_i}(\eta) - f_{\varphi_i}(\zeta)) \le \omega(\eta - \zeta)^T M(\eta - \zeta),\ i = 1, 2, \cdots, m$$

holds, where $M = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_{\mathrm{n}})$. Here, we have

$$\gamma_j = \begin{cases} 1, & j \in \{p_1, p_2, \cdots, p_k\}, \\ 0, & otherwise; \end{cases}$$

(A2) For $j \in \{p_1, p_2, \cdots, p_k\}$, there exist constants $\beta_j \ge \alpha_j > 0$ such that $g_j \in UNI(\alpha_j, \beta_j)$;

(A3) A is irreducible.

Theorem 3.1. *Suppose that (A1), (A2), and (A3) hold. If the following conditions are satisfied:*

$$\omega I_m + \frac{c}{8}(4(\alpha_{p_l} + \beta_{p_l})A + 4A^2 + (\beta_{p_l} - \alpha_{p_l})^2 I_m - 8\alpha_{p_l}D) < 0,\ l = 1, 2, \cdots, k,$$

where $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$, then the pinning controlled network (3.4) achieves clustering component synchronization with respect to the specified components $p_1, p_2, \cdots, p_k$.

Proof. Consider the error dynamic network corresponding to the pinning controlled network (3.4):

$$\dot{e}_i(t) = f_{\varphi_i}(x_i(t)) - f_{\varphi_i}(s_{\varphi_i}(t)) + c\sum_{j=1}^{m} a_{ij}\Big(g(x_j(t)) - g(s_{\varphi_j}(t))\Big) - cd_i\Big(g(x_i(t)) - g(s_{\varphi_i}(t))\Big),$$
$$t \in R^+,\ i = 1, 2, \cdots, m. \tag{3.7}$$

Denote $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_m^T(t))^T$, then (3.7) can be rewritten in the compact form by using the Kronecker product

$$\dot{e}(t) = \widetilde{f}(x_1(t), \cdots, x_m(t)) - \widetilde{f}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))$$

$$+ c((A-D)\otimes I_n)\Big(\widetilde{g}(x_1(t),\cdots,x_m(t)) - \widetilde{g}(s_{\varphi_1}(t),\cdots,s_{\varphi_m}(t))\Big),\ t\in R^+, \tag{3.8}$$

where $\widetilde{f}$ and $\widetilde{g}$ are defined by

$$\widetilde{f}(\vartheta_1,\cdots,\vartheta_m) = (f_{\varphi_1 1}(\vartheta_1),\cdots,f_{\varphi_1 n}(\vartheta_1),\cdots,f_{\varphi_m 1}(\vartheta_m),\cdots,f_{\varphi_m n}(\vartheta_m))^T,$$

and

$$\widetilde{g}(\vartheta_1,\cdots,\vartheta_m) = (g_1(\vartheta_{11}),\cdots,g_n(\vartheta_{1n}),\cdots,g_1(\vartheta_{m1}),\cdots,g_n(\vartheta_{mn}))^T$$

for $\vartheta_i = (\vartheta_{i1},\cdots,\vartheta_{in})^T \in R^n$, $i = 1,2,\cdots,m$, respectively.

Let $\hat{e}(t) = (\hat{e}^T_{p_1}(t),\hat{e}^T_{p_2}(t),\cdots,\hat{e}^T_{p_n}(t))^T$, $t\in R^+$. If we define

$$P = (\xi^T_{p_1},\xi^T_{n+p_1},\cdots,\xi^T_{(m-1)n+p_1},\xi^T_{p_2},\xi^T_{n+p_2},\cdots,\xi^T_{(m-1)n+p_2},\cdots,\xi^T_{p_n},\xi^T_{n+p_n},\cdots,\xi^T_{(m-1)n+p_n})^T,$$

where ξ_j $(j = 1,2,\cdots,mn)$ is the same as in Lemma 2.2, then $\hat{e}(t) = Pe(t)$, which, together with (3.8), implies that

$$\begin{aligned}\dot{\hat{e}}(t) =& \hat{f}(x_1(t),\cdots,x_m(t)) - \hat{f}(s_{\varphi_1}(t),\cdots,s_{\varphi_m}(t)) \\ &+ c(I_n\otimes(A-D))\Big(\hat{g}(x_1(t),\cdots,x_m(t)) - \hat{g}(s_{\varphi_1}(t),\cdots,s_{\varphi_m}(t))\Big),\ t\in R^+,\end{aligned} \tag{3.9}$$

where $\hat{f}$ and $\hat{g}$ are defined by

$$\hat{f}(\vartheta_1,\cdots,\vartheta_m) = (\hat{f}^T_{p_1}(\vartheta_1,\cdots,\vartheta_m),\cdots,\hat{f}^T_{p_n}(\vartheta_1,\cdots,\vartheta_m))^T,$$

and

$$\hat{g}(\vartheta_1,\cdots,\vartheta_m) = (\hat{g}^T_{p_1}(\vartheta_1,\cdots,\vartheta_m),\cdots,\hat{g}^T_{p_n}(\vartheta_1,\cdots,\vartheta_m))^T$$

for $\vartheta_i = (\vartheta_{i1},\cdots,\vartheta_{in})^T \in R^n$, $i = 1,2,\cdots,m$, respectively. Here,

$$\hat{f}_{p_l}(\vartheta_1,\cdots,\vartheta_m) = (f_{\varphi_1 p_l}(\vartheta_1),\cdots,f_{\varphi_m p_l}(\vartheta_m))^T,$$

and

$$\hat{g}_{p_l}(\vartheta_1,\cdots,\vartheta_m) = (g_{p_l}(\vartheta_{1p_l}),\cdots,g_{p_l}(\vartheta_{mp_l}))^T.$$

Now, we construct the Lyapunov function

$$V(t,x) = \frac{1}{2}x^T(\Lambda\otimes I_m)x \text{ for } (t,x)\in R^+\times R^{mn},$$

where $x = (y^T,z^T)^T$. Here, $y = (x_1,\cdots,x_{mk})^T$ and $z = (x_{mk+1},\cdots,x_{mn})^T$, $\Lambda = \mathrm{diag}(\underbrace{1,\cdots,1}_{\mathrm{k}},\underbrace{0,\cdots,0}_{\mathrm{n-k}})$.

First, if we let $\phi(u) = \frac{1}{6}u^2$ and $\psi(u) = u^2$ for $u\in R^+$, then it is obvious that ϕ, $\psi\in K$. Moreover, since $V(t,x) = \frac{1}{2}x^T(\Lambda\otimes I_m)x = \frac{1}{2}\sum_{i=1}^{mk}x_i^2 = \frac{1}{2}y^Ty = \frac{1}{2}\|y\|^2$, we know that $V: R^+\times R^{mn}\to R^+$, $V(t,0) = 0$, $t\in R^+$, and (2.2) of Lemma 2.1 is satisfied.

Differentiating $V(t,x)$ along the trajectories of the error dynamic network (3.9), we have

$$\frac{dV}{dt}\Big|_{(3.9)} = \hat{e}^T(t)(\Lambda\otimes I_m)\dot{\hat{e}}(t)$$

$$
\begin{aligned}
&= \hat{e}^T(t)(\Lambda \otimes I_m)\Big(\hat{f}(x_1(t), \cdots, x_m(t)) - \hat{f}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t)) \\
&\quad + c(I_n \otimes (A - D))\big(\hat{g}(x_1(t), \cdots, x_m(t)) - \hat{g}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big)\Big) \\
&= V_1(t) + V_2(t) + V_3(t), \ t \in R^+,
\end{aligned} \tag{3.10}
$$

where

$$
V_1(t) = \hat{e}^T(t)(\Lambda \otimes I_m)\big(\hat{f}(x_1(t), \cdots, x_m(t)) - \hat{f}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big), \ t \in R^+,
$$

$$
V_2(t) = c\hat{e}^T(t)(\Lambda \otimes A)\big(\hat{g}(x_1(t), \cdots, x_m(t)) - \hat{g}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big), \ t \in R^+,
$$

and

$$
V_3(t) = -c\hat{e}^T(t)(\Lambda \otimes D)\big(\hat{g}(x_1(t), \cdots, x_m(t)) - \hat{g}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big), \ t \in R^+.
$$

By (A1), we get

$$
\begin{aligned}
V_1(t) &= \hat{e}^T(t)(\Lambda \otimes I_m)\big(\hat{f}(x_1(t), \cdots, x_m(t)) - \hat{f}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big) \\
&= \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)\big(\hat{f}_{p_l}(x_1(t), \cdots, x_m(t)) - \hat{f}_{p_l}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big) \\
&= \sum_{i=1}^{m} e_i^T(t)M\big(f_{\varphi_i}(x_i(t)) - f_{\varphi_i}(s_{\varphi_i}(t))\big) \\
&\leq \omega \sum_{i=1}^{m} e_i^T(t)Me_i(t) \\
&= \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)(\omega I_m)\hat{e}_{p_l}(t), \ t \in R^+.
\end{aligned} \tag{3.11}
$$

In view of Lemma 2.3 and (A2), we have

$$
\begin{aligned}
&\sum_{l=1}^{k} \hat{e}_{p_l}^T(t)A\Big(\hat{g}_{p_l}(x_1(t), \cdots, x_m(t)) - \hat{g}_{p_l}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t)) - \frac{\alpha_{p_l} + \beta_{p_l}}{2}\hat{e}_{p_l}(t)\Big) \\
\leq& \frac{1}{2}\sum_{l=1}^{k} \hat{e}_{p_l}^T(t)AA^T\hat{e}_{p_l}(t) + \frac{1}{2}\sum_{l=1}^{k}\Big(\hat{g}_{p_l}(x_1(t), \cdots, x_m(t)) - \hat{g}_{p_l}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t)) - \frac{\alpha_{p_l} + \beta_{p_l}}{2}\hat{e}_{p_l}(t)\Big)^T \\
&\Big(\hat{g}_{p_l}(x_1(t), \cdots, x_m(t)) - \hat{g}_{p_l}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t)) - \frac{\alpha_{p_l} + \beta_{p_l}}{2}\hat{e}_{p_l}(t)\Big) \\
\leq& \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)(\frac{1}{2}A^2)\hat{e}_{p_l}(t) + \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)(\frac{(\beta_{p_l} - \alpha_{p_l})^2}{8}I_m)\hat{e}_{p_l}(t), \ t \in R^+,
\end{aligned}
$$

and so,

$$
\begin{aligned}
V_2(t) &= c\hat{e}^T(t)(\Lambda \otimes A)\big(\hat{g}(x_1(t), \cdots, x_m(t)) - \hat{g}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big) \\
&= c\sum_{l=1}^{k} \hat{e}_{p_l}^T(t)A\big(\hat{g}_{p_l}(x_1(t), \cdots, x_m(t)) - \hat{g}_{p_l}(s_{\varphi_1}(t), \cdots, s_{\varphi_m}(t))\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)(\frac{c(\alpha_{p_l}+\beta_{p_l})}{2}A)\hat{e}_{p_l}(t) \\
&\quad + c\sum_{l=1}^{k} \hat{e}_{p_l}^T(t)A\Big(\hat{g}_{p_l}(x_1(t),\cdots,x_m(t)) - \hat{g}_{p_l}(s_{\varphi_1}(t),\cdots,s_{\varphi_m}(t)) - \frac{\alpha_{p_l}+\beta_{p_l}}{2}\hat{e}_{p_l}(t)\Big) \\
&\le \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)(\frac{c(\alpha_{p_l}+\beta_{p_l})}{2}A + \frac{c}{2}A^2 + \frac{c(\beta_{p_l}-\alpha_{p_l})^2}{8}I_m)\hat{e}_{p_l}(t),\ t\in R^+. \qquad (3.12)
\end{aligned}
$$

By (A2), we know

$$
\begin{aligned}
V_3(t) &= -c\hat{e}^T(t)(\Lambda\otimes D)\Big(\hat{g}(x_1(t),\cdots,x_m(t)) - \hat{g}(s_{\varphi_1}(t),\cdots,s_{\varphi_m}(t))\Big) \\
&= -c\sum_{l=1}^{k} \hat{e}_{p_l}^T(t)D\Big(\hat{g}_{p_l}(x_1(t),\cdots,x_m(t)) - \hat{g}_{p_l}(s_{\varphi_1}(t),\cdots,s_{\varphi_m}(t))\Big) \\
&= -c\sum_{l=1}^{k}\sum_{i=1}^{m} e_{ip_l}(t)d_i\Big(g_{p_l}(x_{ip_l}(t)) - g_{p_l}(s_{\varphi_{i}p_l}(t))\Big) \\
&\le \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)(-c\alpha_{p_l}D)\hat{e}_{p_l}(t),\ t\in R^+. \qquad (3.13)
\end{aligned}
$$

Substituting inequalities (3.11), (3.12), and (3.13) into (3.10), we obtain

$$
\frac{dV}{dt}\Big|_{(3.9)} \le \sum_{l=1}^{k} \hat{e}_{p_l}^T(t)\Big(\omega I_m + \frac{c}{8}(4(\alpha_{p_l}+\beta_{p_l})A + 4A^2 + (\beta_{p_l}-\alpha_{p_l})^2 I_m - 8\alpha_{p_l}D)\Big)\hat{e}_{p_l}(t),\ t\in R^+,
$$

which, together with the fact

$$
\omega I_m + \frac{c}{8}(4(\alpha_{p_l}+\beta_{p_l})A + 4A^2 + (\beta_{p_l}-\alpha_{p_l})^2 I_m - 8\alpha_{p_l}D) < 0,\ l = 1,2,\cdots,k
$$

implies that

$$
\begin{aligned}
\frac{dV}{dt}\Big|_{(3.9)} &\le \sum_{l=1}^{k} \lambda_{\max}\Big(\omega I_m + \frac{c}{8}(4(\alpha_{p_l}+\beta_{p_l})A + 4A^2 + (\beta_{p_l}-\alpha_{p_l})^2 I_m - 8\alpha_{p_l}D)\Big)\hat{e}_{p_l}^T(t)\hat{e}_{p_l}(t) \\
&\le -h\sum_{l=1}^{k} \hat{e}_{p_l}^T(t)\hat{e}_{p_l}(t),\ t\in R^+,
\end{aligned}
$$

where

$$
h = -\max_{1\le l\le k}\Big\{\lambda_{\max}\Big(\omega I_m + \frac{c}{8}(4(\alpha_{p_l}+\beta_{p_l})A + 4A^2 + (\beta_{p_l}-\alpha_{p_l})^2 I_m - 8\alpha_{p_l}D)\Big)\Big\}.
$$

Thus, if we choose $\alpha(u) = hu^2$ for $u \in R^+$, then it is obvious that $\alpha \in K$, and (2.3) of Lemma 2.1 is satisfied. So, it follows from Lemma 2.1 that the trivial solution of the error dynamic network (3.9) is asymptotically stable with respect to the variable y, so $\lim_{t\to+\infty}\sum_{l=1}^{k}\|\hat{e}_{p_l}(t)\| = 0$, which shows that the pinning controlled network (3.4) achieves clustering component synchronization with respect to the specified components $p_1, p_2, \cdots, p_k$. □

Remark 3.1. *In [37], Guo et al. proposed a novel hybrid event-triggered method to realize the group consensus of heterogeneous second-order multi-agent systems with time-varying unknown nonidentical direction faults and stochastic false data injection attacks. By pinning the nodes who can receive state information from other groups, the multi-agent system achieves group consensus, providing it meets the conditions of Theorem 2. Compared with [37], the synchronization investigated in this paper is on any k specified components of a node's state variables, while the consensus realized in [37] is on all components of the multi-agent. Additionally, many scholars employed adaptive pinning control [15], adaptive control [24], periodic secure control [25], aperiodically intermittent pinning control [35], event-triggered impulsive control [36, 51], and so on to realize the cluster synchronization of complex dynamical networks. These synchronizations achieved are also on all components.*

To make Theorem 3.1 more applicable, we give the following corollaries.

Corollary 3.2. *Suppose that (A1), (A2), and (A3) hold. If the following conditions are fulfilled:*

$$\omega + \frac{c}{8}(\beta_{p_l} - \alpha_{p_l})^2 + \frac{c}{2}\lambda_{\max}\big((\alpha_{p_l} + \beta_{p_l})A + A^2 - 2\alpha_{p_l}D\big) < 0, \ l = 1, 2, \cdots, k, \tag{3.14}$$

then the pinning controlled network (3.4) achieves clustering component synchronization with respect to the specified components $p_1, p_2, \cdots, p_k$.

Corollary 3.3. *Suppose that (A1), (A2), and (A3) hold. If the coupling strength is fulfilled:*

$$c > -\frac{8\omega}{\max\limits_{1\le l\le k}\Big\{(\beta_{p_l} - \alpha_{p_l})^2 + 4\lambda_{\max}\big((\alpha_{p_l} + \beta_{p_l})A + A^2 - 2\alpha_{p_l}D\big)\Big\}} > 0,$$

then the pinning controlled network (3.4) achieves clustering component synchronization with respect to the specified components $p_1, p_2, \cdots, p_k$.

4. Numerical simulation

In this section, a numerical simulation is given to illustrate the effectiveness of the results obtained in Section 3.

Example 4.1. *Consider a nonlinearly coupled complex network consisting of* 6 *nonidentical nodes. Suppose that these nodes are divided into 3 clusters:* $U_1 = \{1, 2\}$, $U_2 = \{3, 4, 5\}$, *and* $U_3 = \{6\}$, *and the topology structure is shown in Figure 1.*

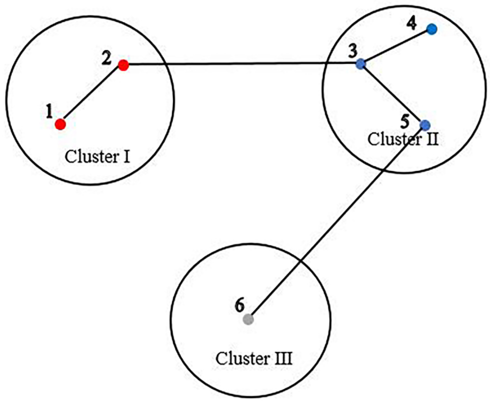

Figure 1. The network with three clusters.

Obviously, the coupling configuration matrix is

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Suppose that the network can be described as

$$\dot{x}_i(t) = f_{\varphi_i}(x_i(t)) + 10\sum_{j=1}^{6} a_{ij} g(x_j(t)),\ t \in R^+,\ i = 1, 2, \cdots, 6,$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T$; f_{φ_i} and g are defined by

$$f_1(v) = (0.3v_1^2 - v_1^3 - \sin v_2 - v_3, 1.3v_1^2 - v_2, 0.11v_1 - 0.1v_3 + 0.02)^T,$$

$$f_2(v) = (0.1v_1^2 - v_1^3 - \tanh v_2 - v_3, 2v_2 + 3, 0.15v_1 - 0.1v_3 + 0.01)^T,$$

$$f_3(v) = (10\tanh v_2 - 3.2v_1 + 2.95(|v_1 + 1| - |v_1 - 1|), -v_1 + v_2 - v_3, -14.87\sin v_2)^T,$$

and

$$g(v) = (4v_1 + \sin v_1, 0.1v_2 + \tanh v_2, 4v_3 + \sin v_3)^T$$

for $v = (v_1, v_2, v_3)^T$, respectively.

Let $s_i(t)$ $(i = 1, 2, 3)$ be the solutions of $\dot{s}_i(t) = f_i(s_i(t))$, satisfying initial conditions $s_1(0) = (1, 1, 1)^T$, $s_2(0) = (0.1, 0.1, 0.1)^T$, and $s_3(0) = (-1, 1, 1)^T$, respectively. Now, we investigate the pinning controlled network

$$\dot{x}_i(t) = f_{\varphi_i}(x_i(t)) + 10\sum_{j=1}^{6} a_{ij} g(x_j(t)) + u_i(t),\ t \in R^+,\ i = 1, 2, \cdots, 6, \tag{4.1}$$

where $u_i(t)$ is defined by (3.6) with $D = \mathrm{diag}(0, 10, 18, 0, 6, 2)$.

Let $p_1 = 1$ and $p_2 = 3$. In what follows, we verify that the pinning controlled network (4.1) can achieve clustering component synchronization with respect to the specified components p_1 and p_2.

In fact, if we choose $\omega = 22.7$, $\alpha_{p_l} = 3$, and $\beta_{p_l} = 5$ $(l = 1, 2)$, then it is not difficult to prove that (A1), (A2), and (3.14) are satisfied. Moreover, it is obvious that A is irreducible. Thus, all the conditions of Corollary 3.2 are fulfilled. So, it follows from Corollary 3.2 that the pinning controlled network (4.1) achieves clustering component synchronization with respect to the specified components p_1 and p_2. Figures 2–4 show the time evolution of the components of the error variables corresponding to the pinning controlled network (4.1). It can be seen that the first and third components of the error variables tend to 0 as $t \to +\infty$, respectively, while the second component does not. Figure 5 illustrates that $S_{i,j}(t) = \|s_i(t) - s_j(t)\|$ does not tend to 0 as $t \to +\infty$, $1 \le i < j \le 3$. These indicate that the pinning controlled network (4.1) achieves clustering component synchronization with respect to the specified components p_1 and p_2.

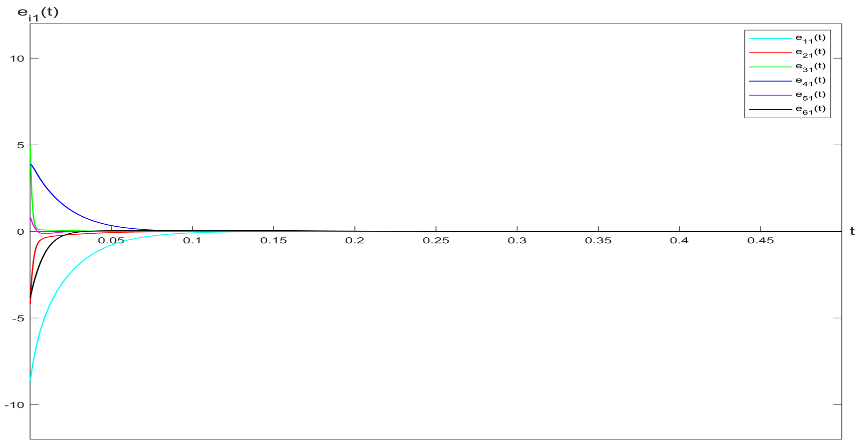

Figure 2. The time evolution of $e_{i1}(t), i = 1, 2, 3, 4, 5, 6$.

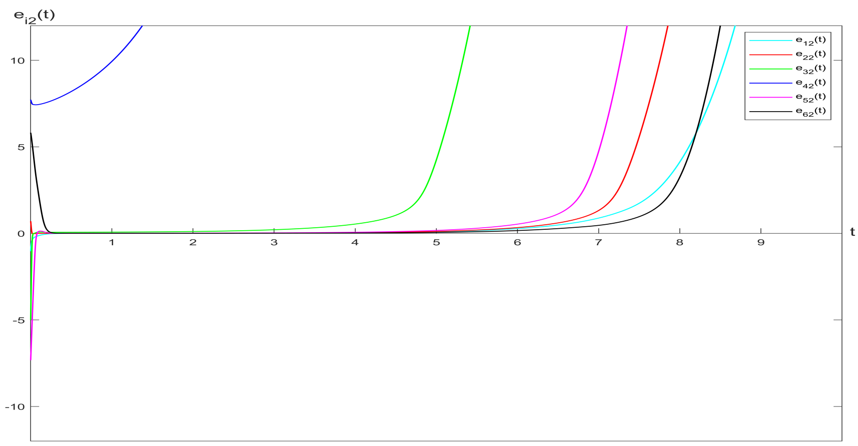

Figure 3. The time evolution of $e_{i2}(t), i = 1, 2, 3, 4, 5, 6$.

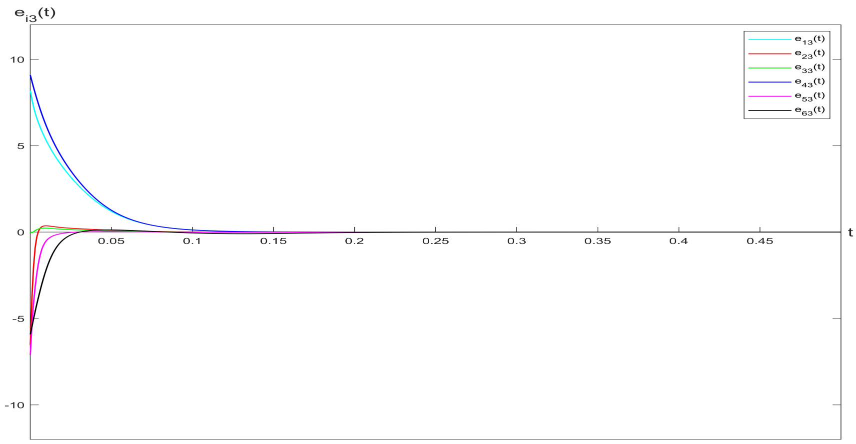

Figure 4. The time evolution of $e_{i3}(t), i = 1, 2, 3, 4, 5, 6$.

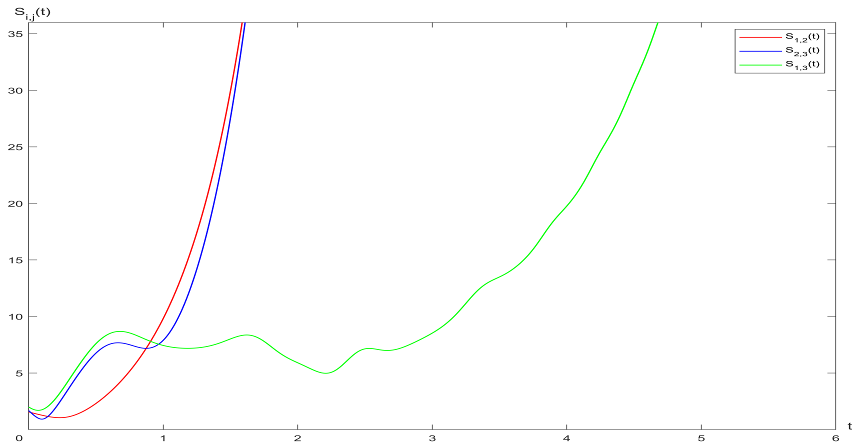

Figure 5. The time evolution of $S_{i,j}(t),\ 1 \leq i < j \leq 3$.

5. Conclusions

Clustering component synchronization is concerned with the convergence of some components of the node's state variables in a network rather than all components. As it has been stated in [48], the research of clustering component synchronization may have potential application in some formation control. For example, in the formation control of multiple unmanned aerial vehicle groups, the motion of them is restricted by many factors (such as their own displacement and velocity, and the wind speed of the environment). However, the control target of the formation is only some components (such as displacement and velocity), and the above factors can be asymptotically convergent, which is essentially a dynamic behavior of clustering component synchronization.

In this paper, the problem of clustering component synchronization for nonlinearly coupled complex dynamical networks with nonidentical nodes is investigated. By applying matrix analysis and stability theory, some sufficient conditions for achieving clustering component synchronization are obtained. A numerical example is also provided to verify the effectiveness of the theoretical results. In reality, complex dynamical networks are often directed and sometimes unconnected. The problem of clustering component synchronization for directed and unconnected networks is the subject of further research in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

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