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Research article

The problem of determining source term in a kinetic equation in an unbounded domain

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Abstract: In this paper, we deal with an inverse problem of determining the source function in a kinetic equation that is considered in an unbounded domain with Cauchy data. We prove the uniqueness of the solution of an inverse problem by means of a pointwise Carleman estimate. In recent years, kinetic equations have occurred in a variety of important fields and applications, such as aerospace engineering, semi-conductor technology, nuclear engineering, chemotaxis, and immunology.

Keywords: kinetic equation; inverse problem; unbounded domain; uniqueness; pointwise Carleman estimate
Mathematics Subject Classification: 35A23, 35R30

1. Introduction

The kinetic equation has become a powerful mathematical tool to describe the dynamics of many interacting particle systems, such as electrons, ions, stars, and galaxy or galactic aggregations. Since the nineteenth century, when Boltzmann formalized the concepts of kinetic equations, they have been used to model a variety of phenomena in different fields, such as rarefied gas dynamics, plasma physics, astrophysics, and socioeconomics. Particularly, in the life and social sciences, kinetic theory is used to model the dynamics of a large number of individuals, for example biological cells, animal flocks, pedestrians, or traders in large economic markets [1–5]. Moreover, it has applications in aerospace engineering [6], semi-conductor technology [7], nuclear engineering [8], chemotaxis, and immunology [9].

In this article, we consider the kinetic equation

$$\partial_t u(x,p,t) + \sum_{j=1}^{n} p_j \partial_{x_j} u(x,p,t) = f(t)g(x,p), \tag{1.1}$$

under the conditions:

$$u(x,p,t)|_{x_1\le 0} = u_1(x,p,t), \tag{1.2}$$
$$u(x,p,0) = u_0(x,p) \tag{1.3}$$

in the domain $\Omega = \{(x,p,t) : x_1 > 0,\ x \in D,\ p \in \mathbb{R}^n,\ t \in \mathbb{R}\}$, where $D \subset \mathbb{R}^n$, $x = (x_1, \bar{x}) \in \mathbb{R}^n$, $\bar{x} = (x_2, ..., x_n) \in \mathbb{R}^{n-1}$. Throughout the paper, we used the following notations:

$$\partial_t u = \frac{\partial u}{\partial t},\ \partial_{x_j} u = \frac{\partial u}{\partial x_j},\ \partial_{x_j}\partial_{y_j} u = \frac{\partial^2 u}{\partial x_j \partial y_j},\ \partial^2_{\eta_j} u = \frac{\partial^2 u}{\partial \eta_j^2},\ \Delta_\eta u = \sum_{j=1}^{n} \partial^2_{\eta_j} u\ (1 \le j \le n) \text{ and } i = \sqrt{-1}.$$

In applications, u represents the number (or the mass) of particles in the unit volume element of the phase space in the neighbourhood of the point (x,p,t), where x, p and t are the space, momentum and time variables, respectively.

Before stating our problem, we reduce Eq (1.1) to a second-order partial differential equation. Toward this aim, we apply the Fourier transform with respect to p, then from (1.1) we obtain

$$\partial_t \hat{u}(x,y,t) + i\sum_{j=1}^{n} \partial_{x_j}\partial_{y_j}\hat{u}(x,y,t) = f(t)\hat{g}(x,y), \tag{1.4}$$

where y is the parameter of the Fourier transform.

By (1.2) and (1.3), we find

$$\hat{u}(x,y,t)|_{x_1\le 0} = \hat{u}_1(x,y,t),\ \hat{u}(x,y,0) = \hat{u}_0(x,y). \tag{1.5}$$

Making the change of variables

$$x - y = \xi,\ x + y = \eta$$

and introducing the function $z(\xi,\eta,t) = \hat{u}(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}, t)$, $h(\xi,\eta) = \hat{g}(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2})$ in (1.4), we get

$$\partial_t z(\xi,\eta,t) + i(\Delta_\eta - \Delta_\xi)z(\xi,\eta,t) = f(t)h(\xi,\eta). \tag{1.6}$$

From (1.5), it follows that

$$z(\xi,\eta,t)|_{\xi_1+\eta_1\le 0} = z_1,\ z(\xi,\eta,0) = z_0(\xi,\eta). \tag{1.7}$$

We introduce the set Z of functions $z \in C^2(\mathbb{R}^{2n+1}) \cap H^2(\mathbb{R}^{2n+1})$ such that $z = 0$ for $x_1 - y_1 \ge \xi_0$, and the Fourier transform of z with respect to t is finite. We assume that $f \in C(\mathbb{R}) \cap H^2(\mathbb{R})$.

Equation (1.6) is called an ultrahyperbolic Schrödinger equation. Here, for technical reasons, we consider the equation

$$\partial_t z(\xi,\eta,t) + i(\Delta_\eta - a^{-1}\Delta_\xi)z(\xi,\eta,t) = f(t)h(\xi,\eta), \tag{1.8}$$

where $a \in C^1(\overline{D})$ and $a > 0$.

Since $z = 0$ for $x_1 - y_1 \ge \xi_0$, by (1.7) we have

$$z(\xi,\xi_0,\bar{\eta},t) = \partial_{\eta_1} z(\xi,\xi_0,\bar{\eta},t) = 0,\ z(\xi,\eta,0) = z_0(\xi,\eta), \tag{1.9}$$

where $\bar{\eta} = (\eta_2, ..., \eta_n)$.

We deal with the following problem:

Problem 1. Determine the pair of functions $z(\xi, \eta, t)$ and $h(\xi, \eta)$ from relations (1.8) and (1.9).

We investigate uniqueness of solution of Problem 1. For the proof, we shall use the Fredholm alternative theorem, so we consider the related homogeneous problem.

Then, by the condition $z(\xi, \eta, 0) = z_0(\xi, \eta) = 0$ from (1.8), we can write

$$h(\xi, \eta) = \frac{i}{f(0)} \int_{\mathbb{R}} \omega \hat{z}(\xi, \eta, \omega) d\omega,$$

so we have

$$\partial_t z(\xi, \eta, t) + \sum_{j=1}^{n} p_j(x) \partial_{x_j} z(\xi, \eta, t) = i \frac{f(t)}{f(0)} \int_{\mathbb{R}} \omega \hat{z}(\xi, \eta, \omega) d\omega. \tag{1.10}$$

The solvability of various inverse problems for kinetic equations was studied by Amirov [10] and Anikonov [11] on a bounded domain, where the problem is reduced to a Dirichlet problem for a third-order partial differential equation. See also [12] for an inverse problem for the transport equation, where the equation is reduced to a second-order differential equation with respect to the time variable t. Moreover, in [10,11], the problem of determining the potential in a quantum kinetic equation was discussed in an unbounded domain. Numerical algorithms to obtain the approximate solutions of some inverse problems were developed in [13,14]. The main difference between the current work and the existing works is that here the problem of finding the source function is considered in an unbounded domain with the Cauchy data which is given on a planar part of the boundary.

The main result of this paper is given below:

Theorem 1.1. Let $\partial_{\eta_1} a > 0$ and $f(0) \neq 0$. Then, Problem 1 has at most one solution (z, h) such that $z \in Z$ and $h \in L^1(\mathbb{R}^{2n})$.

We apply the Fourier transform to Eq (1.10) and for condition (1.9) with respect to (ξ, t), we get

$$-\omega \hat{z} - \Delta_\eta \hat{z} - a^{-1} |s|^2 \hat{z} = -\frac{\hat{f}(\omega)}{f(0)} \int_{\mathbb{R}} \omega \hat{z}(s, \eta, \omega) d\omega, \tag{1.11}$$

$$\hat{z}(s, 0, \bar{\eta}, \omega) = \partial_{\eta_1} \hat{z}(s, 0, \bar{\eta}, \omega) = 0. \tag{1.12}$$

We write $\hat{z} = z_1 + iz_2$ and $\hat{f} = f_1 + if_2$ in (1.11), and so we obtain the following system of equations:

$$\Delta_\eta z_k + a^{-1} |s|^2 z_k = l_k \ (k = 1, 2), \tag{1.13}$$

where

$$\begin{aligned} l_1 &= \frac{1}{f(0)} (f_1 \int_{\mathbb{R}} \omega z_1 d\omega - f_2 \int_{\mathbb{R}} \omega z_2 d\omega) - \omega z_1, \\ l_2 &= \frac{1}{f(0)} (f_1 \int_{\mathbb{R}} \omega z_2 d\omega + f_2 \int_{\mathbb{R}} \omega z_1 d\omega) - \omega z_2. \end{aligned} \tag{1.14}$$

By (1.12), we have

$$z_1(s, 0, \bar{\eta}, \omega) = 0, \ \partial_{\eta_1}(z_1)(s, 0, \bar{\eta}, \omega) = 0, \ z_2(s, 0, \bar{\eta}, \omega) = 0, \ \partial_{\eta_1}(z_2)(s, 0, \bar{\eta}, \omega) = 0. \tag{1.15}$$

Thus, we shall show that this homogeneous problem has only the trivial solution. In the proof of Theorem 1.1, the main tool is a pointwise Carleman estimate, which will be presented in the next section. The proof of Theorem 1.1 will be given in the last section.

2. Carleman estimate

The Carleman estimate is a key tool for proving uniqueness and stability results in determining a source or a coefficient for ill-posed Cauchy problems. Carleman [15] established the first Carleman estimate in 1939 for proving the unique continuation for a two-dimensional elliptic equation. Later, Müller [16], Calderón [17], and Hörmander [18] obtained more general results. In the theory of inverse problems, Carleman estimates were first introduced by Bukgeim and Klibanov in [19]. After that, there have been many works relying on that method with modified arguments. We refer to Puel and Yamamoto [20], Isakov and Yamamoto [21], Imanuvilov and Yamamoto [22,23], Bellassoued and Yamamoto [24], and Klibanov and Yamamoto [25] for hyperbolic equations; Yamamoto [26] for parabolic equations; Amirov [10], Lavrentiev et al. [27], Romanov [28], and Gölgeleyen and Yamamoto [29] for ultrahyperbolic equations; Gölgeleyen and Kaytmaz [30–32] for ultrahyperbolic Schrödinger equations; and Cannarsa et al. [33], Gölgeleyen and Yamamoto [34], and Klibanov and Pamyatnykh [12] for transport equations.

In order to obtain a Carleman estimate for Eq (1.13), we write

$$P_0 z_k = \Delta_\eta z_k + a^{-1}|s|^2 z_k \ (k = 1, 2). \tag{2.1}$$

Next, we define

$$\Omega_0 = \{(s,\eta) : s \in \mathbb{R}^n,\ \eta \in \mathbb{R}^n,\ \eta_1 > 0,\ 0 < \delta\eta_1 < \gamma - \sum_{j=2}^{n}(\eta_j - \eta_j^0)^2\},$$

where $0 < \gamma < 1$, $\delta > 1$, $\eta_0 = (\eta_1^0, ..., \eta_n^0)$, $\gamma + \alpha_0 = \rho < 1$. Moreover, we introduce a Carleman weight function

$$\varphi = e^{\lambda\psi^{-\nu}}, \tag{2.2}$$

where $\alpha_0 > 0$, the parameters δ, λ and ν are positive numbers, and

$$\psi(\eta) = \delta\eta_1 + \frac{1}{2}\sum_{j=2}^{n}(\eta_j - \eta_j^0)^2 + \alpha_0. \tag{2.3}$$

Obviously, $\eta_0 \in \Omega_0$ and $\Omega_0 \subset \Omega$ for sufficiently small $\gamma > 0$.

To establish a Carleman estimate we first present an auxiliary lemma.

Lemma 2.1. The following equality holds for any function $z_k \in Z$:

$$\begin{aligned} z_k(P_0 z_k)\varphi^2 &= a^{-1}|s|^2 z_k^2\varphi^2 - \left|\nabla_\eta z_k\right|^2\varphi^2 + z_k^2\sum_{j=1}^{n}\left(2\lambda^2\nu^2\left(\partial_{\eta_j}\psi\right)^2\psi^{-2\nu-2}\right. \\ &\quad \left. + \lambda\nu(\nu+1)\psi^{-\nu-2}\left(\partial_{\eta_j}\psi\right) - \lambda\nu\psi^{-\nu-1}\left(\partial_{\eta_j}^2\psi\right)\right)\varphi^2 + d_1(z_k), \end{aligned} \tag{2.4}$$

where

$$d_1(z_k) = \sum_{j=1}^{n}\partial_{\eta_j}\left(\left(z_k\left(\partial_{\eta_j} z_k\right) + \psi^{-\nu-1}\lambda\nu\left(\partial_{\eta_j}\psi\right)z_k^2\right)\varphi^2\right).$$

Proof of Lemma 2.1. Since $P_0 z_k = \left(\Delta_\eta + a^{-1}|s|^2\right)z_k$, we can write

$$z_k(P_0 z_k)\varphi^2 = z_k\left(\Delta_\eta z_k\right)\varphi^2 + a^{-1}|s|^2 z_k^2\varphi^2, \tag{2.5}$$

and using the equalities

$$z_k\left(\Delta_\eta z_k\right)\varphi^2=\partial_{\eta_j}\left(z_k\varphi^2\left(\partial_{\eta_j}z_k\right)\right)-\partial_{\eta_j}\left(z_k\varphi^2\right)\left(\partial_{\eta_j}z_k\right), \tag{2.6}$$

$$-\partial_{\eta_j}\left(z_k\varphi^2\right)\left(\partial_{\eta_j}z_k\right)=-\left(\partial_{\eta_j}z_k\right)^2\varphi^2-z_k\partial_{\eta_j}\left(\varphi^2\right)\left(\partial_{\eta_j}z_k\right), \tag{2.7}$$

$$-z_k\partial_{\eta_j}\left(\varphi^2\right)\left(\partial_{\eta_j}z_k\right)=-\partial_{\eta_j}\left(z_k^2\left(-\lambda\nu\left(\partial_{\eta_j}\psi\right)\psi^{-\nu-1}\right)\varphi^2\right)+\frac{1}{2}z_k^2\partial^2_{\eta_j}\left(\varphi^2\right), \tag{2.8}$$

$$\frac{1}{2}z_k^2\partial^2_{\eta_j}\left(\varphi^2\right)=\left(2\lambda^2\nu^2\left(\partial_{\eta_j}\psi\right)^2\psi^{-2\nu-2}+\lambda\nu\left(\nu+1\right)\left(\partial_{\eta_j}\psi\right)^2\psi^{-\nu-2}-\lambda\nu\left(\partial^2_{\eta_j}\psi\right)\psi^{-\nu-1}\right)z_k^2\varphi^2, \tag{2.9}$$

we obtain

$$\begin{aligned} z_k(P_0z_k)\varphi^2 &= -\left|\nabla_\eta z_k\right|^2\varphi^2+z_k^2\sum_{j=1}^n(2\lambda^2\nu^2\left(\partial_{\eta_j}\psi\right)^2\psi^{-2\nu-2}+\lambda\nu(\nu+1)\psi^{-\nu-2}\left(\partial_{\eta_j}\psi\right) \\ &\quad -\lambda\nu\psi^{-\nu-1}\left(\partial^2_{\eta_j}\psi\right))\varphi^2+\sum_{j=1}^n\partial_{\eta_j}\left(\left(z_k\left(\partial_{\eta_j}z_k\right)+\psi^{-\nu-1}\lambda\nu\left(\partial_{\eta_j}\psi\right)z_k^2\right)\varphi^2\right)+a^{-1}\left|s\right|^2z_k^2\varphi^2. \end{aligned}$$

Proposition 2.1. Under the assumptions of Theorem 1.1, the following inequality is valid for all $z_k\in Z$:

$$\begin{aligned} &-2n\lambda\nu z_k(P_0z_k)\varphi^2+\psi^{\nu+1}(P_0z_k)^2\varphi^2 \\ \geq\ & 2\lambda^3\nu^3\psi^{-2\nu-2}z_k^2\varphi^2+2\lambda\nu a^{-1}\left|s\right|^2z_k^2\varphi^2+2\lambda\nu\left|\nabla_\eta z_k\right|^2\varphi^2-2n\lambda\nu d_1\left(z_k\right)+d_2\left(z_k\right), \end{aligned} \tag{2.10}$$

where λ and ν are large parameters to be specified in the proof below. Moreover, $d_2\left(z_k\right)$ denotes the sum of divergence terms, which will be given explicitly later.

Proof of Proposition 2.1. We first define a new function $w=\varphi z_k$, and then we can write

$$\begin{aligned} \psi^{\nu+1}(P_0z_k)^2\varphi^2 &= \psi^{\nu+1}\left(\left(\Delta_\eta z_k\right)\varphi+a^{-1}\left|s\right|^2\varphi z_k\right)^2 \\ &\geq 4\lambda\nu\left(\sum_{j=1}^n\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_j}w\right)\right)\left(\Delta_\eta w+\sum_{r=1}^n\lambda\nu\psi^{-\nu-1}(\lambda\nu\psi^{-\nu-1}\left(\partial_{\eta_r}\psi\right)^2\right. \\ &\quad \left.-\left(\nu+1\right)\psi^{-1}\left(\partial_{\eta_r}\psi\right)^2+\left(\partial^2_{\eta_r}\psi\right))w+a^{-1}\left|s\right|^2w\right) \\ &:=\sum_{j=1}^5K_j. \end{aligned} \tag{2.11}$$

Now, we calculate the terms K_j, $1\leq j\leq 5$ as follows:

$$\begin{aligned} K_1 &= 4\lambda\nu\sum_{j,r=1}^n\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_j}w\right)\partial^2_{\eta_r}w \\ &= 4\lambda\nu\sum_{j,r=1}^n\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_j}w\right)\left(\partial_{\eta_r}w\right)\right)-2\lambda\nu\sum_{j,r=1}^n\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_r}\psi\right)^2\right) \\ &\quad +2\lambda\nu\left(n-1\right)^2\sum_{j=1}^n\left(\partial_{\eta_j}\psi\right)^2-4\lambda\nu\left(n-1\right)^2\sum_{j=2}^n\left(\partial_{\eta_j}\psi\right)^2. \end{aligned} \tag{2.12}$$

Similarly, we can calculate

$$\begin{aligned} K_2 &= 4\lambda^3\nu^3\psi^{-2\nu-2}\sum_{j,r=1}^{n}\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_j}w\right)\left(\partial_{\eta_r}\psi\right)^2 w \\ &= 2\lambda^3\nu^3\psi^{-2\nu-2}\sum_{j,r=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_r}\psi\right)^2 w^2\right) + 4\lambda^3\nu^3\left(\nu+1\right)\psi^{-2\nu-3}\left|\nabla_\eta\psi\right|^4 w^2 \\ &\quad -2\left(n-1\right)\lambda^3\nu^3\psi^{-2\nu-2}\left|\nabla_\eta\psi\right|^2 w^2 - 4\lambda^3\nu^3\psi^{-2\nu-2}\left|\nabla_\eta\psi\right|^2 w^2 \\ &\geq d_{22}\left(w\right) + 4\lambda^3\nu^4\delta^4\psi^{-2\nu-2}w^2. \end{aligned} \tag{2.13}$$

Next, we have

$$\begin{aligned} K_3 &= -4\lambda^2\nu^2\left(\nu+1\right)\psi^{-\nu-2}\sum_{j,r=1}^{n}\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_j}w\right)\left(\partial_{\eta_r}\psi\right)^2 w \\ &= -2\lambda^2\nu^2\left(\nu+1\right)\psi^{-\nu-2}\sum_{j,r=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_r}\psi\right)^2 w^2\right) + 2\lambda^2\nu^2\left(\nu+1\right)\left(n-1\right)\psi^{-\nu-2}\left|\nabla_\eta\psi\right|^2 w^2 \\ &\quad +4\lambda^2\nu^2\left(\nu+1\right)\left|\nabla_\eta\psi\right|^2\psi^{-\nu-2}w^2 - 2\lambda^2\nu^2\left(\nu+1\right)\left(\nu+2\right)\psi^{-\nu-3}\left|\nabla_\eta\psi\right|^4 w^2 \\ &\geq d_{23}\left(w\right) - 2\lambda^2\nu^2\left(\nu+1\right)\left(\nu+2\right)\psi^{-\nu-3}\left|\nabla_\eta\psi\right|^4 w^2. \end{aligned} \tag{2.14}$$

As for the fourth term,

$$\begin{aligned} K_4 &= 4\lambda^2\nu^2\psi^{-\nu-1}\sum_{j,r=1}^{n}\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_j}w\right)\left(\partial^2_{\eta_r}\psi\right)w \\ &= 2\lambda^2\nu^2\psi^{-\nu-1}\left(n-1\right)\sum_{j=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)w^2\right) - 2\lambda^2\nu^2\psi^{-\nu-1}\left(n-1\right)^2 w^2 \\ &\quad +2\lambda^2\nu^2\left(n-1\right)\left(\nu+1\right)\psi^{-\nu-2}\left|\nabla_\eta\psi\right|^2 w^2 \\ &\geq d_{24}\left(w\right) - 2\lambda^2\nu^2\psi^{-\nu-1}\left(n-1\right)^2 w^2. \end{aligned} \tag{2.15}$$

Moreover, we have

$$\begin{aligned} K_5 &= 4\lambda\nu a^{-1}\left|s\right|^2\sum_{j=1}^{n}\left(\partial_{\eta_j}\psi\right)\left(\partial_{\eta_j}w\right)w \\ &= 2\lambda\nu\left|s\right|^2\sum_{j=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)a^{-1}w^2\right) + 4\lambda\nu\left|s\right|^2 a^{-2}w^2\left(\delta\left(\partial_{\eta_1}a\right) + \sum_{j=2}^{n}\left(\partial_{\eta_j}a\right)\left(\eta_j-\eta_j^0\right)\right) \\ &\quad -2\lambda\nu\left(n-1\right)\left|s\right|^2 a^{-1}w^2 \\ &\geq d_{25}\left(w\right) + 2\lambda\nu\left(n+1\right)\left|s\right|^2 a^{-1}w^2, \end{aligned} \tag{2.16}$$

where we choose

$$\delta > \max_{\eta\in\overline{D}}\left\{\sum_{j=2}^{n}\left|\left(\partial_{\eta_j}a\right)\left(\eta_j-\eta_j^0\right)\right| + na\right\}/\min_{\eta\in\overline{D}}\left(\partial_{\eta_1}a\right).$$

If we replace the functions w with z_k, then from (2.12)–(2.16), we can write

$$\begin{aligned}&\psi^{\nu+1}\varphi^2(\left(\Delta_\eta z_k\right)+a^{-1}\left|s\right|^2 z_k)^2\\ \geq\ & d_2\left(z_k\right)-2\lambda\nu\left(n-1\right)\left|\nabla_\eta z_k\right|^2\varphi^2+2\lambda\nu\left(n+1\right)\left|s\right|^2 a^{-1}z_k^2\varphi^2+3\lambda^3\nu^4\delta^4\psi^{-2\nu-2}z_k^2\varphi^2\\ &+\lambda^2\nu^2 z_k^2\varphi^2\left(\lambda\nu^2\delta^4\psi^{-\nu-1}-2\left(\nu+1\right)\left(\nu+2\right)\left|\nabla_\eta\psi\right|^4\psi^{-1}-2\left(n-1\right)^2\right)\psi^{-\nu-1},\end{aligned}\tag{2.17}$$

where

$$\begin{aligned}d_2\left(z_k\right) &= \sum_{j=1}^{5} d_{2j}\left(z_k\right),\\ d_{21}\left(z_k\right) &= 4\lambda\nu\sum_{j,r=1}^{n}\partial_{\eta_r}(\left(\partial_{\eta_j}\psi\right)\left(\left(\partial_{\eta_j}z_k\right)-\lambda\nu\psi^{-\nu-1}\left(\partial_{\eta_j}\psi\right)z_k\right)\left(\left(\partial_{\eta_r}z_k\right)-\lambda\nu\psi^{-\nu-1}\left(\partial_{\eta_r}\psi\right)z_k\right)\varphi^2\\ &\quad -2\lambda\nu\sum_{j,r=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)\left(\left(\partial_{\eta_r}z_k\right)-\lambda\nu\psi^{-\nu-1}\left(\partial_{\eta_r}\psi\right)z_k\right)^2\varphi^2\right)\\ &\quad +2\lambda^2\nu^2\psi^{-\nu-1}\left(n-1\right)\sum_{j=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)z_k^2\varphi^2\right),\\ d_{22}\left(z_k\right) &= 2\lambda^3\nu^3\psi^{-2\nu-2}\partial_{\eta_j}\left(\left|\nabla_\eta\psi\right|^2 z_k^2\sum_{j=1}^{n}\partial_{\eta_j}\psi\right),\\ d_{23}\left(z_k\right) &= -2\lambda^2\nu^2\left(\nu+1\right)\psi^{-\nu-2}\sum_{j=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)\left|\nabla_\eta\psi\right|^2 z_k^2\right),\\ d_{24}\left(z_k\right) &= 2\lambda^2\nu^2\left(n-1\right)\psi^{-\nu-1}\sum_{j=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)z_k^2\right),\\ d_{25}\left(z_k\right) &= 2\lambda\nu\left|s\right|^2\sum_{j=1}^{n}\partial_{\eta_j}\left(\left(\partial_{\eta_j}\psi\right)a^{-1}z_k^2\right).\end{aligned}$$

Here, we can choose $\lambda \geq \lambda_0$ such that

$$\lambda\nu^2\delta^4\psi^{-\nu-1}-2\left(\nu+1\right)\left(\nu+2\right)\left|\nabla_\eta\psi\right|^4\psi^{-1}-2\left(n-1\right)^2\geq 0,$$

and so from (2.17) we obtain

$$\begin{aligned}&\psi^{\nu+1}(P_0 z_k)^2\varphi^2\\ \geq\ & 3\lambda^3\nu^4\delta^4\psi^{-2\nu-2}z_k^2\varphi^2+2\lambda\nu\left(n+1\right)\left|s\right|^2 a^{-1}z_k^2\varphi^2-2\lambda\nu\left(n-1\right)\left|\nabla_\eta z_k\right|^2\varphi^2+d_2\left(z_k\right).\end{aligned}\tag{2.18}$$

Finally, multiplying inequality (2.4) by $-2n\lambda\nu$ and summing with (2.18), we get

$$\begin{aligned}&-2n\lambda\nu z_k(P_0 z_k)\varphi^2+\psi^{\nu+1}(P_0 z_k)^2\varphi^2\\ \geq\ & 2\lambda\nu\left|s\right|^2 a^{-1}z_k^2\varphi^2+2\lambda\nu\left|\nabla_\eta z_k\right|^2\varphi^2+\left(3\lambda^3\nu^4\delta^4\psi^{-2\nu-2}-4n\lambda^3\nu^3\psi^{-2\nu-2}\left|\nabla_\eta\psi\right|^2\right.\\ &\left.-2n\lambda^2\nu^2\left(\nu+1\right)\psi^{-\nu-2}\sum_{j=1}^{n}\left(\partial_{\eta_j}\psi\right)^2+2n\lambda^2\nu^2\psi^{-\nu-1}\left(n-1\right)\right)z_k^2\varphi^2-2n\lambda\nu d_1\left(z_k\right)+d_2\left(z_k\right).\end{aligned}\tag{2.19}$$

By choosing $\nu \geq \nu_1$, we obtain (2.10). Thus, the proof of Proposition 2.1 is complete.

3. Proof of Theorem 1.1

First, for the right-hand side of (1.13), we can write

$$(l_1)^2 + (l_2)^2 \leq 3\frac{(f_1^2 + f_2^2)}{f^2(0)} M_1 \int_{\mathbb{R}} (1+\omega^2)^2 (z_1^2 + z_2^2) d\omega + 3\omega^2 (z_1^2 + z_2^2). \tag{3.1}$$

In (3.1), we used the following expressions:

$$\begin{aligned}
(\int_{\mathbb{R}} \omega z_k d\omega)^2 &= (\int_{\mathbb{R}} (1+\omega^2)^{-1/2}(1+\omega^2)^{1/2}\omega z_k d\omega)^2 \\
&\leq \int_{\mathbb{R}} (1+\omega^2)^{-1} d\omega \int_{\mathbb{R}} \omega^2 (1+\omega^2) z_k^2 d\omega \\
&\leq M_1 \int_{\mathbb{R}} (1+\omega^2)^2 z_k^2 d\omega
\end{aligned}$$

and

$$M_1 = \int_{\mathbb{R}} (1+\omega^2)^{-1} d\omega.$$

Using (2.10), for $k = 1, 2$ we have

$$\begin{aligned}
&((l_k)^2 + \lambda^2 \nu^2 n^2 z_k^2)\varphi^2 + \psi^{\nu+1}(P_0 z_k)^2 \varphi^2 \\
\geq\ & -2\lambda \nu n z_k (P_0 z_k)\varphi^2 + \psi^{\nu+1}(P_0 z_k)^2 \varphi^2 \\
\geq\ & 2\lambda^3 \nu^3 \psi^{-2\nu-2} z_k^2 \varphi^2 + 2\lambda \nu a^{-1} |s|^2 z_k^2 \varphi^2 + 2\lambda \nu \left|\nabla_\eta z_k\right|^2 \varphi^2 - 2\lambda \nu n d_1(z_k) + d_2(z_k).
\end{aligned} \tag{3.2}$$

By (3.1), (3.2) and the equalities $z_1^2 + z_2^2 = |\hat{z}|^2$, $f_1^2 + f_2^2 = \left|\hat{f}\right|^2$, we obtain

$$\begin{aligned}
&(3\frac{\left|\hat{f}\right|^2}{f^2(0)} M_1 \int_{\mathbb{R}} (1+\omega^2)^2 |\hat{z}|^2 d\omega + 3\omega^2 |\hat{z}|^2)\varphi^2 \\
\geq\ & 2\lambda^3 \nu^3 \psi^{-2\nu-2} |\hat{z}|^2 \varphi^2 + 2\lambda \nu a^{-1} |s|^2 |\hat{z}|^2 \varphi^2 + 2\lambda \nu \left|\nabla_\eta \hat{z}\right|^2 \varphi^2 + \sum_{k=1}^{2}(d_2(z_k) - 2\lambda \nu n d_1(z_k)).
\end{aligned} \tag{3.3}$$

If we multiply (3.3) by $(1+\omega^2)^2$ and integrate with respect to the parameters ω over $\mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi^2 (|\hat{z}|^2 (2\lambda^3 \nu^3 \psi^{-2\nu-2} + 2\lambda \nu a^{-1} |s|^2) + 2\lambda \nu \left|\nabla_\eta \hat{z}\right|^2)(1+\omega^2)^2 d\omega \\
\leq\ & (3\overline{f_0} M_2 \varphi^2 \int_{\mathbb{R}} ((1+\omega^2)^2 |\hat{z}|^2) d\omega + 3\varphi^2 \int_{\mathbb{R}} ((1+\omega^2)^2 \omega^2 |\hat{z}|^2) d\omega) \\
&+ \sum_{k=1}^{2} \int_{\mathbb{R}} (1+\omega^2)^2 (2\lambda \nu n d_1(z_k) - d_2(z_k)) d\omega,
\end{aligned} \tag{3.4}$$

where $\overline{f_0} = \max\limits_{\eta \in \overline{D}} \left\{\frac{M_1}{f^2(0)}\right\}$ and $M_2 = \int_{\mathbb{R}} (1+\omega^2)^2 \left|\hat{f}\right|^2 d\omega$.

In inequality (3.4), we can choose the big parameter λ, such that all terms on the right-hand side can be absorbed into the left-hand side. Then, we have

$$\lambda^3 \nu^3 \int_{\mathbb{R}} |\hat{z}|^2 \varphi^2 d\omega \leq div(\hat{z}), \tag{3.5}$$

where

$$div(\hat{z}) = \sum_{k=1}^{2} \int_{\mathbb{R}} (1+\omega^2)^2 (2\lambda\nu n d_1(z_k) - d_2(z_k)) d\omega.$$

Since $\varphi^2 > 1$ on Ω_0, we have

$$\int_{\mathbb{R}} |\hat{z}|^2 d\omega \leq \int_{\mathbb{R}} |\hat{z}|^2 \varphi^2 d\omega \leq \frac{1}{\lambda^3 \nu^3} div(\hat{z}). \tag{3.6}$$

Integrating inequality (3.6) over Ω_0 and passing to the limit as $\lambda \to \infty$, we have

$$\int_{\Omega_0} \int_{\mathbb{R}} |\hat{z}|^2 d\omega ds d\eta \leq 0, \tag{3.7}$$

which means that $\hat{z} = 0$. Therefore, we conclude that $z = 0$. Finally, by (1.8) we obtain $h = 0$, which completes the proof of Theorem 1.1.

4. Conclusions

In this study, we considered an inverse problem for the kinetic equation in an unbounded domain. We reduced the equation to a second-order partial differential equation and proved the uniqueness of the solution of the problem by using the Carleman estimate. The method used in this paper can be applied to a variety of equations, including some first and second-order partial differential equations in mathematical physics, such as transport, ultrahyperbolic, and ultrahyperbolic Schrödinger equations. By similar arguments, a stability estimate can be obtained in an unbounded domain.

Use of AI tools declaration

The author declares that she did not use Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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