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Research article

Long-time dynamics of nonlinear MGT-Fourier system

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Abstract: In this paper, we consider the long-time dynamical behavior of the MGT-Fourier system

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta\Delta u_t - \gamma\Delta u + \eta\Delta\theta + f_1(u, u_t, \theta) = 0, \\ \theta_t - \kappa\Delta\theta - \eta\Delta u_{tt} - \eta\alpha\Delta u_t + f_2(u, u_t, \theta) = 0. \end{cases}$$

First we use the nonlinear semigroup theory to prove the well-posedness of the solutions. Then we establish the existence of smooth finite dimensional global attractors in the system by showing that the solution semigroup is gradient and quasi-stable. Furthermore, we investigate the existence of generalized exponential attractors.

Keywords: MGT-Fourier; well-posedness; global attractors; exponential attractors
Mathematics Subject Classification: 35B41, 35G61, 35L75, 37L05, 37L30

1. Introduction

The field of nonlinear acoustics has received much attention in recent years. There is currently a large number of applications of high intensity ultrasound, ranging from medical therapy (shock wave lithotripsy and thermotherapy) via sonochemistry to ultrasound cleaning and welding. There are some classical models of nonlinear acoustics, such as the Westervelt equation [23], Kuznetsov equation [15] and Jordan-Moore-Gibson-Thompson equation [14].

In this paper, we consider the following system which is related to Jordan-Moore-Gibson-Thompson equation:

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta\Delta u_t - \gamma\Delta u + \eta\Delta\theta + f_1(u, u_t, \theta) = 0, & (1.1) \\ \theta_t - \kappa\Delta\theta - \eta\Delta u_{tt} - \eta\alpha\Delta u_t + f_2(u, u_t, \theta) = 0, & (1.2) \end{cases}$$

with the following initial conditions and boundary conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), u_{tt}(0) = u_2(x), \theta(x,0) = \theta_0(x), & (1.3) \\ u|_{\partial\Omega} = 0, \nabla u|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0, & (1.4) \end{cases}$$

where Ω is a bounded domain with the smooth boundary $\partial\Omega$, and $\alpha, \beta, \gamma, \kappa > 0$ and $\eta \neq 0$ are fixed structural constants, while Δ is the Laplace operator.

The uncoupled case in which $f_1 = 0, f_2 = 0$ and $\eta = 0$, the first equation is the so-called Moore-Gibson-Thompson (MGT) equation that appears in the context of acoustic wave propagation in viscous thermally relaxing fluids [16,21], although it was originally introduced by Stokes in the mid-nineteenth century [20]. It has received much attention in recent years. Quite interestingly, it can be used to model vibration in a standard linear viscoelastic solid, as it can be obtained by differentiating in time the equation of viscoelasticity with an exponential kernel (see [12]). The same equation also arises as a model for temperature evolution in a type III heat conduction model with a relaxation parameter (see [19]). The MGT equation is coupled with the classical Fourier heat equation by means of the coupling constant η which describes the vibrations of a viscoelastic heat conductor that obeys the Fourier thermal law which has been studied in [13]. In [17], the authors proved exponential stability whenever $|\eta| > t$ in the supercritical case. The authors established the exponential stability for the MGT-Fourier system by using the semigroup method. Another meaningful physical model can be obtained by replacing the operator $-\Delta$ with the bi-laplacian Δ^2, yielding thermoviscoelastic plates of MGT type [10, 11, 13].

Regarding the nonlinear MGT equation, it is much more complicated ([4–6, 14, 18]). There are few results for the attractors in the nonlinear MGT system [2, 3]. Recently, we considered the uniform attractors for the nonautonomous MGT-Fourier system in [22]. In this paper, we will investigate the global attractors for the MGT-Fourier system in the one-dimensional case, i.e., $(x,t) \in (0, L) \times (0, +\infty)$. The main goal of this work is to prove the existence of the global attractors for system (1.1)–(1.4). The paper is organized as follows. In Section 2, we will use the semigroup method to prove the global well-posedness. In Section 3, we get the existence of the global attractors by showing that the solution semigroup is gradient and quasi-stable as described in [7, 9]. Indeed, a quasi-stable system is asymptotically compact and its global attractor has a finite fractal dimension. Moreover, we consider the additional regularity for global attractors and the existence of the generalized exponential attractor by referring to [8].

2. Preliminaries and well-posedness

In this section we shall first give some notations, assumptions and energy estimates that will be needed. Then we will establish the well-posedness of the system given by (1.1)–(1.4).

2.1. Notations

Denote $\|X\|_q = \|X\|_{L^q(0,L)}$ and $H_0^1(0, L) = \{X \in H^1(0, L) : X(0) = 0, X(L) = 0\}$; then it follows that

$$\mathcal{H} := H_0^1(0, L) \times H_0^1(0, L) \times L^2(0, L) \times L^2(0, L),$$

equipped with the inner product given by

$$\langle Z, \bar{Z} \rangle_{\mathcal{H}} = \langle w + \alpha v, \bar{w} + \alpha \bar{v} \rangle + \frac{\gamma}{\alpha} \langle \alpha u_x + v_x, \alpha \bar{u}_x + \bar{v}_x \rangle + \chi \langle v_x, \bar{v}_x \rangle + \langle \theta, \bar{\theta} \rangle,$$

where $Z = (u, v, w, \theta)^T$, $\bar{Z} = (\bar{u}, \bar{v}, \bar{w}, \bar{\theta})^T$ and $\chi = \beta - \dfrac{\gamma}{\alpha} > 0$. We can convert (1.1)–(1.4) to an abstract evolution equation

$$\begin{cases} \dfrac{dZ}{dt} = AZ + F, & (2.1)\\ Z(0) = Z_0 = (u_0, u_1, u_2, \theta_0), & (2.2) \end{cases}$$

where

$$AZ = \begin{pmatrix} v \\ w \\ -\alpha w + \beta v_{xx} + \gamma u_{xx} - \eta\theta_{xx} \\ \kappa\theta_{xx} + \eta w_{xx} + \eta\alpha v_{xx} \end{pmatrix}, F = \begin{pmatrix} 0 \\ 0 \\ -f_1 \\ -f_2 \end{pmatrix} \tag{2.3}$$

and

$$D(A) = \{Z \in \mathcal{H} | w \in H_0^2, \beta v + \gamma u - \eta\theta \in H_0^2, \kappa\theta + \eta w + \eta\alpha v \in H_0^2\}. \tag{2.4}$$

By a direct inner product caculation, we also have

$$Re\langle AZ, Z\rangle_{\mathcal{H}} = -\chi\|u_{tx}\|_{L^2}^2 - \kappa\|\theta_x\|_{L^2}^2.$$

By the Poincar$\acute{e}$ inequality, we can get

$$\lambda_0\|u\|_2^2 \le \|u_x\|_2^2, \quad \lambda_0\|\theta\|_2^2 \le \|\theta_x\|_2^2, \tag{2.5}$$

where $\lambda_0 > 0$ is the Poincar$\acute{e}$ constant.

In this work, we apply the forcing terms $f_i(u, u_t, \theta) = f_i(u_t + \alpha u, \theta)$. These are simpler particular choices, representative enough for the case and techniques introduced. The following hypotheses given in the next assumption will be used throughout the paper.

Assumption 2.1. *(i) The forcing terms f_i $(i = 1, 2)$ are assumed to be locally Lipschitz and of gradient type. More precisely, there exists $F \in C(\mathbf{R}^2)$ with*

$$\nabla F = (f_1, f_2). \tag{2.6}$$

(ii) There exist $\lambda, M_F \ge 0$ such that

$$F(u_t + \alpha u, \theta) \ge -\lambda(|u_t + \alpha u|^2 + |\theta|^2) - M_F, \tag{2.7}$$

where

$$0 \le \lambda < \frac{1}{2\lambda_0}. \tag{2.8}$$

(iii) There exist $p \ge 1$ and $C_f > 0$ such that

$$|\nabla f_i(u_t + \alpha u, \theta)| \le C_f(1 + |u_t + \alpha u|^{p-1} + |\theta|^{p-1}), \quad i = 1,\ 2. \tag{2.9}$$

In particular, there exists $C_F > 0$ such that

$$F(u_t + \alpha u, \theta)| \le C_F(1 + |u_t + \alpha u|^{p+1} + |\theta|^{p+1}). \tag{2.10}$$

Moreover,

$$|\nabla F(u_t + \alpha u, \theta) \cdot (u_t + \alpha u, \theta) - F(u_t + \alpha u, \theta)| \ge -\lambda(|u_t + \alpha u|^2 + |\theta|^2) - M_F. \tag{2.11}$$

2.2. Energy of the system

Definition 2.1. *$Z = (u, u_t, u_{tt}, \theta) \in C^0([0, +\infty), \mathcal{H})$ is called a weak solution to (1.1) and (1.2) if it satisfies the initial conditions given by (1.3) and for a.e. $t > 0$,*

$$\frac{d}{dt}(u_{tt} + \alpha u_t, w + \alpha v) + (\beta u_{tx} + \gamma u_x - \eta\theta_x, w + \alpha v) + (f_1, w + \alpha v) = 0,$$
$$\frac{d}{dt}(\theta, \phi) + (\kappa\theta_x + \eta u_{ttx} + \eta\alpha u_{tx}, \phi) + (f_2, \phi) = 0,$$

for $w + \alpha v, \phi \in H_0^1(0, L)$. If a weak solution further satisfies

$$Z \in C^0([0, +\infty), D(A)) \cap C^1([0, +\infty), \mathcal{H}),$$

then it is called a strong solution.

We define the energy of the solutions $z = (u, u_t, u_{tt}, \theta)$ of (1.1)–(1.4) as follows:

$$\mathcal{E}(t) = E(t) + \int_0^L F(u_t + \alpha u, \theta)dx, \tag{2.12}$$

where $E(t) = \frac{1}{2}\|Z(t)\|_{\mathcal{H}}^2$ is the linear energy. We often omit the variable t inside of the integrals.

Lemma 2.1. *Suppose that $z = (u, u_t, u_{tt}, \theta)$ is a strong solution to (1.1)–(1.4). Then*

(i) The total energy satisfies

$$\frac{d}{dt}\mathcal{E}(t) = -\chi\|u_{tx}\|_2^2 - \kappa\|\theta_x\|_2^2 \tag{2.13}$$
$$\leq -M(\|u_t\|_2^2 + \|\theta\|_2^2), \tag{2.14}$$

where the constant $M > 0$.

(ii) There exist constants $\beta_0, K_F > 0$ such that

$$\beta_0\|Z(t)\|_{\mathcal{H}}^2 - LM_F \leq \mathcal{E}(t) \leq K_F(1 + \|Z(t)\|_{\mathcal{H}}^{p+1}). \tag{2.15}$$

Proof. Multiplying (1.1) and (1.2) by $u_{tt} + \alpha u_t$ and θ, respectively, after integration over $(0, L)$ we can obtain (2.13) through integration by parts. Then (2.14) follows from (2.5).

By (2.12) and (2.7), we have

$$\begin{aligned} \mathcal{E}(t) &\geq \frac{1}{2}\|Z(t)\|_{\mathcal{H}}^2 - \lambda\lambda_0(|u_t + \alpha u|^2 + |\theta|^2) - LM_F \\ &\geq (\frac{1}{2} - \lambda\lambda_0)\|Z(t)\|_{\mathcal{H}}^2 - LM_F \\ &\geq \beta_0\|Z(t)\|_{\mathcal{H}}^2 - LM_F, \end{aligned}$$

where $\beta_0 = \frac{1}{2} - \lambda\lambda_0 > 0$. By (2.10), for some $C > 0$,

$$\int_0^L F(u_t + \alpha u, \theta)dx \leq C(1 + |u_t + \alpha u|^{p+1} + |\theta|^{p+1}).$$

Then, following from the embedding of $H_0^1(0,L)$ in $L^p(0,L)$, there exists $K_F > 0$ such that

$$\begin{aligned}\mathcal{E}(t) &\le \frac{1}{2}\|Z(t)\|_{\mathcal{H}}^2 + C(1 + |u_t + \alpha u|^{p+1} + |\theta|^{p+1}) \\ &\le K_F(1 + \|Z(t)\|_{\mathcal{H}}^{p+1}).\end{aligned}$$

□

2.3. Well-posedness

Theorem 2.1. *Assume that Assumption 2.1 holds. Then for any initial data $Z_0 \in \mathcal{H}$, the system given by (1.1)–(1.4) has a unique weak solution*

$$Z \in C([0,+\infty),\mathcal{H}), \quad Z(0) = Z_0,$$

which depends continuously on the initial data. In particular, if $Z_0 \in D(A)$, then the solution is strong.

Proof. It follows from [22], and we already proved that A is m-accrective in $\mathcal{H}$. Then

$$\frac{dZ(t)}{dt} = AZ(t), \quad Z(0) = Z_0 \tag{2.16}$$

has a unique solution. We will show that (2.1) and (2.2) constitute a locally Lipschitz perturbation of (2.16). To show that $F : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz, let B be a bounded set of $\mathcal{H}$ and $Z^1, Z^2 \in B$. For some $R > 0$, $\|Z^i\|_{\mathcal{H}} \le R$. By the definition of the norm in $\mathcal{H}$, we have

$$\|F(Z^1) - F(Z^2)\|_{\mathcal{H}}^2 \le C_1 \sum_{i=1}^{2} \|f_i(u_t^1 + \alpha u^1, \theta^1) - f_i(u_t^2 + \alpha u^2, \theta^2)\|_2^2.$$

By (2.9) and the Sobolev embeddings, for some $C_R > 0$,

$$\|f_i(u_t^1 + \alpha u^1, \theta^1) - f_i(u_t^2 + \alpha u^2, \theta^2)\|_2^2 \le C_R\|Z^1 - Z^2\|_{\mathcal{H}}^2, \quad i = 1,2.$$

Therefore, for some $C_b > 0$,

$$\|F(Z^1) - F(Z^2)\|_{\mathcal{H}} \le C_b\|Z^1 - Z^2\|_{\mathcal{H}}, \quad \forall Z^1,\ Z^2 \in B,$$

which shows that F is locally Lipschitz.

Then from classical results in [1] (see a detailed proof in [8], Theorem 7.2), we can conclude that for $Z_0 \in D(A)$, (2.16) possesses a unique strong solution defined on a maximal interval $[0, t_{max})$, $t_{max} \le \infty$. If $Z_0 \in \mathcal{H}$ then (2.16) possesses a unique weak solution $Z \in C([0, t_{max}), \mathcal{H})$. In addition, if $t_{max} < \infty$, then $\limsup_{t \to t_{max}} \|Z(t)\|_{\mathcal{H}} = \infty$. To prove the existence of the global solutions, we must show that $t_{max} = \infty$. Indeed, let Z be a strong solution defined in $[0, t_{max})$. By (2.15), we have

$$\|Z(t)\|_{\mathcal{H}}^2 \le \frac{1}{\beta_0}(\mathcal{E}(t) + LM_F), \quad t \in [0, t_{max}),$$

which by the density argument, also holds for weak solutions. This shows that the solution does not blow up in finite time. Therefore $t_{max} = \infty$.

Finally, let Z^1, Z^2 be two weak solutions; then, for any $T > 0$, there exists $C_t > 0$, such that

$$\|Z^1(t) - Z^2(t)\|_{\mathcal{H}}^2 \le C_t\|Z^1(0) - Z^2(0)\|_{\mathcal{H}}^2, \quad t \in [0,T],$$

which shows the continuous dependence of solutions on the initial data. □

3. Global attractors

From the well-posedness of weak solutions of (1.1) and (1.4), the semigroup $\{S(t)\}_{t\geq 0}$ is continuous on the phase space $\mathcal{H}$ and thus defines a dynamical system $(\mathcal{H}, S(t))$. In this section, we will investigate the existence of global attractors and their properties.

Definition 3.1. *In a dynamical system (H,S(t)), where H is a complete metric space and $S(t)$ is a C_0−semigroup, a compact set $A \subset H$ is called a global attractor if it is fully invariant and attracts uniformly bounded sets of H, that is*

$$S(t)A = A, \quad \text{and} \quad \lim_{t\to\infty} d(S(t)D, A) = 0,$$

for any bounded set $D \subset H$, where d denotes the Hausdorff semi-distance in H.

3.1. Gradient systems

Definition 3.2. *A dynamical system (H,S(t)) is stated to be of gradient type if it possesses a Lyapunov functional, that is, a functional $\Phi : H \to \mathbf{R}$ such that,*

(i) $t \to \Phi(S(t)Z)$ is decreasing for all $Z \in H$,
(ii) whenever $\Phi(S(t)Z) = \Phi(Z)$ for all $t \geq 0$, then Z is a stationary point of $S(t)$.

Lemma 3.1. *The dynamical system $(\mathcal{H}, S(t))$ corresponding to (1.1)–(1.4) is of gradient type. In addition,*

$$\Phi(Z) \to \infty \Longleftrightarrow \|Z\|_{\mathcal{H}} \to \infty, \tag{3.1}$$

where Φ is the corresponding Lyapunov functional.

Proof. Let $Z = (u_0, u_1, u_2, \theta_0) \in \mathcal{H}$ represent the initial data, the corresponding solution is given by $Z(t) = S(t)Z, t \geq 0$. Let the energy functional defined in (2.12) be the Lyapunov functional $\Phi(S(t)Z) = \mathcal{E}(t)$. By (2.14) we have

$$\frac{d}{dt}\Phi(S(t)Z) \leq -M(\|u_t\|^2 + \|\theta\|^2), \quad t \geq 0,$$

which shows that $t \to \Phi(S(t)Z)$ is a non-increasing function. Suppose that $\Phi(S(t)Z)$ is constant with respect to t. Then

$$\|u_t\|_2^2 = 0, \quad t \geq 0.$$

We can obtain that

$$u_t(x,t) = 0 \quad a.e. \quad in \quad (0, L) \times \mathbf{R}^+.$$

Consequently, $S(t)Z = Z(t) = (u_0, 0, 0, \theta_0)$ is a stationary point of $S(t)$, which shows Φ to be a Lyapunov functional.

From (2.15), we have

$$\Phi(Z) \leq K_F(1 + \|Z(t)\|_{\mathcal{H}}^{p+1}), \quad t \geq 0.$$

Let $\Phi(Z) \to \infty$; we can obtain that $\|Z\|_{\mathcal{H}} \to \infty$. On the other hand, from (2.15), we can obtain

$$\|Z\|_{\mathcal{H}}^2 \leq \frac{\Phi(Z) + LM_F}{\beta_0}.$$

We can infer from $\|Z\|_{\mathcal{H}} \to \infty$ that $\Phi(Z) \to \infty$. □

Lemma 3.2. *The set of stationary points $\mathcal{N}$ of $S(t)$ is bounded in $\mathcal{H}$.*

Proof. Let $Z = (u, 0, 0, \theta) \in \mathcal{N}$ be the stationary solution of the system given by (1.1)–(1.4). Then we have the following:

$$\begin{cases} -\gamma u_{xx} + \eta\theta_{xx} + f_1(\alpha u, \theta) = 0, & (3.2) \\ -\kappa\theta_{xx} + f_2(\alpha u, \theta) = 0. & (3.3) \end{cases}$$

Multiplying (3.2) and (3.3) by αu and θ respectively, and integrating over $[0, L]$, we can obtain

$$\alpha\gamma\|u_x\|_2^2 + \kappa\|\theta_x\|_2^2 - \alpha\eta\int_0^L \theta_x u_x dx = -\int_0^L \nabla F(\alpha u, \theta)\cdot(\alpha u, \theta)dx.$$

By Young's inequality and (2.11), we have

$$\alpha(\gamma - \varepsilon\eta)\|u_x\|_2^2 + (\kappa - C_\varepsilon\alpha\eta)\|\theta_x\|_2^2 \le 2\lambda L(\|\alpha u\|_2^2 + \|\theta\|_2^2) + 2LM_F.$$

Therefore, by the choice of the constant β_0 and the Poincaré inequality, we can conclude that

$$4\beta_0(\alpha\|u_x\|_2^2 + \kappa\|\theta_x\|_2^2) \le 2LM_F.$$

Then we get the boundedness of $\mathcal{N}$ in $\mathcal{H}$. □

3.2. Quasi-stability

Definition 3.3. *Let X, Y be reflexive Banach spaces with X compactly embedded in Y and $H = X \times Y$. Let $(H, S(t))$ be a dynamical system defined by the following evolutionary system*

$$S(t)Z = (u(t), u_t(t)), \quad Z = (u(0), u_t(0)) \in H$$

with regularity $u \in C([0,\infty); X)\bigcap C^1([0,\infty); Y)$. Then $(H, S(t))$ is called quasi-stable on a set $B \subset H$, if there exits a compact semi-norm $[\cdot]_X$ on X (i.e., if $x_j \rightharpoonup 0$ in X then $[\cdot]_X \rightharpoonup 0$) and there are nonnegative scalar functions a, b, c with a, c locally bounded in $[0,\infty)$ and $b \in L^1(\mathbf{R}^+)$ satisfying that $\lim_{t\to\infty} b(t) = 0$, such that

$$\|S(t)y^1 - S(t)y^2\|_H^2 \le a(t)\|y^1 - y^2\|_H^2, \quad t \ge 0$$

and for $S(t)y^i = (u^i(t), u_t^i(t)), i = 1, 2$

$$\|S(t)y^1 - S(t)y^2\|_H^2 \le b(t)\|y^1 - y^2\|_H^2 + c(t)\sup_{0\le s\le t}[u^1(s) - u^2(s)]_X^2 \tag{3.4}$$

for any $y^1, y^2 \in B$.

Lemma 3.3. *Suppose that Assumption 2.1 holds. Let B be a bounded positive invariant set of $\mathcal{H}$ and $S(t)Z^i = (u^1, u_t^i, u_{tt}^i, \theta^i)^T$ be a weak solution of the system given by (1.1)–(1.4) with $Z_0^i \in B, i = 1, 2$. Then there exist constants $K_B, \alpha_B, C_B > 0$, such that*

$$E(t) \le K_B E(0)e^{-\alpha_B t} + C_B \sup_{\tau\in[0,t]}(\|u_t(\tau) + \alpha u(\tau)\|_r^2 + \|\theta(\tau)\|_r^2) \tag{3.5}$$

where $E(t) = \frac{1}{2}\|Z\|_{\mathcal{H}}^2, u = u^1 - u^2, \theta = \theta^1 - \theta^2, r \ge 2$.

Proof. Let $F_i(u_t + \alpha u, \theta) = f_i(u_t^1 + \alpha u^1, \theta^1) - f_i(u_t^2 + \alpha u^2, \theta^2), i = 1, 2$. Then $u = u^1 - u^2, \theta = \theta^1 - \theta^2$ satisfy the following:

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta u_{txx} - \gamma u_{xx} + \eta\theta_{xx} = -F_1(u_t + \alpha u, \theta), & (3.6) \\ \theta_t - \kappa\theta_{xx} - \eta u_{ttxx} - \alpha u_{txx} = -F_2(u_t + \alpha u, \theta), & (3.7) \\ (u(0), u_t(0), u_{tt}(0), \theta(0)) = Z^1 - Z^2, & (3.8) \\ u|_{\partial\Omega} = 0, \nabla u|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0. & (3.9) \end{cases}$$

Multiplying (3.6) by $u_{tt} + \alpha u_t$ and (3.7) by θ, and by integrating over $[0, L]$, we have

$$\frac{d}{dt}E(t) = -\chi\|u_{tx}\|_2^2 - \kappa\|\theta_x\|_2^2 - \int_0^L F_1(u_t + \alpha u, \theta)(u_t + \alpha u, \theta)dx - \int_0^L F_2(u_t + \alpha u, \theta)\theta dx.$$

Integrating the above equation over $[s, T]$, $s > 0$, we can get

$$\begin{aligned} E(T) \;=\; & E(s) - \int_s^T (\chi\|u_{tx}\|_2^2 + \kappa\|\theta_x\|_2^2)dt - \int_s^T \int_0^L F_1(u_t + \alpha u, \theta)(u_t + \alpha u, \theta)dxdt \\ & - \int_s^T \int_0^L F_2(u_t + \alpha u, \theta)\theta dxdt. \end{aligned}$$

Applying (2.9) and $H_0^1(0, L) \hookrightarrow L^p(0, L), p \in (1, \infty)$, together with the Hölder and Young's inequalities, we can establish that there exist $r \geq 2, C_B > 0$ and $\varepsilon > 0$, such that

$$\begin{aligned} -\int_0^L F_1(u_t + \alpha u, \theta)(u_t + \alpha u, \theta)dx \;\leq\; & C_f(1 + |u_t^1 + \alpha u^1|_r^{p+1} + |u_t^2 + \alpha u^2|_r^{p+1} + |\theta^1|_r^{p-1} \\ & + |\theta^2|_r^{p-1})(\|u_t + \alpha u\|_r^{p-1} + \|\theta\|_r^{p-1})\|u_t + \alpha u\|_2 \\ \leq\; & C_B(\|u_t + \alpha u\|_r^2 + \|\theta\|_r^2)\|u_t + \alpha u\|_2 \\ \leq\; & \varepsilon\|u_t + \alpha u\|_2^2 + \frac{C_B}{\varepsilon}(\|u_t + \alpha u\|_r^2 + \|\theta\|_r^2). \end{aligned}$$

Similarly, we have

$$-\int_0^L F_2(u_t + \alpha u, \theta)\theta dx \leq \varepsilon\|\theta\|_2^2 + \frac{C_B}{\varepsilon}(\|u_t + \alpha u\|_r^2 + \|\theta\|_r^2).$$

Therefore,

$$-\int_0^L F_1(u_t + \alpha u, \theta)(u_t + \alpha u, \theta)dx - \int_0^L F_2(u_t + \alpha u, \theta)\theta dx \leq \varepsilon E(t) + \frac{C_B}{\varepsilon}(\|u_t + \alpha u\|_r^2 + \|\theta\|_r^2).$$

Let $\varepsilon = \frac{1}{T}$, we can get

$$E(T) \leq E(s) + \frac{1}{T}\int_0^T E(t)dt + TC_B(\|u_t + \alpha u\|_r^2 + \|\theta\|_r^2).$$

Integrating it over $[0, T]$, there exists a constant $C_T > 0$, such that

$$TE(T) \leq 2\int_0^T E(t)dt + C_T \sup_{s\in[0,t]}(\|u_t + \alpha u\|_r^2 + \|\theta\|_r^2). \tag{3.10}$$

It follows that

$$\begin{aligned}\int_0^T E(t)dt &= \frac{1}{2}\int_0^T \|Z(t)\|_{\mathcal{H}}^2 = \frac{1}{2}\int_0^T\int_0^L (\|u_{tt}+\alpha u_t\|_2^2 + \frac{\gamma}{\alpha}\|u_{tx}+\alpha u_x\|_2^2 + \chi\|u_{tx}\|_2^2 + \|\theta\|_2^2)dxdt \\ &\leq -\frac{1}{2}\int_0^L (u_t+\alpha u)(u_{tt}+\alpha u_t)dx|_0^T - \frac{1}{2}\int_0^L F_1(u_t+\alpha u)dx + \frac{\varepsilon\eta}{2}\int_0^T\int_0^L |\theta_x|^2 dxdt \\ &\quad + \frac{\eta}{2\varepsilon}\int_0^T\int_0^L |u_{tx}+\alpha u_x|^2 dxdt + \frac{1}{2}\int_0^T\int_0^L |\theta|^2 dxdt \\ &\leq C_2(E(T)+E(0)) + C_3\int_0^T (\|u_t+\alpha u\|_r^2 + \|\theta\|_r^2)dt, \end{aligned} \tag{3.11}$$

where $\varepsilon, C_2, C_3 > 0$ are constants.

Combining this with (3.10), for some constant $C_{BT} > 0$, we have

$$TE(T) \leq C_T(E(T)+E(0)) + C_{BT}\sup_{s\in[0,t]}(\|u_t+\alpha u\|_r^2 + \|\theta\|_r^2).$$

Fixing $T > 2C_T$, we can derive

$$E(T) \leq K_T E(0) + C_{BT}\sup_{s\in[0,t]}(\|u_t+\alpha u\|_r^2 + \|\theta\|_r^2), \quad K_T = \frac{C_T}{T-C_T} < 1.$$

Consequently, there exist $K_B, \alpha_B, C_B > 0$, such that (3.5) holds. □

3.3. Main result

We chose to first collect some useful properties of quasi-stable systems from [9] to apply as the following proposition which will help us to reach the main result.

Proposition 3.1. *If the dynamical system $(H, S(t))$ is quasi-stable on a positively invariant bounded set of H, then we have the following:*

(i) The system is asymptotically compact.
(ii) If it has a global attractor A, then A has the finite fractal dimension $\dim_H A$.
(iii) If the coefficient in (3.4) is uniformly bounded, then any bounded full trajectory $(u(t), u_t(t)) \in X\times Y$ has additional regularity, i.e.,

$$u_t \in L^\infty(\mathbf{R}, X)\cap C(\mathbf{R}, Y) \quad \textit{and} \quad u_{tt}\in L^\infty(\mathbf{R}, Y).$$

Moreover, there exists $N > 0$ such that

$$\|u_t(t)\|_X^2 + \|u_{tt}(t)\|_Y^2 \leq N, \quad \forall t\in\mathbf{R}.$$

The main result of this section is stated as below.

Theorem 3.1. *Under the conditions of Assumption 2.1, the dynamical system given by (1.1)-(1.4) has a global attractor A characterized by*

$$A = M_+(\mathcal{N}), \tag{3.12}$$

where $M_+(\mathcal{N})$ denotes the unstable manifold emanating from $\mathcal{N}$, which is the set of stationary points of $S(t)$. Moreover, A is bounded in $(H^2(0,L)\cap H_0^1(0,L))^2 \times (H_0^1(0,L))^2$ and its fractal dimension is finite.

Proof. (1) From Lemma 3.3, we can get that the system is quasi-stable and consequently is asymptotically compact by Proposition 3.4(i). Then we apply the classical result [9] that states that an asymptotically compact gradient system satisfying (3.1) with the bounded stationary set $\mathcal{N}$ possesses a global attractor characterized by (3.12). Hence, by Lemma 3.1, the system has a global attractor A.

(2) From Proposition 3.4(ii) the global attractor A above has the finite fractal dimension $\dim_H A$.

(3) Since the system is quasi-stable with a constant coefficient $c(t) = C_B$, it follows from Proposition 3.4(iii) that any full trajectory (u, u_t, u_{tt}, θ) inside of the attractor has further time regularity

$$u_t \in L^\infty(\mathbf{R}, H_0^1(0,L)) \cap C(\mathbf{R}, L^2(0,L)) \quad \textit{and} \quad u_{tt}, u_{ttt}, \theta_t \in L^\infty(\mathbf{R}, L^2(0,L)).$$

By continuity of the nonlinear terms, we have

$$\beta u_{txx} + \gamma u_{xx} + \eta\theta_{xx} = u_{ttt} + \alpha u_{tt} + f_1(u_t + \alpha u, \theta) \in L^\infty(\mathbf{R}, L^2(0,L)),$$

$$\kappa\theta_{xx} + \eta u_{ttxx} + \eta\alpha u_{txx} = \theta_t + \alpha u_{tt} + f_2(u_t + \alpha u, \theta) \in L^\infty(\mathbf{R}, L^2(0,L)).$$

Then, $u, \theta \in L^\infty(\mathbf{R}, H^2(0,L) \cap H_0^1(0,L))$, i.e., there exists $N > 0$, such that

$$\|(u,\theta)\|_{(H^2(0,L)\cap H_0^1(0,L))^2} + \|(u_t, u_{tt})\|_{(H^2(0,L))^2} + \|(u_{ttt}, \theta_t)\|_{(L^\infty(0,L))^2} \le N^2.$$

Therefore, since the global attractor is the set of all bounded full trajectories, we can obtain that A is bounded in $(H^2(0,L) \cap H_0^1(0,L))^2 \times (H_0^1(0,L))^2$. □

The quasi-stable systems have many other properties such as the existence of generalized exponential attractors. Now that the existence of global attractors have been established, as in Theorem 3.1, we will establish the existence of the generalized exponential attractor in the next theorem.

Theorem 3.2. *The system $(\mathcal{H}, S(t))$ has a generalized fractal exponential attractor. More precisely, for any given $\xi \in [0,1]$, there exists a generalized exponential attractor $A_{exp,\xi}$ in the extended space $\mathcal{H}_{-\xi}$ which is defined as the interpolation of the following:*

$$\mathcal{H}_0 = \mathcal{H}, \quad \mathcal{H}_{-1} = (L^2(0,L))^2 \times (H_0^{-1}(0,L))^2.$$

Proof. Let $\mathcal{B} = \{Z \in \mathcal{H} | \Phi(Z) \le B\}$ where Φ is the strict Lyapunov functional given in Lemma 3.1. Then we can demonstrate that for sufficiently large B, $\mathcal{B}$ is a positively invariant bounded absorbing set, which shows that the system is quasi-stable on the set $\mathcal{B}$.

Then for the solution $Z(t) = S(t)Z_0$ with the initial data $Z(0) \in \mathcal{B}$, we have for any $T > 0$:

$$\int_0^T \|Z_t(s)\|_{\mathcal{H}_{-1}}^2 ds \le C_{\mathcal{B}T},$$

which shows that

$$\|S(t_1)Z - S(t_2)Z\|_{\mathcal{H}_{-1}} \le \int_{t_1}^{t_2} \|Z_t(s)\|_{\mathcal{H}_{-1}}^2 ds \le C_{\mathcal{B}T}|t_1 - t_2|^{\frac{1}{2}},$$

where $C_{\mathcal{B}T}$ is a positive constant and $t_1, t_2 \in [0,T]$. Hence we obtain that for any initial data $Z_0 \in \mathcal{H}$ the map $t \mapsto S(t)Z_0$ is $\frac{1}{2}$−Hölder continuous in the extended phase space $\mathcal{H}_{-1}$. Therefore, it follows from [8] that the dynamical system $(\mathcal{H}, S(t))$ possesses a generalized fractal exponential attractor $A_{exp,\xi}$ with a finite fractal dimension in the extended space $\mathcal{H}_{-1}$.

Furthermore, by the standard interpolation theorem, we can obtain the existence of exponential attractors in the extended space $\mathcal{H}_{-\xi}$ with $\xi \in [0,1]$. □

4. Conclusions

In this work, we obtained the long-time dynamical behavior of the MGT-Fourier system. The main goal was to prove the existence of the global attractors for the system (1.1)–(1.4). Under conditions of the Assumption 2.1, we use the nonlinear semigroup theory in [1] to prove the well-posedness of the solutions, firstly. Next, we established the existence of smooth finite dimensional global attractors in the system by showing that the solution semigroup is gradient and quasi-stable. From Lemma 3.3, we can get that the system is quasi-stable and consequently is asymptotically compact by Proposition 3.4(i). Then we apply the classical result [9] that states that an asymptotically compact gradient system satisfying (3.1) with the bounded stationary set $\mathcal{N}$ possesses a global attractor characterized by (3.12). Hence, by Lemma 3.1, the system has a global attractor A. From Proposition 3.4(ii) the global attractor A also has the finite fractal dimension $\dim_H A$. Since the system is quasi-stable with a constant coefficient $c(t) = C_B$, it follows from Proposition 3.4(iii) that any full trajectory (u, u_t, u_{tt}, θ) inside of the attractor has further time regularity. Furthermore, from [8] that the dynamical system $(\mathcal{H}, S(t))$ possesses a generalized fractal exponential attractor $A_{exp,\xi}$ with a finite fractal dimension in the extended space $\mathcal{H}_{-1}$. By the standard interpolation theorem, we obtained that the existence of exponential attractors in the extended space $\mathcal{H}_{-\xi}$ with $\xi \in [0, 1]$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

This work is not associated with any conflicts of interest.

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