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Research article

Extended existence results for FDEs with nonlocal conditions

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Abstract: This paper discusses the existence of solutions for fractional differential equations with nonlocal boundary conditions (NFDEs) under essential assumptions. The boundary conditions incorporate a point $0 \leq c < d$ and fixed points at the end of the interval $[0, d]$. For $i = 0, 1$, the boundary conditions are as follows: $a_i, b_i > 0$, $a_0 p(c) = -b_0 p(d)$, $a_1 p^{'}(c) = -b_1 p^{'}(d)$. Furthermore, the research aims to expand the usability and comprehension of these results to encompass not just NFDEs but also classical fractional differential equations (FDEs) by using the Krasnoselskii fixed-point theorem and the contraction principle to improve the completeness and usefulness of the results in a wider context of fractional differential equations. We offer examples to demonstrate the results we have achieved.

Keywords: fractional derivatives; differential equations; fractional differential equations; existence of solutions; fixed-point theorem
Mathematics Subject Classification: 26A33, 34A08

1. Introduction

In this paper, we study the existence of solutions for the following NFDE:

$$\begin{cases} {}^{c}D^{\zeta} p(s) = q(s, p(s)), \quad s \in [0, d], \quad 1 < \zeta \leq 2, \ 0 < c < d, \\ a_0 p(c) = -b_0 p(d), \ a_1 p^{'}(c) = -b_1 p^{'}(d), \end{cases} \tag{1.1}$$

where $a_i, b_i \in \mathbb{R}^+$ for $i = 0, 1$, ${}^{c}D^{\zeta}$ represents the Caputo derivative of order ζ, and there is a continuous function $q : [0, d] \times \mathbb{R} \longrightarrow \mathbb{R}$, by using the Krasnoselskii fixed-point theorem and the contraction principle.

Fractional differential equations (FDEs) have loomed as a dominant and masterful mathematical framework. They are a generalization of integer-order of differential equations, which have comprehensive applications over diverse scientific disciplines. Unlike traditional differential equations, FDEs provide a unified framework to address phenomena characterized by fractional-order dynamics due to their incorporation of information in a wider range of points in the domain [1, 2].

The Riemann-Liouville fractional derivative (RLFD) and Caputo fractional derivatives (CFD) are tools in the realm of fractional calculus. They are used to deal with initial and nonlocal conditions involving information, ensuring the description of particular phenomena in diverse applications [1].

Over the years, several methodologies have been used to solve FDEs, such as Laplace transformation, asymptotic methods, and numerical methods. The investigation of solutions under essential conditions has been explored widely concerning various intriguing topics. For some interesting topics, see [3, 4]. In [5], authors achieved an accurate solution for the analyzed model, and methodology has been proposed that combines the Adomian decomposition method with the Laplace transform, while in [6], researchers solved a fractional order partial differential equation (PDE) using the Laplace residual power series method, which was proven to be effective for numerical solutions. These methods have been used widely in single and multi-point initial conditions of fractional differential equations. To the best of our knowledge, very few research papers have delved into the exploration of a nonlocal boundary condition, which presents a promising avenue for further investigation. In [7], Zuo and Wang employed the Krasnosel'skii fixed point theorem with Green's function transformation to demonstrate the existence of positive solutions in a fractional differential equation with periodic boundary conditions.

In [8], existence results are presented using Leray-Schauder theory for the following FDE with order $1 < \zeta \leq 2$:

$$\begin{cases} {}^cD^{\zeta}p(s) = q(s, p(s)), \quad s \in [0, d], \\ p(0) = -p(d), \ p^{'}(0) = -p^{'}(d), \end{cases}$$

where $q : [0, d] \times \mathbb{R} \longrightarrow \mathbb{R}$.

For the same order, Bashir and Espinar [9] consider the following fractional differential inclusion:

$$\begin{cases} {}^cD^{\zeta}p(s) \in Q(s, p(s)), \quad s \in [0, d], \\ p(0) = -p(d), \ p^{'}(0) = -p^{'}(d), \end{cases}$$

where $Q : [0, d] \times \mathbb{R} \longrightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$, to obtain the existence results using the Bohnenblust-Karlin theorem. Most of the studied problems are based on the antiperiodic boundary conditions at the initial point 0, which got researchers attention to study a new kind of problems with a different type of boundary conditions involving a point in the interval $[0, d]$.

For FDEs with parametric type boundary conditions, Agarwal, Bashir, and Nieto [10] devoted the following NFDE:

$$\begin{cases} {}^cD^{\zeta}p(s) = q(s, p(s)), \quad s \in [0, d], \quad 0 < c < d, \quad 1 < \zeta \leq 2, \\ p(c) = -p(d), \ p^{'}(c) = -p^{'}(d), \end{cases}$$

and obtained the existence results for a given function $q : [0, d] \times \mathbb{R} \longrightarrow \mathbb{R}$ using standard fixed-point theorems.

Nonlocal fractional differential equations (NFDEs) serve enchanting and multifaceted applications in physics and mathematical models. These applications and models for characterizing real-world phenomena have been highly influential in capturing the behavior of various systems in engineering, physics, and biology. For some results related to these applications, see [11, 12]. Researchers are delving deeper into mathematical models with non-integer orders to solve practical problems, fostering the advancement of solutions for NFDEs in an engineering context and promising profound insights into the complex and multifaceted dynamics that underpin engineered systems. Such equations are particularly valuable in scenarios where memory effects or correlations beyond neighboring points significantly impact the system's dynamics, such as in modeling anomalous diffusion or complex transport phenomena, and interesting results can be found in [13, 14].

The structure of the paper is outlined as follows. Section 2 delves into the materials and methods employed in this study, particularly the Krasnoselskii's fixed point theorem and the necessary assumptions to investigate the existence of the solution. In Section 3, significant theorems that prove the existence of solutions to NFDE (1.1) are stated and demonstrated. Notably, Section 4 features examples that serve to illustrate the results. Finally, the conclusion is presented in Section 5.

1.1. Preliminaries

Definition 1.1. The Caputo fractional derivative of order $\zeta > 0$, ${}^cD^\zeta$, for a given function $u \in AC^n(0, d)$ is defined as follows:

$$ {}^cD^\zeta u(r) = \frac{1}{\Gamma(m-\zeta)} \int_0^r (r-\tau)^{m-\zeta-1} u^{(m)}(\tau) d\tau, $$

where $m = [\zeta] + 1$, $[\zeta]$ is the integer part of ζ, and Γ is the Gamma function.

Definition 1.2. The Riemann-Liouville fractional integral (RLFI) of order ζ, I^ζ, for a defined function $u \in L^1(0, d)$ is

$$ I^\zeta u(r) = \frac{1}{\Gamma(\zeta)} \int_0^r (r-\tau)^{\zeta-1} u(\tau) d\tau, \quad \zeta > 0. $$

Lemma 1.1. *The general solution of* ${}^cD^\zeta p(s) = 0$*, where* $\zeta > 0$*, is given by*

$$ p(s) = \sum_{j=1}^{j=m} \nu_j s^{j-1}, \quad \nu_j \in \mathbb{R}. \tag{1.2} $$

Lemma 1.2. *The unique solution of*

$$ \begin{cases} {}^cD^\zeta p(s) = \delta(s), \quad s \in [0, d], \quad 1 < \zeta \leq 2, \ 0 < c < d, \\ a_0 p(c) = -b_0 p(d), \ a_1 p^{'}(c) = -b_1 p^{'}(d), \quad a_i, b_i \in \mathbb{R}^+ \ \ for \ \ i = 0, 1, \end{cases} \tag{1.3} $$

is given by:

$$ \begin{aligned} p(s) = \int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)} \delta(r) dr - \frac{1}{a_1 + b_1} [a_1 \int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)} \delta(r) dr + b_1 \int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)} \delta(r) dr] \\ + \frac{(a_1 c + b_1 d) - (a_1 + b_1)s}{(a_0 + b_0)(a_1 + b_1)} [a_0 \int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)} \delta(r) dr + b_0 \int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)} \delta(r) dr]. \end{aligned} \tag{1.4} $$

Proof. In view of Lemma 1.1, $p(s) = I^\zeta \delta(s) - \sum_{j=1}^{j=m} \nu_j s^{j-1}, \quad \nu_j \in \mathbb{R}$ is the solution for (1.3).

$$ p(s) = \int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)} \delta(r) dr - \sum_{j=1}^{j=2} \nu_j s^{j-1}. \tag{1.5} $$

$$\begin{aligned}
\nu_1 =& \frac{1}{(a_1+b_1)\Gamma(\zeta)}\left[a_1\int_0^c (c-r)^{\zeta-1}\delta(r)dr + b_1\int_0^d (d-r)^{\zeta-2}\delta(r)dr\right]\\
&- \frac{a_1c+b_1d}{(a_0+b_0)(a_1+b_1)\Gamma(\zeta-1)}\left[a_0\int_0^c (c-r)^{\zeta-2}\delta(r)dr + b_0\int_0^d (d-r)^{\zeta-2}\delta(r)dr\right],\\
\nu_2 =& \frac{1}{(a_0+b_0)\Gamma(\zeta-1)}\left[a_0\int_0^c (c-r)^{\zeta-2}\delta(r)dr + b_0\int_0^d (d-r)^{\zeta-2}\delta(r)dr\right].
\end{aligned}$$

Substitute values of ν_1 and ν_2 into (1.5) to yield the following:

$$\begin{aligned}
p(s) =& \int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)}\delta(r)dr - \frac{1}{a_1+b_1}\left[a_1\int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)}\delta(r)dr + b_1\int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)}\delta(r)dr\right]\\
&+ \frac{(a_1c+b_1d)-(a_1+b_1)s}{(a_0+b_0)(a_1+b_1)}\left[a_0\int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)}\delta(r)dr + b_0\int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)}\delta(r)dr\right].
\end{aligned}$$

□

Remark 1.1. For $i = 0, 1$, $a_i = b_i = 1$, and $c = 0$, problem (1.3) represents the classical FDE of Lemma 2.5 in [8] . Comparing the solution in [8] and (1.4), we see additional terms are added in (1.4). On the other hand, for $i = 0, 1$, $a_i = b_i = 1$, the solution (1.4) is the same as that of Lemma 2.1 in [10] .

2. Materials and methods

Numerous approaches are used to prove existence of a solution for FDEs, such as Leray-Schauder, Krasnosel'skii, and Schaefer fixed-point theorems. In this research, we use Krasnoseliskii's fixed-point theorem and the contraction principle.

Additionally, let us assume the following:
(A_1): $|q(s, p_1) - q(s, p_2)| \leq K|p_1 - p_2|$, for $p_1, p_2, \in \mathbb{R}$, $K > 0$, and for all $s \in [0, d]$.
(A_2): $|q(s, p)| \leq u(s)$ for all $(s, p) \in [0, 1] \times \mathbb{R}$, and $u \in L^1([0, d], \mathbb{R}^+)$.

Let $\mathcal{B}$ be the Banach space of all continuous functions, $\mathcal{B} = C([0, d], \mathbb{R})$, and define an operator $\mathcal{T} : \mathcal{B} \longrightarrow \mathcal{B}$ as

$$\begin{aligned}
(\mathcal{T}p)(s) =& \int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)}q(r,p(r))dr\\
&- \frac{1}{a_1+b_1}\left[a_1\int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)}q(r,p(r))dr + b_1\int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)}q(r,p(r))dr\right] \qquad (2.1)\\
&+ \frac{(a_1c+b_1d)-(a_1+b_1)s}{(a_0+b_0)(a_1+b_1)}\left[a_0\int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)}q(r,p(r))dr + b_0\int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)}q(r,p(r))dr\right].
\end{aligned}$$

The solution exists for (1.1) iff $\mathcal{T}p = p$, for $p \in [0, d]$.

Theorem 2.1. *[4, Theorem 1.2] If a family $Q\{q(s)\}$ in $C(J, R)$ is uniformly bounded and equicontinuous on J, and if for any $s^* \in J$, $\{q(s^*)\}$ is relatively compact, then, Q has uniformly convergent subsequence $\{q_n\}_{n=1}^{\infty}$.*

Theorem 2.2. *[3, Theorem 1.2.2, Contraction Mapping Theorem] Any contraction mapping of a complete nonempty metric space space Ω into Ω has a unique fixed point in Ω.*

Theorem 2.3. *[3, Theorem 4.4.1, Kasnoseliskii's Theorem] Let $\mathcal{M}$ be a nonempty closed convex subset of a Banach space $\mathcal{B}$. Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ map $\mathcal{M}$ into $\mathcal{B}$ and,*

- $\mathcal{O}_1 p_1 + \mathcal{O}_2 p_2 \in \mathcal{M} \quad (\forall p_1, p_2 \in \mathcal{M})$,
- *$\mathcal{O}_1$ is compact and continuous,*
- *$\mathcal{O}_2$ is a contraction mapping.*

Then there exists p in $\mathcal{M}$ such that $\mathcal{O}_1 p + \mathcal{O}_2 p = p$.

3. Results

In this section we will state and prove important theorems that discuss the existence of solutions to NFDEs (1.1).

Theorem 3.1. *The FBVP (1.1) has a unique solution on $[0, d]$ if, for a continuous function $q : [0, d] \times \mathbb{R} \longrightarrow \mathbb{R}$, assumption (A_1) holds, and $K\theta < 1$, where*

$$\theta = \frac{1}{\Gamma(\zeta+1)}\left[\frac{(a_1+2b_1)d^{\zeta} + a_1 c^{\zeta}}{a_1+b_1} + \frac{(a_1 c + b_1 d)\zeta(a_0 c^{\zeta-1} + b_0 d^{\zeta-1})}{(a_0+b_0)(a_1+b_1)}\right]. \tag{3.1}$$

Proof. Let $\mathcal{T}$ be the operator defined in (2.1) and setting $\sup_{s\in[0,d]} q(s, 0) = \eta < \infty$.

Then choose a suitable radius r for a ball B_r such that

$$r \geq \frac{\eta\theta}{1-K\theta}, \qquad B_r = \{p \in C([0, d], \mathbb{R}) : \ \|p\| \leq r\}.$$

Then,

$$\begin{aligned}
\|\mathcal{T}p_1 - \mathcal{T}p_2\| \leq \max_{s\in[0,d]} \Bigg| &\int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)} |q(r, p_1(r)) - q(r, p_2(r))| dr \\
&+ \frac{1}{a_1+b_1}\left[a_1 \int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)} |q(r, p_1(r)) - q(r, p_2(r))| dr\right. \\
&\left.+ b_1 \int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)} |q(r, p_1(r)) - q(r, p_2(r))| dr\right] \\
&+ \frac{(a_1 c + b_1 d) - (a_1+b_1)s}{(a_0+b_0)(a_1+b_1)}\left[a_0 \int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)} |q(r, p_1(r)) - q(r, p_2(r))| dr\right. \\
&\left.+ b_0 \int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)} |q(r, p_1(r)) - q(r, p_2(r))| dr\right]\Bigg| \\
\leq & \frac{K}{\Gamma(\zeta+1)}\left[\frac{(a_1+2b_1)d^{\zeta} + a_1 c^{\zeta}}{a_1+b_1} + \frac{(a_1 c + b_1 d)\zeta(a_0 c^{\zeta-1} + b_0 d^{\zeta-1})}{(a_0+b_0)(a_1+b_1)}\right] \|p_1 - p_2\|.
\end{aligned}$$

Therefore, $\|\mathcal{T}p_1 - \mathcal{T}p_2\| \leq \theta K \|p_1 - p_2\|$. Thus, $\theta K < 1$, implying that $\mathcal{T}$ is a contraction. Moreover, $\left\|\mathcal{T}\right\| \leq r$, implying that $\mathcal{T}B_r \subset B_r$. Therefore, a unique solution exists for (1.1). □

Remark 3.1. Theorem 3.1 is an extension and a generalization of Theorem 3.1 in [10] . Additionally, when $c = 0$ and for $i = 0, 1$, $a_i = b_i = 1$, the quantity $\frac{d^{\zeta}}{4\Gamma(\zeta+1)}[6 + \zeta]$ is used in Theorem 3.3 [8] as a maximum of the operator.

Theorem 3.2. *Assume there exist constants $\sigma > 0$ and $\mu > 0$, and let $q : [0,d] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with $|q(s, p(s))| \leq \mu|p| + \sigma$, where, $0 < \mu < \frac{1}{\theta}$, and*

$$\theta = \frac{(a_0 + b_0)[a_1 c^{\zeta} + (a_1 + 2b_1)d^{\zeta}] + (a_1 c + b_1 d)\zeta(a_0 c^{\zeta-1} + b_0 d^{\zeta-1})}{(a_0 + b_0)(a_1 + b_1)\Gamma(\zeta + 1)}.$$

Then, problem (1.1) has at least one solution.

Proof. Define the operator $\mathcal{T}$ as in (2.1). Then, the solution exists if and only if $\mathcal{T}p = p$. Choose a ball $B_r = \{p \in C([0,d]) : \max_{s\in[0,d]} |p(s)| < r\}$ with a radius $r > 0$. We need to show that a $p \in C([0,d])$ satisfies $\mathcal{T}p = p$. Show that $p \neq \lambda\mathcal{T}p, \quad \forall p \in \partial B$ and $\lambda \in [0,1]$, where, $\mathcal{T} : \overline{B_r} \longrightarrow C([0,d])$, and setting $G(\lambda, p) = \lambda\mathcal{T}p$ for $p \in C(\mathbb{R})$, and $\lambda[0,1]$. Then $g_\lambda(p) = p - G(\lambda, p) = p - \lambda\mathcal{T}p$ is completely continuous according to Arzelá-Ascoli theorem (Theorem 2.1).

If $p \neq \lambda\mathcal{T}p$, then for $0 \in B_r$ and the unite operator I, we have

$$deg(g_\lambda, B_r, 0) = deg(I - \lambda\mathcal{T}p, B_r, 0) = deg(g_1, B_r, 0) = deg(g_0, B_r, 0) = deg(I, B_r, 0) = 1 \neq 0.$$

Hence, for at least one $p \in B_r$, $g_1(s) = p - \lambda\mathcal{T}p = 0$.

Suppose that $p = \lambda\mathcal{T}p$ for some $\lambda \in [0,1]$, and for all $s \in [0,d]$ and $p \in B_r$. Then,

$$\begin{aligned}
|p(s)| = |\lambda\mathcal{T}p(s)| \leq & \int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)}\Big|q(r,p(r))\Big|dr + \frac{1}{a_1+b_1}\Bigg[a_1\int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)}\Big|q(r,p(r))\Big|dr \\
& + b_1\int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)}\Big|q(r,p(r))\Big|dr\Bigg] \\
& + \frac{(a_1c+b_1d)-(a_1+b_1)s}{(a_0+b_0)(a_1+b_1)}\Bigg[a_0\int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)}\Big|q(r,p(r))\Big|dr \\
& + b_0\int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)}\Big|q(r,p(r))\Big|dr\Bigg] \\
\leq & \Big(\mu\Big|p\Big| + \sigma\Big)\Bigg[\int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)}dr + \frac{1}{a_1+b_1}\Bigg[a_1\int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)}dr + b_1\int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)}|dr\Bigg] \\
& + \frac{\Big|(a_1c+b_1d)-(a_1+b_1)s\Big|}{(a_0+b_0)(a_1+b_1)}\Bigg[a_0\int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)}dr + b_0\int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)}dr\Bigg]\Bigg] \\
= & \Big(\mu\Big|p\Big| + \sigma\Big)\Bigg[\frac{s^{\zeta}}{\Gamma(\zeta+1)} + \frac{a_1c^{\zeta} + b_1d^{\zeta}}{(a_1+b_1)\Gamma(\zeta+1)} \\
& + \frac{\Big|(a_1c+b_1d)-(a_1+b_1)s\Big|(a_0c^{\zeta-1}+b_0d^{\zeta-1})}{(a_0+b_0)(a_1+b_1)\Gamma(\zeta)}\Bigg] \\
\leq & \Big(\mu\Big|p\Big| + \sigma\Big)\Bigg[\frac{(a_0+b_0)[a_1c^{\zeta} + (a_1+2b_1)d^{\zeta}] + (a_1c+b_1d)\zeta(a_0c^{\zeta-1}+b_0d^{\zeta-1})}{(a_0+b_0)(a_1+b_1)\Gamma(\zeta+1)}\Bigg].
\end{aligned}$$

We have,

$$\Big|p\Big| \leq \Big(\mu\Big|p\Big| + \sigma\Big)\theta.$$

Then, applying norm and solving for $\left\|p\right\|$, we have,

$$\left\|p\right\| \leq \frac{\sigma\theta}{1-\mu\theta},$$

thereby completing the proof by choosing $r > \frac{\sigma\theta}{1-\mu\theta}$. □

Theorem 3.3. *Let $q : [0, d] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function for which assumptions (A_1) and (A_2) hold. Then, there exists at least one solution for FBVP (1.1) if*

$$\frac{K}{\Gamma(\zeta+1)}\left[\frac{b_1 d^{\zeta} + a_1 c^{\zeta}}{a_1 + b_1} + \frac{(a_1 c + b_1 d)\zeta(a_0 c^{\zeta-1} + b_0 d^{\zeta-1})}{(a_0 + b_0)(a_1 + b_1)}\right] < 1.$$

Proof. Define a ball $B_r = \{p \in \mathcal{B} : \ \|p\| \leq r\}$, where $r \geq \|u\|_{L_1}\theta$, and θ is defined in (3.1) . Then, define $\sup_{(s,p)\in[0,d]\times B_r}\left\|q(s, p)\right\| = q_{max}$, and the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ as follows:

$$\begin{aligned}
(\mathcal{T}_1 p)(s) &= \int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)} q(r, p(r))dr,\\
\mathcal{T}_2 &= -\frac{1}{a_1+b_1}\left[a_1 \int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)} q(r, p(r))dr + b_1 \int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)} q(r, p(r))dr\right]\\
&\quad + \frac{(a_1 c + b_1 d) - (a_1 + b_1)s}{(a_0 + b_0)(a_1 + b_1)}\left[a_0 \int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)} q(r, p(r))dr + b_0 \int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)} q(r, p(r))dr\right].
\end{aligned}$$

For $p_1, p_2 \in B_r$ we have,

$$\begin{aligned}
\left\|\mathcal{T}_1 p_1 + \mathcal{T}_2 p_2\right\| \leq& \left\|u\right\|\Bigg[\int_0^s \frac{(s-r)^{\zeta-1}}{\Gamma(\zeta)} dr + \frac{1}{a_1+b_1}\left[a_1 \int_0^c \frac{(c-r)^{\zeta-1}}{\Gamma(\zeta)} dr + b_1 \int_0^d \frac{(d-r)^{\zeta-1}}{\Gamma(\zeta)} |dr\right]\\
&+ \frac{\left|(a_1 c + b_1 d) - (a_1 + b_1)s\right|}{(a_0 + b_0)(a_1 + b_1)}\left[a_0 \int_0^c \frac{(c-r)^{\zeta-2}}{\Gamma(\zeta-1)} dr + b_0 \int_0^d \frac{(d-r)^{\zeta-2}}{\Gamma(\zeta-1)} dr\right]\Bigg]\\
\leq& \left\|u\right\|\beta \leq r.
\end{aligned}$$

Therefore, $\mathcal{T}_1 p_1 + \mathcal{T}_2 p_2 \in B_r$. Since q is continuous, $\mathcal{T}_1$ is continuous.

Additionally,

$$\mathcal{T}_1 p \leq \frac{\left\|u\right\| d^{\zeta}}{\Gamma(\zeta+1)},$$

and for $s_1, s_2 \in [0, d]$ we have

$$\begin{aligned}
\left\|(\mathcal{T}_1 p)(s_1) - (\mathcal{T}_1 p)(s_2)\right\| =& \frac{1}{\Gamma(\zeta)}\Bigg\| \int_0^{s_1} [(s_1 - r)^{\zeta-1} - (s_2 - r)^{\zeta-1}] q(r, p(r))dr\\
&+ \int_{s_1}^{s_2} (s_2 - r)^{\zeta-1} q(r, p(r))dr\Bigg\|,\\
\left\|(\mathcal{T}_1 p)(s_1) - (\mathcal{T}_1 p)(s_2)\right\| \leq& \frac{q_{max}}{\Gamma(\zeta+1)}[2(s_2 - s_1)^{\zeta} + s_1^{\zeta} - s_2^{\zeta}].
\end{aligned}$$

As s_2 approaches s_1, the norm will tend to zero. Therefore, we proved that the operator is uniformly bounded on B_r and is relatively compact on B_r. Theorem 2.1 implies the compactness of $\mathcal{T}_1$.

Finally, $\mathcal{T}_2$ is a contraction mapping for

$$\frac{L}{\Gamma(\zeta+1)}\left[\frac{b_1 d^{\zeta}+a_1 c^{\zeta}}{a_1+b_1}+\frac{(a_1 c+b_1 d)\zeta(a_0 c^{\zeta-1}+b_0 d^{\zeta-1})}{(a_0+b_0)(a_1+b_1)}\right]<1.$$

Thus, Theorem 2.2 is satisfied, which implies that FBVP (1.1) has at least one solution in $[0,d]$. □

4. Examples

Example 4.1. Consider the following classical antiperiodic FDE:

$$\begin{cases} {}^{c}D^{\frac{3}{2}}p(s)=\frac{1}{4\pi}sin(2\pi p)+\frac{|p|}{1+|p|}, \quad s\in[0,1], \\ p(0)=-p(1), \; p'(0)=-p'(1). \end{cases} \tag{4.1}$$

Clearly, $|q(s,p)|\le\frac{1}{2}|p|+1$, where $q(s,p)=\frac{1}{4\pi}sin(2\pi p)+\frac{|p|}{1+|p|}$, with $0<\mu=\frac{1}{2}<\frac{2\sqrt{\pi}}{5}$, and $\sigma=1$. Thus, Theorem 3.2 implies that problem (4.1) has at least one solution on [0,1].

Example 4.2. Consider the following NFDE:

$$\begin{cases} {}^{c}D^{\frac{3}{2}}p(s)=\frac{2999\sqrt{\pi}}{15337}[p+ln(p+\sqrt{p^2+1})]+(s+1)^2, \quad s\in[0,1], \\ p(0.01)=-p(1), \; p'(0.01)=-p'(1). \end{cases} \tag{4.2}$$

In this problem we have $q(s,p)=\frac{2999\sqrt{\pi}}{15337}[p+ln(p+\sqrt{p^2+1})]+(s+1)^2$ for $0<p_1\le p_2$, $|q(s,p_2)-q(s,p_1)|\le\frac{5998\sqrt{\pi}}{15337}|p_2-p_1|$, with $K=\frac{5998\sqrt{\pi}}{15337}$. Also, since $\theta=\frac{5}{2\sqrt{\pi}}$, we have, $K\theta\approx 0.9777701<1$. Thus, Theorem 3.1 implies that problem (4.2) has at least one solution on [0,1].

Example 4.3. Consider the following FDE:

$$\begin{cases} {}^{c}D^{\frac{3}{2}}p(s)=\frac{1}{(s+3)^3}tan^{-1}(p)+ln(s+1), \quad s\in[0,1], \\ p(0)=-\frac{1}{2}p(1), \; p'(0)=-\frac{1}{2}p'(1). \end{cases} \tag{4.3}$$

Clearly, $|q(s,p_2)-q(s,p_1)|\le\frac{1}{27}|p_2-p_1|$, with $K=\frac{1}{27}$. Also, since $\theta=\frac{2}{\sqrt{\pi}}$, where, $q(s,p)=\frac{1}{(s+3)^3}tan^{-1}(p)+ln(s+1)$, and we have, $K\theta=\frac{2}{27\sqrt{\pi}}<1$. Thus, Theorem 3.1 implies that problem (4.3) has at least one solution on [0,1].

5. Conclusions

This paper explored a novel type of FDE. For a point $c\in[0,d)$ in problem (1.1), Theorem 3.1 is reduced to Theorem 3.1 in [10]. While when $c=0$ Theorem 3.2 is an extension of Theorem 3.1 in [8]. Considering NFDEs and comparing them with classical FDEs, additional terms will be added to the solution of (1.1). A shift in the position of nonlocal phenomena takes place at the left-end of the interval $[0,d]$ while maintaining a fixed relationship with d. Finally, the obtained results in this paper we consider for $i=0,1$, a_i,b_i need to be positive real numbers to find the maximum of the operator.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author does not have any conflict of interest.

References

1. I. Podlubny, Fractional differential equations, In: *Mathematics in science and engineering*, Elsevier, **198** (1999).
2. S. Abbas, M. Benchohra, G. M. N'Guerekata, *Topics in fractional differential equations*, New York: Springer, 2012. https://doi.org/10.1007/978-1-4614-4036-9
3. D. R. Smart, *Fixed point theorems*, Cambridge University Press, 1980.
4. Y. Zhou, *Fractional evolution equations and inclusions: Analysis and control*, Academic Press, 2016. https://doi.org/10.1016/C2015-0-00813-9
5. A. Ali, I. Suwan, T. Abdeljawad, Abdullah, Numerical simulation of time partial fractional diffusion model by Laplace transform, *AIMS Mathematics*, **7** (2022), 2878–2890. https://doi.org/10.3934/math.2022159
6. M. M. Al-Sawalha, O. Y. Ababneh, R. Shah, N. A. Shah, K. Nonlaopon, Combination of Laplace transform and residual power series techniques of special fractional-order non-linear partial differential equations, *AIMS Mathematics*, **8** (2023), 5266–5280. https://doi.org/10.3934/math.2023264
7. X. Zuo, W. Wang, Existence of solutions for fractional differential equation with periodic boundary condition, *AIMS Mathematics*, **7** (2022), 6619–6633. https://doi.org/10.3934/math.2022369
8. B. Ahmad, J. J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via leray-Schauder degree theory, *Topol. Methods Nonlinear Anal.*, **35** (2010), 295–304.
9. B. Ahmad, V. Otero-Espinar, Existence of solutions for fractional differential inclusions with antiperiodic boundary conditions, *Bound. Value Probl.*, **2009** (2009), 625347. https://doi.org/10.1155/2009/625347
10. R. P. Agarwal, B. Ahmad, J. J. Nieto, Fractional differential equations with nonlocal (parametric type) anti-periodic boundary conditions, *Filomat*, **31** (2017), 1207–1214.
11. A. Hyder, M. A. Barakat, D. Rizk, R. Shah, K. Nonlaopon, Study of HIV model via recent improved fractional differential and integral operators, *AIMS Mathematics*, **8** (2023), 1656–1671. https://doi.org/10.3934/math.2023084

12. M. D. Johansyah, A. K. Supriatna, E. Rusyaman, J. Saputra, Application of fractional differential equation in economic growth model: A systematic review approach, *AIMS Mathematics*, **6** (2021), 10266–10280. https://doi.org/10.3934/math.2021594

13. C. Derbazi, H. Hammouche, Caputo-Hadamard fractional differential equations with nonlocal fractional integro-differential boundary conditions via topological degree theory, *AIMS Mathematics*, **5** (2020), 2694–2709. https://doi.org/10.3934/math.2020174

14. M. Manjula, K. Kaliraj, T. Botmart, K. S. Nisar, C. Ravichandran, Existence, uniqueness and approximation of nonlocal fractional differential equation of sobolev type with impulses, *AIMS Mathematics*, **8** (2023), 4645–4665. https://doi.org/10.3934/math.2023229