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Research article

Best proximity point of α-η-type generalized F-proximal contractions in modular metric spaces

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Abstract: The purpose of this paper is to present a study of α-η-type generalized F-proximal contraction mappings in the framework of modular metric spaces and to prove some best proximity point theorems for these types of mappings. Some examples are given to show the validity of our results. We also apply our results to establish the existence of solutions for a certain type of non-linear integral equation.

Keywords: best proximity point; modular metric space; generalized F-proximal contractions; uniform approximation
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The first attempts to generalize the classical function spaces of the Lebesgue type L_p were made by Birnhaum and Orlicz in 1931 [1]. The more abstract generalization was established by Nakano [2] in 1950 and refined and generalized by Musielak and Orlicz [3] in 1959 under the name of modular and modular spaces. Lately, Chistyakov [4, 5] developed modular spaces and metric spaces by introducing modular metric spaces (or metric modular spaces). The main idea behind this new concept is physical interpretation of the modular metric spaces [6]. Here, we look at modular metric spaces as the nonlinear version of the classical modular spaces as introduced by Nakano [7], on the vector spaces and modular function spaces introduced by Musielack [8] and Orlicz [9]. It is worth noting that there is another similar generalization, i.e., (q_1, q_2)-quasimetric spaces, which were recently introduced and studied by Arutyunov and Greshnov in [10–12].

Over the past hundred years, fixed-point theory, as one of the centers of mathematical analysis, has been used in many different fields of mathematics such as topology, analysis and operator theory; see [13–18]. Let A be a non-empty subset of a metric space (X, d) and $T : A \to A$ be a self-mapping.

A point $x \in A$ is said to be a fixed point of T if $Tx = x$. However, in many practical applications, T does not satisfy the condition for a self-mapping. In other words, the mapping $T : A \to B$ $(A \cap B = \emptyset)$ does not have any fixed point. In this case, it is quite natural to investigate an element $x \in X$ such that $d(x, Tx)$ is minimized. In 2010, Basha [19] introduced the notion for a best proximity point of non-self mappings. Let A, B be two non-empty subsets of a metric space (X, d) and $T : A \to B$ be a non-self mapping. A point $x \in A$ satisfying that $d(x, Tx) = d(A, B)$ is called a best proximity point of the non-self mapping T. If $A \cap B \neq \emptyset$, then the best proximity point becomes a fixed point of T.

Recently, Beg et al. [20] introduced a new type of generalized F-proximal contractions and investigated the unique best proximity point of generalized F-proximal contractions on a complete metric space. Motivated by the recent results, in this paper, we investigate α-η-type generalized F-proximal contractions of the first and second kind in the context of modular metric spaces by focusing on the uniform approximation property of the set. Then, we state several best proximity point theorems for some proximal contractions in modular metric spaces. Some examples are given to demonstrate our theoretical results. Moreover, we give an application of our main results to establish the existence of the solution of a non-linear integral equation.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers respectively. We write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. First, we recall some prerequisites.

Let X be a non-empty set. For any $x, y \in X$, we also write $w_\lambda(x, y) := w(\lambda, x, y)$ for all $\lambda > 0$ and $w = \{w_\lambda\}_{\lambda>0}$ for which $w_\lambda : X \times X \to [0, \infty]$.

Definition 2.1. *[4,5] Let X be a non-empty set and $x, y, z \in X$. A function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be modular (metric) on X if it satisfies the following conditions:*

(i) $w_\lambda(x, y) = 0$ *if and only if* $x = y$ *for all* $\lambda > 0$*;*
(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ *for all* $\lambda > 0$*;*
(iii) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y)$ *for all* $\lambda, \mu > 0$.

If we utilize the condition

(i_p) $w_\lambda(x, x) = 0$ for all $\lambda > 0$,

instead of (i), then w is called pseudomodular on X. If w satisfies i_p and

(i_s) $w_\lambda(x, y) = 0$ if and only if $x = y$ for some $\lambda > 0$,

then w is called strictly modular on X. If condition (iii) is replaced by

(i_c) $w_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \frac{\mu}{\lambda+\mu} w_\mu(z, y)$, for all $\lambda, \mu > 0$,

then w is called convexly modular on X.

Some examples of modular metrics are as follows.

Example 2.1. *[4] If (X, d) is a metric space for any $x, y \in X$ and $\lambda > 0$, then*

$$w_{\lambda_1}(x, y) = d(x, y)$$

is a modular metric and

$$w_{\lambda_2}(x,y) = \frac{d(x,y)}{\lambda}$$

is a convex modular.

Definition 2.2. *[4] Let w be pseudomodular on X and x_0 be a fixed element of X. Then the sets*

$$X_w = X_w(x_0) = \{x \in X : w_\lambda(x,x_0) \to 0 \, as \, \lambda \to \infty\},$$
$$X_w^* = X_w^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0, \text{ such that } w_\lambda(x,x_0) < \infty\}$$

are called modular metric spaces (around x_0).

Obviously, $X_w \subset X_w^*$ holds. If w is a modular metric on X, then the modular space X_w can be equipped with a (nontrivial) metric d_w, was generated by w and given by

$$d_w(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \leq \lambda\}, \ x,y \in X_w.$$

If w is a convex modular metric on X, then $X_w = X_w^*$ and this common set can be endowed with a metric d_w^* given by

$$d_w^*(x,y) = \inf\{\lambda > 0 : w_\lambda(x,y) \leq 1\}, \ x,y \in X_w^*.$$

Given $\lambda, r > 0$ and $x \in X_w^*$, set

$$B_{\lambda,r} \equiv B_{\lambda,r}^w = \{y \in X_w^* : w_\lambda(x,y) < r\}.$$

Definition 2.3. *[21] A non-empty set $A \subset X$ is said to be w-open (or modular open) if, for every $x \in A$ and $\lambda > 0$ there is $r > 0$ (possibly depending on x and λ) such that $B_{\lambda,r} \subset A$.*

Denote by $\pi(w)$ the family of all w-open subsets of X_w^*. Clearly, $\pi(w)$ is a topology on X_w^*; see [21].

Definition 2.4. *[6,21] Let X_w and X_w^* be modular metric spaces and $\{x_n\}$ be in X_w (or X_w^*); then,*

(1) the sequence $\{x_n\}$ is said to be w-convergent to $x \in X$ if and only if $w_\lambda(x_n,x) \to 0$ as $n \to \infty$ for some $\lambda > 0$;
(2) the sequence $\{x_n\}$ is said to be w-Cauchy if $w_\lambda(x_n,x_m) \to 0$ as $m,n \to \infty$ for some $\lambda > 0$;
(3) a subset A of X_w (or X_w^*) is said to be w-complete if any w-Cauchy sequence in A is a w-convergent sequence and its w-limit lies in A.
(4) a subset A of X_w (or X_w^*) is said to be w-closed if the w-limit of a w-convergent sequence of A always belongs to A.

It is easy to see that if w is strict, then we have uniqueness of the w-limit. Indeed, If $x_n \to x$ and $x_n \to y$, then $w_\lambda(x_n,x) \to 0$ and $w_\lambda(x_n,y) \to 0$ for some $\lambda > 0$. By axiom (iii), $w_{2\lambda}(x,y) \leq w_\lambda(x,x_n) + w_\lambda(x_n,y)$; thus, $w_{2\lambda}(x,y) = 0$. If w is strict, then $x = y$.

Definition 2.5. *[22] Let w be a modular metric on X. We say that w satisfies the Fatou property if*

$$w_\lambda(x,y) \leq \liminf_{n\to\infty} w_\lambda(x_n,y_n),$$

for some $\lambda > 0$ whenever $\{x_n\}$ is w-convergent to $x \in X$ and $\{y_n\}$ is w-convergent to $y \in X$.

Samet et al. [23] introduced the notion of α-admissible mappings as follows.

Definition 2.6. *[23] Let T be a self-mapping on X and $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is an α-admissible mapping if*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1,$$

for all $x, y \in X$.

Salimi et al. [24] modified the notion of α-admissible mappings as follows.

Definition 2.7. *[24] Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if*

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow \alpha(Tx,Ty) \geq \eta(Tx,Ty)$$

for all $x, y \in X$.

In 2012, Wardowski [25] introduced the concept of an F-contraction.

Definition 2.8. *[25] Let $F : (0,\infty) \to \mathbb{R}$ be a function such that*

(F1) F is strictly increasing;
(F2) for any sequence $\{\xi_n\} \subseteq (0,\infty)$, then

$$\lim_{n\to\infty} \xi_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\xi_n) = -\infty;$$

(F3) there exists $k \in (0,1)$ such that $\lim_{\xi\to 0^+} \xi^k F(\xi) = 0$ for any $\xi \in (0,\infty)$.

We denote by $\mathfrak{F}$ the set of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying (F1)–(F3).

Example 2.2. *[25] Let $t > 0$. The following functions $F : (0,\infty) \to \mathbb{R}$ belong to $\mathfrak{F}$:*

(1) $F(t) = \ln t$;
(2) $F(t) = t + \ln t$;
(3) $F(t) = -\frac{1}{\sqrt{t}}$;
(4) $F(t) = \ln(t^2 + t)$.

Definition 2.9. *[25] Let (X, d) be a metric space and $T : X \to X$ a mapping. If there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$, then T is called an F-contraction.

Define

$$\begin{aligned} d(A,B) &= \inf\{d(x,y) : x \in A, y \in B\}, \\ A_0 &= \{x \in A : d(x,y) = d(A,B) \ for \ some \ y \in B\} \ and \\ B_0 &= \{y \in B : d(x,y) = d(A,B) \ for \ some \ x \in A\}. \end{aligned}$$

Now, we put forward the definition of a P-property.

Definition 2.10. *[26] Let (A, B) be a pair of non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. We say that the pair (A, B) has the weak P-property if and only if the following holds:*

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Zhang et al. [27] introduced the notion of the weak P-property which is weaker than the P-property.

Definition 2.11. *[27] Let (A, B) be a pair of non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. We say that the pair (A, B) has the weak P-property if and only if the following holds:*

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2),$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Recently, Basha [28] introduced the concept of the uniform approximation of a set.

Definition 2.12. *[28] Let (A, B) be a pair of non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. A is said to have uniform approximation in B if and only if, given $\varepsilon > 0$, there exists $\sigma > 0$ such that the following holds:*

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \\ d(x_1, x_2) < \sigma \end{cases} \Rightarrow d(y_1, y_2) < \varepsilon,$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

It is trivial to see that A (or B) has uniform approximation in B (or A) and the pairs (A, B) and (B, A) do not necessarily have the weak P-property (see [28] and Example 3.1).

3. Main results

Let A and B be two non-empty subsets of a modular metric space (X, w). For all $\lambda > 0$, we set

$$\begin{aligned} w_\lambda(A, B) &= \inf\{w_\lambda(x, y) : x \in A, y \in B\}, \\ A_0^\lambda &= \{x \in A : w_\lambda(x, y) = w_\lambda(A, B) \; for\; some\; y \in B\} \text{ and} \\ B_0^\lambda &= \{y \in B : w_\lambda(x, y) = w_\lambda(A, B) \; for\; some\; x \in A\}. \end{aligned}$$

Definition 3.1. *Let (A, B) be a pair of non-empty subsets of a modular metric space (X, w), and let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. We say that $T : A \to B$ is an α-proximal admissible mapping with respect to η if the following holds:*

$$\begin{cases} \alpha(u_1, u_2) \geq \eta(u_1, u_2) \\ w_\lambda(x_1, Tu_1) = w_\lambda(A, B) \\ w_\lambda(x_2, Tu_2) = w_\lambda(A, B) \end{cases} \Rightarrow \alpha(x_1, x_2) \geq \eta(x_1, x_2),$$

for all $x_1, x_2, u_1, u_2 \in A$ and $\lambda > 0$.

Definition 3.2. *Let (A, B) be a pair of non-empty subsets of a modular metric space (X, w); the subspace A is said to have uniform approximation in the subspace B if and only if, for given $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds:*

$$\begin{cases} w_\lambda(x_1, y_1) = w_\lambda(A, B) \\ w_\lambda(x_2, y_2) = w_\lambda(A, B) \\ w_\lambda(x_1, x_2) < \sigma \end{cases} \Rightarrow w_\lambda(y_1, y_2) < \varepsilon,$$

for all $x_1, x_2 \in A$, $y_1, y_2 \in B$ and $\lambda > 0$.

Here, we introduce the concept of an α-η-type generalized F-proximal contraction of the first and second kind in modular metric spaces.

Definition 3.3. *Let (A, B) be a pair of non-empty subsets of a modular metric space (X, w). A non-self mapping $T : A \to B$ is said to be an α-η-type generalized F-proximal contraction of the first kind if there exist $F \in \mathfrak{F}$, $\lambda_0 > 0$, and $a, b, c, e, \tau > 0$ with $a + b + c + 2e = 1$ such that the following holds:*

$$\begin{cases} \alpha(u_1, u_2) \geq \eta(u_1, u_2) \\ w_\lambda(x_1, Tu_1) = w_\lambda(A, B) \\ w_\lambda(x_2, Tu_2) = w_\lambda(A, B) \end{cases}$$
$$\Rightarrow \tau + F(w_\lambda(x_1, x_2)) \leq F(w_{\frac{\lambda}{a}}(u_1, u_2) + w_{\frac{\lambda}{b}}(x_1, u_1) + w_{\frac{\lambda}{c}}(x_2, u_2) + w_{\frac{\lambda}{e}}(x_1, u_2) + w_{\frac{\lambda}{e}}(x_2, u_1)),$$

for any $x_1, x_2, u_1, u_2 \in A$ with $x_1 \neq x_2$ and $0 < \lambda \leq \lambda_0$.

Definition 3.4. *Let (A, B) be a pair of non-empty subsets of a modular metric space (X, w). A non-self mapping $T : A \to B$ is said to be an α-η-type generalized F-proximal contraction of the second kind if there exist $F \in \mathfrak{F}$, $\lambda_0 > 0$, and $a, b, c, e, \tau > 0$ with $a + b + c + 2e = 1$ such that the following holds:*

$$\begin{cases} \alpha(u_1, u_2) \geq \eta(u_1, u_2) \\ w_\lambda(x_1, Tu_1) = w_\lambda(A, B) \\ w_\lambda(x_2, Tu_2) = w_\lambda(A, B) \end{cases}$$
$$\Rightarrow \tau + F(w_\lambda(Tx_1, Tx_2)) \leq F(w_{\frac{\lambda}{a}}(Tu_1, Tu_2) + w_{\frac{\lambda}{b}}(Tx_1, Tu_1) + w_{\frac{\lambda}{c}}(Tx_2, Tu_2) + w_{\frac{\lambda}{e}}(Tx_1, Tu_2)$$
$$+ w_{\frac{\lambda}{e}}(Tx_2, Tu_1)),$$

for any $x_1, x_2, u_1, u_2 \in A$ with $Tx_1 \neq Tx_2$ and $0 < \lambda \leq \lambda_0$.

Now we state and prove the main results of this section.

Theorem 3.1. *Let w be a strict convex modular metric with the Fatou property on X and X_w^* be a w-complete modular metric space induced by w. Suppose that (A, B) is a pair of non-empty w-closed subsets of X_w^* such that A has uniform approximation in B. Assume that T is an α-η-type generalized F-proximal contraction of the first kind that satisfies the following assertions:*

(1) A_0^λ and B_0^λ are non-empty sets and $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $0 < \lambda \leq \lambda_0$;
(2) T is an α-admissible mapping with respect to η;
(3) there exist elements $x_0, x_1 \in A_0^\lambda$ for all $0 < \lambda \leq \lambda_0$ such that $w_\lambda(x_1, Tx_0) = w_\lambda(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;

(4) *if* $\{x_n\}$ *is a sequence such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}_0$ *and* $x_n \to x$ *as* $n \to \infty$*, then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N}_0$.

If, for every $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) = w_\lambda(A, B)$ *implies that* $w_\lambda(x, Tx) < \infty$*, then* T *has a best proximity point. If, in addition, for any* $x, u \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) = w_\lambda(u, Tu) = w_\lambda(A, B)$ *implies that* $w_\lambda(x, u) < \infty$ *and* $\alpha(x, u) \geq \eta(x, u)$*, then the best proximity point of* T *is unique. Further, for any* $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by*

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B),$$

w-converges to the best proximity point.

Proof. Let $x_0, x_1 \in A_0^\lambda$ such that

$$w_\lambda(x_1, Tx_0) = w_\lambda(A, B) \text{ and } \alpha(x_0, x_1) \geq \eta(x_0, x_1).$$

Given the fact that $T(A_0^\lambda) \subseteq B_0^\lambda$, there exists x_2 in A_0^λ such that

$$w_\lambda(x_2, Tx_1) = w_\lambda(A, B).$$

Since T is an α-admissible mapping with respect to η, we have that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$. Again, in view of the fact that $T(A_0^\lambda) \subseteq B_0^\lambda$, there exists $x_3 \subseteq A_0^\lambda$ such that

$$w_\lambda(x_3, Tx_2) = w_\lambda(A, B).$$

Similarly, we have that $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$. Continuing this process, we get:

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then $w_\lambda(x_{n_0}, Tx_{n_0}) = w_\lambda(A, B)$, which implies that x_{n_0} is a best proximity point of T. Hence, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Given the fact that T is an α-η-type generalized F-proximal contraction of the first kind, we have

$$\tau + F(w_\lambda(x_n, x_{n+1})) \leq F(w_{\frac{\lambda}{a}}(x_{n-1}, x_n) + w_{\frac{\lambda}{b}}(x_n, x_{n-1}) + w_{\frac{\lambda}{c}}(x_{n+1}, x_n) + w_{\frac{\lambda}{e}}(x_n, x_n) + w_{\frac{\lambda}{e}}(x_{n-1}, x_{n+1})). \tag{3.1}$$

By the convexity of w, we get

$$\begin{aligned} w_{\frac{\lambda}{a}}(x_{n-1}, x_n) &= w_{\lambda + \frac{1-a}{a}\lambda}(x_{n-1}, x_n) \\ &\leq \frac{\lambda}{\frac{\lambda}{a}} w_\lambda(x_{n-1}, x_n) + \frac{\frac{1-a}{a}\lambda}{\frac{\lambda}{a}} w_{\frac{1-a}{a}\lambda}(x_n, x_n) \\ &= a w_\lambda(x_{n-1}, x_n) \end{aligned}$$

which implies that

$$w_{\frac{\lambda}{a}}(x_{n-1}, x_n) \leq a w_\lambda(x_{n-1}, x_n). \tag{3.2}$$

Similarly, we can obtain

$$w_{\frac{\lambda}{b}}(x_{n-1}, x_n) \leq b w_\lambda(x_{n-1}, x_n), \tag{3.3}$$

$$w_{\frac{\lambda}{c}}(x_n, x_{n+1}) \leq c w_\lambda(x_n, x_{n+1}). \tag{3.4}$$

Also,

$$\begin{aligned} w_{\frac{\lambda}{e}}(x_{n-1}, x_{n+1}) &= w_{\lambda+\frac{1-e}{e}\lambda}(x_{n-1}, x_{n+1}) \\ &\leq e w_\lambda(x_{n-1}, x_n) + (1-e) w_{\frac{1-e}{e}\lambda}(x_n, x_{n+1}) \\ &\leq e w_\lambda(x_{n-1}, x_n) + e w_\lambda(x_n, x_{n+1}) (as\ e \in (0, \frac{1}{2})). \end{aligned} \tag{3.5}$$

Applying (3.2)–(3.5) in (3.1), we obtain

$$\tau + F\left(w_\lambda(x_n, x_{n+1})\right) \leq F\left((a+b+e) w_\lambda(x_{n-1}, x_n) + (c+e) w_\lambda(x_n, x_{n+1})\right). \tag{3.6}$$

Since F is strictly increasing, we derive

$$w_\lambda(x_n, x_{n+1}) \leq (a+b+e) w_\lambda(x_{n-1}, x_n) + (c+e) w_\lambda(x_n, x_{n+1}).$$

Thus,

$$w_\lambda(x_n, x_{n+1}) \leq \frac{a+b+e}{1-c-e} w_\lambda(x_{n-1}, x_n).$$

Since $a+b+c+2e = 1$ and $a, b, c, e > 0$, we have

$$w_\lambda(x_n, x_{n+1}) \leq w_\lambda(x_{n-1}, x_n),$$

for any $n \in \mathbb{N}_0$ and $0 < \lambda \leq \lambda_0$. Thus, from (3.6), we have

$$\tau + F(w_\lambda(x_n, x_{n+1})) \leq F(w_\lambda(x_{n-1}, x_n)).$$

Therefore,

$$\begin{aligned} F(w_\lambda(x_n, x_{n+1})) &\leq F(w_\lambda(x_{n-1}, x_n)) - \tau \\ &\leq F(w_\lambda(x_{n-2}, x_{n-1})) - 2\tau \leq ... \\ &\leq F(w_\lambda(x_0, x_1)) - n\tau. \end{aligned} \tag{3.7}$$

Denote $\gamma_n^\lambda = w_\lambda(x_n, x_{n+1})$ for any $n \in \mathbb{N}_0$ and $0 < \lambda \leq \lambda_0$. From (3.7), we deduce that $\lim_{n\to\infty} F(\gamma_n^\lambda) = -\infty$. Using (F2), we get

$$\lim_{n\to\infty} \gamma_n^\lambda = 0. \tag{3.8}$$

Taking into account (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \left(\gamma_n^\lambda\right)^k F(\gamma_n^\lambda) = 0. \tag{3.9}$$

It follows from (3.7) that

$$\left(\gamma_n^{\lambda}\right)^k F(\gamma_n^{\lambda}) - \left(\gamma_n^{\lambda}\right)^k F(\gamma_0^{\lambda}) \leq -\left(\gamma_n^{\lambda}\right)^k n\tau \leq 0.$$

Letting $n \to \infty$ in the above inequality, and combining it with (3.8) and (3.9) we get

$$\lim_{n\to\infty} \left(\gamma_n^{\lambda}\right)^k n = 0.$$

So, there exists $n_1 \in \mathbb{N}_0$ such that $\left(\gamma_n^{\lambda}\right)^k n \leq 1$ for all $n \geq n_1$. Consequently, we have

$$\gamma_n^{\lambda} \leq \frac{1}{n^{1/k}},$$

for all $n \geq n_1$ and $0 < \lambda \leq \lambda_0$. Set $\lambda_i = \left(\frac{1}{2}\right)^i \lambda_0$ for any $i \in \mathbb{N}$. It is easy to see that $0 < \lambda_i \leq \lambda_0$. Thus, we have

$$\gamma_n^{\lambda_i} \leq \frac{1}{n^{1/k}} \tag{3.10}$$

for all $n \geq n_1$. For any positive integers m, n with $1 < m < n$, we obtain

$$w_{\lambda_h}(x_m, x_n) \leq \frac{\lambda_m}{\lambda_h} w_{\lambda_m}(x_m, x_{m+1}) + \frac{\lambda_{m+1}}{\lambda_h} w_{\lambda_{m+1}}(x_{m+1}, x_{m+2}) + ... + \frac{\lambda_{n-1}}{\lambda_h} w_{\lambda_n - 1}(x_{n-1}, x_n), \tag{3.11}$$

where $\lambda_h = \lambda_m + \lambda_{m+1} + ... + \lambda_{n-1}$. Since $\lambda_h = \frac{1}{2^m}\lambda_0 + \frac{1}{2^{m+1}}\lambda_0 + ... + \frac{1}{2^{n+1}}\lambda_0 \leq \frac{1}{2^{m-1}}\lambda_0 < \lambda_0$, the inequalities (3.10) and (3.11) imply that

$$\begin{aligned} w_{\lambda_h}(x_m, x_n) &\leq \frac{\lambda_m}{\lambda_h} w_{\lambda_m}(x_m, x_{m+1}) + \frac{\lambda_{m+1}}{\lambda_h} w_{\lambda_{m+1}}(x_{m+1}, x_{m+2}) + ... + \frac{\lambda_{n-1}}{\lambda_h} w_{\lambda_n}(x_{n-1}, x_n) \\ &\leq \gamma_m^{\lambda_m} + \gamma_{m+1}^{\lambda_{m+1}} + ... + \gamma_n^{\lambda_{n-1}} \leq \sum_{j=m}^{n} \frac{1}{j^{1/k}} \leq \sum_{j=m}^{\infty} \frac{1}{j^{1/k}} \to 0. \end{aligned}$$

Hence, $\lim_{m,n\to\infty} w_{\lambda_h}(x_m, x_n) = 0$. Given that $0 < \lambda_h < \lambda_0$, we have that $\lim_{m,n\to\infty} w_{\lambda_0}(x_m, x_n) = 0$, which implies that $\{x_n\}$ is a w-Cauchy sequence. Because the space A has uniform approximation in the space B, it follows that $\{Tx_n\}$ must be a w-Cauchy sequence in B. Since X_w^* is a w-complete modular metric space and (A, B) is a pair of non-empty w-closed subsets of X_w^* , the sequence $\{x_n\}$ w-converges to some element x in A and the sequence $\{Tx_n\}$ w-converges to some element y in B. Noting that w satisfies the Fatou property, then

$$w_{\lambda_0}(A, B) \leq w_{\lambda_0}(x, y) \leq \liminf_{n\to\infty} w_{\lambda_0}(x_{n+1}, Tx_n) = w_{\lambda_0}(A, B),$$

thus, $w_{\lambda_0}(x, y) = w_{\lambda_0}(A, B)$, which implies that x is a member in A_0. Given that $T(A_0) \subseteq B_0$, we have that

$$w_{\lambda_0}(p, Tx) = w_{\lambda_0}(A, B)$$

for some element p in A. If, for some $n \in \mathbb{N}_0$, we have that $x_{n+1} = p$, so $w_{\lambda_0}(x_{n+1}, Tx) = w_{\lambda_0}(A, B)$; then,

$$w_{\lambda_0}(A, B) \leq w_{\lambda_0}(x, Tx) \leq \liminf_{n\to\infty} w_{\lambda_0}(x_{n+1}, Tx_n) = w_{\lambda_0}(A, B),$$

which implies that $w_{\lambda_0}(x, Tx) = w_{\lambda_0}(A, B)$ and the conclusion is immediate. Therefore, we assume that $x_n \neq p$ for all $n \in \mathbb{N}_0$. Again, since T is an α-η-type generalized F-proximal contraction of the first kind, we have

$$\tau + F(w_{\lambda_0}(p, x_{n+1})) \leq F(w_{\frac{\lambda_0}{a}}(x, x_n) + w_{\frac{\lambda_0}{b}}(p, x) + w_{\frac{\lambda_0}{c}}(x_{n+1}, x_n) + w_{\frac{\lambda_0}{e}}(p, x_n) + w_{\frac{\lambda_0}{e}}(x, x_{n+1})).$$

Since F is strictly increasing, we obtain

$$\begin{aligned} w_{\lambda_0}(p, x_{n+1}) &\leq w_{\frac{\lambda_0}{a}}(x, x_n) + w_{\frac{\lambda_0}{b}}(p, x) + w_{\frac{\lambda_0}{c}}(x_{n+1}, x_n) + w_{\frac{\lambda_0}{e}}(p, x_n) + w_{\frac{\lambda_0}{e}}(x, x_{n+1}) \\ &\leq a w_{\lambda_0}(x, x_n) + b w_{\lambda_0}(p, x) + c w_{\lambda_0}(x_{n+1}, x_n) + e w_{\lambda_0}(p, x) + e w_{\lambda_0}(x, x_n) + e w_{\lambda_0}(x_{n+1}, x). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get that $\lim_{n\to\infty} w_{\lambda_0}(p, x_{n+1}) \leq (b + e)w_{\lambda_0}(p, x)$; hence, $w_{\lambda_0}(p, x) \leq (b + e)w_{\lambda_0}(p, x)$, which implies that p and x should be identical. Thus, $w_{\lambda_0}(x, Tx) = w_{\lambda_0}(A, B)$ and x is a best proximity point of T. To prove the uniqueness of the result, suppose that there is another best proximity point u of T such that $w_{\lambda_0}(u, Tu) = w_{\lambda_0}(A, B)$. Given that T is an α-η-type generalized F-proximal contraction of the first kind, we have

$$\tau + F(w_{\lambda_0}(x, u)) \leq F(w_{\frac{\lambda_0}{a}}(x, u) + w_{\frac{\lambda_0}{e}}(u, x) + w_{\frac{\lambda_0}{e}}(x, u)).$$

Since F is strictly increasing, we obtain

$$\begin{aligned} w_{\lambda_0}(x, u) &\leq w_{\frac{\lambda_0}{a}}(x, u) + w_{\frac{\lambda_0}{e}}(u, x) + w_{\frac{\lambda_0}{e}}(x, u) \\ &\leq a w_{\lambda_0}(x, u) + e w_{\lambda_0}(x, u) + e w_{\lambda_0}(x, u) \\ &= (a + 2e) w_{\lambda_0}(x, u). \end{aligned}$$

Since $a + 2e > 0$, it follows that $w_{\lambda_0}(x, u) = 0$, which implies that x and u are identical. This complete the proof.

Let $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$ in Theorem 3.1; we can deduce the following best proximity point theorem in the setting of a modular metric space.

Corollary 3.1. *Let w be a strict convex modular metric with the Fatou property on X and X_w^* be a w-complete modular metric space induced by w. Suppose that (A, B) is a pair of non-empty w-closed subsets of X_w^* such that A has uniform approximation in B. Assume that $T : A \to B$ is a generalized F-proximal contraction of Reich type of the first kind, that is, for any $x_1, x_2, u_1, u_2 \in A$ with $x_1 \neq x_2$, $F \in \mathfrak{F}$ and $\tau > 0$, there exists $\lambda_0 > 0$ such that the following holds:*

$$\begin{cases} w_\lambda(x_1, Tu_1) = w_\lambda(A, B) \\ w_\lambda(x_2, Tu_2) = w_\lambda(A, B) \end{cases} \Rightarrow \tau + F(w_\lambda(x_1, x_2)) \leq F(w_{\frac{\lambda}{a}}(u_1, u_2) + b w_{\frac{\lambda}{b}}(x_1, u_1) + c w_{\frac{\lambda}{c}}(x_2, u_2))$$

for all $0 < \lambda \leq \lambda_0$ and $a, b, c > 0$ with $a + b + c = 1$. Also,

(1) A_0^λ and B_0^λ are non-empty sets and $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $0 < \lambda \leq \lambda_0$;
(2) there exist elements $x_0, x_1 \in A_0^\lambda$ for all $0 < \lambda \leq \lambda_0$ such that $w_\lambda(x_1, Tx_0) = w_\lambda(A, B)$.

If, for every $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) = w_\lambda(A, B)$ *implies that* $w_\lambda(x, Tx) < \infty$*, then* T *has a best proximity point. If in addition, for any* $x, u \in X_w^*$*,* $w_\lambda(x, Tx) = w_\lambda(u, Tu) = w_\lambda(A, B)$ *implies that* $w_\lambda(x, u) < \infty$*, then the best proximity point of* T *is unique. Further, for any fixed element* $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by*

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B),$$

w-converges to the best proximity point.

The following results without the convexity assumption of Corollary 3.1.

Corollary 3.2. *Let* w *be a strict modular metric with the Fatou property on* X *and* X_w^* *be a* w*-complete modular metric space induced by* w*. Suppose that* (A, B) *is a pair of non-empty* w*-closed subsets of* X_w^* *such that* A *has uniform approximation in* B*. Assume that* $T : A \to B$ *is an* F*-proximal contraction of the first kind, that is, for any* $x_1, x_2, u_1, u_2 \in A$ *with* $x_1 \neq x_2$*, there exists* $F \in \mathfrak{F}$*,* $\tau > 0$*, and* $\lambda_0 > 0$ *such that the following holds:*

$$\begin{cases} w_\lambda(x_1, Tu_1) = w_\lambda(A, B) \\ w_\lambda(x_2, Tu_2) = w_\lambda(A, B) \end{cases} \Rightarrow \tau + F(w_\lambda(x_1, x_2)) \leq F(w_\lambda(u_1, u_2))$$

for all $0 < \lambda \leq \lambda_0$*. Also, suppose that* T *satisfies the following assertions:*

(1) A_0^λ *and* B_0^λ *are non-empty sets and* $T(A_0^\lambda) \subseteq B_0^\lambda$ *for all* $0 < \lambda \leq \lambda_0$*;*
(2) there exist elements $x_0, x_1 \in A_0^\lambda$ *for all* $0 < \lambda \leq \lambda_0$ *such that* $w_\lambda(x_1, Tx_0) = w_\lambda(A, B)$*.*

If, for every $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) = w_\lambda(A, B)$ *implies that* $w_\lambda(x, Tx) < \infty$*, then* T *has a best proximity point. If in addition, for any* $x, u \in X_w^*$*,* $w_\lambda(x, Tx) = w_\lambda(u, Tu) = w_\lambda(A, B)$ *implies that* $w_\lambda(x, u) < \infty$*, then the best proximity point of* T *is unique. Further, for any fixed element* $x_0 \in A_0$*, sequence* $\{x_n\}$ *defined by*

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B),$$

w-converges to the best proximity point.

We present an illustrative example.

Example 3.1. *Let us consider the subsets*

$$A = \left\{(x_1, x_2) : x_1^2 + x_2^2 = 4 \text{ and } 0 \leq x_1, x_2 \leq 2\right\},$$
$$B = \left\{(y_1, y_2) : y_1^2 + y_2^2 = 1 \text{ and } 0 \leq y_1, y_2 \leq 1\right\}$$

in the space $X = \mathbb{R}^2$ *with the modular metric* $w_\lambda : (0, \infty) \times X \times X \to [0, \infty]$ *defined by*

$$w_\lambda((x_1, x_2), (y_1, y_2)) = \frac{|x_1 - y_1| + |x_2 - y_2|}{\lambda}$$

for any $\lambda > 0$*. Then, we have that* $w_\lambda(A, B) = 1$*,* $A_0 = A$*, and* $B_0 = B$*. It is easy to see that* (X, w_λ) *is a* w*-complete modular metric space,* w *satisfies the Fatou property, and* A *has uniform approximation in* B*; however, the pair* (A, B) *does not have the weak P-property. Let* $F : (0, \infty) \to \mathbb{R}$ *defined by*

$F(x) = \ln x$ for all $x > 0$. Thus, F belongs to $\mathfrak{F}$. Let $T : A \to B$ be a mapping that satisfies the following for each $(x_1, x_2) \in A$:

$$T(x_1, x_2) = (Px_1, \sqrt{1 - (Px_1)^2}),$$

where $Px_1 = \frac{x_1}{2+x_1}$. We can observe that $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $\lambda > 0$. Assume that u_1, u_2, u_3, u_4 are elements in A such that $w_\lambda(u_1, Tu_2) = w_\lambda(u_3, Tu_4) = w_\lambda(A, B)$. Set $u_2 = (r_1, \sqrt{4 - r_1^2})$ and $u_4 = (r_2, \sqrt{4 - r_2^2})$ for some $1 \leq r_1, r_2 \leq 2$. Then, $Tu_2 = (\frac{r_1}{2+r_1}, \sqrt{1 - (Pr_1)^2})$ and $Tu_4 = (\frac{r_2}{2+r_2}, \sqrt{1 - (Pr_2)^2})$. So $u_1 = (\frac{2r_1}{2+r_1}, 2\sqrt{1 - (Pr_1)^2})$ and $u_3 = (\frac{2r_2}{2+r_2}, 2\sqrt{1 - (Pr_2)^2})$. We obtain

$$\begin{aligned} w_\lambda(u_1, u_3) &= \frac{\left|\frac{2r_1}{2+r_1} - \frac{2r_2}{2+r_2}\right| + \left|2\sqrt{1 - (Pr_1)^2} - 2\sqrt{1 - (Pr_2)^2}\right|}{\lambda} \\ &= \frac{\left|\frac{2r_1}{2+r_1} - \frac{2r_2}{2+r_2}\right| + \left|2\sqrt{1 - (\frac{r_1}{2+r_1})^2} - 2\sqrt{1 - (\frac{r_2}{2+r_2})^2}\right|}{\lambda} \\ &\leq e^{-\tau} \frac{|r_1 - r_2| + \left|\sqrt{4 - r_1^2} - \sqrt{4 - r_2^2}\right|}{\lambda} \\ &= e^{-\tau} w_\lambda(u_2, u_4). \end{aligned}$$

When we set $r_1 = 0$ and $r_2 = 1$, we get that $e^{-\tau} \geq \frac{6-3\sqrt{3}}{8-2\sqrt{2}}$ or $\tau \in (0, \ln \frac{8-2\sqrt{2}}{6-3\sqrt{3}})$. Consequently, T is an F-proximal contraction of the first kind. Thus, all of the conditions of Corollary 3.2 are satisfied. Hence, T has a unique best proximity point $(2, 0)$.

Next, we state and prove the best proximity point theorem for a α-η-type generalized F-proximal contraction of the second kind in a modular metric space.

Theorem 3.2. *Let w be a strict convex modular metric with the Fatou property on X and X_w^* be a w-complete modular metric space induced by w. Suppose that (A, B) is a pair of non-empty w-closed subsets of X_w^* such that B has uniform approximation in A. Assume that T is a continuous α-η-type generalized F-proximal contraction of the second kind that satisfies the following assertions:*

(1) A_0^λ and B_0^λ are non-void and $T(A_0^\lambda) \subseteq B_0^\lambda$;
(2) T is an α-admissible mapping with respect to η;
(3) there exist elements $x_0, x_1 \in A_0^\lambda$ for all $0 < \lambda \leq \lambda_0$ such that $w_\lambda(x_1, Tx_0) = w_\lambda(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$.

If, for every $0 < \lambda \leq \lambda_0$, there exists an $x \in X_w^*$ satisfying that $w_\lambda(x, Tx) = w_\lambda(A, B)$ implies that $w_\lambda(x, Tx) < \infty$, then T has a best proximity point. If in addition, for any $x, u \in X_w^*$ satisfying $w_\lambda(x, Tx) = w_\lambda(u, Tu) = w_\lambda(A, B)$ implies that $w_\lambda(x, u) < \infty$ and $\alpha(x, u) \geq \eta(x, u)$, then the best proximity point of T is unique. Further, for any $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B),$$

w-converges to the best proximity point.

Proof. Similar to Theorem 3.1, we can obtain that there is a sequence $\{x_n\}$ in A_0^λ such that

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}_0$ and $0 < \lambda \leq \lambda_0$. Without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Given the fact that T is an α-η-type generalized F-proximal contraction of the second kind, we have

$$\begin{aligned}\tau + F(w_\lambda(Tx_n, Tx_{n+1})) \leq & F(w_{\frac{\lambda}{a}}(Tx_{n-1}, Tx_n) + w_{\frac{\lambda}{b}}(Tx_n, Tx_{n-1}) + w_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n) \\ & + w_{\frac{\lambda}{e}}(Tx_n, Tx_n) + w_{\frac{\lambda}{e}}(Tx_{n-1}, Tx_{n+1})).\end{aligned}$$

Since F is strictly increasing, we obtain

$$\begin{aligned}w_\lambda(Tx_n, Tx_{n+1}) \leq & w_{\frac{\lambda}{a}}(Tx_{n-1}, Tx_n) + w_{\frac{\lambda}{b}}(Tx_n, Tx_{n-1}) + w_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n) + w_{\frac{\lambda}{e}}(Tx_{n-1}, Tx_{n+1}) \\ \leq & aw_\lambda(Tx_{n-1}, Tx_n) + bw_\lambda(Tx_n, Tx_{n-1}) + cw_\lambda(Tx_{n+1}, Tx_n) + ew_\lambda(Tx_{n-1}, Tx_n) \\ & + ew_\lambda(Tx_n, Tx_{n+1}),\end{aligned}$$

and, thus,

$$w_\lambda(Tx_n, Tx_{n+1}) \leq \frac{a+b+e}{1-c-e} w_\lambda(Tx_{n-1}, Tx_n) \leq w_\lambda(Tx_{n-1}, Tx_n).$$

We can obtain that $\lim_{m,n\to\infty} w_{\lambda_0}(Tx_m, Tx_n) = 0$ and $\{Tx_n\}$ is a w-Cauchy sequence by using a similar technique as in Theorem 3.1. Because the space B has uniform approximation in the space A, it follows that $\{Tx_n\}$ must be a w-Cauchy sequence in A. Since X_w^* is a w-complete modular metric space and A is a non-empty w-closed subset of X_w^*, the sequence $\{x_n\}$ w-converges to some element x in A. By virtue of the fact that w satisfies the Fatou property and T is a continuous mapping, we have

$$w_{\lambda_0}(A, B) \leq w_{\lambda_0}(x, Tx) \leq \liminf_{n\to\infty} w_{\lambda_0}(x_{n+1}, Tx_n) = w_{\lambda_0}(A, B).$$

So, $w_{\lambda_0}(x, Tx) = w_{\lambda_0}(A, B)$, which implies that x is a best proximity point of T. To prove the uniqueness of the result, suppose that there is another best proximity point u of T such that $w_{\lambda_0}(u, Tu) = w_{\lambda_0}(A, B)$. Given that T is an α-η-type generalized F-proximal contraction of the second kind, we have

$$\tau + F(w_{\lambda_0}(Tx, Tu)) \leq F(w_{\frac{\lambda_0}{a}}(Tx, Tu) + w_{\frac{\lambda_0}{e}}(Tu, Tx) + w_{\frac{\lambda_0}{e}}(Tx, Tu)).$$

Since F is strictly increasing, we obtain

$$\begin{aligned}w_{\lambda_0}(Tx, Tu) &\leq w_{\frac{\lambda_0}{a}}(Tx, Tu) + w_{\frac{\lambda_0}{e}}(Tu, Tx) + w_{\frac{\lambda_0}{e}}(Tx, Tu) \\ &\leq aw_{\lambda_0}(Tx, Tu) + ew_{\lambda_0}(Tx, Tu) + ew_{\lambda_0}(Tx, Tu) \\ &= (a+2e)w_{\lambda_0}(Tx, Tu),\end{aligned}$$

Since $a + 2e > 0$, we have that $w_{\lambda_0}(Tx, Tu) = 0$, which implies that $Tx = Tu$. This completes the proof.

Letting $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$ in Theorem 3.2, we can deduce the following best proximity point theorem in the setting of a modular metric space.

Corollary 3.3. *Let w be a strict convex modular metric with the Fatou property on X and X_w^* be a w-complete modular metric space induced by w. Suppose that (A, B) is a pair of non-empty w-closed subsets of X_w^* such that B has uniform approximation in A. Assume that $T : A \to B$ is a generalized F-proximal contraction of Reich type of the second kind, that is, for any $x_1, x_2, u_1, u_2 \in A$ with $Tx_1 \neq Tx_2$, there exist $F \in \mathfrak{F}$, $\tau > 0$, and a $\lambda_0 > 0$ such that the following holds:*

$$\begin{cases} w_\lambda(x_1, Tu_1) = w_\lambda(A, B) \\ w_\lambda(x_2, Tu_2) = w_\lambda(A, B) \end{cases} \Rightarrow \tau + F(w_\lambda(Tx_1, Tx_2)) \leq F(w_{\frac{\lambda}{a}}(Tu_1, Tu_2) + bw_{\frac{\lambda}{b}}(Tx_1, Tu_1) + cw_{\frac{\lambda}{c}}(Tx_2, Tu_2))$$

for all $0 < \lambda \leq \lambda_0$ and $a, b, c > 0$ with $a + b + c = 1$. Also,

(1) A_0^λ and B_0^λ are non-empty sets and $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $0 < \lambda \leq \lambda_0$;
(2) there exist elements $x_0, x_1 \in A_0^\lambda$ for all $0 < \lambda \leq \lambda_0$ such that $w_\lambda(x_1, Tx_0) = w_\lambda(A, B)$.

If, for every $0 < \lambda \leq \lambda_0$, there exists an $x \in X_w^*$ satisfying that $w_\lambda(x, Tx) = w_\lambda(A, B)$ implies that $w_\lambda(x, Tx) < \infty$, then T has a best proximity point. If, in addition, for any $x, u \in X_w^*$, $w_\lambda(x, Tx) = w_\lambda(u, Tu) = w_\lambda(A, B)$ implies that $w_\lambda(x, u) < \infty$, then the best proximity point of T is unique. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B),$$

w-converges to the best proximity point.

Corollary 3.4. *Let w be a strict modular metric with the Fatou property on X and X_w^* be a w-complete modular metric space induced by w. Suppose that (A, B) is a pair of non-empty w-closed subsets of X_w^* such that B has uniform approximation in A. Assume that $T : A \to B$ is an F-proximal contraction of the second kind, that is, for any $x_1, x_2, u_1, u_2 \in A$ with $Tx_1 \neq Tx_2$, there exist $F \in \mathfrak{F}$, $\tau > 0$, and a $\lambda_0 > 0$ such that the following holds:*

$$\begin{cases} w_\lambda(x_1, Tu_1) = w_\lambda(A, B) \\ w_\lambda(x_2, Tu_2) = w_\lambda(A, B) \end{cases} \Rightarrow \tau + F(w_\lambda(Tx_1, Tx_2)) \leq F(w_\lambda(Tu_1, Tu_2))$$

for all $0 < \lambda \leq \lambda_0$. Also,

(1) A_0^λ and B_0^λ are non-empty sets and $T(A_0^\lambda) \subseteq B_0^\lambda$ for all $0 < \lambda \leq \lambda_0$;
(2) there exist elements $x_0, x_1 \in A_0^\lambda$ for all $0 < \lambda \leq \lambda_0$ such that $w_\lambda(x_1, Tx_0) = w_\lambda(A, B)$;

If, for every $0 < \lambda \leq \lambda_0$, there exists an $x \in X_w^*$ satisfying that $w_\lambda(x, Tx) = w_\lambda(A, B)$ implies that $w_\lambda(x, Tx) < \infty$, then T has a best proximity point. If, in addition, for any $x, u \in X_w^*$, $w_\lambda(x, Tx) = w_\lambda(u, Tu) = w_\lambda(A, B)$ implies that $w_\lambda(x, u) < \infty$, then the best proximity point of T is unique. Further, for any fixed element $x_0 \in A_0$, sequence $\{x_n\}$ defined by

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B),$$

w-converges to the best proximity point.

We present an illustrative example.

Example 3.2. *Let us consider the subsets*

$$A = \{(x_1, x_2) : x_1 = 1 \text{ and } x_2 \geq 0\},$$
$$B = \{(y_1, y_2) : y_1 = 0 \text{ and } y_2 \geq 0\}$$

in the space $X = \mathbb{R}^2$ *with the modular metric* $w_\lambda : (0, \infty) \times X \times X \to [0, \infty]$ *defined by*

$$w_\lambda((x_1, x_2), (y_1, y_2)) = \frac{|x_1 - y_1| + |x_2 - y_2|}{\lambda}$$

for any $\lambda > 0$. *Then, we have that* $w_\lambda(A, B) = \sqrt{2}$, $A_0 = A$ *and* $B_0 = B$. *It is easy to see that* (X, w_λ) *is a w-complete modular metric space, w satisfies the Fatou property and B has uniform approximation in A. Let* $F : (0, \infty) \to \mathbb{R}$ *be defined by* $F(x) = \ln x$ *for all* $x > 0$. *Thus, F belongs to* $\mathfrak{F}$. *Let* $T : A \to B$ *be a mapping that satisfies the following for each* $(x_1, x_2) \in A$:

$$T(x_1, x_2) = (0, Px_2),$$

where $Px_2 = \frac{x_2}{1+x_2}$. *We can observe that* $T(A_0^\lambda) \subseteq B_0^\lambda$ *for all* $\lambda > 0$. *Assume that* u_1, u_2, u_3, u_4 *are elements in A such that* $w_\lambda(u_1, Tu_2) = w_\lambda(u_3, Tu_4) = w_\lambda(A, B)$. *Set* $u_2 = (1, i_1)$ *and* $u_4 = (1, i_2)$ *for some* $1 \leq i_1, i_2 \leq 2$. *Then,* $Tu_2 = (0, Pi_1)$ *and* $Tu_4 = (0, Pi_2)$. *So,* $u_1 = (1, Pi_1)$ *and* $u_3 = (1, Pi_2)$. *We obtain*

$$w_\lambda(Tu_1, Tu_3) = \frac{\left|P^2(i_1) - P^2(i_2)\right|}{\lambda} = \frac{\left|\frac{i_1}{1+2i_1} - \frac{i_2}{1+2i_2}\right|}{\lambda} \leq e^{-\tau} \frac{\left|\frac{i_1}{1+i_1} - \frac{i_2}{1+i_2}\right|}{\lambda} = e^{-\tau} w_\lambda(Tu_2, Tu_4).$$

When we set $i_1 = 0$ *and* $i_2 = 1$, *we get that* $e^{-\tau} \geq \frac{2}{3}$ *or* $\tau \in (0, \ln \frac{3}{2})$. *Consequently , T is an F-proximal contraction of the second kind. Thus, all of the conditions of Corollary 3.4 are satisfied. Hence, T has a unique best proximity point* $(1, 0)$.

Our next result is obtained for α-η-type generalized F-proximal contractions of the first kind, as well as α-η-type generalized F-proximal contractions of the second kind without the assumption of uniform approximation of the domains or the co-domain of the mappings.

Theorem 3.3. *Let w be a strict convex modular metric with the Fatou property on X and* X_w^* *be a w-complete modular metric space induced by w. Suppose that* (A, B) *is a pair of non-empty w-closed subsets of* X_w^*. *Moreover, assume the following:*

(1) A_0^λ *and* B_0^λ *are non-empty sets and* $T(A_0^\lambda) \subseteq B_0^\lambda$;
(2) T is an α-η*-type generalized F-proximal contraction of the first kind as well as an* α-η*-type generalized F-proximal contraction of the second kind;*
(3) there exist elements $x_0, x_1 \in A_0^\lambda$ *for all* $0 < \lambda \leq \lambda_0$ *such that* $w_\lambda(x_1, Tx_0) = w_\lambda(A, B)$ *and* $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;
(4) if $\{x_n\}$ *is a sequence such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}_0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N}_0$.

If, for every $0 < \lambda \leq \lambda_0$, *there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) = w_\lambda(A, B)$ *implies that* $w_\lambda(x, Tx) < \infty$, *then T has a best proximity point. If, in addition, for any* $x, y \in X^*$ *such that* $w_\lambda(x, Tx) = w_\lambda(u, Tu) = w_\lambda(A, B)$, *we have that* $w_\lambda(x, u) < \infty$ *and* $\alpha(x, u) \geq \eta(x, u)$, *then the best proximity point of T is unique. Further, for any fixed element* $x_0 \in A_0$, *the sequence* $\{x_n\}$ *defined by*

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B),$$

w-converges to the best proximity point.

Proof. Similar to Theorem 3.1, we can obtain that there is a sequence $\{x_n\}$ in A_0^λ such that

$$w_\lambda(x_{n+1}, Tx_n) = w_\lambda(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$$

for all $n \in \mathbb{N}_0$ and $0 < \lambda \leq \lambda_0$. Proceeding as in Theorem 3.1, we obtain that the sequence $\{x_n\}$ is a w-Cauchy sequence and w-converges to some element x in A. Similar to Theorem 3.2, we obtain that the sequence $\{Tx_n\}$ is a w-Cauchy sequence and w-converges to some element y in B. Noting that w satisfies the Fatou property, it follows that

$$w_{\lambda_0}(A, B) \leq w_{\lambda_0}(x, y) \leq \liminf_{n\to\infty} w_{\lambda_0}(x_{n+1}, Tx_n) = w_{\lambda_0}(A, B),$$

thus, $w_{\lambda_0}(x, y) = w_{\lambda_0}(A, B)$, which implies that x is a member in A_0^λ. Given that $T(A_0^\lambda) \subseteq B_0^\lambda$, we have

$$w_{\lambda_0}(p, Tx) = w_{\lambda_0}(A, B)$$

for some element p in A. If for some $n \in \mathbb{N}_0$ such that $x_{n+1} = p$ we have that $w_{\lambda_0}(x_{n+1}, Tx) = w_{\lambda_0}(A, B)$, then

$$w_{\lambda_0}(A, B) \leq w_{\lambda_0}(x, Tx) \leq \liminf_{n\to\infty} d(x_{n+1}, Tx) = w_{\lambda_0}(A, B)$$

which implies that $w_{\lambda_0}(x, Tx) = w_{\lambda_0}(A, B)$ and the conclusion is immediate. Therefore, we assume that $x_n \neq p$ for all $n \in \mathbb{N}_0$. Again, since T is an α-η-type generalized F-proximal contraction of the first kind, we have

$$\tau + F(w_{\lambda_0}(p, x_{n+1})) \leq F(w_{\frac{\lambda_0}{a}}(x, x_n) + w_{\frac{\lambda_0}{b}}(p, x) + w_{\frac{\lambda_0}{c}}(x_{n+1}, x_n) + w_{\frac{\lambda_0}{e}}(p, x_n) + w_{\frac{\lambda_0}{e}}(x, x_{n+1})).$$

Since F is strictly increasing, we obtain

$$\begin{aligned} w_{\lambda_0}(p, x_{n+1}) \leq& w_{\frac{\lambda_0}{a}}(x, x_n) + w_{\frac{\lambda_0}{b}}(p, x) + w_{\frac{\lambda_0}{c}}(x_{n+1}, x_n) + w_{\frac{\lambda_0}{e}}(p, x_n) + w_{\frac{\lambda_0}{e}}(x, x_{n+1}) \\ \leq& a w_{\lambda_0}(x, x_n) + b w_{\lambda_0}(p, x) + c w_{\lambda_0}(x_{n+1}, x_n) + e w_{\lambda_0}(p, x) + e w_{\lambda_0}(x, x_n) + e w_{\lambda_0}(x, x_{n+1}). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get that $\lim_{n\to\infty} w_{\lambda_0}(p, x_{n+1}) \leq (b + e) w_{\lambda_0}(p, x)$; hence, $w_{\lambda_0}(p, x) \leq (b + e) w_{\lambda_0}(p, x)$, which implies that p and x should be identical. Thus, $w_{\lambda_0}(x, Tx) = w_{\lambda_0}(A, B)$ and x is a best proximity point of T. As in the proof of Theorem 3.1, the uniqueness of the best proximity point of mapping T follows.

4. Applications

Let (X, w_λ) be a metric space and A, B be two subsets of X. If $A \cap B \neq \emptyset$, then $d(A, B) = 0$. In this case, a best proximity result turns to a fixed point result.

Letting $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$ in Theorem 3.1, we can obtain the following:

Corollary 4.1. *Let w be a strict convex modular metric with the Fatou property on X and X_w^* be a w-complete modular metric space induced by w. Assume that T is a self mapping on X if there exist $F \in \mathfrak{F}$, $\lambda_0 > 0$, and $a, b, c, e, \tau > 0$ with $a + b + c + 2e = 1$ such that the following holds:*

$$\tau + F(w_\lambda(Tx, Ty)) \leq F\left(w_{\frac{\lambda}{a}}(x, y) + w_{\frac{\lambda}{b}}(x, Tx) + w_{\frac{\lambda}{c}}(y, Ty) + w_{\frac{\lambda}{e}}(x, Ty) + w_{\frac{\lambda}{e}}(y, Tx)\right),$$

for all $x, y \in X$ *and* $0 < \lambda \leq \lambda_0$*. If, for every* $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) < \infty$*, then* T *has a fixed point. If, in addition, for any* $x, u \in X_w^*$ *satisfying that* $w_\lambda(x, u) < \infty$*, then the fixed point of* T *is unique. Further, for any* $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by* $\{Tx_n\}$ *w-converges to the fixed point.*

Corollary 4.2. *Let* w *be a strict convex modular metric with the Fatou property on* X *and* X_w^* *be a w-complete modular metric space induced by* w*. Assume that* T *is a self-mapping on* X *if there exist* $F \in \mathfrak{F}$*,* $\lambda_0 > 0$*, and* $a, b, c, \tau > 0$ *with* $a + b + c = 1$ *such that the following holds:*

$$\tau + F\left(w_\lambda(Tx, Ty)\right) \leq F\left(w_{\frac{\lambda}{a}}(x, y) + w_{\frac{\lambda}{b}}(x, Tx) + w_{\frac{\lambda}{c}}(y, Ty)\right),$$

for all $x, y \in X$ *and* $0 < \lambda \leq \lambda_0$*. If, for every* $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) < \infty$*, then* T *has a fixed point. If, in addition, for any* $x, u \in X_w^*$ *satisfying that* $w_\lambda(x, u) < \infty$*, then the fixed point of* T *is unique. Further, for any* $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by* $\{Tx_n\}$ *w-converges to the fixed point.*

Corollary 4.3. *Let* w *be a strict convex modular metric with the Fatou property on* X *and* X_w^* *be a w-complete modular metric space induced by* w*. Assume that* T *be a self mapping on* X*, if there exist* $F \in \mathfrak{F}$*,* $\lambda_0 > 0$*, and* $\tau > 0$ *such that*

$$\tau + F\left(w_\lambda(Tx, Ty)\right) \leq F\left(w_\lambda(x, y)\right), \tag{4.1}$$

for all $x, y \in X$ *and* $0 < \lambda \leq \lambda_0$*. If, for every* $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) < \infty$*, then* T *has a fixed point. If, in addition, for any* $x, u \in X_w^*$ *satisfying that* $w_\lambda(x, u) < \infty$*, then the fixed point of* T *is unique. Further, for any* $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by* $\{Tx_n\}$ *w-converges to the fixed point.*

Letting $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$ in Theorem 3.2, we can obtain the following:

Corollary 4.4. *Let* w *be a strict convex modular metric with the Fatou property on* X *and* X_w^* *be a w-complete modular metric space induced by* w*. Assume that* T *is a self-mapping on* X*, if there exist* $F \in \mathfrak{F}$*,* $\lambda_0 > 0$*, and* $a, b, c, e, \tau > 0$ *with* $a + b + c + 2e = 1$ *such that the following holds:*

$$\tau + F\left(w_\lambda(T^2x, T^2y)\right) \leq F\left(w_{\frac{\lambda}{a}}(Tx, Ty) + w_{\frac{\lambda}{b}}(Tx, T^2x) + w_{\frac{\lambda}{c}}(Ty, T^2y) + w_{\frac{\lambda}{e}}(Tx, T^2y) + w_{\frac{\lambda}{e}}(Ty, T^2x)\right),$$

for all $x, y \in X$ *and* $0 < \lambda \leq \lambda_0$*. If, for every* $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) < \infty$*, then* T *has a fixed point. If, in addition, for any* $x, u \in X_w^*$ *satisfying that* $w_\lambda(x, u) < \infty$*, then the fixed point of* T *is unique. Further, for any* $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by* $\{Tx_n\}$ *w-converges to the fixed point.*

Corollary 4.5. *Let* w *be a strict convex modular metric with the Fatou property on* X *and* X_w^* *be a w-complete modular metric space induced by* w*. Assume that* T *is a self-mapping on* X*, if there exist* $F \in \mathfrak{F}$*,* $\lambda_0 > 0$*, and* $a, b, c, \tau > 0$ *with* $a + b + c = 1$ *such that the following holds:*

$$\tau + F\left(w_\lambda(T^2x, T^2y)\right) \leq F\left(w_{\frac{\lambda}{a}}(Tx, Ty) + w_{\frac{\lambda}{b}}(Tx, T^2x) + w_{\frac{\lambda}{c}}(Ty, T^2y)\right),$$

for all $x, y \in X$ *and* $0 < \lambda \leq \lambda_0$*. If, for every* $0 < \lambda \leq \lambda_0$*, there exists an* $x \in X_w^*$ *satisfying that* $w_\lambda(x, Tx) < \infty$*, then* T *has a fixed point. If, in addition, for any* $x, u \in X_w^*$ *satisfying that* $w_\lambda(x, u) < \infty$*, then the fixed point of* T *is unique. Further, for any* $x_0 \in A_0$*, the sequence* $\{x_n\}$ *defined by* $\{Tx_n\}$ *w-converges to the fixed point.*

Corollary 4.6. *Let w be a strict convex modular metric with the Fatou property on X and X_w^* be a w-complete modular metric space induced by w. Assume that T is a self-mapping on X, if there exist $F \in \mathfrak{F}$, $\lambda_0 > 0$, and $\tau > 0$ such that the following holds:*

$$\tau + F\left(w_\lambda(T^2x, T^2y)\right) \leq F\left(w_\lambda(Tx, Ty)\right),$$

for all $x, y \in X$ and $0 < \lambda \leq \lambda_0$. If, for every $0 < \lambda \leq \lambda_0$, there exists an $x \in X_w^*$ satisfying that $w_\lambda(x, Tx) < \infty$, then T has a fixed point. If, in addition, for any $x, u \in X_w^*$ satisfying that $w_\lambda(x, u) < \infty$, then the fixed point of T is unique. Further, for any $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $\{Tx_n\}$ w-converges to the fixed point.

Taking into account Corollary 4.3, we give an existence and uniqueness result for a solution of the Fredholm linear integral equation:

$$x(t) = h(t) + \int_0^{\beta(t)} G(t, \theta)L(\theta, x(\theta))d\theta, \tag{4.2}$$

for $t \in I = [0, 1]$, where $h : I \to X$, $\beta : I \to I$, $G : I \times I \to \mathbb{R}$, and $L : I \times I \to \mathbb{R}$ are continuous functions. Let $C(I, \mathbb{R})$ be the space of all continuous functions on I with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$, and the modular metric $w_\lambda(x, y) = \frac{\|x-y\|}{\lambda}$ for all $x, y \in C(I, \mathbb{R})$. We consider the following assumptions:

(A1) The function $G(t, \theta)$ is continuous and nonnegative on $I \times I$ with $\|G\|_\infty = \sup\{G(t, \theta) : t, \theta \in I\}$;
(A2) $|L(\theta, x(\theta)) - L(\theta, y(\theta))| \leq \delta\, |x(\theta) - y(\theta)|$ for all $\theta \in I$.

Theorem 4.1. *Assume that the hypotheses (A1) and (A2) hold. If $\|G\|_\infty \delta \leq e^{-\frac{1}{\delta}}$ for some $\delta > 0$, then (4.2) has a unique solution in $C(I, \mathbb{R})$.*

Proof. Note that $(C(I, \mathbb{R}), w_\lambda)$ is a w-complete modular metric space. Define a self-map T on $C(I, \mathbb{R})$ by

$$Tx(t) = h(t) + \beta \int_a^b G(t, \theta)L(\theta, x(\theta))d\theta, \;\; for\, all\, x(t) \in C(I, \mathbb{R}).$$

By the hypotheses (H1) and (H2), and by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
w_\lambda(Tx, Ty) &= \frac{1}{\lambda} \sup_{\theta\in I} \left| \int_0^{\beta(\theta)} G(t, \theta)L(\theta, x(\theta))d\theta - \int_0^{\beta(\theta)} G(t, \theta)L(\theta, y(\theta)d\theta \right| \\
&\leq \frac{\delta}{\lambda} \sup_{\theta\in I} \int_0^{\beta(\theta)} |G(t, \theta)|\, |x(\theta) - y(\theta)|\, d\theta \\
&\leq \frac{\delta}{\lambda} \sup_{\theta\in I} \left(\int_0^{\beta(\theta)} G^2(t, \theta)d\theta \right)^{1/2} \left(\int_0^{\beta(\theta)} |x(\theta) - y(\theta)|^2 d\theta \right)^{1/2} \\
&\leq \frac{\delta}{\lambda} \|k\|_\infty \sup_{\theta\in I} |x(\theta) - y(\theta)| \\
&\leq e^{-\frac{1}{\delta}} w_\lambda(x, y).
\end{aligned}$$

This implies that $w_\lambda(Tx, Ty) \leq e^{-\frac{1}{\delta}} w_\lambda(x, y)$. Hence, (4.1) is satisfied for $F(\alpha) = \ln \alpha$, $\alpha > 0$ and $\tau = \frac{1}{\delta}$. Therefore, all conditions of Corollary 4.3 hold; thus, the integral given by (4.2) has a unique positive solution.

5. Conclusions

The main motivation of the current paper is to show that the best proximity point results for α-η-type generalized F-proximal contraction mappings in the framework of modular metric spaces. We have achieved some best proximity point theorems for modular metric spaces. Our new results extend and improve many recent results. We also gave some examples to show the validity of our results and an application to nonlinear integral inclusions. Finally, we plan on looking into two future directions: the first direction is proving the existence of the best proximity points for cyclic mappings in modular metric spaces and the second direction is applying the results of this paper in the settings of other spaces, such as fuzzy metric spaces [29] and (q_1, q_2)-quasimetric spaces [11].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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