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Research article

Stochastic comparisons of second-order statistics from dependent and heterogeneous modified proportional (reversed) hazard rates scale models

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Abstract: In this paper, we investigate the problem of stochastically comparing the second-order statistics from dependent and heterogeneous samples following modified proportional hazard rates scale (MPHRS) and modified proportional reversed hazard rates scale (MPRHRS) models under Archimedean copula. We built some sufficient conditions for the usual stochastic order whenever the samples have different parameter vectors. Finally, some numerical examples were provided to illustrate the theoretical results.

Keywords: second-order statistics; MPHRS and MPRHRS models; archimedean copula; majorization
Mathematics Subject Classification: Primary 90B25, Secondary 60E15, 60K10

1. Introduction

Order statistics play an important role in reliability theory, auction theory, operations research, and many applied probability areas. $X_{k:n}$ denotes the kth smallest of random variables $X_1, \ldots, X_n$, $k = 1, \ldots, n$. In reliability theory, $X_{k:n}$ characterizes the lifetime of a $(n-k+1)$-out-of-n system, which works if at least $n-k+1$ of all the n components function normally. Specifically, $X_{1:n}$ and $X_{n:n}$ denote the lifetimes of series and parallel systems, respectively. In auction theory, $X_{1:n}$ and $X_{n:n}$ represent the final price of the first-price procurement auction and the first-price sealed-bid auction (see [1]), respectively. G. Pledger et al. [2] was the first to deal with the problem of comparing order statistics from heterogeneous exponential random variables. Subsequently, many researchers devoted themselves to stochastic comparisons of order statistics from heterogeneous independent or dependent samples; to name a few, see [3–10].

In this paper, we focus on second-order statistics, which also have a very wide application background. In auction theory, $X_{2:n}$ denotes the winner's price for the bid in the second-price reverse auction [11]. In the reliability context, the second-order statistic $X_{2:n}$ characterizes the lifetime of the

$(n-1)$-out-of-n system in reliability theory (referred to as the fail-safe system; see [12]). E. Păltăne [13] established the hazard rate order for comparing second-order statistics from heterogeneous exponential random variables. P. Zhao et al. [14] further extended the result of [13] from the hazard rate order to the likelihood ratio order. P. Zhao et al. [15] examined the mean residual life order between the second-order statistics from two sets of exponential random variables. P. Zhao et al. [16] studied the stochastic comparison of fail-safe systems with heterogeneous exponential components in terms of the dispersive order. N. Balakrishnan et al. [17] investigated the stochastic comparison of the second-order statistics from independent heterogeneous and homogeneous samples having different sample sizes in the sense of mean residual life, dispersive, hazard rate, and likelihood ratio orderings. X. Cai et al. [18] compared the hazard rate functions of the second-order statistics arising from two sets of independent multiple-outlier proportional hazard rates samples. For dependent and heterogeneous samples, R. Fang et al. [19] conducted stochastic comparisons on sample minimums (maximums) and the second smallest (largest) order statistic from proportional hazard rate and the proportional reversed hazard rates models. Additionly, C. Li et al. [20] obtained the usual stochastic order of the sample extremes and the second smallest order statistic from the scale model. T. Lando et al. [21] dealed with the increasing concave comparison of k-order statistics (iid case) for wide nonparametric families. S. Das et al. [22] considered stochastic comparisons between second-order statistics arising from general exponentiated location-scale models when the random variables are independent, and established usual stochastic and hazard rate orders between second-order statistics. O. Shojaee et al. [23] provided sufficient conditions to compare the smallest and the second smallest (largest and second largest) order statistics of dependent and heterogeneous random variables having the additive hazard model with the Archimedean copula in the sense of usual stochastic order and hazard rate order. R. F. Yan [24] studied the stochastic comparisons of the second-order statistics from dependent or independent and heterogeneous modified proportional hazard rate observations. G. Barmalzan et al. [25] studied the second smallest and the second largest order statistics from a general semiparametric family of distributions.

In reliability theory, to model the lifetime data with different hazard shapes, it is desirable to introduce flexible families of distributions. To this end, A. W. Marshall et al. [26] developed a new method to introduce one parameter to a base distribution, resulting in a new family of distribution with more flexibility. For a baseline distribution function F with support $\mathbb{R}^+ = (0, \infty)$ and corresponding survival function $\bar{F}$, for any $\alpha \in \mathbb{R}^+$,

$$G(x; \alpha) = \frac{F(x)}{1 - \bar{\alpha}\bar{F}(x)}, \quad x \in \mathbb{R}^+, \tag{1.1}$$

and

$$H(x; \alpha) = \frac{\alpha F(x)}{1 - \bar{\alpha}F(x)}, \quad x \in \mathbb{R}^+, \tag{1.2}$$

are two new defined distribution functions, where the parameter α is called a tilt parameter ($\bar{\alpha} = 1 - \alpha$) (see [27]). Note that (1.1) is equivalent to (1.2) if α in (1.1) is changed to $1/\alpha$. The proportional hazard rates (PHR) and the proportional reversed hazard rates (PRHR) models have important applications in reliability engineering and operations research ([28–33]). The random variables $X_1, \cdots, X_n$ are said to follow: (i) PHR model if X_i has the survival function $\bar{F}_{X_i}(x) = \bar{F}^{\lambda_i}(x)$, $i = 1, \cdots, n$, where $\bar{F}$ is the baseline survival function and $(\lambda_1, \cdots, \lambda_n)$ is the frailty vector; (ii) PRHR model if X_i has the distribution function $F_{X_i}(x) = F^{\beta_i}(x)$, $i = 1, \cdots, n$, where F is the baseline distribution and $(\beta_1, \cdots, \beta_n)$ is the resilience vector. It is well-known that the exponential, Weibull, Lomax, and Pareto distributions are special cases of the PHR model, and the Fréchet distribution is a special case of the PRHR model. N. Balakrishnan et al. [34] introduced the modified proportional hazard rates (MPHR) and modified proportional reversed hazard rates (MPRHR) models by adding a parameter to the PHR and PRHR models, respectively. For any $\alpha, \lambda, \beta \in \mathbb{R}^+$, their respective distributions are given by

$$G(x; \alpha, \lambda) = \frac{1 - \bar{F}^{\lambda}(x)}{1 - \bar{\alpha}\bar{F}^{\lambda}(x)}, \quad x \in \mathbb{R}^+,$$

and

$$H(x; \alpha, \beta) = \frac{\alpha F^{\beta}(x)}{1 - \bar{\alpha}F^{\beta}(x)}, \quad x \in \mathbb{R}^+,$$

where λ and β are the PHR and PRHR parameters, respectively. They established some stochastic comparison results between the corresponding order statistics with independent samples. G. Barmalzan et al. [35] discussed the hazard rate order and reversed hazard rate order of series and parallel systems with dependent components following either MPHR or MPRHR models under Archimedean copula. M. M. Zhang et al. [36] investigated stochastic comparisons on extreme order statistics from dependent and heterogeneous samples following MPHR and MPRHR models, and built the usual stochastic order for sample minimums and maximums, the hazard rate order on minimums of sample and the reversed hazard rate order on maximums of sample. S. Das et al. [37] introduced a scale parameter into the MPHR and MPRHR model that leads to new models, which are called modified proportional hazard rate scale (MPHRS) and modified proportional reversed hazard rate scale (MPRHRS) models. For any $\alpha, \lambda, \theta, \beta \in \mathbb{R}^+$,

$$G(x; \alpha, \lambda, \theta) = \frac{1 - \bar{F}^{\lambda}(\theta x)}{1 - \bar{\alpha}\bar{F}^{\lambda}(\theta x)}, \quad x \in \mathbb{R}^+,$$

and

$$H(x; \alpha, \beta, \theta) = \frac{\alpha F^{\beta}(\theta x)}{1 - \bar{\alpha}F^{\beta}(\theta x)}, \quad x \in \mathbb{R}^+,$$

are two newly defined distribution functions, respectively. They also obtained some stochastic comparison results on independent samples in terms of the usual stochastic, (reversed) hazard rate orders. The MPHRS (MPRHRS) model contains the MPHR (MPRHR), PHR (PRH), and scale models as special cases, and the flexibility that it possesses makes it quite suitable for modeling reliability and data analysis.

Motivated by the work of [19, 20], in this paper, we consider two samples from dependent and heterogeneous MPHRS and MPRHRS models. Some sufficient conditions are established to stochastically compare the second smallest (largest) order statistics of two samples with different parameters in the sense of the usual stochastic order.

The remainder of this paper is organized as follows: Section 2 recalls some concepts and notations used in this paper. Section 3 presents the main results and provides some sufficient conditions under which the two samples from the MPHRS model are stochastically comparable in the sense of the usual stochastic order. Also, similar results are obtained for the case of samples following the MPRHRS model in terms of the usual stochastic order. Section 4 presents the applications of the obtained results. Section 5 concludes the paper.

2. Preliminaries

In this section, let us first recall some important concepts and notations related to the main results of this article.

For random variables X and Y, let F and G be distribution functions (f and g be densities when absolutely continuous), and denote $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ as their reliability functions and h and r as their hazard rate and reversed hazard rate functions, respectively. Denote $\mathcal{I}_n = \{1, \ldots, n\}$ and $\mathbf{1} = (\underbrace{1, \ldots, 1}_{n})$.

Definition 1. *For two nonnegative random variables X and Y, X is said to be smaller than Y in the*
(i) *stochastic order (denoted by $X \leq_{\rm st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in \mathbb{R}_+$;*
(ii) *hazard rate order (denoted by $X \leq_{\rm hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in \mathbb{R}_+$;*
(iii) *reversed hazard rate order (denoted by $X \leq_{\rm rh} Y$) if $G(x)/F(x)$ is increasing in $x \in \mathbb{R}_+$.*

For more comprehensive discussions on stochastic orders, please refer to [38, 39].

Next, we introduce the notions of majorization and related orders, which are key tools in establishing various inequalities arising from many research areas. For two real vectors $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{y} = (y_1, ..., y_n) \in \mathbb{R}^n$, denote the increasing arrangement of the components of $\boldsymbol{x}$ and $\boldsymbol{y}$ by $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)}$, respectively.

Definition 2. *The vector $\boldsymbol{x}$ is said to be*
(i) *majorized by the vector $\mathbf{y}$ (write as $\boldsymbol{x} \overset{m}{\leq} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$, for all $j \in \mathcal{I}_{n-1}$, and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;*
(ii) *weakly supermajorized by the vector $\mathbf{y}$ (write as $\boldsymbol{x} \overset{w}{\leq} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$, for all $j \in \mathcal{I}_n$;*
(iii) *p-large than the vector $\mathbf{y}$ (write as $\boldsymbol{x} \overset{p}{\leq} \boldsymbol{y}$) if $\prod_{i=j}^{n} x_{(i)} \geq \prod_{i=1}^{j} y_{(i)}$, for all $j \in \mathcal{I}_n$.*

Definition 3. *A real function $\hbar$ defined on $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathcal{A}$ if*

$$\boldsymbol{x} \overset{m}{\leq} \boldsymbol{y} \text{ on } \mathcal{A} \Rightarrow \hbar(\boldsymbol{x}) \leq (\geq) \hbar(\boldsymbol{y}).$$

It is well-known that $\boldsymbol{x} \overset{m}{\geq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{w}{\geq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{p}{\geq} \boldsymbol{y}$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, while the converse is not always true. For more details on majorization and Schur-convexity (Schur-concavity), please refer to [40].

Now, let us review the concept of Archimedean copulas.

Definition 4. *For a decreasing and continuous function $\psi : [0, +\infty) \mapsto [0, 1]$, such that $\psi(0) = 1$ and $\psi(+\infty) = 0$, let $\phi = \psi^{-1}$ be the pseudo-inverse of ψ, then*

$$C_\psi(u_1, ..., u_n) = \psi(\phi(u_1) + ... + \phi(u_n)), \qquad u_i \in [0, 1], \quad i \in \mathcal{I}_n,$$

is said to be an Archimedean copula with generator ψ if $(-1)^k \psi^k(x) \geq 0$ for $k = 0, \ldots, n-2$ and $(-1)^{n-2} \psi^{n-2}(x)$ is decreasing and convex.

For detailed discussions on copulas and their applications, please refer to [41]. The following lemmas are useful to establish the main results.

Lemma 1. *[42] Let $I \subseteq \mathbb{R}$ be an open interval. A continuously differentiable $\hbar : I^n \to \mathbb{R}$ is Schur-convex (Schur-concave) if and only if $\hbar$ is symmetric on I^n, and for all $i \neq j$*

$$(x_i - x_j)\Big(\frac{\partial \hbar(\boldsymbol{x})}{\partial x_i} - \frac{\partial \hbar(\boldsymbol{x})}{\partial x_j}\Big) \geq (\leq) 0.$$

Lemma 2. *[40] For a real function $\hbar$ on $\mathcal{A} \subseteq \mathbb{R}^n$, $\boldsymbol{x} \overset{w}{\preceq} \boldsymbol{y}$ implies $\hbar(\boldsymbol{x}) \leq (\geq) \hbar(\boldsymbol{y})$ if and only if $\hbar$ is decreasing (increasing) and Schur-convex (Schur-concave) on $\mathcal{A}$.*

Throughout the manuscript, all concerned random variables are assumed to be absolutely continuous and nonnegative, and the terms increasing and decreasing stand for nondecreasing and nonincreasing, respectively.

3. Results

In this section, we present the stochastic comparison results of the second smallest (largest) order statistics from dependent and heterogeneous MPHRS (MPRHRS) samples in the sense of the usual stochastic order.

3.1. Second smallest order statistics

Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be the random vectors. For convenience, denote $\boldsymbol{X} \sim MPHRS(\boldsymbol{\alpha}; \boldsymbol{\theta}; \boldsymbol{\lambda}; \bar{F}_1; \psi)$ and $\boldsymbol{Y} \sim MPHRS(\boldsymbol{\alpha}; \boldsymbol{\theta}; \boldsymbol{\lambda}; \bar{F}_2; \psi)$. The samples follow the MPHRS model, where $\bar{F}_1$ and $\bar{F}_2$ are the baseline survival functions, ψ is generator of the associated Archimedean survival copula, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ are the tilt, scale, and modified proportional hazard rate vectors, respectively. Denote the hazard rate functions of the baseline survival functions $\bar{F}_1$ and $\bar{F}_2$ by h_1 and h_2, respectively.

The first result presents the result for comparing the samples with different tilt parameters in the sense of the usual stochastic order.

Theorem 1. *For $\boldsymbol{X} \sim MPHRS(\boldsymbol{\alpha}; \theta\mathbf{1}; \lambda\mathbf{1}; \bar{F}_1; \psi)$ and $\mathbf{Y} \sim MPHRS(\boldsymbol{\beta}; \theta\mathbf{1}; \lambda\mathbf{1}; \bar{F}_2; \psi)$, if $\bar{F}_1(x) \geq \bar{F}_2(x)$, then $\boldsymbol{\alpha} \overset{w}{\preceq} \boldsymbol{\beta}$ implies*

$$X_{2:n} \geq_{st} Y_{2:n}.$$

Proof. The survival function of $X_{2:n}$ can be written as

$$\begin{aligned}\bar{F}_{X_{2:n}}(x) &= \sum_{i=1}^{n} \psi\left(\sum_{j \neq i} \phi\left(\frac{\alpha_j \bar{F}_1^{\lambda}(\theta x)}{1 - \bar{\alpha}_j \bar{F}_1^{\lambda}(\theta x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha_i \bar{F}_1^{\lambda}(\theta x)}{1 - \bar{\alpha}_i \bar{F}_1^{\lambda}(\theta x)}\right)\right) \\ &= J_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, \bar{F}_1(x)).\end{aligned}$$

It is easy to obtain that

$$J_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, \bar{F}_1(x)) \geq J_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, \bar{F}_2(x)).$$

It is needed to show that $J_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, \bar{F}_2(x))$ is increasing in α_k, $k \in \mathcal{I}_n$ and Schur-concave in $\boldsymbol{\alpha}$. Differentiating $J_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, \bar{F}_2(x))$ with respect to α_k gives rise to

$$\frac{\partial J_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, \bar{F}_2(x))}{\partial \alpha_k}$$

$$= \Big[\sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha_j\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_j\bar{F}_2^\lambda(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^n\phi\left(\frac{\alpha_i\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_i\bar{F}_2^\lambda(\theta x)}\right)\right)\Big] \times\frac{\bar{F}_2^\lambda(\theta x)(1-\bar{F}_2^\lambda(\theta x))}{\psi'\left(\phi\left(\frac{\alpha_k\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_k\bar{F}_2^\lambda(\theta x)}\right)\right)(1-\bar{\alpha}_k\bar{F}_2^\lambda(\theta x))^2}. \quad (3.1)$$

Note that ψ is decreasing and convex. It holds that ψ' is increasing and nonpositive, as a result, (3.1) is nonnegative, which implies that $J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))$ is increasing in α_k. Furthermore, for $k\neq l$, we have

$$\begin{aligned}
&(\alpha_k-\alpha_l)\left(\frac{\partial J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))}{\partial\alpha_k}-\frac{\partial J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))}{\partial\alpha_l}\right)\\
&=\Big\{\Big[\sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha_j\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_j\bar{F}_2^\lambda(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^n\phi\left(\frac{\alpha_i\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_i\bar{F}_2^\lambda(\theta x)}\right)\right)\Big]\frac{1}{\Delta_2(\alpha_k,x)}\\
&-\Big[\sum_{i\neq l}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha_j\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_j\bar{F}_2^\lambda(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^n\phi\left(\frac{\alpha_i\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_i\bar{F}_2^\lambda(\theta x)}\right)\right)\Big]\frac{1}{\Delta_2(\alpha_l,x)}\Big\}\\
&\times\bar{F}_2^\lambda(\theta x)(1-\bar{F}_2^\lambda(\theta x))(\alpha_k-\alpha_l),
\end{aligned}$$

where

$$\Delta_2(\alpha,x)=(1-\bar{\alpha}\bar{F}_2^\lambda(\theta x))^2\psi'\left(\phi\left(\frac{\alpha\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}\bar{F}_2^\lambda(\theta x)}\right)\right).$$

By the decreasing and convex property of ψ, for $\alpha_k\geq(\leq)\alpha_l$, we have

$$\begin{aligned}
&\Big[\sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha_j\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_j\bar{F}_2^\lambda(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^n\phi\left(\frac{\alpha_i\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_i\bar{F}_2^\lambda(\theta x)}\right)\right)\Big]\\
&\geq(\leq)\Big[\sum_{i\neq l}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha_j\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_j\bar{F}_2^\lambda(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^n\phi\left(\frac{\alpha_i\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_i\bar{F}_2^\lambda(\theta x)}\right)\right)\Big]
\end{aligned}$$

and

$$\Big[\sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha_j\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_j\bar{F}_2^\lambda(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^n\phi\left(\frac{\alpha_i\bar{F}_2^\lambda(\theta x)}{1-\bar{\alpha}_i\bar{F}_2^\lambda(\theta x)}\right)\right)\Big]\leq 0.$$

It is easy to verify that $1/\Delta_2(\alpha,x)$ is negative and increasing in α for the given $x\geq 0$. Thus, for $k\neq l$,

$$(\alpha_k-\alpha_l)\left(\frac{\partial J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))}{\partial\alpha_k}-\frac{\partial J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))}{\partial\alpha_l}\right)\leq 0.$$

By Lemma 1, $J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))$ is Schur-concave, which means that $-J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))$ is Schur-convex. By Lemma 2, $\boldsymbol{\alpha}\overset{w}{\preceq}\boldsymbol{\beta}$ implies that $-J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_2(x))\leq -J_2(\boldsymbol{\beta},\theta,\lambda,\psi,\bar{F}_2(x))$. Hence,

$$J_2(\boldsymbol{\alpha},\theta,\lambda,\psi,\bar{F}_1(x))\geq J_2(\boldsymbol{\beta},\theta,\lambda,\psi,\bar{F}_2(x)).$$

That is, $X_{2:n}\geq_{st} Y_{2:n}$. The proof is completed. □

The next corollary follows immediately from Theorem 1.

Corollary 1. *For $\boldsymbol{X} \sim MPHRS(\alpha\mathbf{1};\theta\mathbf{1};\lambda\mathbf{1};\bar{F}_1;\psi)$ and $\boldsymbol{Y} \sim MPHRS(\boldsymbol{\alpha};\theta\mathbf{1};\ \lambda\mathbf{1};\bar{F}_2;\psi)$. If $\bar{F}_1(x) \geq \bar{F}_2(x)$, then*

$$\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i \Rightarrow X_{2:n} \geq_{st} Y_{2:n}.$$

The next theorem investigates the impact of the scale vector on the second smallest samples with respect to the usual stochastic order, whenever other parameters are equal.

Theorem 2. *For $\boldsymbol{X} \sim MPHRS(\alpha\mathbf{1};\boldsymbol{\theta};\lambda\mathbf{1};\bar{F}_1;\psi)$ and $\boldsymbol{Y} \sim MPHRS(\alpha\mathbf{1};\boldsymbol{\eta};\ \lambda\mathbf{1};\bar{F}_2;\psi)$, where $0 < \alpha \leq 1$. If ψ is log-concave, $\bar{F}_1(x) \leq \bar{F}_2(x)$, and $h_2(x)$ is decreasing in x, then $\boldsymbol{\theta} \overset{w}{\preceq} \boldsymbol{\eta}$ implies*

$$X_{2:n} \leq_{st} Y_{2:n}.$$

Proof. The survival function of $X_{2:n}$ can be written as

$$\begin{aligned}\bar{F}_{X_{2:n}}(x) &= \sum_{i=1}^{n}\psi\left(\sum_{j\neq i}\phi\left(\frac{\alpha\bar{F}_1^{\lambda}(\theta_j x)}{1-\bar{\alpha}\bar{F}_1^{\lambda}(\theta_j x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha\bar{F}_1^{\lambda}(\theta_i x)}{1-\bar{\alpha}\bar{F}_1^{\lambda}(\theta_i x)}\right)\right)\\ &= J_3(\alpha,\boldsymbol{\theta},\lambda,\psi,\bar{F}_1(x)).\end{aligned}$$

It is easy to derive that

$$J_3(\alpha,\boldsymbol{\theta},\lambda,\psi,\bar{F}_1(x)) \leq J_3(\alpha,\boldsymbol{\theta},\lambda,\psi,\bar{F}_2(x)).$$

It is needed to show that $J_3(\alpha,\boldsymbol{\theta},\lambda,\psi,\bar{F}_2(x))$ is decreasing in θ_k, $k \in \mathcal{I}_n$ and Schur-convex in $\boldsymbol{\theta}$. Differentiating $J_3(\alpha,\boldsymbol{\theta},\lambda,\psi,\bar{F}_2(x))$ with respect to θ_k gives rise to

$$\begin{aligned}&\frac{\partial J_3(\alpha,\boldsymbol{\theta},\lambda,\psi,\bar{F}_2(x))}{\partial\theta_k}\\ =\ &\left[(n-1)\psi'\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_i x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_i x)}\right)\right) - \sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_j x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_j x)}\right)\right)\right]\\ &\times\frac{\psi\left(\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_k x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x)}\right)\right)\lambda\alpha x h_2(\theta_k x)}{\psi'\left(\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_k x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x)}\right)\right)(1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x))}\\ =\ &\lambda\alpha x\Delta_3(\theta_k,x)\Delta_4(\theta_k,x),\end{aligned} \tag{3.2}$$

where

$$\Delta_3(\theta_k,x) = (n-1)\psi'\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_i x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_i x)}\right)\right) - \sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_j x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_j x)}\right)\right),$$

and

$$\Delta_4(\theta_k,x) = \frac{\psi\left(\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_k x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x)}\right)\right)h_2(\theta_k x)}{\psi'\left(\phi\left(\frac{\alpha\bar{F}_2^{\lambda}(\theta_k x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x)}\right)\right)(1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x))}.$$

Since $\psi'(x) \le 0$ and $\psi'(x)$ is increasing in x, $\phi\left(\frac{\alpha \bar{F}_2^{\lambda}(\theta_k x)}{1-\bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x)}\right)$ is increasing in θ_k, then $\Delta_3(\theta_k, x)$ is decreasing in θ_k and nonnegative by noting that $\phi(x) \in [0, 1]$. Note that ψ is log-concave, and $h_2(\theta_k x)$ and $1/(1 - \bar{\alpha}\bar{F}_2^{\lambda}(\theta_k x))$ are decreasing in θ_k for the given $x \ge 0$ and $0 < \alpha \le 1$. We have that $\Delta_4(\theta_k, x)$ is increasing in θ_k and negative. Therefore, (3.2) is nonpositive, that is, $J_3(\alpha, \boldsymbol{\theta}, \lambda, \psi, \bar{F}_2(x))$ is decreasing in θ_k, and $\Delta_3(\theta_k, x)\Delta_4(\theta_k, x)$ is increasing in θ_k, $k \in \mathcal{I}_n$. Thus, for $k \ne l$,

$$\begin{aligned}&(\theta_k - \theta_l)\Big(\frac{\partial J_3(\alpha, \boldsymbol{\theta}, \lambda, \psi, \bar{F}_2(x))}{\partial \theta_k} - \frac{\partial J_3(\alpha, \boldsymbol{\theta}, \lambda, \psi, \bar{F}_2(x))}{\partial \theta_l}\Big)\\ =\ & \lambda \alpha x(\theta_k - \theta_l)\Big(\Delta_3(\theta_k, x)\Delta_4(\theta_k, x) - \Delta_3(\theta_l, x)\Delta_4(\theta_l, x)\Big) \ge 0.\end{aligned}$$

The desired result follows by Lemma 2. Hence, we complete the proof. □

Remark 1. *Theorem 4.3 of [20] developed a similar result for the scale model, which can be treated as a special case of the above result when $\alpha = \lambda = 1$ and $\bar{F}_1 = \bar{F}_2$. As a further study, it is of interest to obtain the results for the case $\alpha > 1$.*

As a consequence of Theorem 2, we obtain the following corollary immediately.

Corollary 2. *For $\boldsymbol{X} \sim MPHRS(\alpha\mathbf{1}; \theta\mathbf{1}; \lambda\mathbf{1}; \bar{F}_1; \psi)$ and $\boldsymbol{Y} \sim MPHRS(\alpha\mathbf{1}; \boldsymbol{\theta}; \lambda\mathbf{1}; \bar{F}_2; \psi)$, where $0 < \alpha \le 1$, if ψ is log-concave, $\bar{F}_1(x) \le \bar{F}_2(x)$, and $h_2(x)$ is decreasing in x, then*

$$\theta \ge \frac{1}{n}\sum_{i=1}^{n} \theta_i \Rightarrow X_{2:n} \le_{st} Y_{2:n}.$$

Now, we present an example to illustrate the result of Theorem 2.

Example 1. *Let $\bar{F}_1(x) = e^{-3x}$, $\bar{F}_2(x) = e^{-2x}$, and generator $\psi(x) = e^{\frac{1-e^x}{a}}$, $0 < a \le 1$. Take $n = 4, \lambda = 0.2, \alpha = 0.2, a = 0.1$, and $\boldsymbol{\theta} = (0.6, 0.7, 0.8, 0.9) \overset{w}{\le} (0.2, 0.3, 0.5, 0.7) = \boldsymbol{\eta}$. It is easy to check that the conditions of Theorem 2 are all satisfied. To display the whole of survival curves of $X_{2:4}$ and $Y_{2:4}$ on $[0, \infty)$, we perform the transformation $(x + 1)^{-1} : [0, \infty) \longmapsto [0, 1]$. The distribution functions of $(X_{2:4} + 1)^{-1}$ and $(Y_{2:4} + 1)^{-1}$ are plotted in Figure 1, which confirms $X_{2:4} \le_{st} Y_{2:4}$.*

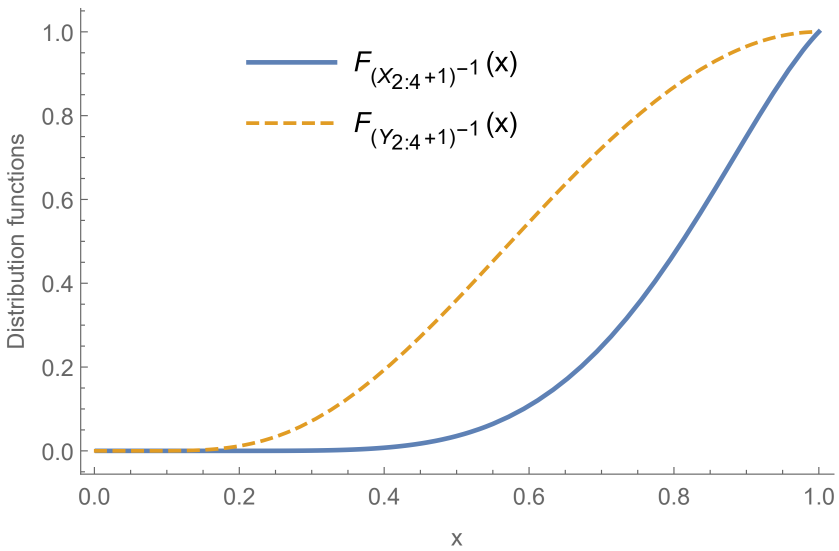

Figure 1. Plots of distribution functions $F_{(X_{2:4}+1)^{-1}}(x)$ and $F_{(Y_{2:4}+1)^{-1}}(x)$.

3.2. Second largest order statistics

In this subsection, we carry out the stochastic comparison of the second largest samples following the MPRHRS model. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be the random vectors. We denote $\boldsymbol{X} \sim MPRHRS(\boldsymbol{\alpha}; \boldsymbol{\theta}; \boldsymbol{\lambda}; F_1; \psi)$ and $\boldsymbol{Y} \sim MPRHRS(\boldsymbol{\alpha}; \boldsymbol{\theta}; \boldsymbol{\lambda}; F_2; \psi)$. The samples follow the MPRHRS model, where F_1 and F_2 are the baseline distribution functions, ψ is generator of the associated Archimedean copula, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ are the tilt, scale, and modified proportional reversed hazard rate vectors, respectively.

First, we establish sufficient conditions for the usual stochastic order when the samples have different modified proportional reversed hazard rate vectors.

Theorem 3. *For $\boldsymbol{X} \sim MPRHRS(\alpha\mathbf{1}; \theta\mathbf{1}; \boldsymbol{\lambda}; F_1; \psi)$ and $\boldsymbol{Y} \sim MPRHRS(\alpha\mathbf{1}; \theta\mathbf{1}; \boldsymbol{\mu}; F_2; \psi)$, where $0 < \alpha \leq 1$. If ψ is log-concave, $F_1(x) \leq F_2(x)$, then $\boldsymbol{\lambda} \overset{w}{\preceq} \boldsymbol{\mu}$ implies*

$$X_{n-1:n} \geq_{st} Y_{n-1:n}.$$

Proof. The distribution function of $X_{n-1:n}$ and $Y_{n-1:n}$ can be expressed as

$$\begin{aligned} F_{X_{n-1:n}}(x) &= \sum_{i=1}^{n} \psi\left(\sum_{j\neq i} \phi\left(\frac{\alpha F_1^{\lambda_j}(\theta x)}{1-\bar{\alpha}F_1^{\lambda_j}(\theta x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha F_1^{\lambda_i}(\theta x)}{1-\bar{\alpha}F_1^{\lambda_i}(\theta x)}\right)\right) \\ &= Q_1(\alpha, \theta, \boldsymbol{\lambda}, \psi, F_1(x)), \end{aligned}$$

and

$$\begin{aligned} F_{Y_{n-1:n}}(x) &= \sum_{i=1}^{n} \psi\left(\sum_{j\neq i} \phi\left(\frac{\mu F_2^{\mu_j}(\theta x)}{1-\bar{\alpha}F_2^{\mu_j}(\theta x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n} \phi\left(\frac{\mu F_2^{\mu_i}(\theta x)}{1-\bar{\alpha}F_2^{\mu_i}(\theta x)}\right)\right) \\ &= Q_1(\alpha, \theta, \boldsymbol{\mu}, \psi, F_2(x)). \end{aligned}$$

We need to show that

$$Q_1(\alpha, \theta, \boldsymbol{\lambda}, \psi, F_1(x)) \leq Q_1(\alpha, \theta, \boldsymbol{\mu}, \psi, F_2(x)).$$

It is easy to obtain that

$$Q_1(\alpha, \theta, \boldsymbol{\lambda}, \psi, F_1(x)) \leq Q_1(\alpha, \theta, \boldsymbol{\lambda}, \psi, F_2(x)).$$

It suffices to show that $Q_1(\alpha, \theta, \boldsymbol{\lambda}, \psi, F_2(x))$ is decreasing in λ_k, $k \in \mathcal{I}_n$ and Schur-convex in $\boldsymbol{\lambda}$. Note that ψ is decreasing and convex. It holds that ψ' is increasing and nonpositive. As a result, differentiating $Q_1(\alpha, \theta, \boldsymbol{\lambda}, \psi, F_2(x))$ with respect to λ_k, we have

$$\begin{aligned} &\frac{\partial Q_1(\alpha, \theta, \boldsymbol{\lambda}, \psi, F_2(x))}{\partial \lambda_k} \\ = &\Big[\sum_{i\neq k} \psi'\left(\sum_{j\neq i} \phi\left(\frac{\alpha F_2^{\lambda_j}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_j}(\theta x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha F_2^{\lambda_i}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_i}(\theta x)}\right)\right)\Big] \\ &\times \frac{\ln F_2(\theta x) \frac{\alpha F_2^{\lambda_k}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_k}(\theta x)}}{\psi'\left(\phi\left(\frac{\alpha F_2^{\lambda_k}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_k}(\theta x)}\right)\right)(1-\bar{\alpha}F_2^{\lambda_k}(\theta x))} \leq 0, \end{aligned}$$

that is, $Q_1(\alpha,\theta,\boldsymbol{\lambda},\psi,F_2(x))$ is decreasing in λ_k. Furthermore, for $k \neq l$, we have

$$\begin{aligned}
&(\lambda_k-\lambda_l)\left(\frac{\partial Q_1(\alpha,\theta,\boldsymbol{\lambda},\psi,F_2(x))}{\partial\lambda_k}-\frac{\partial Q_1(\alpha,\theta,\boldsymbol{\lambda},\psi,F_2(x))}{\partial\lambda_l}\right)\\
=\;&\left\{\left[\sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha F_2^{\lambda_j}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_j}(\theta x)}\right)\right)-(n-1)\psi\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha F_2^{\lambda_i}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_i}(\theta x)}\right)\right)\right]\Lambda_1(\lambda_k,x)\right.\\
&\left.-\left[\sum_{i\neq l}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha F_2^{\lambda_j}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_j}(\theta x)}\right)\right)-(n-1)\psi\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha F_2^{\lambda_i}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_i}(\theta x)}\right)\right)\right]\Lambda_1(\lambda_l,x)\right\}\\
&\times\ln F_2(\theta x)(\lambda_k-\lambda_l),
\end{aligned}$$

where

$$\Lambda_1(\lambda,x)=\frac{\frac{\alpha F_2^{\lambda}(\theta x)}{1-\bar{\alpha}F_2^{\lambda}(\theta x)}}{\psi'\left(\phi\left(\frac{\alpha F_2^{\lambda}(\theta x)}{1-\bar{\alpha}F_2^{\lambda}(\theta x)}\right)\right)(1-\bar{\alpha}F_2^{\lambda}(\theta x))}.$$

By the decreasing and convex property of ψ, for $\lambda_k \geq (\leq)\lambda_l$, it holds that

$$\begin{aligned}
&\left[\sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha F_2^{\lambda_j}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_j}(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha F_2^{\lambda_i}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_i}(\theta x)}\right)\right)\right]\ln F_2(\theta x)\\
&\leq (\geq)\\
&\left[\sum_{i\neq l}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha F_2^{\lambda_j}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_j}(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha F_2^{\lambda_i}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_i}(\theta x)}\right)\right)\right]\ln F_2(\theta x)
\end{aligned}$$

and

$$\left[\sum_{i\neq k}\psi'\left(\sum_{j\neq i}\phi\left(\frac{\alpha F_2^{\lambda_j}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_j}(\theta x)}\right)\right)-(n-1)\psi'\left(\sum_{i=1}^{n}\phi\left(\frac{\alpha F_2^{\lambda_i}(\theta x)}{1-\bar{\alpha}F_2^{\lambda_i}(\theta x)}\right)\right)\right]\ln F_2(\theta x)\geq 0.$$

It is easy to check that $\Lambda_1(\lambda,x)$ is nonpositive and increasing in λ, then, for $k \neq l$,

$$(\lambda_k-\lambda_l)\left(\frac{\partial Q_1(\alpha,\theta,\boldsymbol{\lambda},\psi,F_2(x))}{\partial\lambda_k}-\frac{\partial Q_1(\alpha,\theta,\boldsymbol{\lambda},\psi,F_2(x))}{\partial\lambda_l}\right)\geq 0.$$

Therefore, $Q_1(\alpha,\theta,\boldsymbol{\lambda},\psi,F_2(x))$ is Schur-convex by Lemma 1. According to Lemma 2, $\boldsymbol{\lambda} \overset{w}{\preceq} \boldsymbol{\mu}$ implies $Q_1(\alpha,\theta,\boldsymbol{\lambda},\psi,F_2(x)) \leq Q_1(\alpha,\theta,\boldsymbol{\mu},\psi,F_2(x))$, then, we have $X_{n-1:n} \geq_{st} Y_{n-1:n}$, which completes the proof. □

Remark 2. *Theorem 6.2 of [19] obtained a similar result for the PRH model, which can be regarded as a special case of the above result when $\alpha=\theta=1$ and $F_1=F_2$.*

The following corollary can be derived immediately from Theorem 3.

Corollary 3. *For $\boldsymbol{X} \sim MPRHRS(\alpha\mathbf{1};\theta\mathbf{1};\lambda\mathbf{1};F_1;\psi)$ and $\boldsymbol{Y} \sim MPRHRS\ (\alpha\mathbf{1};\theta\mathbf{1};\boldsymbol{\lambda};F_2;\psi)$, where $0<\alpha\leq 1$, if ψ is log-concave, $F_1(x)\leq F_2(x)$, then*

$$\lambda \geq \frac{1}{n}\sum_{i=1}^{n}\lambda_i \Rightarrow X_{n-1:n} \geq_{st} Y_{n-1:n}.$$

In what follows, we consider the situation in which the two samples possess different tilt parameters.

Theorem 4. *For $\boldsymbol{X} \sim MPRHRS(\boldsymbol{\alpha}; \theta\mathbf{1}; \lambda\mathbf{1}; F_1; \psi)$ and $\boldsymbol{Y} \sim MPRHRS\ (\boldsymbol{\beta}; \theta\mathbf{1}; \lambda\mathbf{1}; F_2; \psi)$. If $F_1(x) \geq F_2(x)$, then $\boldsymbol{\alpha} \overset{w}{\preceq} \boldsymbol{\beta}$ implies*

$$X_{n-1:n} \leq_{st} Y_{n-1:n}.$$

Proof. The distribution function of $X_{n-1:n}$ can be written as

$$\begin{aligned} F_{X_{n-1:n}}(x) &= \sum_{i=1}^{n} \psi\left(\sum_{j\neq i} \phi\left(\frac{\alpha_j F_1^{\lambda}(\theta x)}{1-\bar{\alpha}_j F_1^{\lambda}(\theta x)}\right)\right) - (n-1)\psi\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha_i F_1^{\lambda}(\theta x)}{1-\bar{\alpha}_i F_1^{\lambda}(\theta x)}\right)\right) \\ &= Q_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, F_1(x)). \end{aligned}$$

It is easy to obtain that

$$Q_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, F_1(x)) \geq Q_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, F_2(x)).$$

It is needed to show that $Q_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, F_2(x))$ with respect to a_k is increasing and Schur-concave in $\boldsymbol{\alpha}$, $k \in \mathcal{I}_n$. The derivative of $Q_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, F_2(x))$ with respect to α_k is

$$\begin{aligned} &\frac{\partial Q_2(\boldsymbol{\alpha}, \theta, \lambda, \psi, F_2(x))}{\partial \alpha_k} \\ = &\Big[\sum_{i\neq k} \psi'\left(\sum_{j\neq i} \phi\left(\frac{\alpha_j F_2^{\lambda}(\theta x)}{1-\bar{\alpha}_j F_2^{\lambda}(\theta x)}\right)\right) - (n-1)\psi'\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha_i F_2^{\lambda}(\theta x)}{1-\bar{\alpha}_i F_2^{\lambda}(\theta x)}\right)\right)\Big] \\ &\times \frac{F_2^{\lambda}(\theta x)(1-F_2^{\lambda}(\theta x))}{\psi'\left(\phi\left(\frac{\alpha_k F_2^{\lambda}(\theta x)}{1-\bar{\alpha}_k F_2^{\lambda}(\theta x)}\right)\right)}(1-\bar{\alpha}_k F_2^{\lambda}(\theta x))^2. \end{aligned}$$

The rest of this part can be proven in a similar manner with Theorem 1 and is thus omitted. □

The next corollary follows immediately from Theorem 4.

Corollary 4. *For $\boldsymbol{X} \sim MPRHRS(\alpha\mathbf{1}; \theta\mathbf{1}; \lambda\mathbf{1}; F_1; \psi)$ and $\boldsymbol{Y} \sim MPRHRS\ (\boldsymbol{\alpha}; \theta\mathbf{1}; \lambda\mathbf{1}; F_2; \psi)$. If $F_1(x) \geq F_2(x)$, then*

$$\alpha \geq \frac{1}{n}\sum_{i=1}^{n} \alpha_i \Rightarrow X_{n-1:n} \leq_{st} Y_{n-1:n}.$$

The following example demonstrates the result of Theorem 4.

Example 2. *Let $F_1(x) = 1 - e^{-3x}$ and $F_2(x) = 1 - e^{-2x}$, and generator $\psi(x) = (ax+1)^{-1/a}$, $a > 0$. Take $n = 4, \lambda = 0.5, \theta = 0.1, a = 0.6$, and $\boldsymbol{\alpha} = (0.6, 0.7, 0.8, 0.8) \overset{w}{\preceq} (0.3, 0.4, 0.5, 0.6) = \boldsymbol{\beta}$. It is easy to check that the conditions of Theorem 4 are all satisfied. Thus, $X_{3:4} \leq_{st} Y_{3:4}$, as shown in Figure 2.*

One natural question is whether the condition of weakly supermajorization order can be replaced by p-large order in Theorem 4. The following example gives a negative answer.

Example 3. *Let $F_1(x) = F_2(x) = 1 - e^{-x}$, and generator $\psi(x) = (ax+1)^{-1/a}$, $a > 0$. Take $n = 3$, $\boldsymbol{\alpha} = (3, 4, 5)$, and $\boldsymbol{\beta} = (2, 4.8, 6)$, and it holds that $\boldsymbol{\alpha} \overset{p}{\preceq} \boldsymbol{\beta}$ while $\boldsymbol{\alpha} \overset{w}{\npreceq} \boldsymbol{\beta}$. The curve of $\bar{F}_{(X_{2:3}+1)^{-1}}(x) - \bar{F}_{(Y_{2:3}+1)^{-1}}(x)$ is displayed in Figure 3, from which we confirm that $X_{2:3} \nleq_{st} Y_{2:3}$ and $X_{2:3} \ngeq_{st} Y_{2:3}$.*

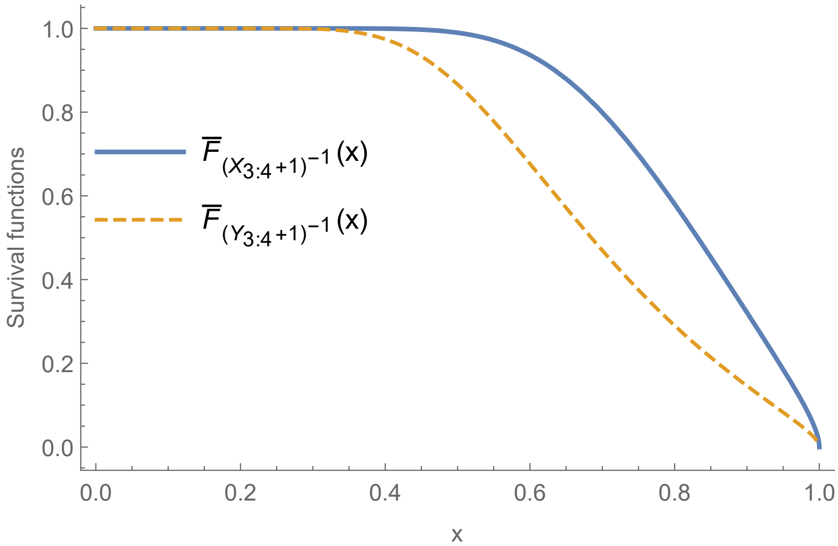

Figure 2. Plots of survival functions $\bar{F}_{(X_{3:4}+1)^{-1}}(x)$ and $\bar{F}_{(Y_{3:4}+1)^{-1}}(x)$.

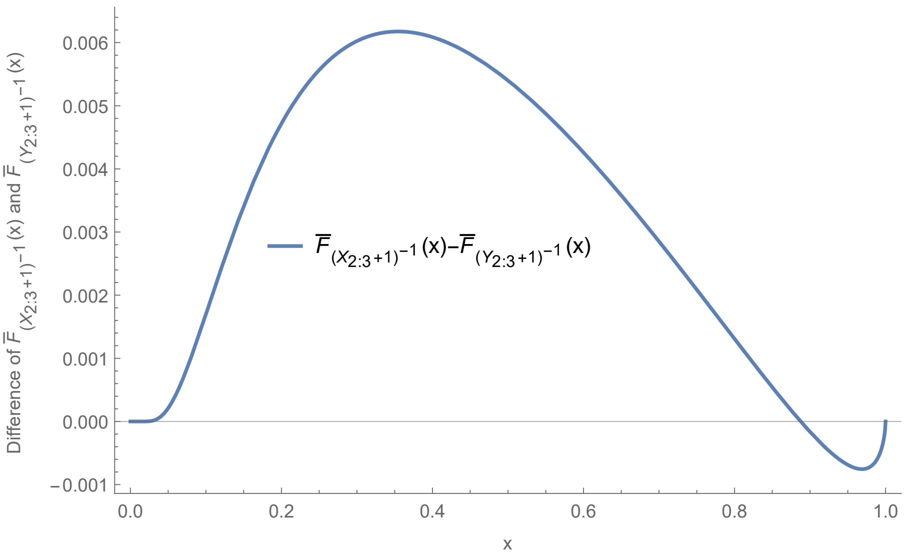

Figure 3. Plots of $\bar{F}_{(X_{2:3}+1)^{-1}}(x) - \bar{F}_{(Y_{2:3}+1)^{-1}}(x)$.

4. Some applications

4.1. Reliability theory

In reliability theory, the k-out-of-n system as the popular fault tolerant system has been widely applied in industrial engineering and system security. Specifically, $X_{1:n}$ and $X_{n:n}$ denote the lifetimes of series and parallel systems, respectively, and the $(n-1)$-out-of-n system is referred to as the fail-safe system.

Consider a fail-safe system with dependent and heterogeneous components lifetime following the MPHRS model. Theorem 2 states that the larger the baseline survival function for the negatively lower orthant dependent components, the larger heterogeneity among the scale parameters, which leads to the better performance of the fail-safe system, respectively. Theorem 1 states that the larger the baseline survival function and the more symmetric the tilt parameter vector leads to a more reliable fail-safe system.

4.2. Auction theory

The second-price sealed-bid auction is of important theoretical and practical interest in auction theory. Several bidders compete to buy a good, and bidders hand in their bids to the auctioneers simultaneously without the knowledge of their rivals' bids. The bidder with the highest bid wins the object and pays the second highest bid in the English auction. While in a second-price reversed auction, the lowest bidder wins and is paid at a price corresponding to the second lowest bid. The winner will pay the rent defined as the difference between his or her bid and the final price for the auctioneer.

In a second-price reversed auction, there are several bidders with dependent and different bids following the MPHRS model with negatively lower orthant dependence. The larger the baseline survival function, the more symmetric scale and modified proportional hazard rate vectors lead to stochastically larger the revenue of the auctioneer, and the more symmetric tilt parameter vector incurs to stochastically lower the revenue of the auctioneer. In the second price auction for bids following the MPRHRS model, Theorem 3 (Theorem 4) states that the more symmetric modified proportional hazard rate vector (tilt parameter vector) with a lower (higher) baseline distribution function will lead to stochastically lower (higher) the revenue of the auctioneer.

5. Conclusions

In this paper, we study the problem of stochastically comparing the second smallest (largest) order statistics from dependent and heterogeneous samples. For the second smallest order statistics from MPHRS samples, sufficient conditions are obtained for the usual stochastic order whenever the sample has different parameters. In addition, similar results are established for the second largest order statistics from MPHRS samples. Lastly, some applications of the obtained results in reliability theory and auction theory are provided.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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