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Research article

Regularity and normality on primal spaces

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Abstract: It is commonly known that some topological spaces include structures that may be used to expand abstract notions. Primal structure is one such sort of structure. We provided the primal Hausdorff class of spaces, which included the class of all Hausdorff spaces. Furthermore, we provide the concepts of primal regular spaces and primal normal spaces. We present new theorems and results.

Keywords: primal; regular space; normal space; $\mathcal{P}$-regular; $\mathcal{P}$-normal; grill; primal space
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1. Introduction

Among the essential fields of mathematics is topology. Numerous related structures, including ideal [1], filter [2], grill [3], etc., have been introduced as a result of its numerous generalized applications in both science and social science. The duality of a filter is the concept of an ideal.

An ideal space is a topic that has been researched by Kuratowski [2] and Vaidyanathaswamy [4]. An ideal $\mathcal{J}$ on a topological space (Z, ρ) is a collection of nonempty subsets of Z that fulfills: (i) $H \in \mathcal{J}$ and $K \subseteq H$ implies $K \in \mathcal{J}$ and (ii) $H \in \mathcal{J}$ and $K \in \mathcal{J}$ implies $H \cup K \in \mathcal{J}$, see [1,5].

Grill is one of the traditional structures in topology that is comparable to ideal. In 1947, Chóquet proposed the term of grill (see [3]).

A family $\mathcal{G}$ of 2^Z is a grill on Z if $\mathcal{G}$ fulfills the requirements listed here:

(1) $\emptyset \notin \mathcal{G}$,
(2) if $N \in \mathcal{G}$ and $N \subseteq M$, then $M \in \mathcal{G}$,
(3) if $N \cup M \in \mathcal{G}$, then $N \in \mathcal{G}$ or $M \in \mathcal{G}$.

Further information can be found in [6–9].

Primal in topological spaces have been considered in 2022 by Acharjee and etc. as the dual concept of a grill (see [10]). This topic has won its importance aspects of interest. The separation axioms in primal topological space have not considered by the authors. These structures are widely used to enlarge abstract concepts, and they also provide an essential tool for addressing certain practical problems, particularly those pertaining to enhancing accuracy measurements and rough approximation operators.

By [10], we know that a primal $\mathcal{P}$ on a topological space (Z,ρ) is a collection $\mathcal{P} \subseteq 2^Z$, such that the below conditions hold:

(1) $Z \notin \mathcal{P}$,
(2) if $M \cap N \in \mathcal{P}$, then $N \in \mathcal{P}$ or $M \in \mathcal{P}$,
(3) if $N \in \mathcal{P}$ and $M \subseteq N$, then $M \in \mathcal{P}$.

A primal space (Z,ρ) is a topological space (Z,ρ) with primal $\mathcal{P}$. The notation $(Z,\rho,\mathcal{P})$ is named a primal topological space or a primal space.

Throughout this paper, (Z,ρ) and (X,σ) (briefly, Z and X) indicate topological spaces unless specified otherwise. For any $A \subseteq Z$, $Cl(A) = \overline{A}$, $Int(A)$ and A^c denote the closure, interior and complement of A, respectively. The family of all open neighborhoods of a point x of Z is symbolized by $\rho(x)$. Also, the set of all closed subsets inside a given space (Z,ρ) will be denoted by ρ^c.

2. Preliminaries

We now obtain the following ideas and results, which the next part will require:

Definition 2.1. *[10] Let $(Z,\rho,\mathcal{P})$ be a primal space. We consider a map $(\cdot)^\diamond : 2^Z \to 2^Z$ as $A^\diamond(Z,\rho,\mathcal{P}) = \{x \in Z : (\forall\ U \in \rho(x))(A^c \cup U^c \in \mathcal{P})\}$ for any subset A of Z. We can also write $A^\diamond_{\mathcal{P}} = A^\diamond$ as $A^\diamond(Z,\rho,\mathcal{P})$ to define the primal in accordance with our specifications.*

Corollary 2.1. *[10] A family $\mathcal{P} \subseteq 2^Z$ is a primal on Z if and only if the conditions below hold:*

(1) $Z \notin \mathcal{P}$,
(2) if $N \notin \mathcal{P}$ and $M \notin \mathcal{P}$, then $M \cap N \notin \mathcal{P}$,
(3) if $N \notin \mathcal{P}$ and $N \subseteq M$, then $M \notin \mathcal{P}$.

Definition 2.2. *[10] Let $(Z,\rho,\mathcal{P})$ be a primal space. Define a map $Cl^\diamond : 2^Z \to 2^Z$ as $Cl^\diamond(A) = A \cup A^\diamond_{\mathcal{P}}$, where A is any subset of Z.*

Definition 2.3. *[10] Let $(Z,\rho,\mathcal{P})$ be a primal space. Then, the collection $\rho^\diamond = \{A \subseteq Z \mid Cl^\diamond(A^c) = A^c\}$ is a topology on Z called primal topology on Z.*

Lemma 2.1. *[10] Let $(Z,\rho,\mathcal{P})$ be a primal space. If $A^c \notin \mathcal{P}$, then $A^\diamond = \emptyset$.*

Corollary 2.2. *[11] Let $(Z,\rho,\mathcal{P})$ be a primal space and let $A, B \subseteq Z$ such that $A \in \rho$. Then, $A \cap B^\diamond = A \cap (A \cap B)^\diamond \subseteq (A \cap B)^\diamond$.*

Lemma 2.2. *[11] Let $(Z,\rho,\mathcal{P})$ be a primal space. Then, the family $\mathcal{B}_P = \{A \cap P : A \in \rho \text{ and } P \notin \mathcal{P}\}$ is a base for the primal topology $\rho^\diamond$ on Z.*

Lemma 2.3. *[11] Let $(Z,\rho,\mathcal{P})$ be a primal space. Then, $Cl(A^\diamond) = A^\diamond$ for all $A \subseteq Z$. If $\rho^c \setminus \{Z\} \subseteq \mathcal{P}$, we get that $U \subseteq U^\diamond$ and $Cl(U) = U^\diamond$ for all $U \in \rho$.*

Lemma 2.4. *[11] Let $(Z,\rho,\mathcal{P})$ be a primal space and let $H,K \subseteq Z$ such that $K^c \notin \mathcal{P}$. Then,*

$$(H \cup K)^\diamond = H^\diamond = (H \setminus K)^\diamond.$$

More about of primal spaces and more details of $\rho^\diamond$ can be found in [9–11, 15].

3. $\mathcal{P}$-Hausdorff spaces

Definition 3.1. *A primal space $(Z,\rho,\mathcal{P})$ is said to be $\mathcal{P}$-Hausdorff if for every distinct points $a,b \in Z$, there exist $H,K \in \rho$ such that $a \in H$, $b \in K$ and $[H \cap K]^c \notin \mathcal{P}$.*

Since $Z \notin \mathcal{P}$, then every Hausdorff space is $\mathcal{P}$-Hausdorff. However, the following examples show that the converse is not true.

Example 3.1. *Consider the set of real numbers $\mathbb{R}$. Fix $p \in \mathbb{R}$ and define ε_p as follows:*

$$U \in \varepsilon_p \Longleftrightarrow U = \mathbb{R} \ \ or \ \ p \notin U.$$

Define $\mathcal{P}$ such that $M \in \mathcal{P}$ if and only if $p \notin M$. Then, $(\mathbb{R},\varepsilon_p,\mathcal{P})$ is $\mathcal{P}$-Hausdorff but not Hausdorff. To show that, let $x,y \in \mathbb{R}$ such that $x \neq y$.
Case 1. Either $x = p$ or $y = p$. Let $x = p$. Then, choose $H = \mathbb{R}$ and $K = \{y\}$. Then, $[H \cap K]^c = [\mathbb{R} \cap \{y\}]^c = [\{y\}]^c = \mathbb{R} \setminus \{y\}$. Since $p \in \mathbb{R} \setminus \{y\}$, then $\mathbb{R} \setminus \{y\} \notin \mathcal{P}$.
Case 2. Neither $x = p$ nor $y = p$. Then, choose $H = \{x\}$ and $K = \{y\}$. Hence, $[H \cap K]^c = [\{x\} \cap \{y\}]^c = [\emptyset]^c = \mathbb{R}$. Since $\mathbb{R} \notin \mathcal{P}$, then $(\mathbb{R},\varepsilon_p,\mathcal{P})$ is $\mathcal{P}$-Hausdorff but not Hausdorff since the only open set contains p in ε_p is $\mathbb{R}$.

Example 3.2. *Let $Z = \{a,b,c\}$, $\rho = \{Z,\emptyset,\{a\},\{b\},\{a,b\},\{b,c\}\}$ and $\mathcal{P} = \{\emptyset,\{b\},\{c\},\{b,c\}\}$. $(Z,\rho,\mathcal{P})$ is not Hausdorff since $b \neq c$ and no any pair of disjoint open sets containing b and c respectively. On the other hand, $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-Hausdorff as shown in the Table 1.*

Table 1. Details on $\mathcal{P}$-Hausdorff.

$a \neq b$	$H = \{a\}$ and $K = \{b\}$	$[H \cap K]^c = Z$	$Z \notin \mathcal{P}$
$a \neq c$	$H = \{a\}$ and $K = \{b,c\}$	$[H \cap K]^c = Z$	$Z \notin \mathcal{P}$
$b \neq c$	$H = \{b\}$ and $K = \{b,c\}$	$[H \cap K]^c = \{a,c\}$	$\{a,c\} \notin \mathcal{P}$

Corollary 3.1. *Let $(Z,\rho,\mathcal{P})$ be a primal space such that $\mathcal{P} = 2^Z \setminus \{Z\}$. Then, $(Z,\rho,\mathcal{P})$ is Hausdorff if and only if $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-Hausdorff.*

Lemma 3.1. *Let $(Z,\rho,\mathcal{P})$ be $\mathcal{P}$-Hausdorff and $\rho^c \setminus \{Z\} \subseteq \mathcal{P}$. Then, $(Z,\rho,\mathcal{P})$ is Hausdorff.*

Proposition 3.1. *Let $(Z,\rho,\mathcal{P})$ be a primal space. Then, the following statements are equivalent.*

(1) $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-Hausdorff.
(2) If $a \in Z$, then for each $a \neq b$, there is $A \in \rho$ such that $a \in A$ and $b \notin A^\diamond$.
(3) For any $a \in Z$, the family $\bigcap\{A^\diamond : A \in \rho(a)\}$ is either $\emptyset$ or $\{a\}$.

Proof. (1) $\Longrightarrow$ (2): Let $a, b \in Z \ni a \neq b$. Then there exist $A, B \in \rho$ such that $a \in A$, $b \in B$ and $[A \cap B]^c \notin \mathcal{P}$. Since $[A \cap B]^c \notin \mathcal{P}$ by Lemma 2.1 implies that $[A \cap B]^\diamond = \emptyset$ and so by Corollary 2.2 $A^\diamond \cap B = \emptyset$. Thus, $b \notin A^\diamond$.

(2) $\Longrightarrow$ (3): Let $a, b \in Z \ni a \neq b$, by hypothesis there is $A \in \rho$ such that $a \in A$ and $b \notin A^\diamond$. This implies that $b \notin \cap\{A^\diamond : A \in \rho(a)\}$ and so $\bigcap\{A^\diamond : A \in \rho(a)\}$ is either $\emptyset$ or $\{a\}$.

(3) $\Longrightarrow$ (1): Let $a, b \in Z$ such that $a \neq b$. By the hypothesis $b \notin \bigcap\{A^\diamond | A \in \rho(a)\}$, we can say that $b \notin A^\diamond$ for some $A \in \rho(a)$. Therefore, there is $B \in \rho(b)$ such that $[A \cap B]^c \notin \mathcal{P}$ so that, $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-Hausdorff. □

Proposition 3.2. *Let $(Z, \rho, \mathcal{P})$ be a primal space. Then, $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-Hausdorff if and only if $(Z, \rho^\diamond, \mathcal{P})$ is $\mathcal{P}$-Hausdorff.*

Proof. Let $(Z, \rho, \mathcal{P})$ be $\mathcal{P}$-Hausdorff. Since $\rho \subseteq \rho^\diamond$, then $(Z, \rho^\diamond, \mathcal{P})$ is $\mathcal{P}$- Hausdorff. Conversely, assume that $(Z, \rho^\diamond, \mathcal{P})$ is $\mathcal{P}$- Hausdorff. Let $a, b \in Z$ such that $a \neq b$. Then, there exist $A, B \in \rho^\diamond$ such that $a \in A$, $b \in B$ and $[A \cap B]^c \notin \mathcal{P}$. Hence, by Lemma 2.2 there exist $U, V \in \rho$ and $P_1, P_2 \notin \mathcal{P}$ such that $U \cap P_1 \subseteq A$ and $V \cap P_2 \subseteq B$. Then, $[A \cap B]^c \subseteq [U \cap V]^c$. Since $[A \cap B]^c \notin \mathcal{P}$ and $[A \cap B]^c \subseteq [U \cap V]^c$, then $[U \cap V]^c \notin \mathcal{P}$. Hence, $(Z, \rho^\diamond, \mathcal{P})$ is $\mathcal{P}$-Hausdorff. □

Definition 3.2. *Let $(Z, \rho, \mathcal{P})$ be a primal space. Let $A \subseteq Z$ be any set such that $A \notin \mathcal{P}$. We define the subprimal space of ρ as follows:*

$$\mathcal{P}_A = \{A \cap N : N \in \mathcal{P}\} = \{N \in \mathcal{P} : N \subseteq A\}.$$

Proposition 3.3. *Let $(Z, \rho, \mathcal{P})$ be $\mathcal{P}$-Hausdorff and let $A \subseteq Z$ such that $A \notin \mathcal{P}$. Then, $(A, \rho_A, \mathcal{P}_A)$ is $\mathcal{P}_A$-Hausdorff.*

Proof. Let a and b be two distant points in A. Since $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-Hausdorff, then there are $H, K \in \rho$ such that $a \in H$ and $b \in K$ with $[H \cap K]^c \notin \mathcal{P}$. Hence, $a \in A \cap H \in \rho_A$ and $b \in A \cap K \in \rho_A$ with $[(A \cap H) \cap (A \cap K)]^c = [(H \cap K) \cap A]^c \notin \mathcal{P}_A$. Therefore, $(A, \rho_A, \mathcal{P}_A)$ is $\mathcal{P}_A$-Hausdorff. □

4. $\mathcal{P}$-regular spaces

Definition 4.1. *A primal space $(Z, \rho, \mathcal{P})$ is said to be $\mathcal{P}$-regular if for every $H \in \rho^c$ and $a \notin H$, there exist $A, B \in \rho$ such that $a \in A$, $A \cap B = \emptyset$ and $[H \setminus B]^c \notin \mathcal{P}$.*

If $\mathcal{P} = 2^Z \setminus \{Z\}$, then the concepts of regularity and $\mathcal{P}$-regularity are equivalent. Moreover, every regular is $\mathcal{P}$-regular but the converse is not true as shown in the following examples:

Example 4.1. *Consider the set of the real numbers with co-finite topology $(\mathbb{R}, \mathcal{CF})$. Let $\mathcal{P}_f$ be the primal of all subsets of the real numbers whose complement is not finite. Then, $(\mathbb{R}, \mathcal{CF}, \mathcal{P}_f)$ is not regular because the co-finite topology has no disjoint nonempty open sets. For $\mathcal{P}$-regularity, let $H \subseteq \mathbb{R}$ and let $x \in \mathbb{R} \setminus H$. As H is a closed proper subset of $\mathbb{R}$, then H is finite. Let $A, B \in \mathcal{CF}$ such that $x \in A$ and $A \cap B = \emptyset$. Then, $B = \emptyset$ and hence $[H \setminus B]^c = H^c \notin \mathcal{P}_f$ because H is finite.*

Example 4.2. *Let $Z = \{a, b, c\}$. Define $\rho = \{\emptyset, Z, \{a\}, \{c\}, \{a, c\}\}$ and $\mathcal{P} = \{\emptyset, \{c\}, \{b\}, \{c, b\}\}$. Let $H = \{b, c\}$. Then, it is closed set and $a \notin H$. Hence, it is clear that $(Z, \rho, \mathcal{P})$ is not regular but it is $\mathcal{P}$-regular as shown in the Table 2.*

Table 2. Details on $\mathcal{P}$-regularity.

$H = \{b, c\}$	$a \notin H$	$N = \{a\}$ and $M = \{c\}$	$[H \setminus M]^c = \{a, c\} \notin \mathcal{P}$
$H = \{a, b\}$	$c \notin H$	$N = \{c\}$ and $M = \{a\}$	$[H \setminus M]^c = \{a, c\} \notin \mathcal{P}$
$H = \{b\}$	$a \notin H$	$N = \{a\}$ and $M = \{c\}$	$[H \setminus M]^c = \{a, c\} \notin \mathcal{P}$
	$c \notin H$	$N = \{c\}$ and $M = \{a\}$	$[H \setminus M]^c = \{a, c\} \notin \mathcal{P}$

Theorem 4.1. *Let $(Z, \rho, \mathcal{P})$ be a primal space . Then, the following statements are equivalent:*

(1) $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular.
(2) For each $a \in Z$ and $a \in U \in \rho$, there is $A \in \rho$ containing a such that $[Cl(A) \setminus U]^c \notin \mathcal{P}$.
(3) For each $a \in Z$ and $F \in \rho^c$ not containing a, there is $A \in \rho$ containing a such that $[Cl(A) \cap F]^c \notin \mathcal{P}$.

Proof. (1) $\Longrightarrow$ (2): Let $U \in \rho$ containing a. Then, $a \notin Z \setminus U = U^c$ which is closed. Since $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular, there exist $A, B \in \rho$ such that $a \in A$, $A \cap B = \emptyset$ and $[U^c \setminus B]^c \notin \mathcal{P}$. Let $U^c \setminus B = I$. Then, $U^c \subseteq B \cup I$. Now, $A \cap B = \emptyset$ and $A \subseteq B^c$, so $Cl(A) \subseteq B^c$ and $Cl(A) \setminus U \subseteq B^c \cap U^c \subseteq B^c \cap (B \cup I) = B^c \cap I \subseteq I = U^c \setminus B$. Therefore, $[U^c \setminus B]^c \subseteq [Cl(A) \setminus U]^c$. By Corollary 2.1, we have $[Cl(A) \setminus U]^c \notin \mathcal{P}$.

(2) $\Longrightarrow$ (3): Let $a \notin F \in \rho^c$. Then, $a \in F^c$ which is open set. Hence, there exist $A \in \rho$ such that $a \in A$ and $[Cl(A) \setminus F^c]^c \notin \mathcal{P}$ which implies that $[Cl(A) \cap F]^c \notin \mathcal{P}$.

(3) $\Longrightarrow$ (1): Let $a \in Z$ and $F \in \rho^c$ such that $a \notin F$. Then, there exists $A \in \rho$ such that $a \in A$ and $[Cl(A) \cap F]^c \notin \mathcal{P}$. Let $B = [Cl(A)]^c$. Since $Cl(A) \in \rho^c$, then $B = (Cl(A))^c \in \rho$ and $A \cap B = \emptyset$. As $[Cl(A) \cap F]^c \notin \mathcal{P}$, then $[F \setminus (Cl(A))^c]^c = [F \setminus B]^c \notin \mathcal{P}$. Hence, $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular. □

We now have the corollary that follows.

Corollary 4.1. *Let $(Z, \rho, \mathcal{P})$ be a primal space and let $\rho^c \setminus \{Z\} \subseteq \mathcal{P}$. Then, the following statements are equivalent.*

(1) $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular.
(2) For each $a \in Z$ and $a \in U \in \rho$, there is $A \in \rho$ containing a such that $[A^\diamond \setminus U]^c \notin \mathcal{P}$.
(3) For each $a \in Z$ and $F \in \rho^c$ not containing a, there is $A \in \rho$ containing a such that $[A^\diamond \cap F]^c \notin \mathcal{P}$.

Proposition 4.1. *Let $(Z, \rho, \mathcal{P})$ be $\mathcal{P}$-regular and let $A \in \rho$. Then, there are $B \in \rho$ and $H^c \notin \mathcal{P}$ such that $a \in B \subseteq Cl(B) \subseteq A \cup H$ for all $a \in A$.*

Proof. Let $a \in A \in \rho$. Since $a \notin A^c$ which is a closed set, there exist $B, U \in \rho$ such that $B \cap U = \emptyset$, $a \in B$ and $[A^c \setminus U]^c \notin \mathcal{P}$. Let $H = A^c \setminus U = A^c \cap U^c = U^c \setminus A$. Then, $a \in B \subseteq U^c = A \cup H$ and so $a \in B \subseteq Cl(B) \subseteq A \cup H$. □

Theorem 4.2. *Let $(Z, \rho, \mathcal{P})$ be a $\mathcal{P}$-regular space and let $z_1, z_2 \in Z$ such that $z_1 \neq z_2$. Then, either $Cl(\{z_1\}) = Cl(\{z_2\})$ or $[Cl(\{z_1\}) \cap Cl(\{z_2\})]^c \notin \mathcal{P}$.*

Proof. Let $z_1 \in Cl(\{z_2\})$ and $z_2 \in Cl(\{z_1\})$. Then $Cl(\{z_1\}) \subseteq Cl(Cl(\{z_2\})) = Cl(\{z_2\}) \subseteq Cl(Cl(\{z_1\})) = Cl(\{z_1\})$ and so $Cl(\{z_1\}) = Cl(\{z_2\})$. Suppose $b \notin Cl(\{z_1\})$. Since $(Z, \rho, \mathcal{P})$ is a $\mathcal{P}$-regular, then by Theorem 4.1 (3) there exists $V \in \rho$ containing z_2 such that $[Cl(V) \cap Cl(\{z_1\})]^c \notin \mathcal{P}$. Since $z_2 \in V$, then $Cl(\{z_2\}) \cap Cl(\{z_1\}) \subseteq Cl(V) \cap Cl(\{z_1\})$, which implies that $[Cl(V) \cap Cl(\{z_1\})]^c \subseteq [Cl(\{z_2\}) \cap Cl(\{z_1\})]^c$ and $[Cl(V) \cap Cl(\{z_1\})]^c \notin \mathcal{P}$. Hence, $[Cl(\{z_1\}) \cap Cl(\{z_2\})]^c \notin \mathcal{P}$. □

Theorem 4.3. *If the primal space $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular, then $(Z, \rho^\diamond, \mathcal{P})$ is $\mathcal{P}$-regular.*

Proof. Let A be $\rho^\diamond$-closed and $z \notin A$. Since $Z \setminus A$ is a $\rho^\diamond$-open set and $z \in Z \setminus A$, there exist $B \in \rho$ and $I \notin \mathcal{P}$ such that $z \in B \cap I \subseteq Z \setminus A$. As $(Z, \rho, \mathcal{P})$ is primal regular, there exists $C \in \rho$ such that $z \in C$ and $[Cl(C) \setminus B]^c \notin \mathcal{P}$ by Theorem 4.1 (2). Let $Cl(C) \setminus B = J$. Then, $Cl(C) \setminus J \subseteq B \subseteq (Z \setminus A) \cup I^c$ and $Cl(C) \setminus (Z \setminus A) = Cl(C) \cap A \subseteq J \cup I^c$. Hence, $[J \cup I^c]^c = J^c \cap I \subseteq [Cl(C) \cap A]^c \subseteq [Cl^\diamond(C) \cap A]^c$. Since $J^c \cap I \notin \mathcal{P}$, we have $[Cl^\diamond(C) \cap A]^c \notin \mathcal{P}$ and since C is also $\rho^\diamond$-open set by Theorem 4.1 (3), $(Z, \rho^\diamond, \mathcal{P})$ is $\mathcal{P}$-regular. □

The converse of Theorem 4.3 is not right as shown in the following examples:

Example 4.3. *Let $(\mathbb{R}, \varepsilon_p, \mathcal{P})$ be defined as in Example 3.1. $(\mathbb{R}, \varepsilon_p, \mathcal{P})$ is not $\mathcal{P}$-regular. To show that, let $H = \{p\}$ and let $a \in \mathbb{R}$ such that $a \neq p$. Now suppose that $A, B \in \varepsilon_p$ such that $a \in A$ and $A \cap B = \emptyset$ which implies that $B \neq \mathbb{R}$.*
Then, $[H \setminus B]^c = [H \cap B^c]^c = [\{p\}]^c = \mathbb{R} \setminus \{p\} \in \mathcal{P}$. Hence, $(\mathbb{R}, \varepsilon_p, \mathcal{P})$ is not $\mathcal{P}$-regular.

Now we want to prove that $(\mathbb{R}, \varepsilon_p^\diamond, \mathcal{P})$ is $\mathcal{P}$-regular.
We know that $\varepsilon_p^\diamond = \{A \subseteq \mathbb{R} \mid Cl^\diamond(A^c) = A^c\}$, where $Cl^\diamond(A) = A \cup A^\diamond$. Let $A \subseteq \mathbb{R}$ be any set. Then, there are two cases:
Case 1. $p \in A$. Hence, $A^\diamond = \{x \in \mathbb{R} \mid U^c \cup A^c \in \mathcal{P},\ \forall U \in \varepsilon_p(x)\}$. $U^c \cup A^c \in \mathcal{P}$ if and only if $p \notin U^c \cup A^c$ which implies that $p \notin U^c$ and $p \notin A^c$. Thus, $p \in U \Rightarrow U = \mathbb{R}$ which is the only open set containing p. Hence, $A^\diamond = \{p\}$. Since $p \in A$, then $Cl^\diamond(A) = A$ which means that $A^c \in \varepsilon_p^\diamond$.
Case 2. $p \notin A$. Hence, $A^\diamond = \{x \in \mathbb{R} \mid U^c \cup A^c \in \mathcal{P},\ \forall U \in \varepsilon_p(x)\}$. Since $p \notin A$, then $p \in A^c$ which implies that $p \in A^c \cup U^c$ for every $U \in \varepsilon_p$. Therefore, $A^c \cup U^c \notin \mathcal{P}$ for all $U \in \varepsilon_p$. Thus, $A^\diamond = \emptyset$, and hence $Cl^\diamond(A) = A$ which means that $A^c \in \varepsilon_p^\diamond$. We conclude that $\varepsilon_p^\diamond = 2^{\mathbb{R}}$. Thus, $(\mathbb{R}, \varepsilon_p^\diamond, \mathcal{P})$ is $\mathcal{P}$-regular.

Example 4.4. *Let $Z = \{a, b, c\}$, $\rho = \{Z, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, $\rho^\diamond = \{Z, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{c\}, \{a, c\}\}$. Then, $(Z, \rho^\diamond, \mathcal{P})$ is $\mathcal{P}$-regular. Let $H = \{c\}$ which is a closed set and $b \notin H$. Then, if A, B are disjoint open sets such that $b \in A$, we have that either $A = \{b\}$ and $B = \{a\}$ or $A = \{b, c\}$ and $B = \{a\}$. In both cases, we have $[H \setminus B]^c \in \mathcal{P}$ which implies that $(Z, \rho, \mathcal{P})$ is not $\mathcal{P}$-regular.*

Theorem 4.4. *Let $(Z, \rho^\diamond, \mathcal{P})$ be $\mathcal{P}$-regular. If $\rho^c \setminus \{Z\} \in \mathcal{P}$, then $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular.*

Proof. Let $A \in \rho^c$ and $z \notin A$. Then, A is $\rho^\diamond$-closed set. Hence, there is a $\rho^\diamond$-open set B containing z such that $[B^\diamond \cap A]^c \notin \mathcal{P}$ by Corollary 4.1 (3). Since B is a $\rho^\diamond$-open set containing z, there exist $U \in \rho$ and $I \notin \mathcal{P}$ such that $z \in U \cap I = U \setminus I^c \subseteq B$. Hence, by Lemma 2.4 $(U \setminus I^c)^\diamond = U^\diamond \subseteq B^\diamond$ and $U^\diamond \cap A \subseteq B^\diamond \cap A$. Also, $[B^\diamond \cap A]^c \subseteq [U^\diamond \cap A]^c$ and $[B^\diamond \cap A]^c \notin \mathcal{P}$. Then, we have that $[U^\diamond \cap A]^c \notin \mathcal{P}$. By Corollary 4.1 (3) $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular. □

Theorem 4.5. *Let $(Z, \rho, \mathcal{P})$ be $\mathcal{P}$-regular and $Y \notin \mathcal{P}$. Then, $(Y, \rho_Y, \mathcal{P}_Y)$ is $\mathcal{P}$-regular.*

Proof. Let $B \in \rho_Y^c$ and $y \in Y$ such that $y \notin B$. Then, $B = Y \cap A$ where $A \in \rho^c$ such that $y \notin A$. Since $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$ regular, there exist $M, N \in \rho$ such that $M \cap N = \emptyset$, $y \in M$ and $[A \setminus N]^c \notin \mathcal{P}$. Now, let $M_1 = Y \cap M$ and $N_1 = Y \cap N$. Then, $M_1, N_1 \in \rho_Y$, $y \in M_1$ and $M_1 \cap N_1 = (Y \cap M) \cap (Y \cap N) = Y \cap (M \cap N) = \emptyset$. Suppose that $A \setminus N = I$. Then, $I^c \notin \mathcal{P}$ and $A \subseteq I \cup N$. Thus, $B = Y \cap A \subseteq Y \cap (I \cup N) = (Y \cap I) \cup N_1$, which implies that $B \setminus N_1 \subseteq Y \cap I$ and hence $[Y \cap I]^c \subseteq [B \setminus N_1]^c$. Since $Y \cap I^c \notin \mathcal{P}_Y$ and $Y \cap I^c = [Y \cap I]^c \cap Y \subseteq [Y \cap I]^c \subseteq [B \setminus N_1]^c$, so that $[B \setminus N_1]^c \notin \mathcal{P}_Y$. Thus, $(Y, \rho_Y, \mathcal{P}_Y)$ is $\mathcal{P}$-regular. □

Definition 4.2. *Let $(Z, \rho, \mathcal{P})$ be a primal space and let Z has a property P. We say that P is a primal topological property if every homeomorphic image for Z has P. That is, if $f : (Z, \rho, \mathcal{P}) \longrightarrow (Y, \tau, f(\mathcal{P}))$ is a homemorphism and $(Z, \rho, \mathcal{P})$ has a property P, then $(Y, \tau, f(\mathcal{P}))$ has the property P.*

Theorem 4.6. *If the primal space $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular and $f : (Z, \rho, \mathcal{P}) \to (Y, \tau, f(\mathcal{P}))$ is a homeomorphism, then $(Y, \tau, f(\mathcal{P}))$ is $\mathcal{P}$- regular. On other word, $\mathcal{P}$-regularity is a primal topological property.*

Proof. Let $B \in \tau^c$ and $y \in Y$ such that $y \notin B$. Let $z = f^{-1}(y)$. Since f is continuous, $f^{-1}(B) \in \rho^c$ not containing z. Since $(Z, \rho, \mathcal{P})$ is primal regular, there exist $A, C \in \rho$ such that $A \cap C = \emptyset$, $z \in C$ and $[f^{-1}(B)\backslash A]^c \notin \mathcal{P}$. Let $I = f^{-1}(B)\backslash A$. Then, $f^{-1}(B) \subseteq A \cup I$ and $B = f(f^{-1}(B)) \subseteq f[A \cup I] \subseteq f(A) \cup f(I)$. Hence, $B\backslash f(A) \subseteq f(I)$ and since f is homeomorphism, we have $f(I^c) \subseteq [f(I)]^c \subseteq [B\backslash f(A)]^c$. Moreover, $f(I^c) \notin f(\mathcal{P})$, then $[B \backslash f(A)]^c \notin f(\mathcal{P})$. Since $f(C) \cap f(A) = \emptyset$ such that $y \in f(C)$ and $[B \backslash f(A)]^c \notin f(\mathcal{P})$, then $(Y, \tau, f(\mathcal{P}))$ is $\mathcal{P}$-regular with respect to $f(\mathcal{P})$. □

Next example shows that Theorem 4.6 does not necessarily hold if the primal topological space defined on Y was different from $f(\mathcal{P})$.

Example 4.5. *Let $(Z, \rho, \mathcal{P}_1)$ be defined as in Example 4.2. Let $Y = \{1, 2, 3\}$. Define $\tau = \{Y, \emptyset, \{1\}, \{3\}, \{1, 3\}\}$ and $\mathcal{P}_2 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$. Let $f : (Z, \rho, \mathcal{P}_1) \longrightarrow (Y, \tau, \mathcal{P}_2)$ be defined as $f(a) = 1, f(b) = 2$, and $f(c) = 3$. Then, $Z \cong Y$. Note that $\mathcal{P}_2 \neq f(\mathcal{P}_1)$. We know that $(Z, \rho, \mathcal{P}_1)$ is $\mathcal{P}$-regular as shown in Example 4.2. However, $(Y, \tau, \mathcal{P}_2)$ is not P-regular. To show that, let $H = \{1, 2\} \in \tau^c$ and $3 \notin H$. Let $A, B \in \tau$ to be any open sets such that $3 \in A$ and $A \cap B = \emptyset$. Then, we have the following cases:*
Case 1. $A = \{1, 3\}$ and $B = \emptyset$. Then, $[H \setminus B]^c = H^c = \{3\} \in \mathcal{P}_2$.
Case 2. $A = \{3\}$ and $B = \emptyset$. Then, $[H \setminus B]^c = H^c = \{3\} \in \mathcal{P}_2$.
Case 3. $A = \{3\}$ and $B = \{1\}$. Then, $[H \setminus B]^c = [\{2\}]^c = \{1, 3\} \in \mathcal{P}_2$.

Definition 4.3. *A primal space $(Z, \rho, \mathcal{P})$ is said to be $\mathcal{P}$-paracompact if every open cover $\mathcal{U}$ of Z has a locally finite open refinement $\mathcal{V}$ such that $\bigcup \mathcal{V} = \bigcup\{H : H \in \mathcal{V}\} \notin \mathcal{P}$.*

Theorem 4.7. *If a primal space $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-paracompact and Hausdorff, then $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular.*

Proof. Let $b \notin A \in \rho^c$. For each $a \in A$, there exist $U_a \in \rho(b)$ and $V_a \in \rho(a)$ such that $U_a \cap V_a = \emptyset$. Then, $b \notin Cl(V_a)$. Let $\mathcal{U} = \{V_a : a \in A\} \bigcup \{Z \setminus A\}$ is an open cover of Z. Hence, there exist a locally finite open refinement $\mathcal{V} = \{V_a^* : a \in A\} \bigcup \{W\}$ such that $V_a^* \subseteq V_a$ for each $a \in A$, $W \subseteq \{Z \setminus A\}$, and $\bigcup \mathcal{V} \notin \mathcal{P}$. Let $U = \{Z \setminus \{\bigcup Cl(V_a^*)\} : a \in A\}$ and $V = \bigcup\{V_a^* : a \in A\}$. Then, $U \cap V = \emptyset$, $b \in U$ and $\bigcup \mathcal{V} = W \cup V \subseteq (Z \setminus A) \cup V = A^c \cup V = [A \setminus V]^c$. Since $\bigcup \mathcal{V} \notin \mathcal{P}$, then $[A \setminus V]^c \notin \mathcal{P}$. Hence, $(Z, \rho, \mathcal{P})$ is $\mathcal{P}$-regular. □

5. $\mathcal{P}$-normal spaces

Definition 5.1. *A primal space $(Z, \rho, \mathcal{P})$ is said to be $\mathcal{P}$-normal if for every disjoint closed sets $H, K \in \rho^c$, there are $A, B \in \rho$ such that $A \cap B = \emptyset$, $[H \setminus A]^c \notin \mathcal{P}$ and $[K \setminus B]^c \notin \mathcal{P}$.*

If $\mathcal{P} = 2^Z \setminus \{Z\}$, then the normality and $\mathcal{P}$-normality are equivalent. Moreover, every normal is $\mathcal{P}$-normal but the converse is not true as shown in the following example.

Example 5.1. *Let $Z = \{a, b, c, d\}$, $\rho = \{\emptyset, Z, \{c\}, \{d\}, \{b, c\}, \{a, c, d\}, \{b, c, d\}\}$ and $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Let $H = \{a\}$ and $F = \{b\}$ which are disjoint closed set and it is clear that $(Z, \rho, \mathcal{P})$ is not normal but it is $\mathcal{P}$-normal as shown in the Table 3.*

Table 3. Details on $\mathcal{P}$-normality.

$H = \{a, d\}$	$F = \{b\}$	$A = \{d\}$ and $B = \{b, c\}$	$(H \setminus A)^c = \{b, c, d\} \notin \mathcal{P}$ and $(F \setminus B)^c = Z \notin \mathcal{P}$
$H = \{a\}$	$F = \{b\}$	$A = \{d\}$ and $B = \{b, c\}$	$(H \setminus A)^c = \{b, c, d\} \notin \mathcal{P}$ and $(F \setminus B)^c = Z \notin \mathcal{P}$

Here is an example to show that $\mathcal{P}$-normality doesn't imply $\mathcal{P}$-regualrity.

Example 5.2. *Let $(\mathbb{R}, \varepsilon_p, \mathcal{P})$ be as defined in Example 3.1. Let $H, K \in \varepsilon_p^c$ such that $H \cap K = \emptyset$. Then, either $H = \emptyset$ or $K = \emptyset$ since if $\emptyset \neq F \in \varepsilon_p^c$, then $p \in F$. Without loss of generality, suppose that $H = \emptyset$ and $K \neq \emptyset$ which implies that $p \in K$. To show that $(\mathbb{R}, \varepsilon_p, \mathcal{P})$ is p-normal, choose $A = \emptyset$ and $B = \mathbb{R}$. Then, $A \cap B = \emptyset$, $[H \setminus A]^c = [\emptyset]^c = \mathbb{R} \notin \mathcal{P}$ and $[K \setminus B]^c = [\emptyset]^c = \mathbb{R} \notin \mathcal{P}$. Hence, $(\mathbb{R}, \varepsilon_p, \mathcal{P})$ is $\mathcal{P}$-normal but not $\mathcal{P}$-regular as shown in Example 4.3.*

Theorem 5.1. *Let $(Z, \rho, \mathcal{P})$ be a primal space. Then, the following are equivalent.*

(1) Z is $\mathcal{P}$-normal.
(2) For every $F \in \rho^c$ and $U \in \rho$ such that $F \subseteq U$, there is $B \in \rho$ such that $[Cl(B) \setminus U]^c \notin \mathcal{P}$ and $[F \setminus B]^c \notin \mathcal{P}$.
(3) For each $H, K \in \rho^c$ such that $H \cap K = \emptyset$, there is $A \in \rho$ such that $[H \setminus A]^c \notin \mathcal{P}$ and $[Cl(A) \cap K]^c \notin \mathcal{P}$.

Proof. (1) $\Longrightarrow$ (2): Let $F \in \rho^c$ and $U \in \rho$ such that $F \subseteq U$. Then, $Z \setminus U \in \rho^c$ and $F \cap (Z \setminus U) = \emptyset$. Thus, there exist $A, B \in \rho$ such that $A \cap B = \emptyset$, $[F \setminus B]^c \notin \mathcal{P}$ and $[(Z \setminus U) \setminus A]^c \notin \mathcal{P}$. Now, since $A \cap B = \emptyset$ implies that $Cl(B) \subseteq Z \setminus A$ and so $(Z \setminus U) \cap Cl(B) \subseteq (Z \setminus U) \cap (Z \setminus A)$ which is equivalent to $[Cl(B) \setminus U] \subseteq [(Z \setminus U) \setminus A]$ and hence $[(Z \setminus U) \setminus A]^c \subseteq [Cl(B) \setminus U]^c$. Since $[(Z \setminus U) \setminus A]^c \notin \mathcal{P}$, then $[Cl(B) \setminus U]^c \notin \mathcal{P}$.

(2) $\Longrightarrow$ (3): Let $H, K \in \rho^c$ such that $H \cap K = \emptyset$ and hence $H \subseteq Z \setminus K$. Then, there is $A \in \rho$ such that $[Cl(A) \cap K]^c = [Cl(A) \setminus (Z \setminus K)]^c \notin \mathcal{P}$ and $[H \setminus A]^c \notin \mathcal{P}$.

(3) $\Longrightarrow$ (1): Let $H, K \in \rho^c$ such that $H \cap K = \emptyset$. Then, there is $A \in \rho$ such that $[H \setminus A]^c \notin \mathcal{P}$ and $[Cl(A) \cap K]^c \notin \mathcal{P}$. Now $[Cl(A) \cap K]^c \notin \mathcal{P}$ implies that $[K \setminus (Z \setminus Cl(A))]^c \notin \mathcal{P}$. If we let $B = (Z \setminus Cl(A))$, then $B \in \rho$ such that $[K \setminus B]^c \notin \mathcal{P}$ and $A \cap B = A \cap [Z \setminus Cl(A)] = \emptyset$. Hence, Z is $\mathcal{P}$-normal. □

We now have the following corollary:

Corollary 5.1. *Let $(Z, \rho, \mathcal{P})$ be a primal space and $\rho^c \setminus \{Z\} \subseteq \mathcal{P}$. Then, the following statements are equivalent.*

(1) Z is $\mathcal{P}$-normal.
(2) For every $F \in \rho^c$ and $U \in \rho$ such that $F \subseteq U$, there is $B \in \rho$ such that $[B^\diamond \setminus U]^c \notin \mathcal{P}$ and $[F \setminus B]^c \notin \mathcal{P}$.
(3) For each $H, K \in \rho^c$ such that $H \cap K = \emptyset$, there is $A \in \rho$ such that $[H \setminus A]^c \notin \mathcal{P}$ and $[A^\diamond \cap K]^c \notin \mathcal{P}$.

Theorem 5.2. *Let $(Z,\rho,\mathcal{P})$ be a primal space and every non improper closed set is primal element that is $\rho^c \setminus \{Z\} \in \mathcal{P}$. Let W be $\rho^\diamond$-open set and $F = Cl^\diamond(W)$. Then, there is $G \in \rho$ such that $W \subseteq G \subseteq Cl(G) \subseteq Cl^\diamond(W) = F$.*

Proof. We can write $W = \bigcup_{\alpha\in A}(G_\alpha \cap I_\alpha)$ and $(Z \setminus F) = \bigcup_{\beta\in B}(G_\beta \cap I_\beta)$, where $\{G_\alpha : \alpha \in A\}$ and $\{G_\beta : \beta \in B\}$ are subsets of ρ and $\{I_\alpha \notin \mathcal{P} : \alpha \in A\}$ and $\{I_\beta \notin \mathcal{P} : \beta \in B\}$. Now for each $\alpha \in A$ and $\beta \in B$ we have $(G_\alpha \cap I_\alpha) \cap (G_\beta \cap I_\beta) = \emptyset$. Thus, $(I_\alpha \cap I_\beta) \subseteq [G_\alpha \cap G_\beta]^c$. Since $I_\alpha, I_\beta \notin \mathcal{P}$, then $I_\alpha \cap I_\beta \notin \mathcal{P}$. Hence, for each $\alpha \in A$ and $\beta \in B$ we get $[G_\alpha \cap G_\beta]^c \notin \mathcal{P}$ and so $G_\alpha \cap G_\beta = \emptyset$. If $G = \bigcup_{\alpha\in A} G_\alpha$ it follows that $G \cap (\bigcup_{\beta\in B} G_\beta) = \emptyset$. Hence, $W \subseteq G \subseteq Cl(G) \subseteq [Z \setminus (\bigcup_{\beta\in B} G_\beta)] \subseteq F = Cl^\diamond(W)$. □

Theorem 5.3. *Let $(Z,\rho,\mathcal{P})$ be a primal space and $\rho^c \setminus \{Z\} \in \mathcal{P}$. Then, $(Z,\rho,\mathcal{P})$ is regular if and only if ρ is regular with respect to $\rho^\diamond$.*

Proof. Since $Cl^\diamond(A) \subseteq Cl(A)$ for any subset A of Z the necessity is obvious. To show the sufficiency, assume ρ is regular with respect to $\rho^\diamond$ and let $z \in Z$, and $U \in \rho(z)$. Then $\exists\ V \in \rho^\diamond(z) \ni\ z \in V \subseteq Cl^\diamond(V) \subseteq U$. By Theorem 5.2, there is $G \in \rho(z)$ such that $z \in V \subseteq G \subseteq Cl(G) \subseteq Cl^\diamond(V) \subseteq U$. Hence, $(Z,\rho,\mathcal{P})$ is regular. □

Theorem 5.4. *If a primal space $(Z,\rho,\mathcal{P})$ is a Lindelöf and $\mathcal{P}$-regular, then $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-normal.*

Proof. Let $K_1, K_2 \in \rho^c$ such that $K_1 \cap K_2 = \emptyset$. Since $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-regular, then there exist $U_a, V \in \rho$ such that $a \in U_a$, $U_a \cap V = \emptyset$ and $[K_2 \setminus V]^c \notin \mathcal{P}$ for each $a \in K_1$. Hence, $Cl(U_a) \cap V = \emptyset$ and $[K_2 \setminus V]^c = [K_2 \cap (Z \setminus V)]^c \subseteq [Cl(U_a) \cap K_2]^c$. Therefore, $[Cl(U_a) \cap K_2]^c \notin \mathcal{P}$. Since the collection $\{U_a \cap K_1 : a \in K_1\}$ is an open cover of K_1. Since K_1 is a Lindelöf subspace of Z, then $K_1 = \bigcup\{U_i \cap K_1 : i \in \mathbb{N}\}$, which implies that $K_1 \subseteq \bigcup\{U_i : i \in \mathbb{N}\}$. Also, $[Cl(U_i) \cap K_2]^c \notin \mathcal{P}$ for all $i \in \mathbb{N}$. Similarly, we can get a countable collection $\{F_t : t \in \mathbb{N}\}$ of open sets such that $K_2 \subseteq \bigcup\{F_t : t \in \mathbb{N}\}$ and $[Cl(F_t) \cap K_1]^c = P_t^c \notin \mathcal{P}$ for all $t \in \mathbb{N}$. For all $k \in \mathbb{N}$, suppose that $G_k = U_k \setminus \bigcup\{Cl(F_t) : t = 1, 2, ..., k\}$ and $H_k = V_k \setminus \bigcup\{Cl(U_t) : t = 1, 2, ..., k\}$. Let $G = \bigcup\{G_k : k \in \mathbb{N}\}$ and $H = \bigcup\{H_k : k \in \mathbb{N}\}$. Since $G_k, H_k \in \rho$ for all $k \in \mathbb{N}$, then $G, H \in \rho$ such that $G \cap H = \emptyset$. Claim that $[K_1 \setminus G]^c \notin \mathcal{P}$. Let $x \in K_1$. Then, $x \in U_m$ for some $m \in \mathbb{N}$. Also, $[Cl(F_k) \cap K_1]^c = P_k^c \notin \mathcal{P}$ for all $k \in \mathbb{N}$ implies that $K_1 \subseteq P_k \cup (Z \setminus Cl(F_k))$ for all $k \in \mathbb{N}$. Since $x \in K_1$, then $x \in P_k \cup (Z \setminus Cl(F_k))$ for all k and so $x \in P_k$ or $x \notin Cl(F_k)$ for all k. Hence, $x \in U_m \setminus \bigcup\{Cl(F_j) : j = 1, 2, ..., m\}$ or $x \in \bigcap\{P_j : j \in \mathbb{N}\} = P$. Thus, $P_j^c \notin \mathcal{P}$ for all $j \in \mathbb{N}$ which implies that $P^c \notin \mathcal{P}$. Since $x \in G_m$, $x \in G$, then $x \in G \cup P$. Therefore, $K_1 \subseteq G \cup P$ and hence $P^c \subseteq [K_1 \setminus G]^c \notin \mathcal{P}$. Similarly, we can prove that $[K_2 \setminus H]^c \notin \mathcal{P}$. Hence, $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-normal. □

Theorem 5.5. *If a primal space $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-paracompact and Hausdorff, then $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-normal.*

Proof. Let $K_1, K_2 \in \rho^c$ such that $K_1 \cap K_2 = \emptyset$. By Theorem 4.7, we have that $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-regular. Then, for each $a \in K_1$, there exist $U_a, V_a \in \rho$ such that $U_a \cap V_a = \emptyset$, $a \in U_a$ and $[K_2 \setminus V_a]^c \notin \mathcal{P}$. Hence, the collection $\mathcal{U} = \{U_a : a \in K_1\} \cup (Z \setminus K_1)$ is an open cover of Z. Since $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-paracompact, there is a locally finite open refinement $\mathcal{V} = \{W_a : a \in K_1\} \cup G$ such that $W_a \subseteq U_a$ for every $a \in K_1$, $G \subseteq K_1^c$ and $\bigcup \mathcal{V} = \bigcup\{A : A \in \mathcal{V}\} \notin \mathcal{P}$. Let $V = \bigcup\{W_a : a \in K_1\}$. Hence, $V \in \rho$. Since $\bigcup\{A : A \in \mathcal{V}\} \notin \mathcal{P}$ and $\bigcup\{A : A \in \mathcal{V}\} \subseteq \bigcup\{A : A \in \mathcal{V}\} \cup K_1^c = [\bigcup\{W_a : a \in K_1\} \cup G] \cup K_1^c = \bigcup\{W_a : a \in K_1\} \cup K_1^c = V \cup K_1^c = [K_1 \setminus V]^c$, then $[K_1 \setminus V]^c \notin \mathcal{P}$. For each $a \in Z$, we have $U_a \cap V_a = \emptyset$, which implies that $Cl(U_a) \subseteq V_a^c$ and hence $Cl(W_a) \subseteq V_a^c$. Then, $\bigcup\{Cl(W_a) : a \in Z\} \subseteq \bigcup\{V_a^c : a \in Z\}$, which implies that $[K_2 \setminus V_a]^c \subseteq \bigcup\{[K_2 \setminus V_a]^c : a \in Z\} \subseteq K_2^c \cup \{\bigcup\{V_a^c : a \in Z\}\}^c \subseteq K_2^c \cup \{\bigcup\{Cl(W_a) : a \in Z\}\}^c = K_2^c \cup [Cl(V)]^c = [K_2 \cap Cl(V)]^c \notin \mathcal{P}$. Hence, by (3) in Theorem 5.1, $(Z,\rho,\mathcal{P})$ is $\mathcal{P}$-normal. □

6. Conclusions

Acharjee et al. [10] and Al-Omari et al. [11, 12] developed the idea of a primal topology, which is the grill's dual structure. Research on this topological generalization is getting more and more intriguing. It is often recognized that one of the most helpful ideas in real analysis [13], summability theory [14], general topology, and other fields is the ideal, dual structure of the filter. Consequently, ideal was a motivation to present the primal structure. Our research was based on primal space regions, and [11] covered a number of fundamental operations on primal spaces. We have introduced three new ideas in primal spaces in this work: The $\mathcal{P}$-Hausdorff, $\mathcal{P}$-regularlrity, and $\mathcal{P}$-normality. As a consequence, we have defined the terms "$\mathcal{P}$-Hausdorff", "$\mathcal{P}$-regular spaces", and "$\mathcal{P}$-normal spaces" and deduced some interesting generalizations and findings about them. Furthermore, we have obtained other theoretical outcomes that illustrate the connections between $\mathcal{P}$-Hausdorff, $\mathcal{P}$-regular, and $\mathcal{P}$-normal spaces. Moreover, we included other examples in addition to a few correlations. However, this research can also be expanded in primal soft topology [15], generalized rough approximation spaces, fuzzy primal space, infra soft topological spaces, and so on. Many characteristics and outcomes of such research may then be deduced and drawn from it based on this notion; these would be left for future discussions. Therefore, we plan to explore this novel idea in general topology as well as other fields. In the future, we hope to relate this concept, if possible, to some concepts of the quantum world [16] from the standpoint of general topology.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. D. Jankovic, T. R. Hamlett, New topologies from old via ideals, *Am. Math. Mon.*, **97** (1990), 295–310. https://doi.org/10.2307/2324512
2. K. Kuratowski, *Topologie: Volume I*, London: Academic Press, 1966.
3. G. Choquet, Sur les notions de filter et grille, *C. R. Math. Acad. Sci. Paris*, **224** (1947), 171–173. https://doi.org/10.2307/2324512
4. R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, 1960.
5. T. R. Hamlett, D. Rose, $*$-topological properties, *Int. J. Math. Math. Sci.*, **13** (1990), 507–512. https://doi.org/10.1155/S0161171290000734

6. A. Al-Omari, T. Noiri, On $\Psi_{\mathcal{G}}$-sets in grill topological spaces, *Filomat*, **25** (2011), 187–196. https://doi.org/10.2298/FIL1102187A
7. B. Roy, M. N. Mukherjee, On a typical topology induced by a grill, *Soochow J. Math.*, **33** (2007), 771–786.
8. S. Modak, Topology on grill-filter space and continuity, *Bol. Soc. Parana. Mat.*, **13** (2013), 219–230. https://doi.org/10.5269/bspm.v31i2.16603
9. H. Al-Saadi, H. Al-Malki, Generalized primal topological spaces. *AIMS Math.*, **8** (2023), 24162–24175. https://doi.org/10.3934/math.20231232
10. S. Acharjee, M. Özkoç, F. Y. Issaka, Primal topological spaces, *arXiv Preprint*, 2022.
11. A. Al-Omari, S. Acharjee, M. Özkoç, A new operator of primal topological spaces, *Mathematica*, **65** (2023), 175–183. https://doi.org/10.24193/mathcluj.2023.2.03
12. A. Al-Omari, M. H. Alqahtani, Primal structure with closure operators and their applications, *Mathematics*, **11** (2023), 4946. https://doi.org/10.3390/math11244946
13. B. C. Tripathy, M. Sen, S. Nath, On generalized difference ideal convergence in generalized probabilistic n-normed spaces, *P. Natl. A. Sci. India A*, **91** (2021), 29–34. https://doi.org/10.1007/s40010-019-00644-1
14. H. Albayrak, Ö. Ölmez, S. Aytar, Some set theoretic operators preserving ideal Hausdorff convergence, *Real Anal. Exch.*, **47** (2022), 179–190. https://doi.org/10.14321/realanalexch.47.1.1618849755
15. T. M. Al-shami, Z. A. Ameen, R. Abu-Gdairi, A. Mhemdi, On primal soft topology, *Mathematics*, **11** (2023), 2329. https://doi.org/10.3390/math11102329
16. C. J. Isham, *An introduction to general topology and quantum topology*, In: Physics, Geometry and Topology, Boston: Springer, 1990, 129–189. https://link.springer.com/chapter/10.1007/978-1-4615-3802-8_5