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Research article

The nearest point problems in fuzzy quasi-normed spaces

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Abstract: Motivated by the fact that the fuzzy quasi-normed space provides a suitable framework for complexity analysis and has important roles in discussing some questions in theoretical computer science, this paper aims to study the nearest point problems in fuzzy quasi-normed spaces. First, by using the theory of dual space and the separation theorem of convex sets, the properties of the fuzzy distance from a point to a set in a fuzzy quasi-normed space are studied comprehensively. Second, more properties of the nearest point are given, and the existence, uniqueness, characterizations, and qualitative properties of the nearest points are obtained. The results obtained in this paper are of great significance for expanding the application fields of optimization theory.

Keywords: best approximation; fuzzy analysis; nearest point; fuzzy distance
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1. Introduction

The best approximation has important applications in many fields such as mathematics, engineering, and economics. It is mainly concerned with the questions of existence, uniqueness, characterizations, and qualitative properties of the nearest points. Therefore, the nearest point theory plays a key role in the best approximation. The extensive and in-depth research, especially in the framework of the normed space, the best approximation theory has achieved rich results, see literature [3-5,8,19,20] and their references. In order to expand the application range, many scholars have studied this problems in different spaces in recent years, for example: probabilistic normed space [18], fuzzy normed space [10], fuzzy 2-normed space [14].

In 1977, Krein and Nudelman [11] first proposed the best approximation problem in asymmetric normed spaces. In 2002 and 2003, Mustăța [15,16] studied the relationship between the existence of the best approximation and the uniqueness of the extension of the bounded linear functional on an asymmetric norm space. In 2004, Cobzas and Mustăța [7] studied the characterization of the best approximating element for the subspace of an asymmetric normed space. In 2013, Cobzas [6] gave a systematic summary of the research on the best approximation in asymmetric normed spaces.

In 2010, Algere and Romaguera [2] put forward the concept of fuzzy quasi-norm (i.e., asymmetric fuzzy norm), which takes fuzzy norm and asymmetric norm (i.e., quasi-norm) as two special cases and has wide applicability. The examples show that a fuzzy quasi-normed space provides a suitable framework for the complexity analysis, with important roles in discussing some questions in optimization, approximation theory, and theoretical computer science.

Therefore, it is a natural direction to study the best approximation problem in the framework of a fuzzy quasi-normed space. Indeed, Wu, et al. [21] carried out the pioneering research in this field in 2023. In the fuzzy quasi-normed space, they put forward the concept of the nearest point and some characteristics of the nearest point, and extended the well-known Arzela distance formula from point to hyperplane in the case of a fuzzy quasi-normed space by using duality.

This paper is a continuation of paper [21]. The main content of this article is as follows: After introducing the basic definitions and conclusions of the fuzzy quasi-normed space in section 2, the relevant properties of the distance between a point and a set are discussed more comprehensively in section 3. In section 4, we focus on the nearest point problem and give some important conclusions, such as the existence, uniqueness, characterizations, and qualitative properties of the nearest points. In section 5, a brief conclusion is given.

2. Preliminaries

In this paper, the symbols R and N represent the set of real numbers and nonnegative integer numbers, respectively, Φ denotes the empty set, X is a real vector space, and θ is the null element in a real vector space. A^c represents the complement of a subset A. If A is a subset of the topological space (X,τ_N), let $\mathrm{cl}_N A$ and $\mathrm{int}_N A$ denote the closure and the interior of A. In order to specify the topology of the space used, a continuous mapping f which is from a topological space (X,τ_1) to another topological space (Y,τ_2), is also called (τ_1,τ_2) continuous.

Definition 2.1. *[17]* A binary operation $*$: $[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions:

(T1) $*$ is associative and commutative,

(T2) $*$ is continuous,

(T3) $a*1=a$ for any $a\in[0,1]$,

(T4) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ with $a,b,c,d\in[0,1]$.

Two paradigmatic examples of continuous t-norms are "$\wedge$" and "$\cdot$", which are defined by $a\wedge b=\min\{a,b\}$ and $a\cdot b=ab$, respectively.

Definition 2.2. *[2]* A fuzzy quasi-norm on a real vector space X is a pair $(N,*)$ such that $*$ is a continuous t-norm and N is a fuzzy set in $X\times[0,+\infty)$ satisfying the following conditions: for every $x,y\in X$,

(FQN1) $N(x,0)=0$;

(FQN2) $N(x,t)=N(-x,t)=1$ for all $t>0 \Leftrightarrow x=\theta$;

(FQN3) $N(\lambda x,t)=N(x,t/\lambda)$ for all $\lambda>0$;

(FQN4) $N(x+y,t+s)\geq N(x,t)*N(y,s)$ for all $t,s\geq 0$;

(FQN5) $N(x,\_)$: $[0,+\infty)\to[0,1]$ is left continuous;

(FQN6) $\lim_{t\to\infty} N(x,t)=1$.

By a fuzzy quasi-normed space, we mean a triple $(X,N,*)$. $(N,*)$ is denoted in short by N, if no confusion arises. Obviously, the function $N(x,\_)$ is increasing for each $x\in X$. Let $\bar{N}(x,t)=N(-x,t)$ for all $x\in X$ and $t\geq 0$, then $\bar{N}$ is also a fuzzy quasi-norm and is called the conjugate of N.

In [2], Alegre and Romaguera pointed out that each fuzzy quasi-norm N on X induces a topology τ_N on X which has as a base the family of open balls

$$B(x)=\{B_N(x,r,t): r\in(0,1), t>0\}$$

at $x\in X$, where

$$B_N(x,r,t)=\{y\in X\text{: } N(y-x,t)>1-r\}.$$

(X,τ_N) is a quasi-metrizable and paratopological vector space [1]. Furthermore, (X,τ_N) is locally convex if the continuous t-norm $*$ is chosen as "$\wedge$". If $\{x_n\}$ is a sequence in X, it is easy to verify that $\{x_n\}$ converges to x with respect to τ_N if and only if $\lim_{n\to\infty} N(x_n-x,t)=1$ for any $t>0$.

For any $x\in X$, $r\in(0,1)$, and $t>0$, set

$$B_N[x,r,t]=\{y\in X: N(y-x,t)\geq 1-r\},$$

$$S_N[x,r,t]=\{y\in X: N(y-x,t)=1-r\}.$$

Remark 2.1. For any $t>0$, it is easy to prove that the mapping $N(\_,t)$ is lower semi-continuous with respect to τ_N, and therefore, is upper semi-continuous with respect to $\tau_{\bar{N}}$. $B[x,r,t]$ is $\tau_{\bar{N}}$-closed.

Definition 2.3. *[9]* Let X be a real vector space, and $*$ be a continuous t-norm. $\mathscr{P}=\{p_\alpha : p_\alpha \text{ is a function from } X \text{ to } [0,\infty), \alpha\in(0,1)\}$ is called a family of star quasi-seminorms if it satisfies the following conditions: for all $x,y\in X$, $\alpha,\beta\in(0,1)$, and $c\in[0,\infty)$,

(*QN1) $p_\alpha(cx)=cp_\alpha(x)$,

(*QN2) $p_{\alpha*\beta}(x+y)\leq p_\alpha(x)+p_\beta(y)$.

If $\mathscr{P}$ satisfies the condition

(*QN3) $p_\alpha(x)=p_\alpha(-x)=0$ for every $\alpha\in(0,1)$ implies $x=\theta$,

then $\mathscr{P}$ is said to be separating.

Theorem 2.1. *[9]* Let $(X,N,*)$ be a fuzzy quasi-normed space, and $\alpha\in(0,1)$. The function $\|\cdot|_\alpha^X$: $X\to[0,\infty)$ is given by

$$\| x|_\alpha^X = \inf\{t>0 : N(x,t)\ge\alpha\}. \tag{2.1}$$

Then, for all $x\in X$ and $t>0$,

(1) $\| x|_\alpha^X$ is increasing with respect to $\alpha\in(0,1)$,

(2) $\| x|_\alpha^X = \sup\{t>0 : N(x,t)<\alpha\}$,

(3) $N(x,t)\ge\alpha$ implies that $\| x|_\alpha^X \le t$,

(4) $N(x,t)>\alpha$ implies that $\| x|_\alpha^X < t$,

(5) $N(x,t)<\alpha$ implies that $\| x|_\alpha^X \ge t$.

Theorem 2.2. *[9]* Let $(X,N,*)$ be a fuzzy quasi-normed space, $x\in X$, and $\alpha\in(0,1)$. Then, the following assertions are equivalent:

(1) N satisfies

(FQN7): For any $x\neq\theta$, $N(x,\_)$ is strictly increasing on $\{t: 0<N(x,t)<1\}$;

(2) $\| x|_\alpha^X = \inf\{t>0 : N(x,t)>\alpha\} = \sup\{t>0 : N(x,t)\le\alpha\}$;

(3) $N(x,t)>\alpha$ if and only if $\| x|_\alpha^X < t$, that is, $N(x,t)\le\alpha$ if and only if $\| x|_\alpha^X \ge t$.

Remark 2.2. It is easy to see that $P_N=\{\|\cdot|_\alpha^X : \alpha\in(0,1)\}$ is a separating family of star quasi-seminorms. If $*$ is chosen as "$\wedge$", then P_N is a family of quasi-norms.

Remark 2.3. If N satisfies condition (FQN7), it follows from Theorems 2.1(3) and 2.2(3) that $N(x,t)=\alpha$, which implies that $\| x|_\alpha^X = t$.

Example 2.1. Let $u(x)=\max\{x,0\}$ for each $x\in\mathbb{R}$. Then, u is a quasi-norm on the field $\mathbb{R}$. Moreover, if

$$N_{\mathrm{st}}(x,t)=\frac{t}{t+u(x)} \text{ for each } x\in\mathbb{R},\ t>0,$$

then it is easy to verify that $(N_{\mathrm{st}},*)$ is a fuzzy quasi-norm satisfying (FQN7) for any continuous t-norm $*$.

In the rest of this paper, the quasi-norm u is always defined as in Example 2.1, its conjugate is denoted by $\bar{u}$, that is, $\bar{u}(x)=\max\{-x,0\}$ for each $x\in\mathbb{R}$. The topology τ_u ($\tau_{\bar{u}}$, resp.) generated by u ($\bar{u}$, resp.) is called the upper (lower, resp.) topology of $\mathbb{R}$. The quasi dual $(X,N,*)^{\#}$ ($(X,N,*)^{\#-}$, resp.) of the fuzzy quasi-normed space $(X,N,*)$ is formed by all continuous linear functionals from (X,τ_N) to (R,τ_u) ($(\mathrm{R},\tau_{\bar{u}})$, resp.), or, equivalently, by all upper (lower resp.) semi-continuous linear functionals from (X,τ_N) to ($\mathbb{R}$, $|\cdot|$). Obviously, $(X,\overline{N},*)^{\#}=(X,N,*)^{\#-}$ and $(X,\overline{N},*)^{\#-}=(X,N,*)^{\#}$.

In the following, $(X,N,*)^{\#}$ and $(X,N,*)^{\#-}$ will be simply denoted by $X^{\#}$ and $X^{\#-}$ resp., if no confusion arises.

Now, for each $f\in X^{\#}$ and $\alpha\in(0,1)$, we define that

$$\| f \|_{\alpha,X}^{\#} = \sup\left\{ f(x) : \| x \|_{1-\alpha}^{X} \leq 1 \right\}. \quad (2.2)$$

It is proved that $P = \left\{ \| \cdot \|_{\alpha,X}^{\#} : \alpha \in (0,1) \right\}$ is a separating family of star quasi-seminorms on $X^{\#}$ in [9]. Obviously, if $f \in X^{\#}$ and $f \neq 0$, then $\| f \|_{\alpha,X}^{\#} > 0$ for any $\alpha \in (0,1)$.

Remark 2.4. $\| f \|_{\alpha,X}^{\#} = \sup\left\{ f(x) : \| x \|_{1-\alpha}^{X} < 1 \right\}$ (see Theorem 3.3 in [12]).

Similar to formulas (2.1) and (2.2), we can define the separating families of star quasi-seminorms $P_{\overline{N}} = \left\{ \| \cdot \|_{\alpha}^{X-} : \alpha \in (0,1) \right\}$ and $\overline{P} = \left\{ \| \cdot \|_{\alpha,X}^{\#-} : \alpha \in (0,1) \right\}$ on the conjugate spaces $\left(X, \overline{N}, * \right)$ and $X^{\#-}$, respectively. It is easy to show that $\| x \|_{\alpha}^{X-} = \| -x \|_{\alpha}^{X}$ and $\| f \|_{\alpha,X}^{\#-} = \| -f \|_{\alpha,X}^{\#}$.

In the rest of the paper, $\| x \|_{1-\alpha}^{X}$, $\| x \|_{1-\alpha}^{X-}$, $\| f \|_{\alpha,X}^{\#}$ and $\| f \|_{\alpha,X}^{\#-}$ will be simply denoted by $\| x \|_{1-\alpha}$, $\| x \|_{1-\alpha}^{-}$, $\| f \|_{\alpha}^{\#}$, and $\| f \|_{\alpha}^{\#-}$, respectively, if no confusion arises.

Definition 2.4. *[21]* Let X be a real vector space, $c \in \mathrm{R}$, and φ be a linear functional on X. The set $H_{\varphi,c} = \left\{ x \in X : \varphi(x) = c \right\}$ is called the hyperplane corresponding to φ and c. The sets

$$H_{\varphi,c}^{<} = \left\{ x \in X : \varphi(x) < c \right\} \text{ and } H_{\varphi,c}^{>} = \left\{ x \in X : \varphi(x) > c \right\}$$

are called the lower open half-space and upper open half-space determined by $H_{\varphi,c}$ or φ, respectively; and the following two sets

$$H_{\varphi,c}^{\leq} = \left\{ x \in X : \varphi(x) \leq c \right\} \text{ and } H_{\varphi,c}^{\geq} = \left\{ x \in X : \varphi(x) \geq c \right\}$$

are called the lower closed half-space and upper closed half-space determined by $H_{\varphi,c}$ or φ, respectively.

Let $x \in X$ and $A \subseteq X$. If x and A lie in opposite closed half-spaces determined by the hyperplane $H_{\varphi,c}$, we say $H_{\varphi,c}$, or that φ separates x and A.

Remark 2.5. *[21]* (1) If $\varphi \in \left(X, N, * \right)^{\#}$, it is obvious that $H_{\varphi,c}^{<} \in \tau_N$ and $H_{\varphi,c}^{>} \in \tau_{\overline{N}}$. Meanwhile, $H_{\varphi,c}^{\leq}$ and $H_{\varphi,c}^{\geq}$ are $\tau_{\overline{N}}$-closed and τ_N-closed, respectively.

(2) If $\phi \in \left(X, N, * \right)^{\#-}$, then, $H_{\phi,c}^{<} \in \tau_{\overline{N}}$ and $H_{\phi,c}^{>} \in \tau_N$. Meanwhile, $H_{\phi,c}^{\leq}$ and $H_{\varphi,c}^{\geq}$ are τ_N-closed and $\tau_{\overline{N}}$-closed, respectively.

(3) Both the open half-spaces and the closed half-spaces determined by H are convex subsets.

3. The fuzzy distance from a point to a set

First, we recall the fuzzy distances from a point to a set defined in [21]. Let A be a nonempty subset of a fuzzy quasi-normed space $\left(X, N, * \right)$. Two fuzzy distances from a point $x \in X$ to A were defined as follows: $t \geq 0$,

$$d_N(x, A, t) = \sup\left\{ N(y - x, t) : y \in A \right\},$$

$$d_N(A, x, t) = \sup\left\{ N(x - y, t) : y \in A \right\}.$$

Remark 3.1. It is easy to verify that

$$d_N(x,A,t) = d_{\overline{N}}(A,x,t),$$

$$d_N(x,A,0) = d_N(A,x,0) = 0,$$

$$\lim_{t\to\infty} d_N(x,A,t) = \lim_{t\to\infty} d_N(A,x,t) = 1.$$

Now, we give some properties of these fuzzy distances.

Theorem 3.1. Let A be a nonempty subset of a fuzzy quasi-normed space $(X,N,*)$. Then, A is τ_N-closed if and only if for any $x_0 \in A^c$ there exists $t_0 > 0$ such that $d_N(x_0,A,t_0) < 1$.

Proof. If A is τ_N-closed, then A^c is τ_N-open. Therefore, for any $x_0 \in A^c$, there is $t_0 > 0$ and $r_0 \in (0,1)$ such that $B_N(x_0,1-r_0,t_0) = \{N(y-x_0,t_0) > r_0 : y \in X\} \subseteq A^c$. That is, $N(y-x_0,t_0) \le r_0$ for any $y \in A$, so that $d_N(x_0,A,t_0) \le r_0 < 1$.

Conversely, if for any $x_0 \in A^c$ there exists $t_0 > 0$ such that $d_N(x_0,A,t_0) < 1$, that is, $\sup\{N(x-x_0,t_0) : x \in A\} = r_0 < 1$, then

$$B_N(x_0,1-r_0,t_0) = \{N(y-x_0,t_0) > r_0 : y \in X\} \subseteq A^c.$$

Therefore, x_0 is a τ_N-interior point of A^c. By the arbitrariness of x_0, we know that A^c is τ_N-open. Thus, A is τ_N-closed.

Theorem 3.2. Let A be a nonempty subset of a fuzzy quasi-normed space $(X,N,*)$, $x_0 \in X$, $t > 0$. Then, $d_N(x_0,A,t) = d_N(x_0,\mathrm{cl}_N A,t)$.

Proof. Take $x \in \mathrm{cl}_N A$ arbitrarily. Then, there is a sequence $\{x_n\} \subseteq A$ such that $\lim_{n\to\infty} N(x_n - x,t) = 1$. So, for any $0 < \varepsilon < t$, we have

$$d_N(x,A,t) \ge N(x_n - x_0,t) \ge N(x_n - x,\varepsilon) * N(x - x_0,t-\varepsilon) \to N(x-x_0,t-\varepsilon) \text{ as } n \to \infty.$$

Since $N(x,\_)$ is increasing and left continuous, we get $d_N(x_0,A,t) \ge N(x-x_0,t)$. By the arbitrariness of $x \in \mathrm{cl}_N A$ we know that $d_N(x_0,A,t) \ge d_N(x_0,\mathrm{cl}_N A,t)$. The inverse of the above inequality follows from $A \subseteq \mathrm{cl}_N A$. So, $d_N(x_0,A,t) = d_N(x_0,\mathrm{cl}_N A,t)$.

Lemma 3.1. Let $(X,N,*)$ be a fuzzy quasi-normed space, $\varphi \in X^{\#} \setminus \{0\}$, and $c \in \mathrm{R}$. Then, $H^{\ge}_{\varphi,c} = \mathrm{cl}_N H^{>}_{\varphi,c}$ and $H^{\le}_{\varphi,c} = \mathrm{cl}_{\overline{N}} H^{<}_{\varphi,c}$.

Proof. We distinguish three cases to prove the conclusion.

Case 1: $c > 0$. For any $x \in H_{\varphi,c}$, take $x_n = \frac{n+1}{n} x$ $(n \in ¥)$, then, $x_n \in H^{>}_{\varphi,c}$. Since $\lim_{n\to\infty} N(x_n - x,t) = \lim_{n\to\infty} N(x,nt) = 1$ for any $t > 0$, we get that $x \in \mathrm{cl}_N H^{>}_{\varphi,c}$. By the arbitrariness of $x \in H_{\varphi,c}$, we know $H_{\varphi,c} \subseteq \mathrm{cl}_N H^{>}_{\varphi,c}$, and hence $H^{\ge}_{\varphi,c} = H^{>}_{\varphi,c} \cup H_{\varphi,c} \subseteq \mathrm{cl}_N H^{>}_{\varphi,c}$. By Remark 2.5, $H^{\ge}_{\varphi,c}$ is τ_N-closed, so $H^{\ge}_{\varphi,c} \supseteq \mathrm{cl}_N H^{>}_{\varphi,c}$. Thus, $H^{\ge}_{\varphi,c} = \mathrm{cl}_N H^{>}_{\varphi,c}$.

Again, for any $x \in H_{\varphi,c}$, take $y_n = \frac{n}{n+1} x$ $(n \in ¥)$, then, $y_n \in H^{<}_{\varphi,c}$. Since $\lim_{n\to\infty} \overline{N}(y_n - x,t) = \lim_{n\to\infty} N(x,(n+1)t) = 1$ for any $t > 0$, we have $x \in \mathrm{cl}_{\overline{N}} H^{<}_{\varphi,c}$. By the arbitrariness of

$x\in H_{\varphi,c}$, we know $H_{\varphi,c}\subseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$, and hence $H^{\leq}_{\varphi,c}=H^{<}_{\varphi,c}\cup H_{\varphi,c}\subseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$. By Remark 2.5, $H^{\leq}_{\varphi,c}$ is $\tau_{\overline{N}}$ -closed, so $H^{\leq}_{\varphi,c}\supseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$. Thus, $H^{\leq}_{\varphi,c}=\mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$.

Case 2: $c<0$. For any $x\in H_{\varphi,c}$, take $x_n=\dfrac{n}{n+1}x$ $(n\in\mathbb{N})$, then $x_n\in H^{>}_{\varphi,c}$. Since $\lim\limits_{n\to\infty}N(x_n-x,t)=\lim\limits_{n\to\infty}N(-x,nt)=1$ for any $t>0$, we get $x\in \mathrm{cl}_N H^{>}_{\varphi,c}$. By the arbitrariness of $x\in H_{\varphi,c}$, we know $H_{\varphi,c}\subseteq \mathrm{cl}_N H^{>}_{\varphi,c}$, and hence $H^{\geq}_{\varphi,c}=H^{>}_{\varphi,c}\cup H_{\varphi,c}\subseteq \mathrm{cl}_N H^{>}_{\varphi,c}$. By Remark 2.5, $H^{\geq}_{\varphi,c}$ is τ_N -closed, so that $H^{\geq}_{\varphi,c}\supseteq \mathrm{cl}_N H^{>}_{\varphi,c}$. Thus $H^{\geq}_{\varphi,c}=\mathrm{cl}_N H^{>}_{\varphi,c}$.

Again, for any $x\in H_{\varphi,c}$, take $y_n=\dfrac{n+1}{n}x$ $(n\in\mathbb{N})$, then $y_n\in H^{<}_{\varphi,c}$. Since $\lim\limits_{n\to\infty}\overline{N}(y_n-x,t)=\lim\limits_{n\to\infty}\overline{N}(x,nt)=1$ for any $t>0$, we get $x\in \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$. By the arbitrariness of $x\in H_{\varphi,c}$, we have $H_{\varphi,c}\subseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$, and hence $H^{\leq}_{\varphi,c}=H^{<}_{\varphi,c}\cup H_{\varphi,c}\subseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$. By Remark 2.5, $H^{\leq}_{\varphi,c}$ is $\tau_{\overline{N}}$ -closed, so $H^{\leq}_{\varphi,c}\supseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$. Thus, $H^{\leq}_{\varphi,c}=\mathrm{cl}_{\overline{N}}H^{<}_{\varphi,c}$.

Case 3: $c=0$. Take $x\in H_{\varphi,c}=H_{\varphi,0}$, then $\varphi(x)=c=0$. For any $h\in H^{>}_{\varphi,0}$, take $x_n=\dfrac{1}{n}h+\dfrac{n-1}{n}x$ $(n\in\mathbb{N})$, then $x_n\in H^{>}_{\varphi,0}$. Since $\lim\limits_{n\to\infty}N(x_n-x,t)=\lim\limits_{n\to\infty}N(h-x,nt)=1$ for any $t>0$, we get $x\in \mathrm{cl}_N H^{>}_{\varphi,0}$. By the arbitrariness of $x\in H_{\varphi,0}$, we know $H_{\varphi,0}\subseteq \mathrm{cl}_N H^{>}_{\varphi,0}$, and hence $H^{\geq}_{\varphi,0}=H^{>}_{\varphi,0}\cup H_{\varphi,0}\subseteq \mathrm{cl}_N H^{>}_{\varphi,0}$. Since $H^{\geq}_{\varphi,0}$ is τ_N -closed, $H^{\geq}_{\varphi,0}\supseteq \mathrm{cl}_N H^{>}_{\varphi,0}$. Thus, $H^{\geq}_{\varphi,0}=\mathrm{cl}_N H^{>}_{\varphi,0}$.

For any $g\in H^{<}_{\varphi,0}$, take $y_n=\dfrac{1}{n}g+\dfrac{n-1}{n}x$ $(n\in\mathbb{N})$, then $y_n\in H^{<}_{\varphi,0}$. Since $\lim\limits_{n\to\infty}\overline{N}(y_n-x,t)=\lim\limits_{n\to\infty}\overline{N}(g-x,nt)=1$ for any $t>0$, we get $x\in \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,0}$. By the arbitrariness of $x\in H_{\varphi,0}$, we know $H_{\varphi,0}\subseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,0}$, and hence $H^{\leq}_{\varphi,0}=H^{<}_{\varphi,0}\cup H_{\varphi,0}\subseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,0}$. Since $H^{\leq}_{\varphi,0}$ is $\tau_{\overline{N}}$ -closed, $H^{\leq}_{\varphi,0}\supseteq \mathrm{cl}_{\overline{N}}H^{<}_{\varphi,0}$. Thus, $H^{\leq}_{\varphi,0}=\mathrm{cl}_{\overline{N}}H^{<}_{\varphi,0}$.

Theorem 3.3. Let $(X,N,\wedge)$ be a fuzzy quasi-normed space, $\varphi\in X^{\#}\setminus\{0\}$, $c\in \mathrm{R}$, $t\geq 0$. Then,

(1) $d_N\left(x_0,H_{\varphi,c},t\right)=d_N\left(x_0,H^{\geq}_{\varphi,c},t\right)=d_N\left(x_0,H^{>}_{\varphi,c},t\right),\ \ \forall x_0\in H^{\leq}_{\varphi,c}$;

(2) $d_{\overline{N}}\left(x_0,H_{\varphi,c},t\right)=d_{\overline{N}}\left(x_0,H^{\leq}_{\varphi,c},t\right)=d_{\overline{N}}\left(x_0,H^{<}_{\varphi,c},t\right),\ \ \forall x_0\in H^{\geq}_{\varphi,c}$.

Proof. It follows from Remark 3.1 that the conclusion holds for $t=0$. Now we suppose that $t>0$. Theorem 3.2 and Lemma 3.1 imply that

$$d_N\left(x_0,H^{\geq}_{\varphi,c},t\right)=d_N\left(x_0,H^{>}_{\varphi,c},t\right)\ \text{ and }\ d_{\overline{N}}\left(x_0,H^{\leq}_{\varphi,c},t\right)=d_{\overline{N}}\left(x_0,H^{<}_{\varphi,c},t\right).$$

Moreover, by the definition of fuzzy distance from a point to a set, we have

$$d_N\left(x_0,H_{\varphi,c},t\right)=d_N\left(x_0,H^{\geq}_{\varphi,c},t\right)=1\ \text{ and }\ d_{\overline{N}}\left(x_0,H_{\varphi,c},t\right)=d_{\overline{N}}\left(x_0,H^{\leq}_{\varphi,c},t\right)=1,$$

when $x_0\in H_{\varphi,c}$. So, to complete the proof, we need only to show

(1) $d_N\left(x_0,H_{\varphi,c},t\right)=d_N\left(x_0,H^{\geq}_{\varphi,c},t\right),\ \ \forall x_0\in H^{<}_{\varphi,c}$;

(2) $d_{\overline{N}}\left(x_0,H_{\varphi,c},t\right)=d_{\overline{N}}\left(x_0,H^{\leq}_{\varphi,c},t\right),\ \ \forall x_0\in H^{>}_{\varphi,c}$.

We give only the proof of (1) as the proof of (2) is similar. To this end, take $x\in H^{\geq}_{\varphi,c}=H^{>}_{\varphi,c}\cup H_{\varphi,c}$

arbitrarily. If $x\in H_{\varphi,c}$, it is obvious that $d_N\left(x_0,H_{\varphi,c},t\right)\ge N\left(x-x_0,t\right)$.

Now, we suppose that $x\in H_{\varphi,c}^{>}$. Set

$$f(\lambda)=\lambda\varphi(x_0)+(1-\lambda)\varphi(x),\lambda\in[0,1],$$

then $f(0)=\varphi(x)>c$ and $f(1)=\varphi(x_0)<c$. Since $f(\lambda)$ is continuous with respect to λ, there is $0<\lambda_1<1$ such that $f(\lambda_1)=c$. Let $x_1=\lambda_1x_0+(1-\lambda_1)x$, then $x_1\in H_{\varphi,c}$ and

$$N(x_1-x_0,t)=N(\lambda_1x_0+(1-\lambda_1)x-x_0,t)=N((1-\lambda_1)(x-x_0),t)$$
$$=N(x-x_0,t/(1-\lambda_1))\ge N(x-x_0,t),$$

hence

$$d_N\left(x_0,H_{\varphi,c},t\right)\ge N\left(x_1-x_0,t\right)\ge N\left(x-x_0,t\right).$$

By the arbitrariness of $x\in H_{\varphi,c}^{\ge}$, we get $d_N\left(x_0,H_{\varphi,c},t\right)\ge d_N\left(x_0,H_{\varphi,c}^{\ge},t\right)$. The inverse of the above inequality is obvious. Thus, conclusion (1) holds.

Lemma 3.2. *[13]* Let $(X,N,*)$ be a fuzzy quasi-normed space, and C a nonempty either τ_N-open or $\tau_{\overline{N}}$-open subset of X. If f is a linear functional X and is not identically equal to 0, then

$$\inf f(C)<f(c)<\sup f(C),\quad \forall c\in C.$$

Theorem 3.4. Let A be a nonempty subset of a fuzzy quasi-normed space $(X,N,*)$, $x_0\in X$, $c\in\mathbb{R}$, $f\in X^{\#}\setminus\{0\}$, $H_0=\{x: f(x)=c\}$.

(1) If H_0 separates x_0 and A, then

(i) $x_0\in H_0^{\le}$ implies that $d_N(x_0,A,t)\le d_N(x_0,H_0,t)$, $\forall t\ge 0$;

(ii) $x_0\in H_0^{\ge}$ implies that $d_{\overline{N}}(x_0,A,t)\le d_{\overline{N}}(x_0,H_0,t)$, $\forall t\ge 0$.

(2) If U is a τ_N or $\tau_{\overline{N}}$-open neighborhood of $h_0\in H_0$, then there exist x_1 and x_2 in U such that $f(x_1)<c<f(x_2)$, that is, x_1 and x_2 can be separated by H_0 strictly.

Proof. (1) If H_0 separates x_0 and A, and $x_0\in H_0^{\le}$, then $A\subseteq H_0^{\ge}$. It follows from Theorem 3.3 (1) that $d_N\left(x_0,A,t\right)\le d_N\left(x_0,H_0^{\ge},t\right)=d_N\left(x_0,H_0,t\right)$ for any $t\ge 0$. Similarly, we can prove (ii).

(2) Without loss of generality, we suppose that U is a τ_N-open ball containing h_0, that is, $U=B_N(h_0,1-r,t)=\{x\in X: N(x-h_0,t)>r\}$ where $t>0$ and $r\in(0,1)$. By Lemma 3.2, we get $\inf f(U)<f(h_0)<\sup f(U)$. Since $f(h_0)=c$, there exist x_1 and $x_2\in U$ such that $f(x_1)<c$ $<f(x_2)$.

Lemma 3.3. *[13]* Let $(X,N,\wedge)$ be a fuzzy quasi-normed space and C_1, C_2 two nonempty convex subsets of X with $\operatorname{int}_N C_1\ne\Phi$. Then, the following are equivalent:

(1) for each $\zeta\in C_2-\operatorname{int}_N C_1$ and $\alpha\in(0,1)$, there is an $f_\alpha\in X^{\#}\setminus\{0\}$ such that $f_\alpha(\zeta)=\|\zeta|_\alpha$, $\|f_\alpha|_\alpha^{\#}\ge 1$ and $\sup f_\alpha(C_1)\le\inf f_\alpha(C_2)$;

(2) $C_2\cap\operatorname{int}_N C_1=\Phi$.

Theorem 3.5. Let A be a nonempty convex subset of a fuzzy quasi-normed space $(X, N, \wedge)$, $x_0 \in X$, $t \geq 0$.

(1) If $d_N(x_0, A, t) < 1$, then, for any $\alpha \in (0,1)$, there is $f_\alpha \in X^\# \setminus \{0\}$ such that $\| f_\alpha |_\alpha^\# \geq 1$ and $d_N(x_0, H, t) = d_N(x_0, A, t)$, where $H = \{y : f_\alpha(y) = \inf f_\alpha(A)\}$.

(2) If $d_{\overline{N}}(x_0, A, t) < 1$, then, for any $\alpha \in (0,1)$, there is $g_\alpha \in X^\# \setminus \{0\}$ such that $\| g_\alpha |_\alpha^\# \geq 1$ and $d_{\overline{N}}(x_0, H, t) = d_{\overline{N}}(x_0, A, t)$, where $H = \{y : g_\alpha(y) = \sup g_\alpha(A)\}$.

Proof. (1) Set $P = \{y \in X : N(y - x_0, t) > d_N(x_0, A, t)\}$, then $x_0 \in P$, $P \cap A = \Phi$, and $P \in \tau_N$. Take $x, y \in P$, $\lambda \in [0,1]$ arbitrarily, then

$$N(\lambda x + (1-\lambda)y - x_0, t) \geq N(\lambda(x - x_0), \lambda t) \wedge N((1-\lambda)(y - x_0), (1-\lambda)t)$$

$$= N(x - x_0, t) \wedge N(y - x_0, t) > d_N(x_0, A, t).$$

Thus, $\lambda x + (1-\lambda)y \in P$, and hence P is convex. By Lemma 3.3, for any $\alpha \in (0,1)$, there is $f_\alpha \in X^\# \setminus \{0\}$ such that $\| f_\alpha |_\alpha^\# \geq 1$ and

$$f_\alpha(x_0) \leq \sup f_\alpha(P) \leq \inf f_\alpha(A). \tag{3.1}$$

That is, the hyperplane $H = \{y : f_\alpha(y) = \inf f_\alpha(A)\}$ separates x_0 and A. By Theorem 3.4 (1), we get $d_N(x_0, H, t) \geq d_N(x_0, A, t)$.

If $d_N(x_0, H, t) > d_N(x_0, A, t)$, then there is $h \in H$ such that $N(h - x_0, t) > d_N(x_0, A, t)$. Thus, $h \in P$, and hence P is a τ_N-open neighborhood of h. However, the inequalities (3.1) indicates that all points in P are on the same side of the hyperplane H, which contradicts Theorem 3.4 (2). Thus, $d_N(x_0, H, t) \not> d_N(x_0, A, t)$. Thus $d_N(x_0, H, t) = d_N(x_0, A, t)$.

(2) Set $P = \{y \in X : \overline{N}(y - x_0, t) > d_{\overline{N}}(x_0, A, t)\}$. By a similar method as that used in (1), we can show that $x_0 \in P$, $P \cap A = \Phi$, and P is $\tau_{\overline{N}}$-open convex. Therefore, Lemma 3.3 implies that, for any $\alpha \in (0,1)$, there exists $f_\alpha \in (X, \overline{N}, *)^\# \setminus \{0\} = (X, N, *)^{\#-} \setminus \{0\}$ such that $\| f_\alpha |_\alpha^{\#-} \geq 1$ and

$$f_\alpha(x_0) \leq \sup f_\alpha(P) \leq \inf f_\alpha(A).$$

Let $g_\alpha = -f_\alpha$, then $g_\alpha \in (X, N, \wedge)^\# \setminus \{0\}$ and

$$\| g_\alpha |_\alpha^\# = \sup\{g_\alpha(x) : \| x |_{1-\alpha} \leq 1\} = \sup\{-f_\alpha(x) : \| x |_{1-\alpha} \leq 1\}$$

$$= \sup\{f_\alpha(-x) : \| -x |_{1-\alpha}^- \leq 1\} = \| f_\alpha |_\alpha^{\#-} \geq 1,$$

moreover

$$\sup g_\alpha(A) \leq \inf g_\alpha(P) \leq g_\alpha(x_0). \tag{3.2}$$

So, the hyperplane $H = \{y : g_\alpha(y) = \sup g_\alpha(A)\}$ separates x_0 and A. Using Theorem 3.4 (1), we get $d_{\overline{N}}(x_0, H, t) \geq d_{\overline{N}}(x_0, A, t)$.

If $d_{\overline{N}}(x_0, H, t) > d_{\overline{N}}(x_0, A, t)$, then there is $h \in H$ such that $N(x_0 - h, t) > d_{\overline{N}}(x_0, A, t)$. Thus, $h \in P$, and hence P is a $\tau_{\overline{N}}$-open neighborhood of h. However, the inequalities (3.2) indicate that all points in P are on the same side of the hyperplane H, which contradicts Theorem 3.4 (2). Thus, $d_{\overline{N}}(x_0, H, t) \not> d_{\overline{N}}(x_0, A, t)$. Thus, $d_{\overline{N}}(x_0, H, t) = d_{\overline{N}}(x_0, A, t)$.

Formulas (3.3) and (3.4) can be understood as the fuzzy distance formulas from the point x to the set G.

Theorem 3.6. Let $(X, N, \wedge)$ be a fuzzy quasi-normed space satisfying (FQN7), G be a nonempty subset of X, $x \in X$, $\alpha \in (0,1)$, and $\varphi \in X^{\#} \setminus \{0\}$.

(1) If $c = \inf \varphi(G) > -\infty$ and $\varphi(x) \le c$, then

$$d_N\left(x, G, \frac{c - \varphi(x)}{\| \varphi |_\alpha^{\#}}\right) \le 1 - \alpha; \tag{3.3}$$

(2) If $d = \sup \varphi(G) < \infty$ and $\varphi(x) \ge d$, then

$$d_{\overline{N}}\left(x_0, G, \frac{\varphi(x) - d}{\| \varphi |_\alpha^{\#}}\right) \le 1 - \alpha. \tag{3.4}$$

Proof. (1) Since $c = \inf \varphi(G) > -\infty$ and $\varphi(x) \le c$, for any $g \in G$ we have

$$0 \le c - \varphi(x) \le \varphi(g - x) \le \| \varphi |_\alpha^{\#} \cdot \| g - x |_{1-\alpha},$$

and therefore,

$$\| g - x |_{1-\alpha} \ge \frac{c - \varphi(x)}{\| \varphi |_\alpha^{\#}}.$$

Because $(X, N, \wedge)$ satisfies (FQN7), it follows from Theorem 2.2 (3) that

$$N\left(g - x, \frac{c - \varphi(x)}{\| \varphi |_\alpha^{\#}}\right) \le N\left(g - x, \| g - x |_{1-\alpha}\right) \le 1 - \alpha,$$

which together with the arbitrariness of $g \in G$ implies (3.3).

(2) Since $d = \sup \varphi(G) < \infty$ and $\varphi(x) \ge d$, for any $g \in G$, we get

$$0 \le \varphi(x) - d \le \varphi(x - g) \le \| \varphi |_\alpha^{\#} \cdot \| x - g |_{1-\alpha},$$

and therefore

$$\| x - g |_{1-\alpha} \ge \frac{\varphi(x) - d}{\| \varphi |_\alpha^{\#}} \ge 0.$$

Because $(X, N, \wedge)$ satisfies (FQN7), it follows from Theorem 2.2 (3) that

$$N\left(x - g, \frac{\varphi(x) - \mathrm{s}}{\| f_\alpha |_\alpha^{\#}}\right) \le N\left(x - g, \| x - g |_{1-\alpha}\right) \le 1 - \alpha,$$

which together with the arbitrariness of $g \in G$ implies (3.4).

4. The nearest point from a set to a point

Recently, Wu et al. [21] initiated the concept of nearest point and began research on the best approximation problems in the framework of fuzzy quasi-normed space. Based further study of the properties of nearest points, this section focuses on the existence and equivalent characterization of the nearest point.

Definition 4.1. *[21]* Let A be a nonempty subset of a fuzzy quasi-normed space $(X,N,*)$, $x\in X$, and $t>0$. An element $y_0\in A$ is said to be a N-t-nearest point to x from A if $d_N(x,A,t)=N(y_0-x,t)$.

We denote by $P_A^N(x,t)$ the set of all N-t-nearest points to x from A. For $t>0$, a subset A of a fuzzy quasi-normed space $(X,N,*)$ is called N-t-proximinal if $P_A^N(x,t)\neq\Phi$ for every point $x\in X$.

First, we give the representation of $P_A^N(x,t)$.

Theorem 4.1. Let A be a nonempty subset of a fuzzy quasi-normed space $(X,N,*)$, $x_0\in X$, $t>0$, and $r_0=d_N(x_0,A,t)\in(0,1)$. Set $S=S[x_0,1-r_0,t]$, and $B=B[x_0,1-r_0,t]$. Then,

(1) $P_A^N(x_0,t)=A\cap S=A\cap B$,

(2) $P_A^N(x_0,t)=P_{A\cap S}^N(x_0,t)$.

Proof. (1) If $y_0\in A\cap B$, then $y_0\in A$ and $N(y_0-x_0,t)\geq r_0$. Noting that $d_N(x_0,A,t)=r_0$, we have $y_0\in P_A^N(x_0,t)$. Thus, $A\cap B\subseteq P_A^N(x_0,t)$. Obviously, $S\subseteq B$. Therefore,

$$A\cap S\subseteq A\cap B\subseteq P_A^N(x_0,t). \tag{4.1}$$

Let us suppose that $P_A^N(x_0,t)\neq\Phi$ (otherwise, (1) obviously holds). For any $y\in P_A^N(x_0,t)$, we know that $y\in A$ and $N(y-x_0,t)=r_0$, which means $y\in A\cap S$. Therefore, $P_A^N(x_0,t)\subseteq A\cap S$. Which, together with (4.1), implies that (1).

(2) Take any $y_0\in P_A^N(x_0,t)$, then $N(y_0-x_0,t)=r_0$. It follows from (1) that $y_0\in A\cap S$. Therefore,

$$N(y_0-x_0,t)\leq d_N(x_0,A\cap S,t)\leq d_N(x_0,A,t)=N(y_0-x_0,t).$$

Hence, $d_N(x_0,A\cap S,t)=N(y_0-x_0,t)$, that is, $y_0\in P_{A\cap S}^N(x_0,t)$. So, $P_A^N(x_0,t)\subseteq P_{A\cap S}^N(x_0,t)$. The inverse of the above inequality follows from $A\cap S\subseteq A$. Thus, (2) holds.

Next, we show some basic properties of the set of all N-t-nearest points.

Theorem 4.2. Let A be a nonempty subset of a fuzzy quasi-normed space $(X,N,*)$, $x_0\in X$, and $t>0$.

(1) If A is convex, $*=\wedge$, then $P_A^N(x_0,t)$ is convex.

(2) If A is $\tau_{\overline{N}}$-closed, then $P_A^N(x_0,t)$ is $\tau_{\overline{N}}$-closed.

(3) $P_{A+y}^N(x_0+y,t)=y+P_A^N(x_0,t)$, $\forall y\in X$.

(4) $P_{\alpha A}^N(x_0,t)=\alpha P_A^N(x_0/\alpha,t/\alpha)$, $\forall\alpha>0$.

Proof. (1) Suppose that $P_A^N(x_0,t)\neq\Phi$. Take $y,z\in P_A^N(x_0,t)$ arbitrarily. Since A is convex, so $\alpha y+(1-\alpha)z\in A$ for any $0<\alpha<1$. Hence,

$$
\begin{aligned}
d_N(x_0,A,t) &\geq N(\alpha y+(1-\alpha)z-x_0,t)\\
&\geq N(\alpha y-\alpha x_0,\alpha t)\wedge N((1-\alpha)z-(1-\alpha)x_0,(1-\alpha)t)\\
&= N(y-x_0,t)\wedge N(z-x_0,t)\\
&= d_N(x_0,A,t)\wedge d_N(x_0,A,t)\\
&= d_N(x_0,A,t),
\end{aligned}
$$

and hence $d_N(x_0,A,t)=N(\alpha y+(1-\alpha)z-x_0,t)$. Therefore, $\alpha y+(1-\alpha)z\in P_A^N(x_0,t)$. Thus, $P_A^N(x_0,t)$ is convex.

(2) can be proved by using Theorem 4.1(1) and Remark 2.1.

(3) Since

$$
\begin{aligned}
d_N(x_0+y,A+y,t) &= \sup\{N(x-x_0-y,t): x\in A+y\}\\
&= \sup\{N(a-x_0,t): a\in A\}\\
&= d_N(x_0,A,t),
\end{aligned}
$$

then

$$
\begin{aligned}
z\in P_{A+y}^N(x_0+y,t) &\Leftrightarrow z\in A+y \text{ and } N(z-x_0-y,t)=d_N(x_0+y,A+y,t)\\
&\Leftrightarrow z-y\in A \text{ and } N(z-x_0-y,t)=d_N(x_0,A,t)\\
&\Leftrightarrow z-y\in P_A^N(x_0,t)\\
&\Leftrightarrow z\in y+P_A^N(x_0,t).
\end{aligned}
$$

Thus, $P_{A+y}^N(x_0+y,t)=y+P_A^N(x_0,t)$.

(4) The proof is as follows:

$$
\begin{aligned}
P_{\alpha A}^N(x_0,t) &= \left\{y_0\in\alpha A: N(y_0-x_0,t)=\sup_{y\in\alpha A}N(y-x_0,t)\right\}\\
&= \alpha\left\{z_0\in A: N(\alpha z_0-x_0,t)=\sup_{z\in A}N(\alpha z-x_0,t)\right\}\\
&= \alpha\left\{z_0\in A: N(z_0-x_0/\alpha,t/\alpha)=\sup_{z\in A}N(z-x_0/\alpha,t/\alpha)\right\}\\
&= \alpha P_A^N(x_0/\alpha,t/\alpha).
\end{aligned}
$$

Definition 4.2. A fuzzy quasi-normed space $(X,N,*)$ is said to be strictly convex, if $N(\lambda x+(1-\lambda)y,t)>N(x,t)*N(y,t)$ for any $\lambda\in(0,1)$, $t>0$, and $x,y\in X$ with $x\neq y$, $N(x,t)<1$, and $N(y,t)<1$.

Example 4.1. The fuzzy quasi-normed space $(\mathrm{R},N_{\mathrm{st}},\wedge)$ is strictly convex. In fact, if for any

$x, y \in \mathrm{R}$ with $x \neq y$, $N_{\mathrm{st}}(x,t) < 1$ and $N_{\mathrm{st}}(y,t) < 1$, it follows from the definition of N_{st} that both $x > 0$ and $y > 0$. Without loss of generality, we suppose that $x > y$. Then,

$$N_{\mathrm{st}}\left(\lambda x + (1-\lambda)y, t\right) = \frac{t}{t + \lambda x + (1-\lambda)y} > \frac{t}{t+x} = \frac{t}{t+x} \wedge \frac{t}{t+y} = N_{\mathrm{st}}(x,t) \wedge N_{\mathrm{st}}(y,t)$$

for any $\lambda \in (0,1)$, $t > 0$.

Theorem 4.3. If a fuzzy quasi-normed space $(X, N, \wedge)$ is strictly convex, A is a nonempty convex subset of X, $x_0 \in X$, $t > 0$, and $d_N(x_0, A, t) < 1$. Then, $P_A^N(x_0, t)$ is either an empty set or a single point set.

Proof. Suppose there exist $y, z \in P_A^N(x_0, t) \subseteq A$ with $y \neq z$. Then,

$$N(y - x_0, t) = N(z - x_0, t) = d_N(x_0, A, t) < 1.$$

By Theorem 4.2 (1), we get $(y+z)/2 \in P_A^N(x_0, t)$. Since $(X, N, \wedge)$ is strictly convex, we have

$$N((y+z)/2 - x_0, t) = N((y - x_0)/2 + (z - x_0)/2, t) > N(y - x_0, t) \wedge N(z - x_0, t)$$

$$= d_N(x_0, A, t) \wedge d_N(x_0, A, t) = d_N(x_0, A, t),$$

which contradicts that $(y+z)/2 \in P_A^N(x_0, t)$. The proof is complete.

Now, we investigate the existence of the nearest point.

Theorem 4.4. Let $(X, N, *)$ be a fuzzy quasi-normed space, and A be a nonempty subset of X. If A is $\tau_{\overline{N}}$-closed and $\tau_{\overline{N}}$-compact, then $P_A^N(x,t) \neq \Phi$ for any $x \in X$ and $t > 0$.

Proof. Set $r = d_N(x, A, t)$. Let us suppose that $r > 0$ (otherwise, $r = 0$, and we have $P_A^N(x,t) = A$). For any $n \in \mathbb{N}$, we get $0 < 1 - \frac{nr}{n+1} < 1$. Set

$$A_n^t = A \cap B[x, 1 - \frac{nr}{n+1}, t],$$

then $A_1^t \supseteq A_2^t \supseteq A_3^t \supseteq \cdots \supseteq A_n^t \supseteq \cdots$, and A_n^t is $\tau_{\overline{N}}$-closed. Since $d_N(x, A, t) = r > \frac{nr}{n+1}$, there exists $g_n^t \in A$ such that $N(g_n^t - x, t) > \frac{nr}{n+1}$. Therefore, $g_n^t \in A_n^t \subseteq A$, and hence $g_n^t \in \bigcap_{k=1}^{n} A_K^t \neq \Phi$. That is, $\{A_n^t\}$ has the property of finite intersection. Since A is $\tau_{\overline{N}}$-compact, we get $\cap A_n^t \neq \Phi$.

Take $g_0 \in \cap A_n^t$, then $d_N(x, A, t) \geq N(g_0 - x, t) \geq \frac{nr}{n+1}$, $\forall n \in \mathbb{N}$. Letting $n \to \infty$, we have $d_N(x, A, t) \geq N(g_0 - x, t) \geq r = d_N(x, A, t)$, which means $g_0 \in P_A^N(x,t)$. Thus, $P_A^N(x,t) \neq \Phi$.

In the rest of this section, we characterize nearest points by the dual space.

Theorem 4.5. Let $(X, N, \wedge)$ be a fuzzy quasi-normed space satisfying (FQN7), G is a nonempty convex subset of X, $x_0 \in G^C$, $g_0 \in G$, $t_0 > 0$, $N(g_0 - x_0, t_0) \in (0,1)$, and $\alpha_0 = 1 - N(g_0 - x_0, t_0)$. Then, $g_0 \in P_G^N(x_0, t_0)$ if and only if there is $\varphi_0 \in X^{\#}$ with $\| \varphi_0 \|_{\alpha_0}^{\#} = 1$ such that $\varphi_0(g - x_0) \geq t_0$ for any $g \in G$.

Proof. Since $N(g_0-x_0,t_0)=1-\alpha_0$ and $(X,N,\wedge)$ satisfies (FQN7), we get $\|g_0-x_0\|_{1-\alpha_0}=t_0$ from Remark 2.3.

Necessity: Suppose $g_0\in P_G^N(x_0,t_0)$, then $G\cap B(x_0,\alpha_0,t_0)=\Phi$. Since $g_0-x_0 \in G-B(x_0,\alpha_0,t_0)$, it follows from Lemma 3.3 that there is $\varphi_0\in X^{\#}$ such that $\varphi_0(g_0-x_0) =\|g_0-x_0\|_{1-\alpha_0}=t_0$, $\|\varphi_0\|_{\alpha_0}^{\#}\geq 1$ and

$$\sup\varphi_0(B(x_0,\alpha_0,t_0))\leq\inf\varphi_0(G). \tag{4.2}$$

Take $b_n=\frac{n}{n+1}g_0+\frac{x_0}{n+1}$ for each $n\in¥$, then

$$N(b_n-x_0,t_0)=N\left(\frac{n}{n+1}g_0-\frac{n}{n+1}x_0,t_0\right)=N\left(g_0-x_0,\frac{n+1}{n}t_0\right)$$

$$>N(g_0-x_0,t_0)=1-\alpha_0,$$

and therefore $\{b_n\}\in B(x_0,\alpha_0,t_0)$. By (4.2), we get $\varphi_0(b_n)\leq\varphi_0(g)$ for any $g\in G$. So,

$$\varphi_0(g_0)=\lim_{n\to\infty}\left[\frac{n}{n+1}\varphi_0(g_0)+\frac{1}{n+1}\varphi_0(x_0)\right]=\lim_{n\to\infty}\varphi_0(b_n)\leq\varphi_0(g).$$

Hence $\varphi_0(g-x_0)\geq\varphi_0(g_0-x_0)=t_0$ for any $g\in G$.

Now, we show that $\|\varphi_0\|_{\alpha_0}^{\#}=1$. If it does not hold, then $\|\varphi_0\|_{\alpha_0}^{\#}>1$. It follows from Remark 2.4 there is $y\in X$ such that $\|y\|_{1-\alpha_0}<1$ and $\varphi_0(y)>1$. Let $b_y=x_0+t_0\cdot y$, then

$$\|b_y-x_0\|_{1-\alpha_0}=\|t_0\cdot y\|_{1-\alpha_0}=t_0\|y\|_{1-\alpha_0}<t_0.$$

Which, together with Theorem 2.2 (3), implies that $N(b_y-x_0,t_0)>1-\alpha_0$, that is, $b_y\in B(x_0,\alpha_0,t_0)$. Noting that

$$\varphi_0(b_y-x_0)=t_0\cdot\varphi_0(y)>t_0=\varphi_0(g_0-x_0),$$

we obtain $\varphi_0(b_y)>\varphi_0(g_0)$, which contradicts (4.2). Thus, $\|\varphi_0\|_{\alpha_0}^{\#}=1$.

Sufficiency: Suppose there is $\varphi_0\in X^{\#}$ with $\|\varphi_0\|_{\alpha_0}^{\#}=1$ such that $\varphi_0(g-x_0)\geq t_0$ for any $g\in G$. Then,

$$t_0\leq\varphi_0(g-x_0)\leq\|\varphi_0\|_{\alpha_0}^{\#}\cdot\|g-x_0\|_{1-\alpha_0}=\|g-x_0\|_{1-\alpha_0}.$$

It follows from Theorem 2.1(4) that $N(g-x_0,t_0)\leq 1-\alpha_0$, and therefore,

$$d_N(x_0,G,t_0)=\sup\{N(g-x_0,t_0):g\in G\}\leq 1-\alpha_0.$$

Since $1-\alpha_0=N(g_0-x_0,t_0)$, then $g_0\in P_G^N(x_0,t_0)$.

Theorem 4.6. Let $(X,N,\wedge)$ be a fuzzy quasi-normed space satisfying (FQN7), G is a nonempty convex subset of X, $x_0\in X$, $t_0>0$, $g_0\in G$, $r=N(g_0-x_0,t_0)$, and $r'=\overline{N}(g_0-x_0,t_0)$.

(1) If $0<r<1$, then $g_0 \in P_G^N(x_0,t_0)$ if and only if there is $\varphi \in (X,N,\wedge)^\#$ such that (i) $\varphi(g_0 - x_0) = t_0$, (ii) $\varphi(g_0) = \inf \varphi(G)$, (iii) $\| \varphi |_{1-r}^\# = 1$;

(2) If $0<r'<1$, then $g_0 \in P_G^{\overline{N}}(x_0,t_0)$ if and only if there is $\phi \in (X,N,\wedge)^\#$ such that (i) $\phi(x_0 - g_0) = t_0$, (ii) $\phi(g_0) = \sup \phi(G)$, (iii) $\| \phi |_{1-r'}^\# = 1$.

Proof. (1) Sufficiency: Suppose there is $\varphi \in (X,N,\wedge)^\#$ satisfying (i)-(iii). Then, for each $g \in G$,

$$t_0 = \varphi(g_0 - x_0) = \varphi(g_0) - \varphi(x_0) \le \varphi(g) - \varphi(x_0) \le \| \varphi |_{1-r}^\# \cdot \| g - x_0 |_r = \| g - x_0 |_r .$$

It follows from Theorem 2.1 (4) that $N(g - x_0, t_0) \le r$. Therefore, $d_N(x_0, G, t_0) \le r$. Since $N(g_0 - x_0, t_0) = r$, we get $d_N(x_0, G, t_0) = r$. Thus, $g_0 \in P_G^N(x_0,t_0)$.

Necessity: Suppose that $g_0 \in P_G^N(x_0,t_0)$. Then, for each $g \in G$,

$$r = N(g_0 - x_0, t_0) = \sup_{g \in G} N(g - x_0, t_0) \ge N(g - x_0, t_0).$$

It follows from Theorem 2.2 (3) that $\| g - x_0 |_r \ge t_0$. By Remark 2.3, we get $\| g_0 - x_0 |_r = t_0$. Hence, $\inf_{g \in G} \| g - x_0 |_r = \| g_0 - x_0 |_r = t_0$.

For $g_0 - x_0$, the following facts are known from the proof of necessity of Theorem 4.5: there is $\varphi \in X^\#$ such that $\| \varphi |_{1-r}^\# = 1$, $\varphi(g_0 - x_0) = \| g_0 - x_0 |_r = t_0$ and $\varphi(g - x_0) \ge \varphi(g_0 - x_0) = t_0$ for any $g \in G$. So, φ satisfies (i) and (iii), and $\varphi(g) \ge \varphi(g_0)$ for any $g \in G$. Therefore, $\varphi(g_0) = \inf \varphi(G)$, that is, φ satisfies (ii).

(2) If we replace the fuzzy quasi-norm N in (1) with its conjugate $\overline{N}$, we obtain: $g_0 \in P_G^{\overline{N}}(x_0,t_0)$ if and only if there is $\eta \in (X,\overline{N},\wedge)^\#$ such that it satisfies (i)–(iii) in (1). Let $\phi = -\eta$. Then, it is easy to see that $\phi \in (X,N,\wedge)^\#$ and satisfies (i)–(iii) in (2).

5. Conclusions

The present paper gives some properties of the distance from a point to a set in a fuzzy quasi-normed space, and obtains some important results about nearest points. Obviously, there are many problems which we consider could lead to further research in the topic developed in the present article. For example: (1) How to get some existence conclusions of the nearest point under the weaker condition? (2) How to find the nearest points if they exist? How to design the specific algorithms? (3) How to apply the results obtained in this paper to convex programming and optimization problems in the framework of the fuzzy quasi-normed space.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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