AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 7589–7609.
DOI: 10.3934/math.2024368
Received: 22 December 2023
Revised: 30 January 2024
Accepted: 18 February 2024
Published: 22 February 2024

Research article

Fixed point and endpoint theorems of multivalued mappings in convex b-metric spaces with an application

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Abstract: In this paper, we investigated several new fixed points theorems for multivalued mappings in the framework b-metric spaces. We first generalized S-iterative schemes for multivalued mappings to above spaces by means of a convex structure and then we developed the existence of fixed points and approximate endpoints of the multivalued contraction mappings using iteration techniques. Furthermore, we introduced the modified S-iteration process for approximating a common endpoint of a multivalued α_s-nonexpansive mapping and a multivalued mapping satisfying conditon (E'). We also showed that this new iteration process converges faster than the S-iteration process in the sense of Berinde. Some convergence results for this iterative procedure to a common endpoint under some certain additional hypotheses were proved. As an application, we applied the S-iteration process in finding the solution to a class of nonlinear quadratic integral equations.

Keywords: fixed point and endpoint; multivalued α-nonexpansive mapping; S-iterative process; b-metric space; quadratic integral equation
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The concept of metric space, introduced by Fróchet in 1906, plays an important role in mathematics and several quantitative sciences. The notion of b-metric spaces, as a generalization of metric spaces, was proposed by Bakhtin [1] and Czerwik [2].

Definition 1. *Let U be a nonempty set. If a function $d : U \times U \to [0, +\infty)$, for all $u, v, z \in U$, satisfies the following conditions:*

(1) $d(u, v) = 0$ *if, and only if,* $u = v$ *;*
(2) $d(u, v) = d(v, u)$ *;*

(3) there exists a real number $s \geq 1$ such that $d(u, v) \leq s[d(u, z) + d(z, v)]$.

Then d is called a b-metric on U and (U, d) is called a b-metric space with coefficient $s \geq 1$.

There are many examples of b-metric spaces: Normed linear spaces, Banach spaces, Hilbert spaces, l^p (or L^p)$(p > 0)$ spaces, l^∞ (or L^∞) spaces, CAT(0) spaces [3], and $\mathbb{R}$-trees.

The study of fixed points for multivalued mappings using the Hausdorff metric was first studied by Markin [4]. Successively, Nadler [5] extended the Banach contraction principle for multivalued contractive maps in complete metric spaces. Since then, many authors have devoted themselves to study the validity of the multivalued version of the classical fixed point theorems for some mappings; see [6–13]. Endpoints for multivalued mappings were first studied by Aubin [14], as an important concept of fixed points, which lies between single valued mappings and multivalued mappings. For some results of endpoint theory in metric spaces, we refer to [15–18].

Meanwhile, during the years, the idea of approximating the fixed points for single valued and multivalued nonexpanded mappings by means of an iterative process has attracted much attention. The most used iteration procedure to approximate fixed points is the Picard iteration, that is, $u_n = Tu_n, n \in \mathbb{N}$. It is well known that the Picard iteration does not converge to a fixed point for all kinds of contractive mappings; see [19, 20]. However, other iteration processes have been developed to approximate the fixed points, such as Mann iteration, Ishikawa iteration, Noor iteration, etc. (see [21–23]). One of the interesting and important iterative processes is the S-iterative process, which was first studied by Agarwal et al. [24] in 2007, as follows:

$$\begin{cases} u_1 \in U \\ v_n = \beta_n u_n + (1 - \beta_n)Tu_n \\ u_{n+1} = \alpha_n Tu_n + (1 - \alpha_n)Tv_n \end{cases}$$

where $\alpha_n, \beta_n \in [0, 1]$ and $n \in \mathbb{N}$. They also showed that S-iterative converges at that same rate as Picard iteration and faster than Mann iteration for contractive mappings.

In particular, due to iterative construction of fixed points of nonexpansive mappings depending on the linear structure, many attempts have been made to introduce convex structures in metric spaces. As an important result in this direction, in 1970, Takahashi [25] introduced the concepts of the convex structure and the convex metric space as follows: Let (U, d) be a metric space. A map $w : U \times U \times [0, 1] \to U$ is said to be a convex structure in U if

$$d(z, w(u, v; \alpha)) \leq \alpha d(z, u) + (1 - \alpha)d(z, v)$$

holds for each $z, u, v \in U$, and $\alpha \in [0, 1]$. A metric space embedded with a convex structure is called a convex metric space. After that, numerous researchers were attracted in this direction and developed the iterative process to approximate fixed point in convex metric spaces; see [26–31].

The main purpose of this paper is to show some new results about fixed points and approximate endpoints of multivalued mappings in the setting of b-metric spaces. To begin, we present the existence and uniqueness results of fixed points and approximate endpoints for multivalued contractions mappings in a convex b-metric space by virtue of S-iterative techniques, and an example for supporting the result is presented. Moreover, we introduce the modified the S-iteration process and prove that this new iterative process converges faster than the S-iteration process in the sense of Berinde [19]. We shall establish the results of convergence of the modified S-iteration procedure to a common endpoint of a multivalued α_s-nonexpansive mapping and a multivalued mapping satisfying conditon $(E^{'})$. Finally, we study the existence of a solution to a nonlinear quadratic integral equation to support our main results.

2. Preliminaries

In what follows, $\mathbb{N}$ and $\mathbb{R}$ stand for the set of positive integers and the set of real numbers, respectively. Let U be a nonempty set. For any $u \in U$, set

$$d(u, U) = \inf \{d(u, z) : z \in U\}$$
$$D(u, U) = \sup \{d(u, z) : z \in U\}.$$

The set U is called proximinal if for any $u \in U$, there exists an element $z \in U$ such that $d(u, z) = d(u, U)$. We shall denote the family of closed and bounded subsets and compact subsets of U by $CB(U)$ and $C(U)$. The Hausdorff metric on $CB(U)$ is defined by

$$H(A, B) = \max \left\{ \sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A) \right\}$$

for all $A, B \in CB(U)$.

A point $u \in U$ is called a fixed point of a multivalued mapping T if $u \in Tu$ and u is said to be an endpoint of a multivalued mapping T if $Tu = \{u\}$. We will denote the set of all fixed points of T by $F(T)$ and the set of all endpoints of T by $E(T)$. Obviously, $E(T) \subseteq F(T)$. In addition, let $E(T, S) := E(T) \cap E(S)$. A mapping $T : U \to 2^U$ has the approximation endpoint property [15], if $\inf_{u \in U} D(u, Tu) = 0$.

A multivalued mapping $T : U \to 2^U$ is said to be:

(1) a contraction mapping if there exists an $\lambda \in [0, 1)$ such that

$$H(Tu, Tv) \leq \lambda d(u, v)$$

for all $u, v \in U$;

(2) a nonexpansive mapping if

$$H(Tu, Tv) \leq d(u, v)$$

for all $u, v \in U$;

(3) a quasi nonexpansive such that

$$d(Tu, p) \leq d(u, p)$$

for all $u \in U$ and $p \in F(T)$.

Aoyama et al. [32] introduced the notion of α-nonexpansive mappings. Followed by the multivalued form. For $\alpha < 1$, a multivalued mapping $T : U \to 2^u$ is said to be α-nonexpansive if

$$H^2(Tu, Tv) \leq \alpha d^2(u, Tv) + \alpha d^2(v, Tu) + (1 - 2\alpha) d^2(u, v),$$

for all $u, v \in U$.

Next, we study the following multivalued mappings in b-metric spaces. For $0 \leq \alpha < 1$, a multivalued mapping $T : U \to 2^u$ is said to be α_s-nonexpansive if

$$H^2(Tu, Tv) \leq \frac{1}{s^2}\left[\alpha d^2(u, Tv) + \alpha d^2(v, Tu)\right] + (1 - 2\alpha)d^2(u, v)$$

for all $u, v \in U$.

Remark 1. *It is clear that a multivalued α_s-nonexpansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive.*

In 2011, Llorens-Fuster [33] gave a new class of nonlinear mappings called (L)-type. We introduce (L_s)-type multivalued mapping as follows.

Definition 2. *Let U be a nonempty closed and convex subset of a b-metric space (X, d) and $\{u_n\} \in U$. For any $u \in U$, a multivalued mapping $T : U \to 2^U$ is said to be (L_s)-type if for any $v \in U$ and some $\kappa \geq s^2$, $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ implies*

$$\limsup_{n\to\infty} d(u_n, Tv) \leq \kappa \limsup_{n\to\infty} d(u_n, v).$$

It is natural to study the relationship between multivalued α_s-nonexpansive mappings and (L_s)-type mappings.

Lemma 1. *Let U be a nonempty closed convex subset of a b-metric space (X, d) and $\{u_n\}$ be a bounded sequence in U. For a multivalued α_s-nonexpansive mapping $T : U \to CB(X)$, then T satisfies condition (L_s).*

Proof. For any $v \in U$, we have

$$\begin{aligned} d^2(u_n, Tv) \leq& s^2[d(u_n, Tu_n) + H(Tu_n, Tv)]^2 \\ \leq& s^2 d^2(u_n, Tu_n) + 2s^2 d(u_n, Tu_n)H(Tu_n, Tv) + s^2 H^2(Tu_n, Tv) \\ \leq& s^2 d^2(u_n, Tu_n) + 2s^2 d(u_n, Tu_n)H(Tu_n, Tv) + \alpha d^2(u_n, Tv) + \alpha d^2(v, Tu_n) + s^2(1 - 2\alpha)d^2(u_n, v) \end{aligned}$$

and

$$\begin{aligned} d^2(v, Tu_n) \leq& s^2[d(v, u_n) + d(u_n, Tu_n)]^2 \\ \leq& s^2 d^2(v, u_n) + 2s^2 d(v, u_n)d(u_n, Tu_n) + s^2 d^2(u_n, Tu_n). \end{aligned}$$

Thus, we get

$$d^2(u_n, Tv) \leq \frac{(s^2 + \alpha s^2)d(u_n, Tu_n) + 2s^2 H(Tu_n, Tv) + 2\alpha s^2 d(u_n, v)}{1 - \alpha} d(u_n, Tu_n) + \frac{s^2(1 - 2\alpha) + \alpha s^2}{1 - \alpha} d^2(u_n, v).$$

Taking limit superior as $n \to \infty$, we have

$$\limsup_{n\to\infty} d(u_n, Tv) \leq s^2 \limsup_{n\to\infty} d(u_n, v),$$

which concludes the proof. □

We consider a general class of multivalued nonexpansive type mappings in b-metric spaces. A multivalued mapping $T: U \to 2^u$ satisfies the condition $(E^{'})$ (see [34]) if for some $\mu \geq 1$ such that

$$H(Tu, Tv) \leq \mu d(u, Tu) + d(u, v), for\ all\ u, v \in U.$$

Remark 2. *Let U be a nonempty closed and convex subset of a b-metric space (X, d), then, a multivalued α_s-nonexpansive mapping $T: U \to C(U)$ satisfies condition $(E^{'})$. Indeed, if $H(Tu, Tv) \leq d(u, v)$, it is clear that the claim holds. Suppose that $H(Tu, Tv) > d(u, v)$, then we have*

$$\begin{aligned} H^2(Tu, Tv) \leq & \alpha[d(u, Tu) + H(Tu, Tv)]^2 + \alpha[d(v, u) + d(u, Tu)]^2 + (1 - 2\alpha)d^2(u, v) \\ \leq & \alpha[d^2(u, Tu) + 2d(u, Tu)H(Tu, Tv) + H^2(Tu, Tv)] \\ & + \alpha[d^2(u, v) + 2d(u, v)d(u, Tu) + d^2(u, Tu)] + (1 - 2\alpha)d^2(u, v) \\ = & 2\alpha d(u, Tu)[d(u, Tu) + H(Tu, Tv) + d(u, v)] + \alpha H^2(Tu, Tv) + (1 - \alpha)d^2(u, v), \end{aligned}$$

it follows that

$$H(Tu, Tv) \leq \frac{2\alpha}{1 - \alpha}\frac{d(u, Tu) + H(Tu, Tv) + d(u, v)}{H(Tu, Tv)}d(u, Tu) + \frac{d(u, v)}{H(Tu, Tv)}d(u, v)$$

which means that

$$H(Tu, Tv) \leq \mu d(u, Tu) + d(u, v)$$

where $\mu = 2\frac{1+\alpha}{1-\alpha}\frac{d(u,Tu)+H(Tu,Tv)+d(u,v)}{H(Tu,Tv)}$.

Definition 3. *[1, 2] Let (U, d) be a b-metric space and a sequence $\{u_n\}$ in U, then*

(1) the sequence $\{u_n\}$ is said to be convergent in U to u if, for any $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $u \in U$ such that $d(u_n, u) < \varepsilon$ for all $n \geq n_0$, or, equivalently, $\lim_{n\to\infty} u_n = u$;

(2) the sequence $\{u_n\}$ is said to be Cauchy sequence in U if, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(u_n, u_m) < \varepsilon$ for all $n, m \geq n_0$, or, equivalently, $\lim_{n,m\to\infty} d(u_n, u_m) = 0$;

(3) the b-metric space (U, d) is called complete if every Cauchy sequence is convergent in U.

Chen et al. [35] extended the idea of convexity into b-metric spaces.

Definition 4. *Let (U, d) be a b-metric space and $I = [0, 1]$. A function $w: U \times U \times [0, 1] \to U$ is said to be a convex structure on U if for each $z, u, v \in U$, and $\alpha \in I$ satisfy*

$$d(z, w(u, v, \alpha)) \leq \alpha d(z, u) + (1 - \alpha)d(z, v). \tag{2.1}$$

A b-metric space (U, d) with a convex structure is called a convex b-metric space and it is denoted by the triplet $(U, d; w)$.

Remark 3. *It is worth mentioning that any linear normed space and any of its convex subsets are convex metric spaces, with the natural convex structure*

$$w(u, v; \alpha) = \alpha u + (1 - \alpha)v,$$

but it is not valid for some b-metric spaces [35].

Lemma 2. *[7, 36, 37] Let (U, d) be a b-metric space. For any $A, B, C \in CB(U)$ and any $u, v \in U$, we have the following facts:*

(1) $d(u, B) \leq d(u, v)$ *for all* $v \in B$*;*
(2) $d(u, B) \leq H(A, B)$ *for all* $u \in A$*;*
(3) $H(A, A) = 0$*;*
(4) $H(A, B) = H(B, A)$*;*
(5) $H(A, B) \leq s\,(H(A, C) + H(C, B))$*;*
(6) $d(u, A) \leq s\,(d(u, v) + d(v, A))$*;*
(7) $d(u, A) \leq s\,(d(u, B) + H(B, A))$.

Lemma 3. *[36] Let (U, d) be a b-metric space and $A, B \in CB(U)$, then, for each $u \in U$ and $\varepsilon > 0$, there exists $v \in B$ such that*

$$d(u, v) \leq H(A, B) + \varepsilon.$$

The proof technique of the following lemma can refer to the proof of Proposition 2.1 of Laokul [38].

Lemma 4. *Let (U, d) be a b-metric space and $A, B, C \in CB(U)$. For any $u, v \in U$, then*

(1) $D(u, A) = H(\{u\}, A)$*;*
(2) $D(u, A) \leq s[D(u, B) + H(B, A)]$*;*
(3) $D(u, A) \leq s[d(u, v) + D(v, A)]$.

The following lemma is easy to verify.

Lemma 5. *Let $T : U \to CB(U)$ be a multivalued mapping, then*

(1) $u \in F(T) \Leftrightarrow d(u, Tu) = 0$*;*
(2) $u \in E(T) \Leftrightarrow D(u, Tu) = 0$.

3. Existence of fixed points and endpoints results

Let (U, d, w) be a convex b-metric space and $T : U \to 2^U$ be a given multivalued mapping. For any $n \in \mathbb{N}$, we say $\{u_n\}$ is the sequence generated by the S-iterative procedure involving the mapping T, as follows

$$\begin{cases} u_1 \in U \\ v_n = w(u_n, z_n; \beta_n) \\ u_{n+1} = w(z_n, h_n; \alpha_n) \end{cases} \tag{3.1}$$

where $z_n \in Tu_n$, $h_n \in Tv_n$, and $\alpha_n, \beta_n \in (0, 1)$.

We now establish the existence of fixed points and approximate endpoints of multivalued contraction mappings in a convex b-metric space.

Theorem 1. *Let (U, d, w) be a complete convex b-metric space and let $T : U \to CB(U)$ be a multivalued contraction mapping. For arbitrary chosen $u_1 \in U$, let the iterative sequence $\{u_n\}$ be defined by (3.1). Assume that $0 < \lambda < \frac{1}{s^2}$, then T has a fixed point in U, that is, there exists $u^* \in U$ such that $u^* \in Tu^*$. Moreover, if T has the approximate endpoint property, then $F(T) = E(T) = \{u^*\}$.*

Proof. For any $n \in \mathbb{N}$, if $d(u_n, z_n) = 0$, then we have

$$d(u_n, Tu_n) = d(u_n, z_n) = 0,$$

which implies that $u_n \in Tu_n$ and u_n is a fixed point of T. For the rest, suppose that $d(u_n, z_n) > 0$ for all $n \in \mathbb{N}$. Let $\varepsilon_n = (1-\lambda)(1-\alpha_n)(1-\beta_n)\lambda\alpha_n\beta_n > 0$ and take $\varepsilon = \frac{\varepsilon_n}{2} d(u_n, z_n) > 0$ for each $z_n \in Tu_n$. By Lemma 3, there exist $z_{n-1} \in Tu_{n-1}$ and $h_{n-1} \in Tv_{n-1}$, such that

$$d(z_n, z_{n-1}) \leq H(Tu_n, Tu_{n-1}) + \varepsilon \tag{3.2}$$

and

$$d(z_n, h_{n-1}) \leq H(Tu_n, Tv_{n-1}) + \varepsilon. \tag{3.3}$$

We shall suppose that $u_n \neq u_{n-1}$ for all $n \in \mathbb{N}$. In fact, if $u_n = u_{n-1}$, by the help of triangle inequality and inequalities (3.2) and (3.3), we get

$$\begin{aligned}
d(u_n, z_n) &= d(w(z_{n-1}, h_{n-1}; \alpha_{n-1}), z_n) \\
&\leq \alpha_{n-1} d(z_{n-1}, z_n) + (1-\alpha_{n-1}) d(h_{n-1}, z_n) \\
&\leq \alpha_{n-1}[H(Tu_{n-1}, Tu_n) + \varepsilon] + (1-\alpha_{n-1})[H(Tv_{n-1}, Tu_n) + \varepsilon] \\
&\leq \alpha_{n-1}\lambda d(u_{n-1}, u_n) + (1-\alpha_{n-1})\lambda d(v_{n-1}, u_n) + \varepsilon \\
&\leq (1-\alpha_{n-1})\lambda[\beta_{n-1} d(u_{n-1}, u_n) + (1-\beta_{n-1}) d(z_{n-1}, u_n)] + \varepsilon \\
&\leq (1-\alpha_{n-1})(1-\beta_{n-1})\lambda s[d(z_{n-1}, z_n) + d(z_n, u_n)] + \varepsilon \\
&\leq (1-\alpha_{n-1})(1-\beta_{n-1})\lambda s[H(Tu_{n-1}, Tu_n) + \varepsilon + d(z_n, u_n)] + \varepsilon \\
&\leq (1-\alpha_{n-1})(1-\beta_{n-1})\lambda s[\lambda d(u_{n-1}, u_n) + \varepsilon + d(z_n, u_n)] + \varepsilon \\
&\leq (1-\alpha_{n-1})(1-\beta_{n-1})\lambda s d(z_n, u_n) + ((1-\alpha_{n-1})(1-\beta_{n-1})\lambda s + 1)\varepsilon \\
&\leq (1-\beta_n)\lambda s d(z_n, u_n) < d(u_n, z_n)
\end{aligned}$$

which is a contradiction. Employing Lemma 3 again, for all $z_n \in Tu_n$, there exists $h_n \in Tv_n$ such that

$$d(z_n, h_n) \leq H(Tu_n, Tv_n) + \varepsilon,$$

and for $h_n \in Tv_n$, there exists $z_{n+1} \in Tu_{n+1}$ such that

$$d(h_n, z_{n+1}) \leq H(Tv_n, Tu_{n+1}) + \varepsilon.$$

By the help of the triangle inequality, we obtain

$$\begin{aligned}
d(u_{n+1}, z_{n+1}) \leq& s[d(u_{n+1}, h_n) + d(h_n, z_{n+1})] \\
\leq& s[\alpha_n d(z_n, h_n) + H(Tv_n, Tu_{n+1}) + \varepsilon] \\
\leq& s[\alpha_n H(Tu_n, Tv_n) + \alpha_n\varepsilon + \lambda d(v_n, u_{n+1}) + \varepsilon] \\
\leq& s[\alpha_n\lambda d(u_n, v_n) + \lambda s[d(v_n, z_n) + d(z_n, u_{n+1})] + (1+\alpha_n)\varepsilon] \\
\leq& s[\alpha_n\lambda(1-\beta_n) d(u_n, z_n) + \lambda s[\beta_n d(u_n, z_n) + (1-\alpha_n) d(z_n, h_n)] + (1+\alpha_n)\varepsilon] \\
\leq& s[\alpha_n\lambda(1-\beta_n) d(u_n, z_n) + \lambda s[\beta_n d(u_n, z_n) + (1-\alpha_n) H(Tu_n, Tv_n) + (1-\alpha_n)\varepsilon] + (1+\alpha_n)\varepsilon]
\end{aligned}$$

$$
\begin{aligned}
&\leq s[\alpha_n\lambda(1-\beta_n)d(u_n,z_n)+\lambda s[\beta_n d(u_n,z_n)+(1-\alpha_n)\lambda d(u_n,v_n)+\varepsilon]+2\varepsilon]\\
&\leq s[\alpha_n\lambda(1-\beta_n)d(u_n,z_n)+\lambda s[\beta_n d(u_n,z_n)+(1-\alpha_n)\lambda(1-\beta_n)d(u_n,z_n)]+2\varepsilon]\\
&\leq s[\alpha_n\lambda(1-\beta_n)+\lambda s\beta_n+\lambda^2 s(1-\alpha_n)(1-\beta_n)]d(u_n,z_n)+2\varepsilon\\
&\leq s^2\lambda[\alpha_n(1-\beta_n)+\beta_n+(1-\alpha_n)(1-\beta_n)]d(u_n,z_n)-s^2\lambda(1-\lambda)(1-\alpha_n)(1-\beta_n)d(u_n,z_n)+2\varepsilon\\
&\leq s^2\lambda d(u_n,z_n).
\end{aligned}
$$

Denote that $\gamma = s^2\lambda < 1$. It follows from the above inequality that

$$d(u_n,z_n) \leq \gamma d(u_{n-1},z_{n-1}) \leq \gamma^2 d(u_{n-2},z_{n-2}) \leq \ldots \leq \gamma^n d(u_1,z_1),$$

which shows that $\lim_{n\to\infty} d(u_n,z_n) = 0$. Next, we shall prove that $\{u_n\}$ is a Cauchy sequence. Similarly, for any $z_n \in Tu_n$, by Lemma 3, there exists $z_m \in Tu_m$ such that

$$d(z_n,z_m) \leq H(Tu_n,Tu_m) + \varepsilon.$$

Using the triangle inequality, we have

$$
\begin{aligned}
d(u_n,u_m) &\leq s[d(u_n,z_n)+d(z_n,u_m)]\\
&\leq s[d(u_n,z_n)+sd(z_n,z_m)+sd(z_m,u_m)]\\
&\leq s[d(u_n,z_n)+sH(Tu_n,Tu_m)+s\varepsilon+sd(z_m,u_m)]\\
&\leq 2sd(u_n,z_n)+s^2d(z_m,u_m)+\gamma d(u_n,u_m).
\end{aligned}
$$

Notice that $\gamma < 1$; thus, we derive that

$$d(u_n,u_m) \leq \frac{1}{1-\gamma}[2sd(u_n,z_n)+s^2d(u_m,z_m)].$$

Hence, we conclude that $\lim_{n,m\to\infty} d(u_n,u_m) = 0$, which shows that $\{u_n\}$ is a Cauchy sequence in U. By completeness of U, there exists point $u^* \in U$ such that $\lim_{n\to\infty} u_n = u^*$. Now, we will show that u^* is a fixed point of T.

$$
\begin{aligned}
d(u^*,Tu^*) &\leq s[d(u^*,u_n)+d(u_n,Tu^*)]\\
&\leq s^2[d(u^*,u_n)+d(u_n,z_n)]+sH(Tu_n,Tu^*)\\
&\leq s^2[d(u^*,u_n)+d(u_n,z_n)]+s\lambda d(u_n,u^*).
\end{aligned}
$$

Consequently, we conclude that $\lim_{n\to\infty} d(u^*,Tu^*) = 0$, so u^* is a fixed point of T. Assume that T has the approximate endpoint property and there exists a sequence $\{u_n\}$ such that $\lim_{n\to\infty} D(u_n,Tu_n) = 0$, then we have

$$
\begin{aligned}
D(u^*,Tu^*) &\leq s[d(u^*,u_n)+D(u_n,Tu^*)]\\
&\leq s^2d(u^*,u_n)+s^2[D(u_n,Tu_n)+H(Tu_n,Tu^*)]\\
&\leq s^2d(u^*,u_n)+s^2[D(u_n,Tu_n)+\lambda d(u_n,u^*)].
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} D(u^*, Tu^*) = 0$ which implies that $u^* \in E(T)$. Assume that $u^{**} \in E(T)$, that is, $Tu^{**} = \{u^{**}\}$. We deduce

$$d(u^*, u^{**}) \leq H(Tu^*, Tu^{**}) \leq \lambda d(u^*, u^{**}),$$

hence, the above inequality is true unless $d(u^*, u^{**}) = 0$, i.e., $u^* = u^{**}$. Let $v \in F(T)$. We have

$$d(u^*, v) \leq H(Tu^*, Tv) \leq \lambda d(u^*, v).$$

Thus, $d(u^*, v) = 0$ and $u^* = v$. This complete the proof. □

Remark 4. *Note the well-posedness property of fixed point problem [7] for T with respect to D, if*

(i) $E(T) = \{u^*\}$ *;*
(ii) $\{x_n\}$ *is a sequence that satisfies* $\lim_{n\to\infty} D(u_n, Tu_n) = 0$*, then* $u_n \to u \in U$*, as* $n \to \infty$*.*

The proof of Theorem 1 shows that the fixed point problem of T with respect to D is well-posed.

Let us give an example to illustrate the above theorem.

Example 1. *Let* $U = \mathbb{R}^+ \cup \{0\}$ *and mapping T be defined by* $Tu = [0, \frac{u}{3}]$ *for all* $u = (u_1, u_2, ..., u_n) \in U$*. For any* $u = (u_1, u_2, ..., u_n), v = (v_1, v_2, ..., v_n) \in U$*, we define function* $d : U \times U \to [0, +\infty)$ *by the formula* $d(u, v) = (u - v)^2$*, while the mapping* $w : U \times U \times [0, 1] \to U$ *is defined as* $w(u, v; \alpha) = \alpha u + (1 - \alpha)v$ *for all* $u, v \in U$*. Let the iterative sequence* $\{u_n\}$ *be defined by (3.1) and let* $\alpha_n = \beta_n = \frac{1}{2}$*. By Example 1 in [35], we get that* (U, d, w) *is a complete convex b−metric space with* $s = 2$*. On the other hand, let* $\lambda \in [\frac{1}{9}, \frac{1}{4})$*, then we have*

$$H(Tu, Tv) = H([0, \frac{u}{3}], [0, \frac{v}{3}]) = \frac{(u - v)^2}{9} \leq \lambda(u - v)^2 = \lambda d(u, v).$$

Thus, all the conditions of Theorem 1 are fulfilled. We choose $u_0 \in U\setminus\{0\}$*. According to* $u_{n+1} = w(z_n, h_n; \alpha_n)$ *and* $v_n = w(u_n, z_n; \beta_n)$*, we have* $u_{n+1} = w(z_n, h_n; \alpha_n) = \frac{1}{2}z_n + \frac{1}{2}h_n$ *and* $v_n = w(u_n, z_n; \beta_n) = \frac{1}{2}u_n + \frac{1}{2}z_n$*. Combining with* $z_n \in Tu_n = [0, \frac{u_n}{3}]$*,* $v_n \in [\frac{1}{2}u_n, \frac{2u_n}{3}]$ *and* $h_n \in Tv_n = [0, \frac{u_n}{3}]$*, we obtain that* $u_{n+1} \leq \frac{1}{3}u_n$*. Hence,* $u_{n+1} \leq \left(\frac{1}{3}\right)^n u_0$ *and* $\lim_{n\to\infty} u_{n+1} = 0$*. Since* $T0 = 0$*, 0 is a unique fixed point of T.*

4. Convergence results

In this section, we will use modified S-iteration procedures to approximate common endpoints of multivalued mappings in uniformly convex b-metric spaces.

The notion of uniform convexity in a metric space due to Takahashi [25] is based on an inequality and one parameter only. Now, we present the notion of uniform convexity in a b-metric space.

Definition 5. *A convex b-metric space* (U, d, w) *is said to be uniformly convex if for any* $\varepsilon > 0$*, there exists* $\delta(\varepsilon) > 0$ *such that for all* $\xi > 0$ *and* $u, v, z \in U$ *with* $d(u, z) \leq \xi$*,* $d(v, z) \leq \xi$*, and* $d(u, v) \geq \xi\varepsilon$ *implies that*

$$d\left(z, w(u, v; \frac{1}{2})\right) \leq (1 - \delta)\xi < \xi.$$

A convex b-metric space together with a uniformly convex structure is called a uniformly convex b-metric space.

The proofs of the following lemma are independent of the property of the triangle inequality of the metric d. Here, we only state the results without the proof.

Lemma 6. *[39] Let (U, d, w) ba a uniformly convex b-metric space and $\{\alpha_n\}$ be a sequence in $[n, m]$ for some $n, m \in (0, 1)$. Suppose that $\{u_n\}, \{v_n\}$ are sequences in U and $u \in U$. If $\limsup_{n\to\infty} d(u_n, u) \leq \xi$, $\limsup_{n\to\infty} d(v_n, u) \leq \xi$, and $\lim_{n\to\infty} d(w(u_n, v_n; \alpha_n), u) = \xi$ hold for some $\xi \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

Definition 6. *[16] A multivalued map $T : U \to 2^U$ is said to*

(i) satisfy condition (I) in relation to endpoint, if there is a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $g(a) > 0$ for $a \in (0, \infty)$ such that $D(u, Tu) \geq g(d(u, E(T)))$ for all $u \in U$;

(ii) be semicompact, if for any sequence $\{u_n\}$ in U such that $\lim_{n\to\infty} D(u_n, Tu_n)=0$, there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\lim_{k\to\infty} u_{n(k)} = u^* \in U$.

The following definitions are due to Berinde [19, 20].

Definition 7. *Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes converging to u and v respectively, if*

$$\lim_{n\to\infty} \frac{d(u_n, u)}{d(v_n, v)} = 0,$$

then we say $\{u_n\}$ converges faster than $\{v_n\}$.

Definition 8. *Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes that both converge to the same fixed point p and*

$$\begin{cases} \lim_{n\to\infty} d(u_n, p) \leq a_n \\ \lim_{n\to\infty} d(v_n, p) \leq b_n \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then we say $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Let $T : U \to 2^U$ be a given multivalued mapping. In order to develop an iteration process which rata of convergence for contractive mappings is faster than the S-iteration process, we construct an iteration process in convex b-metric spaces as follows:

$$\begin{cases} u_1 \in U \\ v_n = w(u_n, t_n; \beta_n) \\ u_{n+1} = w(z_n, h_n; \alpha_n) \end{cases} \tag{4.1}$$

where $t_n \in Tu_n$, $z_n \in Tt_n$, $h_n \in Tv_n$, and $\alpha_n, \beta_n \in (0, 1)$.

Next, we prove that the iteration process (4.1) converges faster than iteration process (3.1) in the sense of Berinde.

Theorem 2. *Let (U, d, w) be a complete convex b-metric space and $T : U \longrightarrow 2^U$ a multivalued contractive mapping and fixed point p. If the sequence $\alpha_n \in (0, 1)$ converges $c \neq 0$, then the iterative process (4.1) converges to p faster than the S-iterative process (3.1).*

Proof. Given the iteration process (4.1), we get

$$\begin{aligned} d(v_n, p) &= d(w(u_n, t_n; \beta_n), p) \\ &\leq \beta_n d(u_n, p) + (1 - \beta_n) d(t_n, p) \\ &= \beta_n d(u_n, p) + (1 - \beta_n) H(T u_n, T p) \\ &\leq (\beta_n + (1 - \beta_n)\lambda) d(u_n, p) \end{aligned}$$

and

$$\begin{aligned} d(u_{n+1}, p) &= d(w(z_n, h_n; \alpha_n), p) \\ &\leq \alpha_n d(z_n, p) + (1 - \alpha_n) d(h_n, p) \\ &\leq \alpha_n H(T t_n, T p) + (1 - \alpha_n) H(T v_n, T p) \\ &\leq \alpha_n \lambda d(t_n, p) + (1 - \alpha_n)\lambda d(v_n, T p) \\ &\leq \alpha_n \lambda^2 d(u_n, p) + (1 - \alpha_n)\lambda d(v_n, p), \end{aligned}$$

so that

$$\begin{aligned} d(u_{n+1}, p) &\leq [\alpha_n \lambda^2 + (1 - \alpha_n)(\beta_n + (1 - \beta_n)\lambda)\lambda] d(u_n, p) \\ &\leq [\alpha_n \lambda^2 + (1 - \alpha_n)(\beta_n + (1 - \beta_n)\lambda)\lambda]^2 d(u_{n-1}, p) \\ &\leq ... \leq [\alpha_n \lambda^2 + (1 - \alpha_n)(\beta_n + (1 - \beta_n)\lambda)\lambda]^n d(u_1, p). \end{aligned}$$

Since $\alpha_n, \beta_n \in (0, 1)$, then

$$\begin{aligned} &\alpha_n \lambda^2 + (1 - \alpha_n)(\beta_n + (1 - \beta_n)\lambda)\lambda \\ =&\alpha_n \lambda^2 + (1 - \alpha_n)(\lambda + (1 - \lambda)\beta_n)\lambda \\ =&\lambda^2 + (1 - \alpha_n)(1 - \lambda)\beta_n \lambda \\ \leq&\lambda^2 + (1 - \lambda)\lambda. \end{aligned}$$

Set

$$a_n = [\lambda^2 + (1 - \lambda)\lambda]^n d(u_1, p).$$

Also, given the iteration (3.1), we get

$$\begin{aligned} d(v_n, p) &= d(w(u_n, z_n; \beta_n), p) \\ &\leq \beta_n d(u_n, p) + (1 - \beta_n) d(z_n, p) \\ &\leq \beta_n d(u_n, p) + (1 - \beta_n)\lambda d(u_n, p) \\ &= (\beta_n + (1 - \beta_n)\lambda) d(u_n, p) \end{aligned}$$

and

$$d(u_{n+1}, p) = d(w(z_n, h_n; \alpha_n), p)$$

$$\leq \alpha_n d(z_n, p) + (1-\alpha_n) d(h_n, p)$$
$$\leq \alpha_n \lambda d(u_n, p) + (1-\alpha_n)\lambda d(v_n, p),$$

so that

$$\begin{aligned} d(u_{n+1}, p) &\leq [\alpha_n\lambda + (1-\alpha_n)(\beta_n + (1-\beta_n)\lambda)\lambda] d(u_n, p) \\ &\leq [\alpha_n\lambda + (1-\alpha_n)(\beta_n + (1-\beta_n)\lambda)\lambda]^2 d(u_{n-1}, p) \\ &\leq ... \\ &\leq [\alpha_n\lambda + (1-\alpha_n)(\beta_n + (1-\beta_n)\lambda)\lambda]^n d(u_1, p). \end{aligned}$$

Since $\alpha_n, \beta_n \in (0, 1)$, then

$$\begin{aligned} &\alpha_n\lambda + (1-\alpha_n)(\beta_n + (1-\beta_n)\lambda)\lambda \\ =&\alpha_n\lambda + (1-\alpha_n)(\lambda + (1-\lambda)\beta_n)\lambda \\ =&\alpha_n\lambda + (1-\alpha_n)\lambda^2 + (1-\alpha_n)(1-\lambda)\beta_n\lambda \\ \leq&\alpha_n\lambda + (1-\alpha_n)\lambda^2 + (1-\lambda)\lambda \\ =&\lambda^2 + (1-\lambda)\alpha_n\lambda + (1-\lambda)\lambda. \end{aligned}$$

Set

$$b_n = [\lambda^2 + (1-\lambda)\alpha_n\lambda + (1-\lambda)\lambda]^n d(u_1, p).$$

Hence,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{[\lambda^2 + (1-\lambda)\lambda]^n d(u_1, p)}{[\lambda^2 + (1-\lambda)c\lambda + (1-\lambda)\lambda]^n d(u_1, p)} = 0$$

which completes the proof. □

Motivated by recent research in [16] and combining with iteration process (4.1), we will use the following modified S-iteration process to approximate common endpoints of a multivalued α-nonexpansive mapping and a multivalued mapping satisfying conditon $(E^{'})$ in a uniformly convex b-metric space.

Let $T, S : U \to 2^U$ be two given multivalued mappings. For any $n \in \mathbb{N}$, we say $\{u_n\}$ is the sequence generated by the modified S-iterative procedure involving the mapping T, S, as follows

$$\begin{cases} u_1 \in U \\ v_n = w(u_n, t_n; \beta_n) \\ u_{n+1} = w(z_n, h_n; \alpha_n) \end{cases} \tag{4.2}$$

where $t_n \in Tu_n$, where $d(u_n, t_n) = D(u_n, Tu_n)$, $z_n \in Tt_n$, where $d(t_n, z_n) = D(t_n, Tt_n)$, $h_n \in Sv_n$; therefore, $d(v_n, h_n) = D(v_n, Sv_n)$ and $\alpha_n, \beta_n \in (0, 1)$.

Two multivalued versions of Condition (I) in relation to endpoint are defined as follows.

Definition 9. *Two multivalued mappings $T, S : U \to 2^U$ are said to satisfy condition (II) if there is a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and $g(a) > 0$ for $a \in (0,\infty)$ such that $\frac{1}{2}[D(u, Tu) + D(u, Su)] \geq g(d(u, E(T, S)))$ for all $u \in U$.*

Lemma 7. *Let U be a nonempty closed convex subset of a uniformly convex b-metric space (X, d). Suppose that $T : U \to C(U)$ is a multivalued α_s-nonexpansive mappings and $S : U \to C(U)$ is a multivalued mapping satisfying condition (E'). Assume that $E(T, S) \neq \emptyset$ and $\{u_n\}$ is the sequence of modified S-iterative defined by* (4.2) *and $\alpha_n, \beta_n \in [a, b] \in (0, 1)$, then $\lim_{n\to\infty} d(u_n, p)$ exists for all $p \in E(T, S)$. Moreover, $\lim_{n\to\infty} D(u_n, Tu_n) = \lim_{n\to\infty} D(u_n, Su_n) = 0$.*

Proof. Let $p \in E(T, S)$, and we get

$$d(t_n, p) \leq H(Tu_n, Tp) \leq d(u_n, p). \tag{4.3}$$

Thus,

$$d(z_n, p) \leq H(Tt_n, Tp) \leq d(t_n, p) \leq d(u_n, p). \tag{4.4}$$

Also,

$$\begin{aligned} d(v_n, p) &= d(w(u_n, t_n; \beta_n), p) \\ &\leq \beta_n d(u_n, p) + (1 - \beta_n) d(t_n, p) \\ &\leq d(u_n, p), \end{aligned}$$

and

$$d(h_n, p) \leq H(Sv_n, Sp) \leq d(v_n, p) \leq d(u_n, p). \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\begin{aligned} d(u_{n+1}, p) &= d(w(z_n, h_n; \alpha_n), p) \\ &\leq \alpha_n d(z_n, p) + (1 - \alpha_n) d(h_n, p) \\ &\leq d(u_n, p). \end{aligned}$$

Thus, $\{d(u_n, p)\}$ is nonincreasing. Hence, $\lim_{n\to\infty} d(u_{n+1}, p)$ exists for all $p \in E(T, S)$. Assume that

$$\lim_{n\to\infty} d(u_n, p) = \xi. \tag{4.6}$$

Clearly, we get that

$$\limsup_{n\to\infty} d(z_n, p) \leq \xi, \limsup_{n\to\infty} d(h_n, p) \leq \xi, \tag{4.7}$$

and

$$\limsup_{n\to\infty} d(v_n, p) \leq \xi, \limsup_{n\to\infty} d(t_n, p) \leq \xi. \tag{4.8}$$

As $\lim_{n\to\infty} d(w(z_n, h_n; \alpha_n), p) = \lim_{n\to\infty} d(u_{n+1}, p) = \xi$, by Lemma 6, we obtain

$$\lim_{n\to\infty} d(z_n, h_n) = 0. \tag{4.9}$$

On the other hand, we have the following inequalities

$$\begin{aligned} d(u_{n+1}, p) &\leq \alpha_n d(z_n, p) + (1-\alpha_n) d(h_n, p) \\ &= \alpha_n H(Tt_n, Tp) + (1-\alpha_n) H(Sv_n, Sp) \\ &\leq \alpha_n d(t_n, p) + (1-\alpha_n) d(v_n, p) \\ &\leq \alpha_n d(u_n, p) + (1-\alpha_n) d(v_n, p). \end{aligned}$$

We get

$$d(v_n, p) \geq \frac{\alpha_n}{1-\alpha_n}[d(u_{n+1}, p) - d(u_n, p)] + d(u_{n+1}, p),$$

which gives

$$\lim_{n\to\infty} \inf d(v_n, p) \geq \xi.$$

Hence,

$$\lim_{n\to\infty} d(w(u_n, t_n; \beta_n), p) = \lim_{n\to\infty} d(v_n, p) = \xi.$$

Combining (4.6) and (4.8) and using Lemma 6, we deduce that

$$\lim_{n\to\infty} d(u_n, t_n) = 0. \tag{4.10}$$

Hence,

$$\lim_{n\to\infty} D(u_n, Tu_n) = 0. \tag{4.11}$$

Next, we shall prove that $\lim_{n\to\infty} D(u_n, Su_n) = 0$. Since

$$\begin{aligned} H^2(Tu_n, Tt_n) \leq& \alpha d^2(u_n, Tt_n) + \alpha d^2(t_n, Tu_n) + (1-2\alpha) d^2(u_n, t_n) \\ =& \alpha d^2(u_n, Tt_n) + (1-2\alpha) d^2(u_n, t_n), \end{aligned}$$

from (4.10) and Lemma 1, we have

$$\lim_{n\to\infty} H(Tu_n, Tt_n) = 0. \tag{4.12}$$

Using (4.11) and (4.12) in

$$d(u_n, z_n) \leq D(u_n, Tt_n) \leq s[D(u_n, Tu_n) + H(Tu_n, Tt_n)],$$

we obtain

$$\lim_{n\to\infty} d(u_n, z_n) = 0. \tag{4.13}$$

Similarly,

$$\begin{aligned} d(v_n, h_n) &\leq s[d(v_n, t_n) + d(t_n, h_n)] \\ &\leq s\beta_n d(u_n, t_n) + s^2[d(t_n, u_n) + d(u_n, h_n)] \\ &\leq (s\beta_n + s^2) d(u_n, t_n) + s^3[d(u_n, z_n) + d(z_n, h_n)] \end{aligned}$$

$$= (s\beta_n + s^2)D(u_n, Tu_n) + s^3 d(z_n, h_n) + s^3 d(u_n, z_n).$$

From (4.9), (4.11), and (4.13), we obtain

$$\lim_{n\to\infty} d(v_n, h_n) = 0. \tag{4.14}$$

Note that

$$\begin{aligned} D(u_n, Su_n) &\leq s[d(u_n, v_n) + D(v_n, Su_n)] \\ &\leq sd(u_n, v_n) + s^2[D(v_n, Sv_n) + H(Sv_n, Su_n)] \\ &\leq sd(u_n, v_n) + s^2 d(v_n, h_n) + \mu s^2 d(v_n, Sv_n) + s^2 d(u_n, v_n) \\ &= (s + s^2)d(u_n, v_n) + (s^2 + \mu s^2)d(v_n, h_n) \\ &\leq (s + s^2)(1 - \beta_n)D(u_n, Tu_n) + (s^2 + \mu s^2)d(v_n, h_n), \end{aligned}$$

and it follows from (4.11) and (4.14) that

$$\lim_{n\to\infty} D(u_n, Su_n) = 0.$$

Hence, $\lim_{n\to\infty} D(u_n, Tu_n) = \lim_{n\to\infty} D(u_n, Su_n) = 0$ for $n \in \mathbb{N}$. □

Theorem 3. *Let U be a nonempty closed convex subset of a uniformly convex b-metric space (X, d). Let $T : U \to C(U)$ be a multivalued α_s-nonexpansive mapping and $S : U \to C(U)$ be a multivalued mapping satisfying condition (E') with $E(T, S) \neq \emptyset$. If T and S satisfy condition (II), then the sequence of modified S-iterates $\{u_n\}$ defined by (4.2) with $\alpha_n, \beta_n \in [a, b] \in (0, 1)$ converges to an element of $E(T, S)$.*

Proof. By the condition (II), we have

$$g(d(u_n, E(T, S))) \leq \frac{1}{2}[D(u_n, Tu_n) + D(u_n, Su_n)].$$

By Lemma 7, we get $\lim_{n\to\infty} g(d(u_n, E(T, S))) = 0$, which implies that $\lim_{n\to\infty} d(u_n, E(S, T)) = 0$. We shall claim that $\{u_n\}$ is a Cauchy sequence in U. Let $m, n \in N$. Without loss of generality, we assume that $n < m$. Therefore, Lemma 7 gives that $d(u_m, p) \leq d(u_n, p)$. Now, the estimate is

$$d(u_n, u_m) \leq sd(u_n, p) + sd(p, u_m) \leq 2sd(u_n, p).$$

Taking the infimum for all $p \in E(T, S)$, we have

$$d(u_n, u_m) \leq 2sd(u_n, E(T, S)),$$

which implies that $\lim_{n,m\to\infty} d(u_n, u_m) = 0$. Thus, $\{u_n\}$ is a Cauchy sequence in U and $\{u_n\}$ converges to $u^* \in U$. By using the triangle inequality, we have

$$d(u^*, E(T, S)) \leq s[d(u^*, u_n) + d(u_n, E(T, S))],$$

and letting $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} d(u^*, E(T, S)) = 0$. Therefore, $u^* \in E(T, S)$. This completes the proof. □

Theorem 4. *Let U be a nonempty closed convex subset of a uniformly convex b-metric space (X, d). Suppose that $T : U \to C(U)$ is a multivalued α_s-nonexpansive mapping and $S : U \to C(U)$ is a multivalued mapping satisfying condition (E') with $E(T, S) \neq \emptyset$. If T, S are hemi-compact, then the sequence of modified S-iterates $\{u_n\}$ defined by (4.2) with $\alpha_n, \beta_n \in [a, b] \in (0, 1)$ converges to an element of $E(T, S)$.*

Proof. By the proof of Lemma 7, we have $\lim_{n\to\infty} D(u_n, Tu_n) = \lim_{n\to\infty} D(u_n, Su_n) = 0$. Since T, S are hemi-compact, there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\lim_{k\to\infty} u_{n(k)} = p$ for some $p \in U$. Now we show that $p \in E(T)$. Indeed, by using the triangle inequality, we have

$$\begin{aligned} D(p, Tp) &\leq s[d(p, u_{n(k)}) + D(u_{n(k)}, Tp)] \\ &\leq sd(p, u_{n(k)}) + s^2[D(u_{n(k)}, Tu_{n(k)}) + H(Tu_{n(k)}, Tp)] \\ &\leq sd(p, u_{n(k)}) + s^2[D(u_{n(k)}, Tu_{n(k)}) + d(u_{n(k)}, p)]. \end{aligned}$$

On letting $n \to \infty$, we get $\lim_{k\to\infty} D(p, Tp) = 0$, which implies that $p \in E(T)$. Similarly, we can derive that $p \in E(S)$ for some $p \in U$. Therefore, $p \in E(T, S)$. It follows from Lemma 7 that $\lim_{n\to\infty} d(u_n, p)$ exists for all $p \in E(T, S)$ and, hence, $\{u_n\}$ converges to p. □

5. Application

In this section, we apply Theorem 1 to guarantee the existence of solution to the following integral equation:

$$u(t) \in f(t) + \gamma \int_a^b x(t, \tau)K_1(\tau, u(\tau))d\tau \int_a^b x(t, \tau)K_2(\tau, u(\tau))d\tau, \tag{5.1}$$

for $t \in [a, b]$, where γ is a constant, $f : [a, b] \to \mathbb{R}$, $x : [a, b]\times[a, b] \to [0, \infty)$ and $K_1, K_2 : [a, b]\times\mathbb{R} \to \mathbb{R}$ are continuous functions. Let $U = C([a, b], \mathbb{R})$ denote the space of all continuous functions on $[a, b]$. Suppose that $d : U \times U \longrightarrow [0, \infty)$ be defined by $d(u, v) = \sup_{a\leq t\leq b} |u(t) - v(t)|^2$ and the function $w : U \times U \times (0, 1) \to U$ is defined as $w(u, v; \alpha) = \alpha u + (1 - \alpha)v$. It is known that (X, d, w) is a complete convex $b-$metric spaces with $s = 2$; see [35]. Define multivalued mapping $T : U \to U$ by

$$Tu(t) = \left\{ p(t) \in U : p(t) \in f(t) + \gamma \int_a^b x(t, \tau)K_1(\tau, u(\tau))d\tau \int_a^b x(t, \tau)K_2(\tau, u(\tau))d\tau \right\}.$$

Obviously, T is well defined. Next, we state the following consequence.

Theorem 5. *Assume that the following conditions are satisfied:*

(1) $\gamma \leq \frac{1}{2s}$;
(2) $\int_a^b |x(t, \tau)|^2 d(\tau) \leq 1$;
(3) for all $p, q \in U$, and $k_i(t, u(t)) \in K_i(t, u(t))$, there exists $k_i(t, v(t)) \in K_i(t, v(t))$ such that
$|k_i(\tau, u(t)) - k_i(\tau, v(t))| \leq |u - v|, i = 1, 2$ *and* $\left(\int_a^b x(t, \tau)\,|k_1(\tau, v(\tau))|d\tau\right)^2 + \left(\int_a^b x(t, \tau)\,|k_2(\tau, u(\tau))|\,d\tau\right)^2 = 1.$

The integral equation (5.1) has a solution $u(t) \in U$.

Proof. It is sufficient to find a fixed point of T to find a solution for integral equation (5.1). By Michael's selection theorem, there exist two continuous mappings $k_1(\tau, u(\tau))$ and $k_2(\tau, u(\tau))$ such that

$$p(t) = f(t) + \gamma \int_a^b x(t,\tau)k_1(\tau, u(\tau))d\tau \int_a^b x(t,\tau)k_2(\tau, u(\tau))d\tau.$$

By hypothesis (3), there exist $k_1(\tau, v(\tau))$ and $k_2(\tau, v(\tau))$ such that

$$|k_i(\tau, u(\tau)) - k_i(\tau, v(\tau))| \le |u - v|, \tau \in [a, b], i = 1, 2.$$

Let

$$q(t) = f(t) + \gamma \int_a^b x(t,\tau)k_1(\tau, v(\tau))d\tau \int_a^b x(t,\tau)k_2(\tau, v(\tau))d\tau,$$

which yields that

$$q(t) \in f(t) + \gamma \int_a^b x(t,\tau)K_1(\tau, v(\tau))d\tau \int_a^b x(t,\tau)K_2(\tau, v(\tau))d\tau.$$

Therefore, $q \in Tv$. Notice that

$$\begin{aligned}
d(p,q) =& \sup_{t\in[a,b]} |p(t) - q(t)|^2 \\
=& \sup_{t\in[a,b]} \bigg|\gamma \int_a^b x(t,\tau)k_1(\tau, u(\tau))d\tau \int_a^b x(t,\tau)k_2(\tau, u(\tau))d\tau \\
& -\gamma \int_a^b x(t,\tau)k_1(\tau, v(\tau))d\tau \int_a^b x(t,\tau)k_2(\tau, v(\tau))d\tau\bigg| \\
\le& \gamma^2 \sup_{t\in[a,b]} \bigg|\int_a^b x(t,\tau) |k_1(\tau, u(\tau)) - k_1(\tau, v(\tau))| d\tau \int_a^b x(t,\tau)k_2(\tau, u(\tau))d\tau \\
& + \int_a^b x(t,\tau)k_1(\tau, v(\tau))d\tau \int_a^b x(t,\tau) |k_2(\tau, u(\tau)) - k_2(\tau, v(\tau))| d\tau\bigg|^2 \\
\le& 2\gamma^2 \sup_{t\in[a,b]} \bigg\{\bigg(\int_a^b x(t,\tau) |k_1(\tau, u(\tau)) - k_1(\tau, v(\tau))| d\tau \int_a^b x(t,\tau) |k_2(\tau, u(\tau))| d\tau\bigg)^2 \\
& + \bigg(\int_a^b x(t,\tau) |k_1(\tau, v(\tau))| d\tau \int_a^b x(t,\tau) |k_2(\tau, u(\tau)) - k_2(\tau, v(\tau))| d\tau\bigg)^2\bigg\} \\
\le& 2\gamma^2 \sup_{t\in[a,b]} \bigg(\int_a^b x(t,\tau) |u(\tau) - v(\tau)| d\tau\bigg)^2 \bigg(\int_a^b x(t,\tau) |k_2(\tau, u(\tau))| d\tau\bigg)^2 \\
& + \sup_{t\in[a,b]} \bigg(\int_a^b x(t,\tau) |k_1(\tau, v(\tau))| d\tau\bigg)^2 \bigg(\int_a^b x(t,\tau) |u(\tau) - v(\tau)| d\tau\bigg)^2 \\
\le& 2\gamma^2 \sup_{\tau\in[a,b]} |u(\tau) - v(\tau)| \bigg\{\sup_{t\in[a,b]} \bigg(\int_a^b x(t,\tau)d\tau\bigg)^2 \bigg(\int_a^b x(t,\tau) |k_2(\tau, u(\tau))| d\tau\bigg)^2 \\
& + \sup_{t\in[a,b]} \bigg(\int_a^b x(t,\tau) |k_1(\tau, v(\tau))| d\tau\bigg)^2 \bigg(\int_a^b x(t,\tau)d\tau\bigg)^2\bigg\}
\end{aligned}$$

$$
\leq 2\gamma^2 d(u,v) \sup_{t\in[a,b]} \left(\int_a^b x(t,\tau)d\tau\right)^2 \left\{\left(\int_a^b x(t,\tau)\,|k_2(\tau,u(\tau))|d\tau\right)^2 + \left(\int_a^b x(t,\tau)\,|k_1(\tau,v(\tau))|\,d\tau\right)^2\right\}
$$

$$
\leq 2\left(\frac{1}{2s}\right)^2 d(u,v) = \frac{1}{2s^2}d(u,v),
$$

that is, $d(p,q) \leq \frac{1}{2s^2}d(u,v)$. By just interchanging the role of u and v, we can conclude that $H(Tu,Tv) \leq \lambda d(u,v)$ with $\lambda \in [\frac{1}{2s^2}, \frac{1}{s^2})$. Hence, from Theorem 1, we can get that the integral equation (5.1) has a solution $u(t)$ satisfying $\lim_{n\to\infty} u_n(t) = u(t)$ and the sequence $\{u_n\}$ generated by

$$
\begin{cases} v_n = \beta_n u_n + (1-\beta_n)z_n \\ u_{n+1} = \alpha_n z_n + (1-\alpha_n)h_n \end{cases}
$$

where $z_n \in Tu_n$, $h_n \in Tv_n$, and $\alpha_n, \beta_n \in (0,1)$. □

6. Conclusions

In this paper, we have presented the definitions of the S-iterative schemes and the modified S-iteration schemes for multivalued mappings in b-metric spaces by means of the convex structure. By the S-iteration technique, we developed the existence of fixed points and approximate endpoints of the multivalued contraction mappings in convex b-metric spaces. We also used the modified S-iteration procedures to approximate common endpoints of multivalued mappings in uniformly convex b-metric spaces. Lastly, we gave an application to show the applicability of our obtained results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Natural Science Foundation of Heilongjiang Province of China (Grant No. YQ2021C025).

Conflict of interest

The authors declare no conflict of interest.

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