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Research article

A Legendre spectral method based on a hybrid format and its error estimation for fourth-order eigenvalue problems

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Abstract: In this paper, we developed and studied an efficient Legendre spectral method for fourth order eigenvalue problems with the boundary conditions of a simply supported plate. Initially, a new variational formulation based on a hybrid format and its discrete variational form were established. We then employed the spectral theory of complete continuous operators to establish the prior error estimates of the approximate solutions. By integrating approximation results of some orthogonal projection operators in weighted Sobolev spaces, we further gave the error estimation for the approximating eigenvalues and eigenfunctions. In addition, we developed an effective set of basis functions by utilizing the orthogonal properties of Legendre polynomials, and subsequently derived the matrix eigenvalue system of the discrete variational form for both two-dimensional and three-dimensional cases, based on a tensor product. Finally, numerical examples were provided to demonstrate the exponential convergence and efficiency of the algorithm.

Keywords: fourth order eigenvalue problem; hybrid format; Legendre spectral methods; error estimation
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1. Introduction

Fourth-order equations and eigenvalue problems have been extensively utilized in many scientific and engineering fields [1,2], and the numerical computation of numerous intricate and nonlinear fourth-order equations and eigenvalue problems, such as the Cahn-Hilliard equation and the transmission eigenvalue problem [3–7], stem from repeatedly solving a linear fourth-order equation and eigenvalue problem. Numerous theoretical and numerical computing results have been obtained on fourth-order equations and eigenvalue problems, primarily involving the finite element methods [8–11] and spectral methods [12–17].

For the finite element method applied to fourth-order equations and eigenvalue problems, a C^1 continuous finite element space is typically required. This not only complicates the construction of basis functions but also leads to a significant number of degrees of freedom. The spectral method, which is known to all, is a high-order numerical method with a spectral accuracy and plays a vital role in finding numerical solutions of many differential equations [18–22]. However, it is necessary for the computational domain to be square or cubic. To address this limitation, some spectral element methods are commonly employed to solve differential equations on general domains. For the spectral element methods applied to second-order problems, both their theoretical foundation and numerical calculations are well-established. However, for the spectral element methods applied to fourth-order problems on general domains, the construction of basis functions is also intricate and the computational load is substantial. Therefore, it is highly significant to introduce spectral methods based on hybrid format for the fourth-order equations and eigenvalue problems. To our knowledge, there are few reports on Legendre spectral methods based on a hybrid format for fourth-order eigenvalue problems with the boundary conditions of simply supported plates.

Thus, our aim of the current paper is to propose an efficient Legendre spectral method for fourth order eigenvalue problems with the boundary conditions of a simply supported plate. Initially, a new variational formulation based on a hybrid format and its discrete variational form are established. We then employ the spectral theory of complete continuous operators to establish the prior error estimates of the approximate solutions. By integrating approximation results of some orthogonal projection operators in weighted Sobolev spaces, we further give the error estimation for approximating eigenvalues and eigenfunctions. In addition, we developed an effective set of basis functions by utilizing the orthogonal properties of Legendre polynomials, and subsequently derive the matrix eigenvalue system of the discrete variational form for both two-dimensional and three-dimensional cases, based on tensor product. Finally, numerical examples are provided to demonstrate the exponential convergence and efficiency of the algorithm.

The rest of this article is arranged as follows. In Section 2, we derive the equivalent hybrid format and its Legendre spectral approximation. In Section 3, we provide the error estimation of the approximating eigenvalues and eigenfunctions. In Section 4, we carry on a detailed description for the efficient implementation of the discrete variational form. In Section 5, we present some numerical examples to validate the theoretical findings and the effectiveness of the algorithm. Finally, we give in Section 6 a concluding remark.

2. Hybrid format and its Legendre spectral approximation

In this paper, we consider the fourth-order eigenvalue problem as follows:

$$\Delta^2 u(\mathbf{x}) - \alpha\Delta u(\mathbf{x}) + \beta u(\mathbf{x}) = \lambda u(\mathbf{x}), \quad \text{in } \Omega, \tag{2.1}$$

$$u(\mathbf{x}) = 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

$$\Delta u(\mathbf{x}) = 0, \quad \text{on } \partial\Omega, \tag{2.3}$$

where both α and β are non-negative constants, and $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a bounded domain. Let us introduce an auxiliary variable:

$$\Delta u(\mathbf{x}) + w(\mathbf{x}) = 0. \tag{2.4}$$

The Eqs (2.1)–(2.3) can be restated as:

$$
\begin{aligned}
&-\Delta w(\mathbf{x})+\alpha w(\mathbf{x})+\beta u(\mathbf{x})=\lambda u(\mathbf{x}), && \text{in } \Omega, && (2.5)\\
&-\Delta u(\mathbf{x})=w(\mathbf{x}), && \text{in } \Omega, && (2.6)\\
&w(\mathbf{x})=u(\mathbf{x})=0, && \text{on } \partial\Omega. && (2.7)
\end{aligned}
$$

Without loss of generality, we assume $\beta > 0$. If $\beta = 0$, we can add $u(\mathbf{x})$ to both sides of the equation (2.5). At this time, only the corresponding eigenvalue becomes $\lambda + 1$, and the structure of the equation remains unchanged. By multiplying both sides of equation (2.6) with β, equations (2.5)-(2.7) can be rewritten as

$$
\begin{aligned}
&-\Delta w(\mathbf{x})+\alpha w(\mathbf{x})+\beta u(\mathbf{x})=\lambda u(\mathbf{x}), && \text{in } \Omega, && (2.8)\\
&-\beta\Delta u(\mathbf{x})=\beta w(\mathbf{x}), && \text{in } \Omega, && (2.9)\\
&w(\mathbf{x})=u(\mathbf{x})=0, && \text{on } \partial\Omega. && (2.10)
\end{aligned}
$$

Let $H^m(\Omega)$ and $H_0^m(\Omega)$ be the usual Sobolev spaces of order m, and their norms and seminorms are denoted by $\|\cdot\|_m$ and $|\cdot|_m$, respectively. Especially, we denote $L^2(\Omega)$ by $H^0(\Omega)$, equipped by the inner product $\langle\sigma,\varrho\rangle := \int_\Omega \sigma\bar{\varrho}d\mathbf{x}$ and norm $\|\sigma\|_0 = \sqrt{\langle\sigma,\sigma\rangle}$, here $\bar{\varrho}$ denotes the complex conjugate of ϱ.

Define the product Sobolev spaces as follows:

$$\mathbf{H}_0^1(\Omega) := H_0^1(\Omega)\times H_0^1(\Omega), \mathbf{H}^0(\Omega) := L^2(\Omega)\times L^2(\Omega),$$

and the corresponding norms are given by

$$\|(w,u)\|_{1,\Omega} = (\|w\|_1^2+\|u\|_1^2)^{\frac{1}{2}}, \|(w,u)\|_{0,\Omega} = (\|w\|_0^2+\|u\|_0^2)^{\frac{1}{2}}.$$

Then a variational formulation of (2.8)–(2.10) is: Find $\lambda\in\mathbb{C}$ and $\mathbf{0}\neq(w,u)\in\mathbf{H}_0^1(\Omega)$ such that

$$\mathcal{A}((w,u),(v,\varphi)) = \lambda\mathcal{B}((w,u),(v,\varphi)), \forall(v,\varphi)\in\mathbf{H}_0^1(\Omega), \tag{2.11}$$

where

$$
\begin{aligned}
\mathcal{A}((w,u),(v,\varphi)) &= \int_\Omega \nabla w\nabla\bar{v}+\alpha w\bar{v}+\beta u\bar{v}+\beta\nabla u\nabla\bar{\varphi}-\beta w\bar{\varphi}d\mathbf{x},\\
\mathcal{B}((w,u),(v,\varphi)) &= \int_\Omega u\bar{v}d\mathbf{x}.
\end{aligned}
$$

Define the finite element space $\mathbf{W}_N^d = \mathbf{H}_0^1(\Omega)\cap(P_N^d\times P_N^d)$, here P_N denotes the space of Nth-order polynomials. Then the corresponding discrete variational form of (2.11) reads: Find $\lambda_N\in\mathbb{C}$ and $\mathbf{0}\neq(w_N,u_N)\in\mathbf{W}_N^d$ such that

$$\mathcal{A}((w_N,u_N),(v_N,\varphi_N)) = \lambda_N\mathcal{B}((w_N,u_N),(v_N,\varphi_N)), \quad \forall(v_N,\varphi_N)\in\mathbf{W}_N^d. \tag{2.12}$$

3. Error estimation of the approximate solutions

For simplicity, we limit our consideration to the case where $\Omega = (-1,1)^d$ with $d = 2,3$, and $a \lesssim b$ means that $a \leq cb$, where c is a positive constant independent of N.

Lemma 1. *$\mathcal{A}((w,u),(v,\varphi))$ is a continuous and coercive bilinear functionals defined on $\mathbf{H}_0^1(\Omega)\times\mathbf{H}_0^1(\Omega)$. That is, for any given $((w,u),(v,\varphi))\in\mathbf{H}_0^1(\Omega)\times\mathbf{H}_0^1(\Omega)$, there hold*

$$|\mathcal{A}((w,u),(v,\varphi))| \lesssim \|(w,u)\|_{1,\Omega}\|(v,\varphi)\|_{1,\Omega}, \tag{3.1}$$

$$\mathcal{A}((w,u),(w,u)) \gtrsim \|(w,u)\|_{1,\Omega}^2. \tag{3.2}$$

Proof. Using Cauchy-Schwarz inequality, we derive that

$$\begin{aligned}
|\mathcal{A}((w,u),(v,\varphi))| =& |\int_\Omega \nabla w\nabla\bar{v} + \alpha w\bar{v} + \beta u\bar{v} + \beta\nabla u\nabla\bar{\varphi} - \beta w\bar{\varphi}d\mathbf{x}| \\
&\leqslant \int_\Omega |\nabla w\nabla v| + \alpha|wv| + \beta|uv| + \beta|\nabla u\nabla\varphi| - \beta|w\varphi|d\mathbf{x} \\
&\leqslant \left(\int_\Omega |\nabla w|^2 + \alpha|w|^2 + \beta|u|^2 + \beta|\nabla u|^2 + \beta|w|^2 d\mathbf{x}\right)^{\frac{1}{2}} \\
&\times \left(\int_\Omega |\nabla v|^2| + \alpha|v|^2 + \beta|v|^2 + \beta|\nabla\varphi|^2 + \beta|\varphi|^2 d\mathbf{x}\right)^{\frac{1}{2}} \\
&\lesssim \|(w,u)\|_{1,\Omega}\|(v,\varphi)\|_{1,\Omega}.
\end{aligned}$$

Thus, (3.1) holds. On the other hand, from Poincaré inequality, we have

$$\begin{aligned}
\mathcal{A}((w,u),(w,u)) &= \int_\Omega |\nabla w|^2 + \alpha|w|^2 + \beta uw + \beta|\nabla u|^2 - \beta wud\mathbf{x} \\
&= \int_\Omega |\nabla w|^2 + \alpha|w|^2 + \beta|\nabla u|^2 d\mathbf{x} \\
&= |w|_1^2 + \alpha\|w\|_0^2 + \beta|u|_1^2 \gtrsim \|(w,u)\|_{1,\Omega}^2.
\end{aligned}$$

This proof is completed.

Lemma 2. *$\mathcal{B}((w,u),(v,\varphi))$ is a continuous bilinear functional defined on $\mathbf{H}^0(\Omega)\times\mathbf{H}^0(\Omega)$. That is, for any $(w,u)\in\mathbf{H}^0(\Omega)$ and $(v,\varphi)\in\mathbf{H}^0(\Omega)$, it holds*

$$|\mathcal{B}((w,u),(v,\varphi))| \lesssim \|(w,u)\|_{0,\Omega}\|(v,\varphi)\|_{0,\Omega}. \tag{3.3}$$

Proof. Using Cauchy-Schwarz inequality and the definition of $\mathcal{B}$, we have

$$\begin{aligned}
|\mathcal{B}((w,u),(v,\varphi))| = |\int_\Omega uvd\mathbf{x}| &\le (\int_\Omega |u|^2 d\mathbf{x})^{\frac{1}{2}} \times (\int_\Omega |v|^2 d\mathbf{x})^{\frac{1}{2}} \\
&\le (\int_\Omega |u|^2 + |w|^2 d\mathbf{x})^{\frac{1}{2}} \times (\int_\Omega |v|^2 + |\varphi|^2 d\mathbf{x})^{\frac{1}{2}} \\
&= \|(w,u)\|_{0,\Omega}\|(v,\varphi)\|_{0,\Omega}.
\end{aligned}$$

Note that the soure problem associated with (2.11) is to find $(w,u)\in\mathbf{H}_0^1(\Omega)$ such that

$$\mathcal{A}((w,u),(v,\varphi)) = \mathcal{B}((f,g),(v,\varphi)), \quad \forall(f,g)\in\mathbf{H}^0(\Omega), \quad \forall(v,\varphi)\in\mathbf{H}_0^1(\Omega). \tag{3.4}$$

Using Lemmas 1-2 and Lax-Milgram Theorem, the soure problem (3.4) exists an unique solution. Thus, we can define a solution operator $T : \mathbf{H}^0(\Omega) \to \mathbf{H}_0^1(\Omega)$ such that

$$\mathcal{A}(T(f,g),(v,\varphi)) = \mathcal{B}((f,g),(v,\varphi)), \quad \forall (f,g) \in \mathbf{H}^0(\Omega), \quad \forall (v,\varphi) \in \mathbf{H}_0^1(\Omega). \tag{3.5}$$

Thus, we can obtain the equivalent operator form of (2.11) as follows

$$T(w,u) = \lambda^{-1}(w,u). \tag{3.6}$$

Note that the corresponding adjoint problem of (2.11) is: Find $\lambda^* \in \mathbb{C}$ and $\mathbf{0} \neq (w^*,u^*) \in \mathbf{H}_0^1(\Omega)$ such that

$$\mathcal{A}((v,\varphi),(w^*,u^*)) = \overline{\lambda^*}\mathcal{B}((v,\varphi),(w^*,u^*)), \quad \forall (v,\varphi) \in \mathbf{H}_0^1(\Omega). \tag{3.7}$$

Similarly, the solution operator $T^* : \mathbf{H}^0(\Omega) \to \mathbf{H}_0^1(\Omega)$ can be defined by

$$\mathcal{A}((v,\varphi),T^*(f,g)) = \mathcal{B}((v,\varphi),(f,g)), \quad \forall (f,g) \in \mathbf{H}^0(\Omega), \quad \forall (v,\varphi) \in \mathbf{H}_0^1(\Omega). \tag{3.8}$$

Note that $\mathbf{H}_0^1(\Omega)$ is embedded in $\mathbf{H}^0(\Omega)$, together with (3.8), we can obtain the equivalent operator formulation of (3.7):

$$T^*(w^*,u^*) = (\lambda^*)^{-1}(w^*,u^*).$$

Theorem 1. *Both $T : \mathbf{H}_0^1(\Omega) \to \mathbf{H}_0^1(\Omega)$ and $T^* : \mathbf{H}_0^1(\Omega) \to \mathbf{H}_0^1(\Omega)$ are complete continuous operators.*

Proof. We can obtain by taking $(v,\varphi) = T(f,g)$ in (3.5) that

$$\mathcal{A}(T(f,g),T(f,g)) = \mathcal{B}((f,g),T(f,g)).$$

From Lemmas 1-2 and Poincaré inequality, we have

$$\begin{aligned}\|T(f,g)\|_{1,\Omega}^2 &\lesssim \mathcal{A}(T(f,g),T(f,g)) = \mathcal{B}((f,g),T(f,g)) \\ &\lesssim \|(f,g)\|_{0,\Omega}\|T(f,g)\|_{0,\Omega} \leq \|(f,g)\|_{0,\Omega}\|T(f,g)\|_{1,\Omega},\end{aligned}$$

which implies that

$$\|T(f,g)\|_{1,\Omega} \lesssim \|(f,g)\|_{0,\Omega}. \tag{3.9}$$

Assuming S is a bounded subset in $\mathbf{H}_0^1(\Omega)$, since $\mathbf{H}_0^1(\Omega)$ is compactly embeded in $\mathbf{H}^0(\Omega)$, then S is the sequentially compact set in $\mathbf{H}^0(\Omega)$. It follows from (3.9) that TS is a sequentially compact set in $\mathbf{H}_0^1(\Omega)$. Therefore, $T : \mathbf{H}_0^1(\Omega) \to \mathbf{H}_0^1(\Omega)$ is a complete continuous operator. We derive from (3.4) and (3.7) that

$$\mathcal{A}(T(f,g),(v,\varphi)) = \mathcal{B}((f,g),(v,\varphi)) = \mathcal{A}((f,g),T^*(v,\varphi)), \quad \forall (f,g),\ (v,\varphi) \in \mathbf{H}_0^1(\Omega),$$

which means that in the sense of inner product $\mathcal{A}(\cdot,\cdot)$, T^* is the adjoint operator of T. Thus, $T^* : \mathbf{H}_0^1(\Omega) \to \mathbf{H}_0^1(\Omega)$ is also a complete continuous operator.

We can similarly define the discrete solution operator T_N by

$$\mathcal{A}(T_N(f,g),(v_N,\varphi_N)) = \mathcal{B}((f,g),(v_N,\varphi_N)), \quad \forall (f,g), \in \mathbf{H}^0(\Omega), \quad \forall\ (v_N,\varphi_N) \in \mathbf{W}_N^d. \tag{3.10}$$

It is obvious that both $T_N : \mathbf{H}_0^1(\Omega) \to \mathbf{W}_N^d$ and $\mathbf{H}^0(\Omega) \to \mathbf{W}_N^d$ are all finite rank operators. Using (3.10), we can obtain the equivalent operator formulation of (2.12):

$$T_N(w_N,u_N) = \lambda_N^{-1}(w_N,u_N). \tag{3.11}$$

Let us define a projection operator $\Pi_N : \mathbf{H}_0^1(\Omega) \to \mathbf{W}_N^d$ such that

$$\mathcal{A}((w,u) - \Pi_N(w,u),(v_N,\varphi_N)) = 0, \quad \forall (w,u) \in \mathbf{H}_0^1(\Omega), \quad \forall (v_N,\varphi_N) \in \mathbf{W}_N^d. \tag{3.12}$$

Lemma 3. *Let T and T_N be the operators defined by* (3.5) *and* (3.10)*, respectively. Then, it holds:*

$$T_N = \Pi_N T. \tag{3.13}$$

Proof. For any $(w,u) \in \mathbf{H}_0^1(\Omega)$ and $(v_N,\varphi_N) \in \mathbf{W}_N^d$, we derive that

$$\begin{aligned}
&\mathcal{A}(\Pi_N T(w,u) - T_N(w,u),(v_N,\varphi_N))\\
&= \mathcal{A}(\Pi_N T(w,u) - T(w,u) + T(w,u) - T_N(w,u),(v_N,\varphi_N))\\
&= \mathcal{A}(\Pi_N T(w,u) - T(w,u),(v_N,\varphi_N)) + \mathcal{A}(T(w,u) - T_N(w,u),(v_N,\varphi_N)) = 0.
\end{aligned}$$

By taking $(v_N,\varphi_N) = \Pi_N T(w,u) - T_N(w,u)$, we have

$$\mathcal{A}(\Pi_N T(w,u) - T_N(w,u), \Pi_N T(w,u) - T_N(w,u)) = 0.$$

Then, (3.13) follows from Lemma 1.

It is clear that $T_N|_{\mathbf{W}_N^d} : \mathbf{W}_N^d \to \mathbf{W}_N^d$ is a finite rank operator. Let

$$\xi_N = \sup_{(w,u)\in\mathbf{H}_0^1(\Omega),\|(w,u)\|_{1,\Omega}=1} \ \inf_{(v_N,\varphi_N)\in\mathbf{W}_N^d} \|T(w,u) - (v_N,\varphi_N)\|_{1,\Omega}.$$

With the theory of approximation, we have

$$\lim_{N\to\infty} \xi_N = 0. \tag{3.14}$$

Let us define

$$\|(w,u)\|_{\mathcal{A}} = [\mathcal{A}((w,u),(w,u))]^{\frac{1}{2}}.$$

Then, Lemma 1 implies that $\|(w,u)\|_{\mathcal{A}}$ is a equivalent norm in $\mathbf{H}_0^1(\Omega)$.

Theorem 2. *There holds:*

$$\lim_{N\to\infty} \|T - T_N\|_{\mathcal{L}(\mathbf{H}_0^1(\Omega),\mathbf{H}_0^1(\Omega))} = 0. \tag{3.15}$$

Proof. Based on the definition of operator norm, we have

$$\|T - T_N\|_{\mathcal{L}(\mathbf{H}_0^1(\Omega),\mathbf{H}_0^1(\Omega))} = \sup_{(w,u)\in\mathbf{H}_0^1(\Omega),\|(w,u)\|_{1,\Omega}=1} \|T(w,u) - \Pi_N T(w,u)\|_{1,\Omega}\,.$$

For $\forall(v_N,\varphi_N) \in \mathbf{W}_N^d$, we derive from Lemma 1 and (3.12) that

$$\begin{aligned}
&\|T(w,u) - \Pi_N T(w,u)\|_{1,\Omega}^2 \lesssim \|T(w,u) - \Pi_N T(w,u)\|_{\mathcal{A}}^2 \\
&= \mathcal{A}(T(w,u) - \Pi_N T(w,u), T(w,u) - \Pi_N T(w,u)) \\
&= \mathcal{A}(T(w,u) - \Pi_N T(w,u), T(w,u) - (v_N,\varphi_N)) \\
&\lesssim \|T(w,u) - \Pi_N T(w,u)\|_{1,\Omega}\,\|T(w,u) - (v_N,\varphi_N)\|_{1,\Omega}\,.
\end{aligned}$$

That is,

$$\|T(w,u) - \Pi_N T(w,u)\|_{1,\Omega} \lesssim \|T(w,u) - (v_N,\varphi_N)\|_{1,\Omega}\,.$$

Thus, we have

$$\|T - T_N\|_{\mathcal{L}(\mathbf{H}_0^1(\Omega),\mathbf{H}_0^1(\Omega))} \lesssim \sup_{(w,u)\in\mathbf{H}_0^1(\Omega),\|(w,u)\|_{1,\Omega}=1}\ \inf_{(v_N,\varphi_N)\in\mathbf{W}_N^d} \|T(w,u) - (v_N,\varphi_N)\|_{1,\Omega}.$$

Then, (3.15) holds from (3.14).

Note that the discrete variational form of (2.12) is: Find $\lambda_N^* \in \mathbb{C}, \mathbf{0} \neq (v_N^*,\varphi_N^*) \in \mathbf{W}_N^d$ such that

$$\mathcal{A}((v,\varphi),(v_N^*,\varphi_N^*)) = \overline{\lambda_N^*}\mathcal{B}((v,\varphi),(v_N^*,\varphi_N^*)), \quad \forall(v_N,\varphi_N) \in \mathbf{W}_N^d, \quad \forall(v,\varphi) \in \mathbf{W}_N^d. \tag{3.16}$$

Define the discrete solution operator $T_N^* : \mathbf{H}^0(\Omega) \to \mathbf{W}_N^d$ by

$$\mathcal{A}((v,\varphi),T_N^*(f,g)) = \mathcal{B}((v,\varphi),(f,g)), \quad \forall(f,g) \in \mathbf{H}^0(\Omega), \quad \forall(v,\varphi) \in \mathbf{W}_N^d. \tag{3.17}$$

Then, (3.17) implies that (3.16) has the following equivalent operator form:

$$T_N(w_N^*,u_N^*) = (\lambda_N^*)^{-1}(w_N^*,u_N^*).$$

Let λ and μ be the nonzero eigenvalue with algebraic multiplicity g and the ascent of $(\lambda^{-1} - T)$, respectively. It follows from (3.15) that g eigenvalues $\lambda_{j,N}(j = 1,2,\cdots,g)$ will converge to λ. Let $\rho(T)$ and $\sigma(T)$ be the resolvent set and the spectrum set, respectively. Define the spectral projection operators:

$$E = \frac{1}{2\pi i}\int_\Gamma R_z(T)dz, \quad E_N = \frac{1}{2\pi i}\int_{\Gamma_N} R_z(T_N)dz,$$

where $R_z(T) = (z - T)^{-1}$, and Γ lies in $\rho(T)$ and is a circle centered at λ^{-1} that does not enclose any other points within $\sigma(T)$.

According to [23], E is a projection onto the generalized eigenvectors space corresponding to λ^{-1} and T, that is, $R(E) = \mathcal{N}((\lambda^{-1} - T)^\mu)$, where R and $\mathcal{N}$ denote the range and the null space, respectively.

Similarly, $R(E_N) = \sum\limits_{j=1}^{g} \mathcal{N}((\lambda_{j,N}^{-1} - T_N)^{\mu_j})$, where μ_j is the ascent of $\lambda_{j,N}^{-1} - T_N$. For the dual problems (3.7) and (3.16), we can similarly define $E^*, R(E^*), E_N^*$ and $R(E_N^*)$.

For two closed subspaces Q_1, Q_2, let

$$d(Q_1, Q_2) = \sup_{(w,u)\in Q_1, \|(w,u)\|_{1,\Omega}=1} \inf_{(v,\varphi)\in Q_2} \|(w,u) - (v,\varphi)\|_{1,\Omega}.$$

Defining the gaps between $R(E)$ and $R(E_N)$ in $\mathbf{H}_0^1(\Omega)$ as follows

$$\delta(R(E), R(E_N)) = \max\{d(R(E), R(E_N)), d(R(E_N), R(E))\}.$$

Denote

$$\varepsilon_N(\lambda) = \sup_{(w,u)\in R(E), \|(w,u)\|_{1,\Omega}=1} \inf_{(v_N,\varphi_N)\in \mathbf{W}_N^d} \|(w,u) - (v_N,\varphi_N)\|_{1,\Omega},$$

$$\varepsilon_N^*(\lambda^*) = \sup_{(w^*,u^*)\in R(E^*), \|(w^*,u^*)\|_{1,\Omega}=1} \inf_{(v_N,\varphi_N)\in \mathbf{W}_N^d} \|(w^*,u^*) - (v_N,\varphi_N)\|_{1,\Omega}.$$

Based on the Theorems 8.1–8.4 in [23], we have the following prior error estimates.

Theorem 3. *There exists a constant C such that*

$$\delta(R(E), R(E_N)) \le C\varepsilon_N(\lambda),$$

$$|\lambda - (\frac{1}{g}\sum_{j=1}^{g} \lambda_{j,N}^{-1})^{-1}| \le C\varepsilon_N(\lambda)\varepsilon_N^*(\lambda^*),$$

$$|\lambda - \lambda_{j,N}| \le C[\varepsilon_N(\lambda)\varepsilon_N^*(\lambda^*)]^{\frac{1}{\mu}}.$$

Theorem 4. *If* $\lim\limits_{N\to\infty} \lambda_N = \lambda$. *Suppose for each N that* (w_N, u_N) *satisfy* $\|(w_N, u_N)\|_{1,\Omega} = 1$ *and* $(\lambda_N^{-1} - T_N)^k(w_N, u_N) = 0$ *for some positive integer* $k \le \mu$. *Then, for any integer l with* $k \le l \le \mu$, *there exists a vector* (w, u) *such that* $(\lambda^{-1} - T)^l(w, u) = 0$ *and*

$$\|(w,u) - (w_N, u_N)\|_{1,\Omega} \le C[\varepsilon_N(\lambda)]^{\frac{l-k+1}{\mu}}.$$

In order to offer the error estimates for the approximation of eigenvalues and eigenfunctions, we begin by introducing the d-dimensional Jacobian polynomial and weight function:

$$\boldsymbol{J}_{\boldsymbol{n}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) = \prod_{j=1}^{d} \tilde{J}_{n_j}^{\alpha_j,\beta_j}(x_j), \boldsymbol{\omega}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) = \prod_{j=1}^{d} \omega^{\alpha_j,\beta_j}(x_j), \forall \boldsymbol{x} \in I^d, \tag{3.18}$$

where $\boldsymbol{n} = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_d)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_d)$, $I = (-1, 1)$. Define the non-uniformly weighted Sobolev space:

$$B_{\boldsymbol{\alpha},\boldsymbol{\beta}}^s(I^d) := \{\rho : \partial_{\boldsymbol{x}}^{\boldsymbol{k}}\rho \in L^2_{\boldsymbol{\omega}^{\boldsymbol{\alpha}+\boldsymbol{k},\boldsymbol{\beta}+\boldsymbol{k}}}(I^d),\ 0 \le |\boldsymbol{k}|_1 \le s\},\quad \forall s \in \mathbb{N},$$

with the following norm and semi-norm

$$\|\rho\|_{B_{\boldsymbol{\alpha},\boldsymbol{\beta}}^s(I^d)} = \Big(\sum_{0\le|\boldsymbol{k}|_1\le s} \|\partial_{\boldsymbol{x}}^{\boldsymbol{k}}\rho\|^2_{\boldsymbol{\omega}^{\boldsymbol{\alpha}+\boldsymbol{k},\boldsymbol{\beta}+\boldsymbol{k}}}\Big)^{\frac{1}{2}}, |\rho|_{B_{\boldsymbol{\alpha},\boldsymbol{\beta}}^s(I^d)} = \Big(\sum_{j=1}^{d} \|\partial_{x_j}^s\rho\|^2_{\boldsymbol{\omega}^{\boldsymbol{\alpha}+s\boldsymbol{e}_j,\boldsymbol{\beta}+s\boldsymbol{e}_j}}\Big)^{\frac{1}{2}},$$

where $\boldsymbol{e}_j$ is the jth unit vector in $\mathbb{R}^d$, $\boldsymbol{k} = (k_1, k_2, \cdots, k_d)$, $|\boldsymbol{k}|_1 = k_1 + k_2 + \cdots + k_d$. From the theorem 8.1 and remark 8.14 in [18], we have following lemma.

Lemma 4. *There exists a projection operator* $\mathbf{\Pi}_N^{-1,-1} : \mathbf{H}_0^1(\Omega) \to \mathbf{W}_N^d$*, such that for any* $\sigma \in B^s_{-1,-1}(\mathcal{D})$*, it holds for* $1 \le s \le N+1$ *that*

$$|\mathbf{\Pi}_N^{-1,-1}\sigma - \sigma|_{B^1_{-1,-1}(I^d)} \le c\sqrt{\frac{(N-s)!}{(N-1)!}}\Big(N+s\Big)^{\frac{1-s}{2}} |\sigma|_{B^s_{-1,-1}(I^d)},$$

where $c \simeq \sqrt{2}$ *for* $N \gg 1$.

Theorem 5. *There exists a constant C such that*

$$\varepsilon_N(\lambda) \le CN^{1-m} \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} [|w|_{B^m_{-1,-1}(I^d)} + |u|_{B^m_{-1,-1}(I^d)}], \tag{3.19}$$

$$\varepsilon^*_N(\lambda^*) \le CN^{1-m} \sup_{(w^*,u^*)\in R(E^*),\|(w^*,u^*)\|_{1,\Omega}=1} [|w^*|_{B^m_{-1,-1}(I^d)} + |u^*|_{B^m_{-1,-1}(I^d)}]. \tag{3.20}$$

Proof. We only give the proof for (3.19), and the same argument can be applied to (3.20). Using Poincaré inequality, we derive that

$$\begin{aligned}\varepsilon_N(\lambda) =& \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} \inf_{(v_N,\varphi_N)\in \mathbf{W}_N^d} \|(w,u)-(v_N,\varphi_N)\|_{1,\Omega} \\ \le& C \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} \inf_{(v_N,\varphi_N)\in \mathbf{W}_N^d} [|w-v_N|_1^2 + |u-\varphi_N|_1^2]^{\frac{1}{2}} \\ \le& C \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} [|w-\mathbf{\Pi}_N^{-1,-1}w|_1^2 + |u-\mathbf{\Pi}_N^{-1,-1}u|_1^2]^{\frac{1}{2}}.\end{aligned}$$

Note that

$$\begin{aligned}|w-\mathbf{\Pi}_N^{-1,-1}w|_1^2 &= \sum_{j=1}^d \int_{I^d} [\partial_{x_j}(w-\mathbf{\Pi}_N^{-1,-1}w)]^2 d\mathbf{x} \\ \le \sum_{j=1}^d \int_{I^d} &[\partial_{x_j}(w-\mathbf{\Pi}_N^{-1,-1}w)]^2 \prod_{i=1,i\neq j}^d \frac{1}{(1-x_i)(1+x_i)} d\mathbf{x} \\ = \sum_{j=1}^d \int_{I^d} &[\partial_{x_j}(w-\mathbf{\Pi}_N^{-1,-1}w)]^2 \omega^{-\mathbf{1}+\mathbf{e}_j,-\mathbf{1}+\mathbf{e}_j} d\mathbf{x} = |w-\mathbf{\Pi}_N^{-1,-1}w|^2_{B^1_{-1,-1}(I^d)}.\end{aligned}$$

Through a similar derivation, we can obtain that

$$|u-\mathbf{\Pi}_N^{-1,-1}u|_1^2 \le |u-\mathbf{\Pi}_N^{-1,-1}u|^2_{B^1_{-1,-1}(I^d)}.$$

From Lemma 4, we derive that

$$\begin{aligned}\varepsilon_N(\lambda) \le& C \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} [|w-\mathbf{\Pi}_N^{-1,-1}w|_1^2 + |u-\mathbf{\Pi}_N^{-1,-1}u|_1^2]^{\frac{1}{2}} \\ \le& C \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} [|w-\mathbf{\Pi}_N^{-1,-1}w|_{B^2_{-1,-1}}(I^d) + |u-\mathbf{\Pi}_N^{-1,-1}u|_{B^1_{-1,-1}}(I^d)] \\ \le& C\sqrt{\frac{(N-m)!}{(N-1)!}}(N+m)^{\frac{1-m}{2}} \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} [|w|_{B^m_{-1,-1}}(I^d) + |u|_{B^m_{-1,-1}}(I^d)].\end{aligned}$$

Then, (3.19) follows from (3.5.32) in [18]. The proof is completed.

Denote

$$\mathcal{M}_m(w,u) = \sup_{(w,u)\in R(E),\|(w,u)\|_{1,\Omega}=1} [|w|_{B^m_{-1,-1}(I^d)} + |u|_{B^m_{-1,-1}(I^d)}],$$
$$\mathcal{M}^*_m(w*,u*) = \sup_{(w^*,u^*)\in R(E^*),\|(w^*,u^*)\|_{1,\Omega}=1} [|w^*|_{B^m_{-1,-1}(I^d)} + |u^*|_{B^m_{-1,-1}(I^d)}].$$

We can obtain from Theorems 3-5 the error estimation of the approximating eigenvalues and eigenfunctions.

Theorem 6. *There exists a constant C such that*

$$\delta(R(E),R(E_N)) \le CN^{1-m}\mathcal{M}_m(w,u),$$
$$|\tau - (\frac{1}{g}\sum_{j=1}^{g}\tau^{-1}_{j,N})^{-1}| \le CN^{2(1-m)}\mathcal{M}_m(w,u)\mathcal{M}^*_m(w*,u*),$$
$$|\lambda - \lambda_{j,N}| \le CN^{\frac{2(1-m)}{\mu}}[\mathcal{M}_m(w,u)\mathcal{M}^*_m(w*,u*)]^{\frac{1}{\mu}}.$$

Theorem 7. *If* $\lim_{N\to\infty}\lambda_N = \lambda$. *Suppose for each N that* (w_N,u_N) *satisfy* $\|(w_N,u_N)\|_{1,\Omega} = 1$ *and* $(\lambda_N^{-1} - T_N)^k(w_N,u_N) = 0$ *for some positive integer* $k \le \mu$. *Then, for any integer l with* $k \le l \le \mu$, *there exists a vector* (w,u) *such that* $(\lambda^{-1} - T)^l(w,u) = 0$ *and*

$$\|(w,u) - (w_N,u_N)\|_{1,\Omega} \le C[N^{1-m}\mathcal{M}_m(w,u)]^{\frac{l-k+1}{\mu}}.$$

4. Efficient implementation of the discrete variational form.

In order to effectively solve problem (2.12), we first construct a set of basis functions for the approximation space $\mathbf{W}^d_N$. Denote by $L_m(x)$ the Legendre polynomial of degree m. Let

$$\varphi_m(x) = L_m(x) - L_{m+2}(x), (m = 0,1,\cdots,N-2),$$

It is obvious that

$$\mathbf{W}^d_N = \{\prod_{k=1}^{d}\varphi_{ik}(x_k) : ik = 0,1,\cdots,N-2\}\times\{\prod_{k=1}^{d}\varphi_{jk}(x_k) : jk = 0,1,\cdots,N-2\}.$$

Denote

$$a_{ij} = \int_I \varphi'_j(x)\varphi'_i(x)dx, b_{ij} = \int_I \varphi_j(x)\varphi_i(x)dx.$$

• Case $d = 2$. We can expand the eigenfunctions as follows:

$$(w_N,u_N) = (\sum_{i,j=0}^{N-2} w_{ij}\varphi_i(x_1)\varphi_j(x_2), \sum_{i,j=0}^{N-2} u_{ij}\varphi_i(x_1)\varphi_j(x_2)), \tag{4.1}$$

where w_{ij}, u_{ij} are the expansion coefficients of the w_N and u_N, respectively.

Denote

$$\mathbf{w} = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0,N-2} \\ w_{10} & w_{11} & \cdots & w_{1,N-2} \\ \vdots & \vdots & \cdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \cdots & w_{N-2,N-2} \end{pmatrix},$$

$$\mathbf{u} = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0,N-2} \\ u_{10} & u_{11} & \cdots & u_{1,N-2} \\ \vdots & \vdots & \cdots & \vdots \\ u_{N-2,0} & u_{N-2,1} & \cdots & u_{N-2,N-2} \end{pmatrix}.$$

We use $\bar{\mathbf{u}}$ to denote the vector formed by the columns of $\mathbf{u}$. Now, plugging the expressions of (4.1) in (2.12), and taking (v_N, φ_N) through all the basis functions in $\mathbf{W}_N^2$, the discrete variational form (2.12) is equivalent to the following matrix form:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{w}} \\ \bar{\mathbf{u}} \end{bmatrix} = \lambda_N \begin{bmatrix} \mathbf{0} & \mathcal{E} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{w}} \\ \bar{\mathbf{u}} \end{bmatrix}, \tag{4.2}$$

where

$$\mathcal{A} = A \otimes B + B \otimes A - \alpha B \otimes B,\ \mathcal{B} = \beta B \otimes B,$$
$$\mathcal{C} = -\beta B \otimes B,\ \mathcal{D} = \beta(A \otimes B + B \otimes A),\ \mathcal{E} = B \otimes B,$$

where $A = (a_{ij}), B = (b_{ij})$, $\otimes$ represents the tensor product symbol of the matrix.

• Case $d = 3$. Here, we can expand the eigenfunctions as follows:

$$(w_N, u_N) = (\sum_{i,j,k=0}^{N-2} w_{ijk}\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3), \sum_{i,j,k=0}^{N-2} u_{ijk}\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)). \tag{4.3}$$

Denote

$$\mathbf{w}^k = \begin{pmatrix} w^k_{00} & w^k_{01} & \cdots & w^k_{0,N-2} \\ w^k_{10} & w^k_{11} & \cdots & w^k_{1,N-2} \\ \vdots & \vdots & \cdots & \vdots \\ w^k_{N-2,0} & w^k_{N-2,1} & \cdots & w^k_{N-2,N-2} \end{pmatrix},$$

$$\mathbf{u}^k = \begin{pmatrix} u^k_{00} & u^k_{01} & \cdots & u^k_{0,N-2} \\ u^k_{10} & u^k_{11} & \cdots & u^k_{1,N-2} \\ \vdots & \vdots & \cdots & \vdots \\ u^k_{N-2,0} & u^k_{N-2,1} & \cdots & u^k_{N-2,N-2} \end{pmatrix}.$$

Denote by $\bar{\mathbf{w}}^k$ and $\bar{\mathbf{u}}^k$ the vectors formed by the columns of $\mathbf{w}^k$ and $\mathbf{u}^k$, respectively. Let $\mathbf{W} = (\bar{\mathbf{w}}^0, \bar{\mathbf{w}}^1, \cdots, \bar{\mathbf{w}}^{N-2})$, $\mathbf{U} = (\bar{\mathbf{u}}^0, \bar{\mathbf{u}}^1, \cdots, \bar{\mathbf{u}}^{N-2})$. Denote by $\bar{\mathbf{W}}$ and $\bar{\mathbf{U}}$ the vectors formed by the columns of $\mathbf{W}$ and $\mathbf{U}$. Now, plugging the expressions of (4.3) in (2.12), and taking (v_N, φ_N) through all the basis functions in $\mathbf{W}_N^3$, we obtain the matrix form of the discrete variational form (2.12) as follows:

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{U} \end{bmatrix} = \lambda_N \begin{bmatrix} \mathbf{0} & \mathbb{E} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{U} \end{bmatrix}, \tag{4.4}$$

where

$$
\begin{aligned}
&\mathbb{A} = B \otimes B \otimes A + B \otimes A \otimes B + A \otimes B \otimes B - \alpha B \otimes B \otimes B,\\
&\mathbb{B} = \beta B \otimes B \otimes B,\ \mathbb{C} = -\beta B \otimes B \otimes B,\\
&\mathbb{D} = \beta(B \otimes B \otimes A + B \otimes A \otimes B + A \otimes B \otimes B),\ \mathbb{E} = B \otimes B \otimes B.
\end{aligned}
$$

It is noted that each block matrix in (4.2) and (4.4) is sparse, and each non-zero element in them can be precisely calculated by utilizing the orthogonal properties of Legendre polynomials [18]. Therefore, we can employ the sparse solver $eigs(A, B, k,' sm')$ to effectively solve (4.2) and (4.4).

5. Numerical experiment

In this section, a series of numerical experiments will be presented to confirm the theoretical findings and demonstrate the efficiency of our algorithm. Our program is compiled and executed in MATLAB R2019a.

Example 1 We take $\Omega = (-1, 1)^2$, $\alpha = \beta = 1$. The numerical results of the first fourth eigenvalues λ_N^1, λ_N^2 (double eigenvalue), λ_N^3, λ_N^4 for different N are listed in Table 1. To intuitively demonstrate the spectral accuracy of our algorithm, we employ the numerical solution with $N = 40$ as a reference solution and plot the absolute error curves of approximate eigenvalues as well as corresponding error curves under a log-log scale in Figure 1. Additionally, we also give an image of the reference solution for the eigenfunction and an error image between the reference solution and the approximate solution with $N = 30$ in Figure 2.

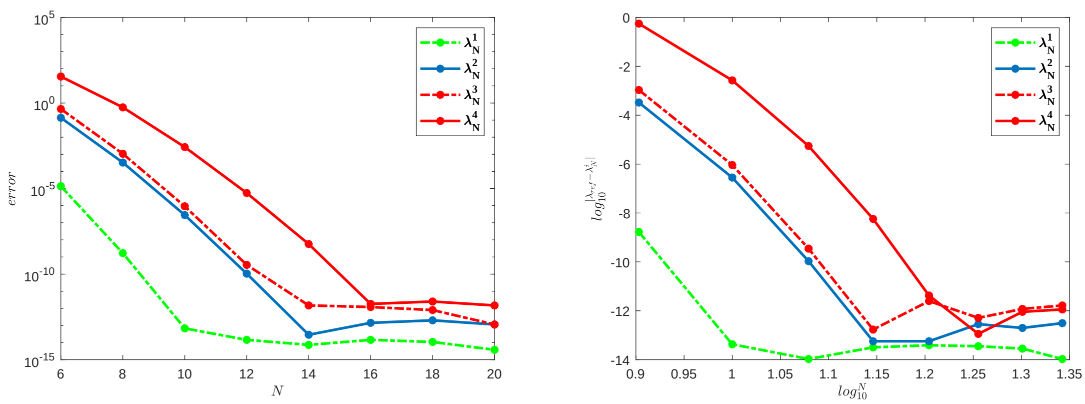

Figure 1. Absolute error curves (left) between the numerical solution and the reference solution and the errors curves (right) under log-log scale.

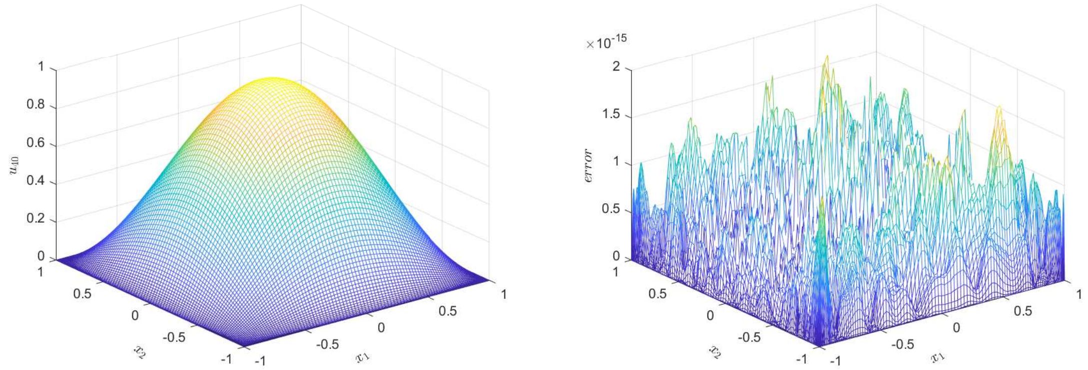

Figure 2. Image (left) of the reference solution $u_{40}(\mathbf{x})$ and the error image (right) between reference solution and numerical solution $u_{30}(\mathbf{x})$.

Table 1. Numerical results of the first four approximation eigenvalues for different N.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
5	30.287864207529200	163.1204205202208	402.7547327452907	813.0000000000005
10	30.287074959045135	165.5387105473934	410.3755739012024	634.4780697093954
15	30.287074959045280	165.5387102419903	410.3755729381884	634.4808299653813
20	30.287074959045280	165.5387102419902	410.3755729381863	634.4808299652357
25	30.287074959045283	165.5387102419905	410.3755729381896	634.4808299652358

We observe from Table 1 that the first four eigenvalues achieve at least 13-digit accuracy with $N \geq 25$. Likewise, as shown in Figures 1–2, our algorithm is both convergent and spectral accurate.

As a comparison, we list in Table 2 the numerical results of the first four approximate eigenvalues obtained by directly solving the fourth-order eigenvalue problem using the classical Legendre spectral method.

Table 2. Numerical results of the first four approximation eigenvalues for different N.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
5	30.295236624303840	165.2737789038451	407.8269176502486	1247.000000000008
10	30.287074959045206	165.5387151406006	410.3755834469236	634.4806909655549
15	30.287074959045313	165.5387102419901	410.3755729381873	634.4808299678238
20	30.287074959045288	165.5387102419903	410.3755729381873	634.4808299652353
25	30.287074959045280	165.5387102419902	410.3755729381866	634.4808299652374

From Tables 1 and Table 2, we can observe that the convergence orders of the two numerical methods are almost the same. However, for the fourth-order eigenvalue problem in general domain, if the spectral element method is directly applied to solve it, not only does the construction of the basis function become complex, but the computational load is also significant. On the contrary, the spectral element method for second-order problems is relatively mature in theoretical analysis and numerical calculation. Therefore, by transforming a fourth-order eigenvalue problem into a second-order coupled system, not only are the difficulties of theoretical analysis overcome, but the construction of the basis function is also relatively simple, facilitating efficient programming.

Although our theoretical analysis is based on the case where α and β are all positive constants, our algorithm is applicable to the case where α and β are variable coefficients. Thus, we provide a two-dimensional numerical example with variable coefficients in Example 2. Initially, we derive the equivalent matrix form for the case where α and β are all variable coefficients. Let us denote

$$\begin{aligned} q_{(N-1)j+(n+1),(N-1)k+(m+1)} &= \int_{-1}^{1}\int_{-1}^{1} \varphi_k'\varphi_j'\varphi_m\varphi_n + \varphi_k\varphi_j\varphi_m'\varphi_n' dx_1dx_2, \\ c_{(N-1)j+(n+1),(N-1)k+(m+1)} &= \int_{-1}^{1}\int_{-1}^{1} \alpha\varphi_k\varphi_j\varphi_m\varphi_n dx_1dx_2. \end{aligned}$$

Let $\beta_1 = \partial_{x_1}\beta, \beta_2 = \partial_{x_2}\beta$. We further denote

$$\begin{aligned} d_{(N-1)j+(n+1),(N-1)k+(m+1)} &= \int_{-1}^{1}\int_{-1}^{1} \beta_1\varphi_k'\varphi_j\varphi_m\varphi_n dx_1dx_2, \\ e_{(N-1)j+(n+1),(N-1)k+(m+1)} &= \int_{-1}^{1}\int_{-1}^{1} \beta_2\varphi_k\varphi_j\varphi_m'\varphi_n dx_1dx_2, \\ f_{(N-1)j+(n+1),(N-1)k+(m+1)} &= \int_{-1}^{1}\int_{-1}^{1} \beta(\varphi_k'\varphi_j'\varphi_m\varphi_n + \varphi_k\varphi_j\varphi_m'\varphi_n') dx_1dx_2, \\ g_{(N-1)j+(n+1),(N-1)k+(m+1)} &= \int_{-1}^{1}\int_{-1}^{1} \beta\varphi_k\varphi_j\varphi_m\varphi_n dx_1dx_2, \\ h_{(N-1)j+(n+1),(N-1)k+(m+1)} &= \int_{-1}^{1}\int_{-1}^{1} \varphi_k\varphi_j\varphi_m\varphi_n dx_1dx_2. \end{aligned}$$

Similar to the deduction of (4.2), we can obtain the equivalent matrix form of the discrete variational form (2.12) as follows:

$$\begin{bmatrix} \mathcal{A}_v & \mathcal{B}_v \\ C_v & \mathcal{D}_v \end{bmatrix} \begin{bmatrix} \bar{\mathbf{w}} \\ \bar{\mathbf{u}} \end{bmatrix} = \lambda_N \begin{bmatrix} \mathbf{0} & \mathcal{E}_v \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{w}} \\ \bar{\mathbf{u}} \end{bmatrix}, \tag{5.1}$$

where

$$\mathcal{A}_v = Q + C,\ \mathcal{B}_v = G,\ C_v = D + E + F,\ \mathcal{D}_v = -G,\ \mathcal{E}_v = H,$$

where

$$\begin{aligned} &C = (c_{(N-1)j+(n+1),(N-1)k+(m+1)}), D = (d_{(N-1)j+(n+1),(N-1)k+(m+1)}), \\ &E = (e_{(N-1)j+(n+1),(N-1)k+(m+1)}), F = (f_{(N-1)j+(n+1),(N-1)k+(m+1)}), \\ &G = (g_{(N-1)j+(n+1),(N-1)k+(m+1)}), H = (h_{(N-1)j+(n+1),(N-1)k+(m+1)}), \\ &Q = (q_{(N-1)j+(n+1),(N-1)k+(m+1)}). \end{aligned}$$

Example 2 We take $\Omega = (-1, 1)^2$, $\alpha = 1, \beta = e^{\sin(x_1+x_2)}$. The numerical results of the first fourth eigenvalues $\lambda_N^j (j = 1, 2, 3, 4)$ for different N are listed in Table 3. Similarly, in order to intuitively demonstrate the spectral accuracy of our algorithm, we employ the numerical solution with $N = 40$ as a reference solution and plot the absolute error curves of approximate eigenvalues as well as corresponding error curves under a log-log scale in Figure 3. Additionally, we also give an image of the reference solution for the eigenfunction and an error image between the reference solution and the approximate solution with $N = 30$ in Figure 4.

Table 3. Numerical results of the first four approximation eigenvalues for different N.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
5	20.516080766173257	140.7497111800699	140.9491722973433	370.2555125438320
10	20.523346905996156	140.9328397952852	141.1180042820287	371.0947733339725
15	20.523346901558348	140.9328457799942	141.1180110253167	371.0947913721103
20	20.523346901558394	140.9328457799939	141.1180110253170	371.0947913721107
25	20.523346901558362	140.9328457799946	141.1180110253167	371.0947913721118

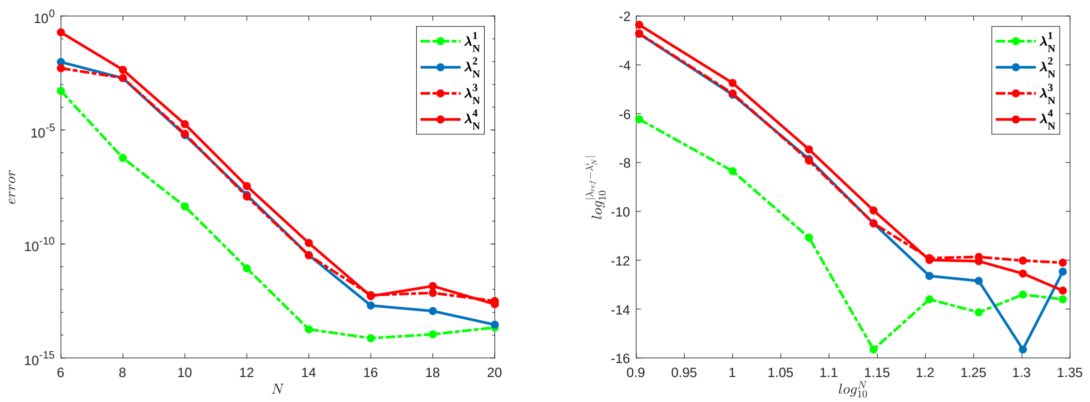

Figure 3. Absolute error curves (left) between the numerical solution and the reference solution and the errors curves (right) under log-log scale.

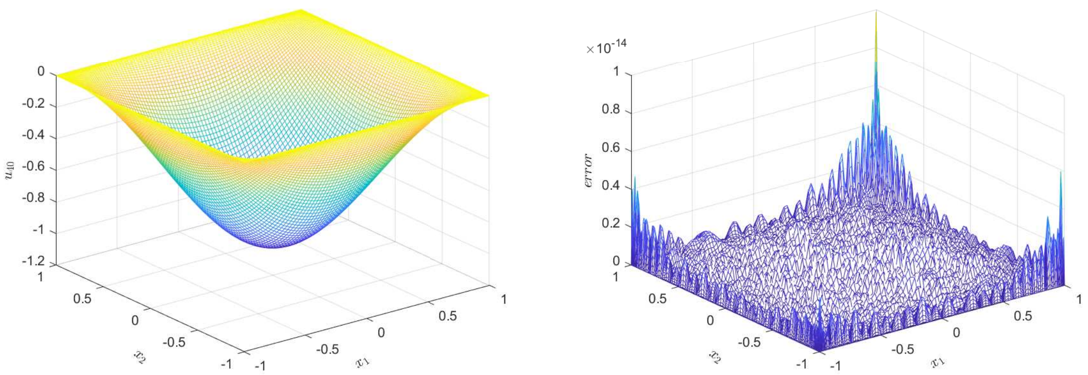

Figure 4. Image (left) of the reference solution $u_{40}(\mathbf{x})$ and the error image (right) between reference solution and numerical solution $u_{30}(\mathbf{x})$.

From Table 3, it is observed once again that the first four numerical eigenvalues arrive at an accuracy of approximately 14-digits when $N \geq 25$. Additionally, it is observed from Figures 3-4 that our algorithm is convergent and spectral accurate.

Next, we provide a three-dimensional numerical example in Example 3.

Example 3 We take $\Omega = (-1, 1)^3$, $\alpha = \beta = 1$. The numerical results of the first fourth eigenvalues $\lambda_N^j (j = 1, 2, 3, 4)$ for different N are listed in Table 4. Again, in order to intuitively demonstrate the spectral accuracy of our algorithm, we employ the numerical solution with $N = 40$ as a reference solution and plot absolute error curves of approximate eigenvalues as well as corresponding error curves under a log-log scale in Figure 5.

Table 4. Numerical results of the first four approximation eigenvalues for different N.

N	λ_N^1	λ_N^2	λ_N^3	λ_N^4
5	48.391913926971800	202.6713333320417	463.7525086494870	831.6354398793130
10	48.390410405809130	205.3660485651845	471.9269144900231	710.5119176894398
15	48.390410405809410	205.3660482248708	471.9269134571854	710.5148388418000
20	48.390410405809240	205.3660482248708	471.9269134571845	710.5148388416472
25	48.390410405809270	205.3660482248702	471.9269134571852	710.5148388416419

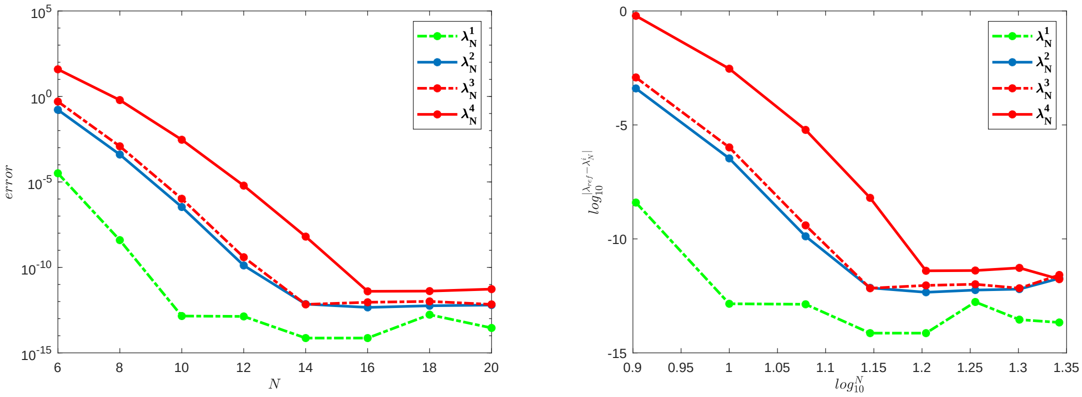

Figure 5. Absolute error curves (left) between the numerical solution and the reference solution and the errors curves (right) under log-log scale.

As shown in Table 4, the first four eigenvalues achieve at least 14-digit accuracy when $N \geq 25$. Furthermore, from Figure 5, we can see that our algorithm is also convergent and spectral accurate.

6. Conclusions

In this paper, an efficient Legendre spectral method is proposed and studied for fourth order eigenvalue problems with the boundary conditions of a simply supported plate. By introducing an auxiliary variable, the fourth order eigenvalue problem is transformed into a coupled second-order eigenvalue problem. By utilizing the hybrid format, a fresh weak formulation and its corresponding discrete variational form are formulated. Error estimates for the eigenvalues and eigenfunction approximations are also derived. In addition, numerical results validate the effectiveness of the algorithm and the correctness of theoretical results.

The algorithm proposed in this paper can be combined with the spectral element method to be applied to the numerical computation of fourth-order problems on more general domains.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare to have no competing interests.

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