AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 7492–7510.
DOI: 10.3934/math.2024363
Received: 10 January 2024
Revised: 02 February 2024
Accepted: 05 February 2024
Published: 21 February 2024

Research article

Some identities of the generalized bi-periodic Fibonacci and Lucas polynomials

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Abstract: In this paper, we considered the generalized bi-periodic Fibonacci polynomials, and obtained some identities related to generalized bi-periodic Fibonacci polynomials using the matrix theory. In addition, the generalized bi-periodic Lucas polynomial was defined by $L_n(x) = bp(x)L_{n-1}(x) + q(x)L_{n-2}(x)$ (if n is even) or $L_n(x) = ap(x)L_{n-1}(x) + q(x)L_{n-2}(x)$ (if n is odd), with initial conditions $L_0(x) = 2$, $L_1(x) = ap(x)$, where $p(x)$ and $q(x)$ were nonzero polynomials in $Q[x]$. We obtained a series of identities related to the generalized bi-periodic Fibonacci and Lucas polynomials.

Keywords: generalized bi-periodic Fibonacci polynomial; generalized bi-periodic Lucas polynomial; Binet formula; matrix theory
Mathematics Subject Classification: 11B37, 11B39

1. Introduction

The Fibonacci and Lucas polynomials are important in various fields such as number theory, probability, numerical analysis, and physics. In addition, many famous polynomials, such as the Pell polynomials, Pell Lucas polynomials, Tribonacci polynomials, etc., are generalizations of the Fibonacci and Lucas polynomials. Many scholars discussed the Fibonacci polynomials and its generalization; see [1–5]. This paper mainly extends linear recursive polynomials to nonlinearity and discusses some basic properties of the generalized bi-periodic Fibonacci and Lucas polynomials.

The Fibonacci $\{u_n(x)\}$ and Lucas $\{v_n(x)\}$ polynomials are defined by

$$u_0(x) = 0, \quad u_1(x) = 1, \quad u_n(x) = xu_{n-1}(x) + u_{n-2}(x), \quad n \geq 2$$

and

$$v_0(x) = 2, \quad v_1(x) = x, \quad v_n(x) = xv_{n-1}(x) + v_{n-2}(x), \quad n \geq 2.$$

When $x = 1$, we obtain Fibonacci $\{u_n\}$ and Lucas $\{v_n\}$ sequences defined by

$$u_0 = 0, \quad u_1 = 1, \quad u_n = u_{n-1} + u_{n-2}, \quad n \geq 2$$

and

$$v_0 = 2, \quad v_1 = 1, \quad v_n = v_{n-1} + v_{n-2}, \quad n \geq 2.$$

The Fibonacci $\{u_n\}$ ($\{u_n(x)\}$) or Lucas $\{v_n\}$ ($\{v_n(x)\}$) sequences (polynomials) have more interesting properties and applications; see [6–10].

In [11], the generalized Fibonacci $\{U_n(x)\}$ and Lucas $\{V_n(x)\}$ polynomials are defined by

$$U_0(x) = 0, \quad U_1(x) = 1, \quad U_n(x) = p(x)\,U_{n-1}(x) + q(x)\,U_{n-2}(x), \quad n \geq 2$$

and

$$V_0(x) = 2, \quad V_1(x) = p(x), \quad V_n(x) = p(x)\,V_{n-1}(x) + q(x)\,V_{n-2}(x), \quad n \geq 2,$$

where $p(x)$ and $q(x)$ are nonzero polynomials in $Q[x]$. For more consideration of generalized polynomials $\{U_n(x)\}$ or $\{V_n(x)\}$, see [12–14].

In [15], the bi-periodic Fibonacci $\{f_n(x)\}$ and Lucas $\{l_n(x)\}$ polynomials are defined by

$$f_0(x) = 0, \quad f_1(x) = 1, \quad f_n(x) = \begin{cases} axf_{n-1}(x) + f_{n-2}(x), & \text{if } n \text{ is even and } n \geq 2, \\ bxf_{n-1}(x) + f_{n-2}(x), & \text{if } n \text{ is odd and } n \geq 3, \end{cases}$$

and

$$l_0(x) = 2, \quad l_1(x) = ax, \quad l_n(x) = \begin{cases} bxl_{n-1}(x) + l_{n-2}(x), & \text{if } n \text{ is even and } n \geq 2, \\ axl_{n-1}(x) + l_{n-2}(x), & \text{if } n \text{ is odd and } n \geq 3, \end{cases}$$

where a and b are any nonzero real numbers. For more discussions of bi-periodic polynomials $\{f_n(x)\}$ and $\{l_n(x)\}$; see [16, 17].

In [18], the author defined a new kind of Fibonacci polynomials called the generalized bi-periodic Fibonacci polynomial $\{F_n(x)\}$, which is defined by

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_n(x) = \begin{cases} ap(x)\,F_{n-1}(x) + q(x)\,F_{n-2}(x), & \text{if } n \text{ is even and } n \geq 2, \\ bp(x)\,F_{n-1}(x) + q(x)\,F_{n-2}(x), & \text{if } n \text{ is odd and } n \geq 3, \end{cases} \tag{1.1}$$

where a, b are nonzero real numbers and $p(x)$ and $q(x)$ are nonzero polynomials in $Q[x]$. They obtained the following Binet formula:

$$F_m(x) = \left(\frac{a^{1-\zeta(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right) \frac{\sigma^m(x) - \tau^m(x)}{\sigma(x) - \tau(x)}, \quad m \geq 0, \tag{1.2}$$

where

$$\sigma(x) = \frac{abp(x) + \sqrt{a^2b^2p^2(x) + 4abq(x)}}{2},$$

$$\tau(x) = \frac{abp(x) - \sqrt{a^2b^2p^2(x) + 4abq(x)}}{2},$$

and

$$\zeta(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$$

is the parity function, with $\lfloor \cdot \rfloor$ denoting the floor function.

In addition, they obtained a series of classical identities of the generalized bi-periodic Fibonacci polynomial as follows:

(a) Generated functions

$$G_n(x,t) = \frac{t + ap(x)t^2 - q(x)t^3}{1 - (abp^2(x) + 2q(x))t^2 + q^2(x)t^4};$$

(b) Generalized Catalan's identity

$$a^{\zeta(m-r)}b^{1-\zeta(m-r)}F_{m-r}(x)F_{m+r}(x) - a^{\zeta(m)}b^{1-\zeta(m)}F_m^2(x) = -(-q(x))^{m-r}a^{\zeta(r)}b^{1-\zeta(r)}F_r^2(x);$$

(c) Generalized Cassini's identity

$$a^{\zeta(m-1)}b^{\zeta(m)}F_{m-1}(x)F_{m+1}(x) - a^{\zeta(m)}b^{1-\zeta(m)}F_m^2(x) = -a(-q(x))^{m-1};$$

(d) Generalized d'Ocagne's identity

$$a^{\zeta(mr+m)}b^{\zeta(mr+r)}F_m(x)F_{r+1}(x) - a^{\zeta(mr+r)}b^{\zeta(mr+m)}F_{m+1}(x)F_r(x) = -(-q(x))^r a^{\zeta(m-r)}F_{m-r}(x);$$

(e) Negative subscript terms

$$F_{-m}(x) = (-1)^{m+1}(q(x))^{-m}F_m(x).$$

2. The generalized bi-periodic Fibonacci polynomial by matrix methods

Recently, some scholars considered the identities of recursive sequences (polynomials) by the matrix theory. For example, in [19], the author defined the Fibonacci Q-matrix as follows:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so that

$$Q^m = \begin{pmatrix} u_{m+1} & u_m \\ u_m & u_{m-1} \end{pmatrix}, \quad n \geq 1,$$

where $\{u_n\}$ is a Fibonacci sequence. For more on considering recursive sequences (polynomials) by the matrix theory; see [20–22].

In this section, we consider the generalized bi-periodic Fibonacci polynomial defined by a 2×2 matrix S and we give the mth power S^m for any integer m.

Theorem 2.1. Let

$$S = \begin{pmatrix} abp(x) & bq(x) \\ a & 0 \end{pmatrix}, \tag{2.1}$$

then, we have

$$S^m = (ab)^{\frac{m-\zeta(m)}{2}} \begin{pmatrix} b^{\zeta(m)}F_{m+1}(x) & a^{-\zeta(m+1)}bq(x)F_m(x) \\ a^{\zeta(m)}F_m(x) & b^{\zeta(m)}q(x)F_{m-1}(x) \end{pmatrix}, \quad n \geq 1, \tag{2.2}$$

where a and b are nonzero real numbers, and $p(x)$ and $q(x)$ are nonzero polynomials in $Q[x]$,

$$\zeta(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$$

is the parity function, with $\lfloor \cdot \rfloor$ denoting the floor function. $\{F_n(x)\}$ is the generalized bi-periodic Fibonacci polynomial.

Proof. We prove (2.2) by mathematical induction. Obviously, the identity is true when $m = 1$,

$$S^1 = \begin{pmatrix} bF_2(x) & bq(x) F_1(x) \\ aF_1(x) & bq(x) F_0(x) \end{pmatrix} = \begin{pmatrix} abp(x) & bq(x) \\ a & 0 \end{pmatrix} = S.$$

We assume that the identity is true with m. Next, we prove that the identity is true when $m+1$.

$$\begin{aligned} S^{m+1} = S^m \cdot S &= (ab)^{\frac{m-\zeta(m)}{2}} \begin{pmatrix} b^{\zeta(m)} F_{m+1}(x) & a^{-\zeta(m+1)} bq(x) F_m(x) \\ a^{\zeta(m)} F_m(x) & b^{\zeta(m)} q(x) F_{m-1}(x) \end{pmatrix} \cdot \begin{pmatrix} abp(x) & bq(x) \\ a & 0 \end{pmatrix} \\ &= \begin{pmatrix} a^{\frac{2+m-\zeta(m)}{2}} b^{\frac{2+m+\zeta(m)}{2}} p(x) F_{m+1}(x) + a^{\frac{m+1-\zeta(m+1)}{2}} b^{\frac{2+m-\zeta(m)}{2}} q(x) F_m(x) & a^{\frac{m-\zeta(m)}{2}} b^{\frac{2+m+\zeta(m)}{2}} q(x) F_{m+1}(x) \\ a^{\frac{2+m+\zeta(m)}{2}} b^{\frac{2+m-\zeta(m)}{2}} p(x) F_m(x) + a^{\frac{m+2-\zeta(m)}{2}} b^{\frac{m+\zeta(m)}{2}} q(x) F_{m-1}(x) & a^{\frac{m+\zeta(m)}{2}} b^{\frac{2+m-\zeta(m)}{2}} q(x) F_m(x) \end{pmatrix} \\ &= (ab)^{\frac{m+1-\zeta(m+1)}{2}} \begin{pmatrix} a^{\zeta(m+1)} bp(x) F_{m+1}(x) + b^{\zeta(m+1)} q(x) F_m(x) & a^{-\zeta(m)} bq(x) F_{m+1}(x) \\ ab^{\zeta(m+1)} p(x) F_m(x) + a^{\zeta(m+1)} q(x) F_{m-1}(x) & b^{\zeta(m+1)} q(x) F_m(x) \end{pmatrix} \\ &= (ab)^{\frac{m+1-\zeta(m+1)}{2}} \begin{pmatrix} b^{\zeta(m+1)} F_{m+2}(x) & a^{-\zeta(m)} bq(x) F_{m+1}(x) \\ a^{\zeta(m+1)} F_{m+1}(x) & b^{\zeta(m+1)} q(x) F_m(x) \end{pmatrix}, \end{aligned}$$

where the generalized bi-periodic Fibonacci polynomial is given by

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_m(x) = a^{\zeta(m+1)} b^{\zeta(m)} p(x) F_{m-1}(x) + q(x) F_{m-2}(x), \quad m \geq 2, \tag{2.3}$$

where a, b are nonzero real numbers and $p(x)$, $q(x)$ are nonzero polynomials in $Q[x]$,

$$\zeta(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$$

is the parity function, with $\lfloor \cdot \rfloor$ denoting the floor function. □

Remark 2.1. The characteristic polynomial of S is

$$\lambda^2 - abp(x)\lambda - abq(x) = 0.$$

Thus

$$\sigma(x) = \frac{abp(x) + \sqrt{a^2b^2p^2(x) + 4abq(x)}}{2}$$

and

$$\tau(x) = \frac{abp(x) - \sqrt{a^2b^2p^2(x) + 4abq(x)}}{2}$$

are the eigenvalues of S. We can diagonalize S to get $S = P^{-1}DP$, so $S^m = P^{-1}D^mP$, where P is invertible and

$$D = \begin{pmatrix} \sigma(x) & 0 \\ 0 & \tau(x) \end{pmatrix},$$

The generalized bi-periodic Fibonacci polynomial may be expressed in the form

$$F_m(x) = A\sigma^m(x) + B\tau^m(x),$$

where A and B are constants. When $m = 0$ and $m = 1$, we use the special value method to get the explicit identity of the generalized bi-periodic Fibonacci polynomial.

Next, we get a series of identities of the generalized bi-periodic Fibonacci polynomial by the matrix S.

Theorem 2.2. (Generalized Cassinis's identity) Let $\{F_m(x)\}$ be the generalized bi-periodic Fibonacci polynomial. We have

$$a^{\zeta(m+1)}b^{\zeta(m)}F_{m+1}(x)F_{m-1}(x) - a^{\zeta(m)}b^{\zeta(m+1)}F_m^2(x) = -a(-q(x))^{m-1}. \tag{2.4}$$

Proof. According to the identity (2.2),

$$\begin{aligned} det(S^m) &= (ab)^{m-\zeta(m)}\left(b^{2\zeta(m)}q(x)F_{m+1}(x)F_{m-1}(x) - a^{\zeta(m)-\zeta(m+1)}bq(x)F_m^2(x)\right) \\ &= a^{m-1}b^m q(x)\left(a^{\zeta(m+1)}b^{\zeta(m)}F_{m+1}(x)F_{m-1}(x) - a^{\zeta(m)}b^{\zeta(m+1)}F_m^2(x)\right) \\ &= det(S)^m = (-abq(x))^m. \end{aligned}$$

Thus,

$$a^{\zeta(m+1)}b^{\zeta(m)}F_{m+1}(x)F_{m-1}(x) - a^{\zeta(m)}b^{\zeta(m+1)}F_m^2(x) = -a(-q(x))^{m-1}.$$

□

Since S is invertible, and S^m is also invertible, we have

$$(S^m)^{-1} = S^{-m} = (-q(x))^{-m}(ab)^{\frac{-n-\zeta(m)}{2}}\begin{pmatrix} b^{\zeta(m)}q(x)F_{m-1}(x) & -a^{-\zeta(m+1)}bq(x)F_m(x) \\ -a^{\zeta(m)}F_m(x) & b^{\zeta(m)}F_{m+1}(x) \end{pmatrix}. \tag{2.5}$$

Theorem 2.3. (Generalized d'Ocagne's identity) Let $\{F_n(x)\}$ be the generalized bi-periodic Fibonacci polynomial. We have

$$F_{m+n+1}(x) = a^{-\zeta(mn)}b^{\zeta(mn)}F_{m+1}(x)F_{n+1}(x) + a^{\zeta(mn+m+n)-1}b^{1-\zeta(mn+m+n)}q(x)F_m(x)F_n(x), \tag{2.6}$$

$$F_{m+n}(x) = a^{-\zeta(mn+n)}b^{\zeta(mn+n)}F_m(x)F_{n+1}(x) + a^{-\zeta(mn+m)}b^{\zeta(mn+m)}q(x)F_{m-1}(x)F_n(x), \tag{2.7}$$

$$F_{m+n-1}(x) = a^{\zeta(mn+m+n)-1}b^{1-\zeta(mn+m+n)}F_m(x)F_n(x) + a^{-\zeta(mn)}b^{\zeta(mn)}q(x)F_{m-1}(x)F_{n-1}(x), \tag{2.8}$$

$$F_{m-n+1}(x) = -(-q(x))^{-n+1}\left[a^{-\zeta(mn)}b^{\zeta(mn)}F_{m+1}(x)F_{n-1}(x) - a^{\zeta(mn+m+n)-1}b^{1-\zeta(mn+m+n)}F_m(x)F_n(x)\right], \tag{2.9}$$

$$F_{m-n}(x) = (-q(x))^{-n}\left[a^{-\zeta(mn+n)}b^{\zeta(mn+n)}F_m(x)F_{n+1}(x) - a^{-\zeta(mn+m)}b^{\zeta(mn+m)}F_{m+1}(x)F_n(x)\right], \tag{2.10}$$

$$F_{m-n}(x) = -(-q(x))^{-n+1}\left[a^{-\zeta(mn+n)}b^{\zeta(mn+n)}F_m(x)F_{n-1}(x) - a^{-\zeta(mn+m)}b^{\zeta(mn+m)}F_{m-1}(x)F_n(x)\right], \tag{2.11}$$

$$F_{m-n-1}(x) = (-q(x))^{-n}\left[a^{-\zeta(mn)}b^{\zeta(mn)}F_{m-1}(x)F_{n+1}(x) - a^{\zeta(mn+m+n)-1}b^{1-\zeta(mn+m+n)}F_m(x)F_n(x)\right]. \tag{2.12}$$

Proof. According to the identity (2.2),

$$S^{m+n} = (ab)^{\frac{m+n-\zeta(m+n)}{2}}\begin{pmatrix} b^{\zeta(m+n)}F_{m+n+1}(x) & a^{-\zeta(m+n+1)}bq(x)F_{m+n}(x) \\ a^{\zeta(m+n)}F_{m+n}(x) & b^{\zeta(m+n)}q(x)F_{m+n-1}(x) \end{pmatrix} \tag{2.13}$$

and

$$S^m \cdot S^n = (ab)^{\frac{m+n-\zeta(m)-\zeta(n)}{2}} \times \begin{pmatrix} b^{\zeta(m)+\zeta(n)} F_{m+1}(x) F_{n+1}(x) & a^{-\zeta(m+1)} b^{1+\zeta(n)} q(x) F_m(x) F_{n+1}(x) \\ +a^{\zeta(m)-\zeta(n+1)} bq(x) F_m(x) F_n(x) & +a^{-\zeta(n+1)} b^{1+\zeta(m)} q^2(x) F_{m-1}(x) F_n(x) \\ & \\ a^{\zeta(n)} b^{\zeta(m)} F_{m+1}(x) F_n(x) & a^{\zeta(n)-\zeta(m+1)} bq(x) F_m(x) F_n(x) \\ +a^{\zeta(m)} b^{\zeta(n)} q(x) F_m(x) F_{n-1}(x) & +b^{\zeta(m)+\zeta(n)} q^2(x) F_{m-1}(x) F_{n-1}(x) \end{pmatrix}. \tag{2.14}$$

Since

$$S^{m+n} = S^m S^n,$$

the corresponding entries in identities (2.13) and (2.14) are equal, so we obtain (2.6)–(2.8), where

(f) $\zeta(m+n) - \zeta(m) - \zeta(n) = -2\zeta(mn)$,

(g) $\zeta(m+n) + \zeta(m) + \zeta(n) = 2\zeta(mn+m+n)$,

(h) $\zeta(m+n) - \zeta(m) + \zeta(n) = 2\zeta(mn+n)$,

(i) $\zeta(m+n) + \zeta(m) - \zeta(n) = 2\zeta(mn+m)$,

(j) $\zeta(m+n+1) - \zeta(m+1) - \zeta(n) = -2\zeta(mn+n)$,

(k) $\zeta(m+n+1) - \zeta(m) - \zeta(n+1) = -2\zeta(mn+m)$.

In addition, we have

$$S^{m-n} = (ab)^{\frac{m-n-\zeta(m-n)}{2}} \begin{pmatrix} b^{\zeta(m-n)} F_{m-n+1}(x) & a^{-\zeta(m-n+1)} bq(x) F_{m-n}(x) \\ a^{\zeta(m-n)} F_{m-n}(x) & b^{\zeta(m-n)} q(x) F_{m-n-1}(x) \end{pmatrix} \tag{2.15}$$

and

$$S^m \cdot S^{-n} = (-q(x))^{-n+1} (ab)^{\frac{m-n-\zeta(m)-\zeta(n)}{2}} \times \begin{pmatrix} b^{\zeta(m)+\zeta(n)} F_{m+1}(x) F_{n-1}(x) & a^{-\zeta(m+1)} b^{1+\zeta(n)} F_m(x) F_{n+1}(x) \\ -a^{\zeta(n)-\zeta(m+1)} bF_m(x) F_n(x) & -a^{-\zeta(n+1)} b^{1+\zeta(m)} F_{m+1}(x) F_n(x) \\ & \\ a^{\zeta(m)} b^{\zeta(n)} F_m(x) F_{n-1}(x) & b^{\zeta(n)+\zeta(m)} F_{m-1}(x) F_{n+1}(x) \\ -a^{\zeta(n)} b^{\zeta(m)} F_{m-1}(x) F_n(x) & -a^{\zeta(m)-\zeta(n+1)} bF_m(x) F_n(x) \end{pmatrix}. \tag{2.16}$$

Since

$$S^{m-n} = S^m S^{-n},$$

the corresponding entries in identities (2.15) and (2.16) are equal, so we obtain (2.9)–(2.12), where

(l) $\zeta(m-n) - \zeta(m) - \zeta(n) = -2\zeta(mn)$,

(m) $\zeta(m-n) + \zeta(m) + \zeta(n) = 2\zeta(mn+m+n)$,

(n) $\zeta(m-n) + \zeta(m) - \zeta(n) = 2\zeta(mn+m)$,

(o) $\zeta(m-n) - \zeta(m) + \zeta(n) = 2\zeta(mn+n)$. □

Theorem 2.4. (Sum involving binomial coefficients) Let $\{F_m(x)\}$ be the generalized bi-periodic Fibonacci polynomial. We have

$$F_{2m}(x) = \sum_{k=0}^{m} \binom{m}{k} a^{\frac{k+\zeta(k)}{2}} b^{\frac{k-\zeta(k)}{2}} p^k(x) q^{m-k}(x) F_k(x), \tag{2.17}$$

$$F_{2m+1}(x) = \sum_{k=0}^{m} \binom{m}{k} a^{\frac{k-\zeta(k)}{2}} b^{\frac{k+\zeta(k)}{2}} p^k(x) q^{m-k}(x) F_{k+1}(x), \tag{2.18}$$

$$F_{2m-1}(x) = \sum_{k=0}^{m} \binom{m}{k} a^{\frac{k-\zeta(k)}{2}} b^{\frac{k+\zeta(k)}{2}} p^k(x) q^{m-k}(x) F_{k-1}(x), \tag{2.19}$$

$$F_{-2m}(x) = q^{-m}(x) \sum_{k=0}^{m} \binom{m}{k} (-1)^k a^{\frac{k+\zeta(k)}{2}} b^{\frac{k-\zeta(k)}{2}} p^k(x) F_{-k}(x), \tag{2.20}$$

$$F_{-2m+1}(x) = q^{-m}(x) \sum_{k=0}^{m} \binom{m}{k} (-1)^k a^{\frac{k-\zeta(k)}{2}} b^{\frac{k+\zeta(k)}{2}} p^k(x) F_{-k+1}(x), \tag{2.21}$$

$$F_{-2m-1}(x) = q^{-m}(x) \sum_{k=0}^{n} \binom{m}{k} (-1)^k p^k(x) a^{\frac{k-\zeta(k)}{2}} b^{\frac{k+\zeta(k)}{2}} F_{-k-1}(x). \tag{2.22}$$

Proof. According to Cayley Hamilton's theorem, the following matrix S identity is obtained:

$$S^2 - abp(x) S - abq(x) I = 0,$$

then

$$\left(S^2\right)^m = (abp(x) S + abq(x) I)^m = (ab)^m \sum_{k=0}^{m} \binom{m}{k} (p(x) S)^k (q(x))^{m-k}.$$

We obtain

$$(ab)^m \begin{pmatrix} F_{2m+1}(x) & a^{-1}bq(x) F_{2m}(x) \\ F_{2m}(x) & q(x) F_{2m-1}(x) \end{pmatrix} = (ab)^m \sum_{k=0}^{m} \binom{m}{k} p^k(x) q^{m-k}(x) (ab)^{\frac{k-\zeta(k)}{2}} \begin{pmatrix} b^{\zeta(k)} F_{k+1}(x) & a^{-\zeta(k+1)} bq(x) F_k(x) \\ a^{\zeta(k)} F_k(x) & b^{\zeta(k)} q(x) F_{k-1}(x) \end{pmatrix}. \tag{2.23}$$

The corresponding entries in identity (2.23) are equal, so we obtain (2.17)–(2.19).

Thus,

$$S^{-1} = \begin{pmatrix} 0 & a^{-1} \\ b^{-1} q^{-1}(x) & -p(x) q^{-1}(x) \end{pmatrix}.$$

According to Cayley Hamilton's theorem, the following matrix S^{-1} identity is obtained:

$$S^{-2} + p(x) q^{-1}(x) S^{-1} - a^{-1} b^{-1} q^{-1}(x) I = 0,$$

then,

$$\left(S^{-2}\right)^m = \left(a^{-1} b^{-1} q^{-1}(x) I - p(x) q^{-1}(x) S^{-1}\right)^m = q^{-m}(x) \sum_{k=0}^{m} \binom{m}{k} (-1)^k p^k(x) S^{-k} (ab)^{k-m}.$$

We obtain

$$(ab)^{-m} \begin{pmatrix} F_{-2m+1}(x) & a^{-1}bq(x) F_{-2m}(x) \\ F_{-2m}(x) & q(x) F_{-2m-1}(x) \end{pmatrix} = (abq(x))^{-m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k p^k(x) (ab)^{\frac{k-\zeta(k)}{2}} \begin{pmatrix} b^{\zeta(k)} F_{-k+1}(x) & a^{-\zeta(k+1)} bq(x) F_{-k}(x) \\ a^{\zeta(k)} F_{-k}(x) & b^{\zeta(k)} q(x) F_{-k-1}(x) \end{pmatrix}. \tag{2.24}$$

The corresponding entries in identity (2.24) are equal, so we obtain (2.20)–(2.22). □

3. The generalized bi-periodic Lucas polynomial

Inspired by [18], in this section, we define the generalized bi-periodic Lucas polynomial $\{L_n(x)\}$ as follows:

Definition 3.1. The generalized bi-periodic Lucas polynomial is defined by

$$L_0(x) = 2, \quad L_1(x) = ap(x)$$

and

$$L_n(x) = \begin{cases} bp(x) L_{n-1}(x) + q(x) L_{n-2}(x), & \text{if } n \text{ is even and } n \geq 2, \\ ap(x) L_{n-1}(x) + q(x) L_{n-2}(x), & \text{if } n \text{ is odd and } n \geq 3, \end{cases}$$

where a and b are nonzero real numbers, and $p(x)$, $q(x)$ are nonzero polynomials in $Q[x]$.

According to the definition, we obtain another expression of the generalized bi-periodic Lucas polynomial as follows:

$$L_m(x) = a^{\zeta(m)} b^{1-\zeta(m)} p(x) L_{m-1}(x) + q(x) L_{m-2}(x), \quad m \geq 2, \tag{3.1}$$

where

$$\zeta(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$$

is the parity function, with $\lfloor \cdot \rfloor$ denoting the floor function. The characteristic polynomial of the generalized bi-periodic Lucas polynomial is

$$t^2 - p(x) abt - q(x) ab = 0,$$

and the roots are

$$\sigma(x) = \frac{abp(x) + \sqrt{a^2 b^2 p^2(x) + 4abq(x)}}{2}$$

and

$$\tau(x) = \frac{abp(x) - \sqrt{a^2 b^2 p^2(x) + 4abq(x)}}{2}.$$

We have:

(p) $\sigma(x) + \tau(x) = abp(x)$,

(q) $\sigma(x) - \tau(x) = \sqrt{p^2(x) a^2 b^2 + 4q(x) ab}$,

(r) $\sigma(x) \tau(x) = -abq(x)$.

Theorem 3.1. The generating functions of the generalized bi-periodic Lucas polynomial $\{L_m(x)\}$ are

$$\begin{aligned} T_m(x,t) &= \sum_{m=0}^{\infty} L_m(x) t^m \\ &= \frac{2 + ap(x) t - \left(abp^2(x) + 2q(x)\right) t^2 + ap(x) q(x) t^3}{1 - (abp^2(x) + 2q(x)) t^2 + q^2(x) t^4}. \end{aligned}$$

Lemma 3.1. The generalized bi-periodic Lucas $\{L_m(x)\}$ polynomial satisfy the following identities

$$L_{2m}(x) = \left(abp^2(x) + 2q(x)\right) L_{2m-2}(x) - q^2(x) L_{2m-4}(x) \tag{3.2}$$

and

$$L_{2m+1}(x) = \left(abp^2(x) + 2q(x)\right) L_{2m-1}(x) - q^2(x) L_{2m-3}(x). \tag{3.3}$$

Proof. By identity (3.1),

$$\begin{aligned}
L_{2m}(x) &= bp(x) L_{2m-1}(x) + q(x) L_{2m-2}(x) \\
&= bp(x)\left[ap(x) L_{2m-2}(x) + q(x) L_{2m-3}(x)\right] + q(x) L_{2m-2}(x) \\
&= \left[abp^2(x) + q(x)\right] L_{2m-2}(x) + bp(x) q(x) L_{2m-3}(x) \\
&= \left[abp^2(x) + q(x)\right] L_{2m-2}(x) + q(x) L_{2m-2}(x) - q^2(x) L_{2m-4}(x) \\
&= \left[abp^2(x) + 2q(x)\right] L_{2m-2}(x) - q^2(x) L_{2m-4}(x)
\end{aligned}$$

and

$$\begin{aligned}
L_{2m+1}(x) &= ap(x) L_{2m}(x) + q(x) L_{2m-1}(x) \\
&= ap(x)\left[bp(x) L_{2m-1}(x) + q(x) L_{2m-2}(x)\right] + q(x) L_{2m-1}(x) \\
&= \left[abp^2(x) + q(x)\right] L_{2m-1}(x) + ap(x) q(x) L_{2m-2}(x) \\
&= \left[abp^2(x) + q(x)\right] L_{2m-1}(x) + q(x) L_{2m-1}(x) - q^2(x) L_{2m-3}(x) \\
&= \left[abp^2(x) + 2q(x)\right] L_{2m-1}(x) - q^2(x) L_{2m-3}(x).
\end{aligned}$$

□

Proof of Theorem 3.1. According to the definition of the generating functions of the generalized bi-periodic Lucas polynomial, we have

$$\begin{aligned}
T_m(x,t) &= T_m^e(x,t) + T_m^o(x,t) \\
&= \sum_{k=0}^{\infty} L_{2k}(x) t^{2k} + \sum_{k=0}^{\infty} L_{2k+1}(x) t^{2k+1}.
\end{aligned}$$

To begin, we consider $T_m^e(x,t)$,

$$T_m^e(x,t) = \sum_{k=0}^{\infty} L_{2k}(x) t^{2k} = L_0(x) + L_2(x) t^2 + L_4(x) t^4 + \cdots, \tag{3.4}$$

$$-\left(abp^2(x) + 2q(x)\right) t^2 T_m^e(x,t) = -\left(abp^2(x) + 2q(x)\right) \sum_{k=0}^{\infty} L_{2k}(x) t^{2k+2}, \tag{3.5}$$

$$q^2(x) t^4 T_m^e(x,t) = q^2(x) \sum_{k=0}^{\infty} L_{2k}(x) t^{2k+4}. \tag{3.6}$$

Contact (3.4)–(3.6) and Lemma 3.1. We obtain

$$\begin{aligned}
&\left\{1-\left(abp^2(x)+2q(x)\right)t^2+q^2(x)t^4\right\}T_m^e(x,t)\\
&=L_0(x)+L_2(x)t^2+\sum_{k=2}^{\infty}L_{2k}(x)t^{2k}-\left(abp^2(x)+2q(x)\right)\sum_{k=0}^{\infty}L_{2k}(x)t^{2k+2}+q^2(x)\sum_{k=0}^{\infty}L_{2k}(x)t^{2k+4}\\
&=2+\left(abp^2(x)+2q(x)\right)t^2+\sum_{k=2}^{\infty}L_{2k}(x)t^{2k}-\left(abp^2(x)+2q(x)\right)2t^2\\
&\quad-\left(abp^2(x)+2q(x)\right)\sum_{k=2}^{\infty}L_{2k-2}(x)t^{2k}+q^2(x)\sum_{k=2}^{\infty}L_{2k-4}(x)t^{2k}\\
&=2-\left(abp^2(x)+2q(x)\right)t^2+\sum_{k=2}^{\infty}\left\{L_{2k}(x)-\left(abp^2(x)+2q(x)\right)L_{2k-2}(x)+q^2(x)L_{2k-4}(x)\right\}t^{2k}\\
&=2-\left(abp^2(x)+2q(x)\right)t^2.
\end{aligned}$$

Therefore,

$$T_m^e(x,t)=\frac{2-\left(abp^2(x)+2q(x)\right)t^2}{1-\left(abp^2(x)+2q(x)\right)t^2+q^2(x)t^4}. \tag{3.7}$$

Next, we consider $T_m^o(x,t)$,

$$T_m^o(x,t)=\sum_{k=0}^{\infty}L_{2k+1}(x)t^{2k+1}=L_1(x)t+L_3(x)t^3+L_5(x)t^5+\cdots, \tag{3.8}$$

$$-\left(abp^2(x)+2q(x)\right)t^2T_m^o(x,t)=-\left(abp^2(x)+2q(x)\right)\sum_{k=0}^{\infty}L_{2k+1}(x)t^{2k+3}, \tag{3.9}$$

$$q^2(x)t^4T_m^o(x,t)=q^2(x)\sum_{k=0}^{\infty}L_{2k+1}(x)t^{2k+5}. \tag{3.10}$$

Contact (3.8)–(3.10) and Lemma 3.1. We obtain

$$\begin{aligned}
&\left\{1-\left(abp^2(x)+2q(x)\right)t^2+q^2(x)t^4\right\}T_m^o(x,t)\\
&=L_1(x)t+L_3(x)t^3+\sum_{k=2}^{\infty}L_{2k+1}(x)t^{2k+1}-\left(abp^2(x)+2q(x)\right)\sum_{k=0}^{\infty}L_{2k+1}(x)t^{2k+3}+q^2(x)\sum_{k=0}^{\infty}L_{2k+1}(x)t^{2k+5}\\
&=ap(x)t+\left(a^2bp^3(x)+3ap(x)q(x)\right)t^3+\sum_{k=2}^{\infty}L_{2k+1}(x)t^{2k+1}-\left(a^2bp^3(x)+2ap(x)q(x)\right)t^3\\
&\quad-\left(abp^2(x)+2q(x)\right)\sum_{k=2}^{\infty}L_{2k-1}(x)t^{2k+1}+q^2(x)\sum_{k=2}^{\infty}L_{2k-3}(x)t^{2k+1}\\
&=ap(x)t+ap(x)q(x)t^3+\sum_{k=2}^{\infty}\left\{L_{2k+1}(x)-\left(abp^2(x)+2q(x)\right)L_{2k-1}(x)+q^2(x)L_{2k-3}(x)\right\}t^{2k+1}\\
&=ap(x)t+ap(x)q(x)t^3.
\end{aligned}$$

Thus,

$$T_m^o(x,t)=\frac{ap(x)t+ap(x)q(x)t^3}{1-(abp^2(x)+2q(x))t^2+q^2(x)t^4}. \tag{3.11}$$

By (3.7) and (3.11), we have

$$T_m(x,t)=\frac{2+ap(x)t-\left(abp^2(x)+2q(x)\right)t^2+ap(x)q(x)t^3}{1-(abp^2(x)+2q(x))t^2+q^2(x)t^4}.$$

□

Theorem 3.2. The Binet identity of the generalized bi-periodic Lucas polynomial is

$$L_m(x)=\frac{a^{\zeta(m)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}\left(\sigma^m(x)+\tau^m(x)\right), \tag{3.12}$$

where

$$\sigma(x)=\frac{abp(x)+\sqrt{a^2b^2p^2(x)+4abq(x)}}{2},$$
$$\tau(x)=\frac{abp(x)-\sqrt{a^2b^2p^2(x)+4abq(x)}}{2},$$

and

$$\zeta(m)=m-2\left\lfloor\frac{m}{2}\right\rfloor$$

is the parity function, with $\lfloor\cdot\rfloor$ denoting the floor function.

Proof. We prove (3.12) by mathematical induction. Obviously, the identity is true when $m=0$ and $m=1$. We assume that the identity is true with m. Next, we prove that the identity is true when $m+1$.

According to the identity (3.1) and mathematical induction, we have

$$\begin{aligned}
L_{m+1}(x)&=a^{\zeta(m+1)}b^{1-\zeta(m+1)}p(x)L_m(x)+q(x)L_{m-1}(x)\\
&=a^{\zeta(m+1)}b^{1-\zeta(m+1)}p(x)\left\{\frac{a^{\zeta(m)}}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}\left(\sigma^m(x)+\tau^m(x)\right)\right\}+q(x)\left\{\frac{a^{\zeta(m-1)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}}\left(\sigma^{m-1}(x)+\tau^{m-1}(x)\right)\right\}\\
&=a^{\zeta(m+1)}\sigma^{m-1}(x)\left(\frac{a^{\zeta(m)}b^{1-\zeta(m+1)}p(x)\sigma(x)}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}+\frac{q(x)}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}}\right)\\
&\quad+a^{\zeta(m+1)}\tau^{m-1}(x)\left(\frac{a^{\zeta(m)}b^{1-\zeta(m+1)}p(x)\tau(x)}{(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}+\frac{q(x)}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor}}\right)\\
&=a^{\zeta(m+1)}\sigma^{m-1}(x)\left(\frac{abp(x)\sigma(x)}{a^{1-\zeta(m)}b^{\zeta(m+1)}(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}+\frac{abq(x)}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+1}}\right)\\
&\quad+a^{\zeta(m+1)}\tau^{m-1}(x)\left(\frac{abp(x)\tau(x)}{a^{1-\zeta(m)}b^{\zeta(m+1)}(ab)^{\left\lfloor\frac{m+1}{2}\right\rfloor}}+\frac{abq(x)}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+1}}\right)\\
&=a^{\zeta(m+1)}\sigma^{m-1}(x)\left\{\frac{ab\left(p(x)\sigma(x)+q(x)\right)}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+1}}\right\}+a^{\zeta(m+1)}\tau^{m-1}(x)\left\{\frac{ab\left(p(x)\tau(x)+q(x)\right)}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+1}}\right\}\\
&=\frac{a^{\zeta(m+1)}}{(ab)^{\left\lfloor\frac{m}{2}\right\rfloor+1}}\left[\sigma^{m+1}(x)+\tau^{m+1}(x)\right],
\end{aligned}$$

where

(s) $a^{1-\zeta(m)}b^{\zeta(m+1)}(ab)^{\lfloor\frac{m+1}{2}\rfloor}=(ab)^{\lfloor\frac{m}{2}\rfloor+1}$,

(t) $p(x)\sigma(x)+q(x)=\frac{\sigma^2(x)}{ab}$,

(u) $p(x)\tau(x)+q(x)=\frac{\tau^2(x)}{ab}$. □

Theorem 3.3. Negative subscript terms of the generalized bi-periodic Lucas polynomial $\{L_n(x)\}$ are

$$L_{-m}(x)=(-1)^m q^{-m}(x)L_m(x).$$

Proof. According to the identity (3.12),

$$\begin{aligned}L_{-m}(x)&=\frac{a^{\zeta(-m)}}{(ab)^{\lfloor\frac{-m+1}{2}\rfloor}}\left(\sigma^{-m}(x)+\tau^{-m}(x)\right)=(-1)^m\cdot\frac{a^{\zeta(-m)}}{(ab)^{\lfloor\frac{-m+1}{2}\rfloor}}\left\{\frac{\sigma^m(x)+\tau^m(x)}{(abq(x))^m}\right\}\\&=(-1)^m q^{-m}(x)\left(\frac{a^{\zeta(m)}}{(ab)^{\lfloor\frac{m+1}{2}\rfloor}}\right)(\sigma^m(x)+\tau^m(x))=(-1)^m q^{-m}(x)L_m(x).\end{aligned}$$

□

Theorem 3.4. The generalized Catalan's identity of the generalized bi-periodic Lucas polynomial $\{L_n(x)\}$ is

$$a^{1-\zeta(m-r)}b^{\zeta(m-r)}L_{m-r}(x)L_{m+r}(x)-a^{1-\zeta(m)}b^{\zeta(m)}L_m^2(x)=a^{\zeta(r+1)}b^{\zeta(r)}(-q(x))^{m-r}L_r^2(x)-4a(-q(x))^m.$$

Proof. According to the identity (3.12),

$$\begin{aligned}&a^{1-\zeta(m-r)}b^{\zeta(m-r)}L_{m-r}(x)L_{m+r}(x)-a^{1-\zeta(m)}b^{\zeta(m)}L_m^2(x)\\&=a^{1-\zeta(m-r)}b^{\zeta(m-r)}\cdot\frac{a^{\zeta(m-r)}}{(ab)^{\lfloor\frac{m-r+1}{2}\rfloor}}\frac{a^{\zeta(m+r)}}{(ab)^{\lfloor\frac{m+r+1}{2}\rfloor}}\left(\sigma^{m-r}(x)+\tau^{m-r}(x)\right)\left(\sigma^{m+r}(x)+\tau^{m+r}(x)\right)\\&\quad-a^{1-\zeta(m)}b^{\zeta(m)}\left(\frac{a^{\zeta(m)}}{(ab)^{\lfloor\frac{m+1}{2}\rfloor}}\right)^2(\sigma^m(x)+\tau^m(x))^2\\&=\frac{a^{1+\zeta(m+r)}b^{\zeta(m-r)}}{(ab)^{m+1-\zeta(m+1-r)}}\left\{\sigma^{2m}(x)+(\sigma(x)\tau(x))^{m-r}\left(\sigma^{2r}(x)+\tau^{2r}(x)\right)+\tau^{2m}(x)\right\}\\&\quad-\frac{a^{1+\zeta(m)}b^{\zeta(m)}}{(ab)^{m+1-\zeta(m+1)}}\left(\sigma^{2m}(x)+2\sigma^m(x)\tau^m(x)+\tau^{2m}(x)\right)\\&=\frac{a}{(ab)^m}\left\{\sigma^{2m}(x)+(\sigma(x)\tau(x))^{m-r}\left(\sigma^{2r}(x)+\tau^{2r}(x)\right)+\tau^{2m}(x)\right\}\\&\quad-\frac{a}{(ab)^m}\left(\sigma^{2m}(x)+2\sigma^m(x)\tau^m(x)+\tau^{2m}(x)\right)\\&=\frac{a}{(ab)^m}\left\{[\sigma(x)\tau(x)]^{m-r}\left(\sigma^{2r}(x)+\tau^{2r}(x)\right)-2\sigma^m(x)\tau^m(x)\right\}\\&=\frac{a(\sigma(x)\tau(x))^{m-r}}{(ab)^m}\left(\sigma^{2r}(x)+\tau^{2r}(x)-2\sigma^r(x)\tau^r(x)\right)\\&=\frac{a(\sigma(x)\tau(x))^{m-r}}{(ab)^m}\left\{(\sigma^r(x)+\tau^r(x))^2-4\sigma^r(x)\tau^r(x)\right\}\\&=\frac{a(-q(x))^{m-r}}{(ab)^r}(\sigma^r(x)+\tau^r(x))^2-4a(-q(x))^m\\&=a^{\zeta(r+1)}b^{\zeta(r)}(-q(x))^{m-r}L_r^2(x)-4a(-q(x))^m,\end{aligned}$$

where

(v) $\left\lfloor \frac{m-r+1}{2} \right\rfloor + \left\lfloor \frac{m+r+1}{2} \right\rfloor = m + 1 - \zeta(m + 1 - r).$ $\square$

When $r = 1$, we have:

Corollary 3.1. The generalized Cassini's identity of the generalized bi-periodic Lucas polynomial $\{L_n(x)\}$ is

$$a^{\zeta(m)} b^{1-\zeta(m)} L_{m-1}(x) L_{m+1}(x) - a^{1-\zeta(m)} b^{\zeta(m)} L_n^2(x) = a^2 b(-q(x))^{m-1} p^2(x) - 4a(-q(x))^m .$$

Theorem 3.5. Let $\{F_n(x)\}$ be the generalized bi-periodic Fibonacci and $\{L_n(x)\}$ be the Lucas polynomials. We get the relations between $\{F_n(x)\}$ and $\{L_n(x)\}$ as follows:

$$F_{m+1}(x) + q(x) F_{m-1}(x) = L_m(x), \tag{3.13}$$

$$L_{m+1}(x) + q(x) L_{m-1}(x) = \left(p^2(x) ab + 4q(x)\right) F_m(x), \tag{3.14}$$

$$F_{m+2}(x) - q^2(x) F_{m-2}(x) = a^{\zeta(m+1)} b^{\zeta(m)} p(x) L_m(x), \tag{3.15}$$

$$L_{m+2}(x) - q^2(x) L_{m-2}(x) = a^{\zeta(m)} b^{\zeta(m+1)} \left(p^2(x) ab + 4q(x)\right) p(x) F_m(x). \tag{3.16}$$

Proof. We prove only (3.13), and other identities are proved similarly. According to the identities (1.2) and (3.12),

$$\begin{aligned}
\frac{(ab)^{\lfloor \frac{m+1}{2} \rfloor}}{a^{\zeta(m)}} F_{m+1}(x) + abq(x) \cdot \frac{(ab)^{\lfloor \frac{m-1}{2} \rfloor}}{a^{\zeta(m)}} F_{m-1}(x) &= \frac{\sigma^{m+1}(x) - \tau^{m+1}(x)}{\sigma(x) - \tau(x)} + abq(x) \cdot \frac{\sigma^{m-1}(x) - \tau^{m-1}(x)}{\sigma(x) - \tau(x)} \\
&= \frac{\sigma^m(x)\left(\sigma(x) + \frac{abq(x)}{\sigma(x)}\right) - \tau^m(x)\left(\tau(x) + \frac{abq(x)}{\tau(x)}\right)}{\sigma(x) - \tau(x)} \\
&= \sigma^m(x) + \tau^m(x) = \frac{(ab)^{\lfloor \frac{m+1}{2} \rfloor}}{a^{\zeta(n)}} L_m(x).
\end{aligned}$$

$\square$

Theorem 3.6. Let $\{F_n(x)\}$ be the generalized bi-periodic Fibonacci and $\{L_n(x)\}$ be the Lucas polynomials. We have the following identity:

$$\left(\frac{b}{a}\right)^{\zeta(mn+n)} F_m(x) L_n(x) + \left(\frac{b}{a}\right)^{\zeta(mn+m)} F_n(x) L_m(x) = 2F_{m+n}(x), \tag{3.17}$$

$$\left(\frac{b}{a}\right)^{\zeta(mn)} L_m(x) L_n(x) + \left(\frac{a}{b}\right)^{\zeta(mn+m+n)} \left(\frac{a^2 b^2 p^2(x) + 4abq(x)}{a^2}\right) F_m(x) F_n(x) = 2L_{m+n}(x). \tag{3.18}$$

Proof. According to the identities (1.2) and (3.12),

$$\frac{(ab)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor}}{a^{1-\zeta(m)+\zeta(n)}} F_m(x) L_n(x) + \frac{(ab)^{\lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}}{a^{1-\zeta(n)+\zeta(m)}} F_n(x) L_m(x) = \frac{2\left(\sigma^{m+n}(x) - \tau^{m+n}(x)\right)}{\sigma(x) - \tau(x)} = \frac{2(ab)^{\lfloor \frac{m+n}{2} \rfloor}}{a^{1-\zeta(m+n)}} F_{m+n}(x).$$

Similary, we get

$$
\begin{aligned}
&\frac{(ab)^{\left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor}}{a^{\zeta(m)+\zeta(n)}} L_m(x) L_n(x) + \frac{(ab)^{\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor} (\sigma(x) - \tau(x))^2}{a^{2-\zeta(n)-\zeta(m)}} F_m(x) F_n(x) \\
&= (\sigma^m(x) + \tau^m(x))(\sigma^n(x) + \tau^n(x)) + (\sigma^m(x) - \tau^m(x))(\sigma^n(x) - \tau^n(x)) \\
&= 2(\sigma^{n+m}(x) + \tau^{n+m}(x)) = \frac{2(ab)^{\left\lfloor \frac{m+n+1}{2} \right\rfloor}}{a^{\zeta(m+n)}} L_{m+n}(x),
\end{aligned}
$$

where

(w) $\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{m+n}{2} \right\rfloor = \zeta(mn + n)$,

(x) $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m+1}{2} \right\rfloor - \left\lfloor \frac{m+n}{2} \right\rfloor = \zeta(mn + m)$,

(y) $\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{m+1}{2} \right\rfloor - \left\lfloor \frac{m+n+1}{2} \right\rfloor = \zeta(mn)$,

(z) $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m+n+1}{2} \right\rfloor = -\zeta(mn + m + n)$. □

Theorem 3.7. Let $\{F_n(x)\}$ be the generalized bi-periodic Fibonacci and $\{L_n(x)\}$ be the Lucas polynomials, then we obtain the following identities

$$
\sum_{k=0}^{m} \binom{m}{k} a^{\zeta(k)} (ab)^{\left\lfloor \frac{k}{2} \right\rfloor} p^k(x) q^{m-k}(x) F_k(x) = F_{2m}(x) \tag{3.19}
$$

and

$$
\sum_{k=0}^{m} \binom{m}{k} a^{\zeta(k+1)} (ab)^{\left\lfloor \frac{k+1}{2} \right\rfloor} p^k(x) q^{m-k}(x) L_k(x) = aL_{2m}(x). \tag{3.20}
$$

Proof. We prove only (3.19), and (3.20) is proved similarly. According to the identity (1.2),

$$
\begin{aligned}
&\sum_{k=0}^{m} \binom{m}{k} a^{\zeta(k)} (ab)^{\left\lfloor \frac{k}{2} \right\rfloor} p^k(x) q^{m-k}(x) F_k(x) \\
&= \sum_{k=0}^{m} \binom{m}{k} a^{\zeta(k)} (ab)^{\left\lfloor \frac{k}{2} \right\rfloor} p^k(x) q^{m-k}(x) \cdot \frac{a^{1-\zeta(k)}}{(ab)^{\left\lfloor \frac{k}{2} \right\rfloor}} \cdot \frac{\sigma^k(x) - \tau^k(x)}{\sigma(x) - \tau(x)} \\
&= \sum_{k=0}^{m} \binom{m}{k} a p^k(x) q^{m-k}(x) \cdot \frac{\sigma^k(x) - \tau^k(x)}{\sigma(x) - \tau(x)} \\
&= \frac{a}{\sigma(x) - \tau(x)} \left(\sum_{k=0}^{m} \binom{m}{k} p^k(x) \sigma^k(x) q^{m-k}(x) - \sum_{k=0}^{m} \binom{m}{k} p^k(x) \tau^k(x) q^{m-k}(x) \right) \\
&= \frac{a}{\sigma(x) - \tau(x)} \{(\sigma(x) p(x) + q(x))^m - (\tau(x) p(x) + q(x))^m\} \\
&= \frac{a}{\sigma(x) - \tau(x)} \left(\left(\frac{\sigma^2(x)}{ab} \right)^m - \left(\frac{\tau^2(x)}{ab} \right)^m \right) \\
&= \frac{a}{(ab)^m} \left(\frac{\sigma^{2m}(x) - \tau^{2m}(x)}{\sigma(x) - \tau(x)} \right) = F_{2m}(x).
\end{aligned}
$$

□

Theorem 3.8. The sum of binomial coefficients of generalized bi-periodic Fibonacci $\{F_m(x)\}$ and Lucas $\{L_m(x)\}$ polynomials are

$$F_m(x) = \frac{2a^{1-\zeta(m)}}{2^m (ab)^{\lfloor \frac{m}{2} \rfloor}} \cdot \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1} (abp(x))^{m-2k-1} \left(a^2b^2p^2(x) + 4abq(x)\right)^k, \tag{3.21}$$

$$L_m(x) = \frac{2a^{\zeta(m)}}{2^m (ab)^{\lfloor \frac{m+1}{2} \rfloor}} \cdot \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} (abp(x))^{m-2k} \left(a^2b^2p^2(x) + 4abq(x)\right)^k. \tag{3.22}$$

Proof. By

$$\sigma(x) = \frac{abp(x) + \sqrt{a^2b^2p^2(x) + 4abq(x)}}{2}$$

and

$$\tau(x) = \frac{abp(x) - \sqrt{a^2b^2p^2(x) + 4abq(x)}}{2},$$

we have

$$\begin{aligned}
&\sigma^m(x) - \tau^m(x) \\
&= 2^{-m}\left(\left(abp(x) + \sqrt{a^2b^2p^2(x) + 4abq(x)}\right)^m - \left(abp(x) - \sqrt{a^2b^2p^2(x) + 4abq(x)}\right)^m\right) \\
&= 2^{-m}\left(\sum_{k=0}^{m} \binom{m}{k} p^{m-k}(x) a^{m-k} b^{m-k} \left(\sqrt{a^2b^2p^2(x) + 4abq(x)}\right)^k \right.\\
&\quad \left. - \sum_{k=0}^{m} \binom{m}{k} p^{m-k}(x) a^{m-k} b^{m-k} \left(-\sqrt{a^2b^2p^2(x) + 4abq(x)}\right)^k\right) \\
&= 2^{-m+1} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1} p^{m-2k-1}(x) a^{m-2k-1} b^{m-2k-1} \left(\sqrt{a^2b^2p^2(x) + 4abq(x)}\right)^{2k+1}.
\end{aligned}$$

According to the identity (1.2),

$$\begin{aligned}
F_m(x) &= \frac{a^{1-\zeta(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}} \cdot \frac{\sigma^m(x) - \tau^m(x)}{\sigma(x) - \tau(x)} \\
&= \frac{2a^{1-\zeta(m)}}{2^m (ab)^{\lfloor \frac{m}{2} \rfloor}} \cdot \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1} [abp(x)]^{m-2k-1} \left(a^2b^2p^2(x) + 4abq(x)\right)^k.
\end{aligned}$$

Similarly, we show that

$$\sigma^m(x) + \tau^m(x) = 2^{-m+1} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} p^{m-2k}(x) a^{m-2k} b^{m-2k} \left(\sqrt{p^2(x) a^2b^2 + 4q(x) ab}\right)^{2k}.$$

According to the identity (3.12),

$$L_m(x) = \frac{2a^{\zeta(m)}}{2^m (ab)^{\lfloor \frac{m+1}{2} \rfloor}} \cdot \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} (p(x) ab)^{m-2k} \left(p^2(x) a^2b^2 + 4q(x) ab\right)^k.$$

□

Theorem 3.9. Let $\{F_n(x)\}$ be the generalized bi-periodic Fibonacci and $\{L_n(x)\}$ be the Lucas polynomials. We have the following identities

$$F_{2m}(x)F_{2n}(x)=\left(\frac{a}{b}\right)^{\zeta(m+n)}\left(F_{m+n}^2(x)-q^{2n}(x)F_{m-n}^2(x)\right), \tag{3.23}$$

$$F_{2m}(x)F_{2n}(x)=\left(\frac{a}{b}\right)^{\zeta(m+n)}F_{m+n}^2(x)-\frac{a^2q^{2n}(x)}{(\sigma(x)-\tau(x))^2}\cdot\left(\frac{b}{a}\right)^{\zeta(m+n)}L_{m-n}^2(x)+\frac{4a^2(-q(x))^{m+n}}{(\sigma(x)-\tau(x))^2}, \tag{3.24}$$

$$F_{2m}(x)F_{2n}(x)=-q^{2n}(x)\left(\frac{a}{b}\right)^{\zeta(m+n)}F_{m-n}^2(x)+\frac{a^2}{(\sigma(x)-\tau(x))^2}\cdot\left(\frac{b}{a}\right)^{\zeta(m+n)}L_{m+n}^2(x)-\frac{4a^2(-q(x))^{m+n}}{(\sigma(x)-\tau(x))^2}, \tag{3.25}$$

$$L_{2m}(x)L_{2n}(x)=\left(\frac{b}{a}\right)^{\zeta(m+n)}\left(L_{m+n}^2(x)-q^{2n}(x)L_{m-n}^2(x)\right)-4(-q(x))^{m+n}, \tag{3.26}$$

$$L_{2m}(x)L_{2n}(x)=\frac{(\sigma(x)-\tau(x))^2}{a^2}\left(\frac{a}{b}\right)^{\zeta(m+n)}F_{m+n}^2(x)-q^{2n}\left(\frac{b}{a}\right)^{\zeta(m+n)}L_{m-n}^2(x), \tag{3.27}$$

$$L_{2m}(x)L_{2n}(x)=\frac{(\sigma(x)-\tau(x))^2q^{2n}(x)}{a^2}\cdot\left(\frac{a}{b}\right)^{\zeta(m+n)}F_{m-n}^2(x)+\left(\frac{b}{a}\right)^{\zeta(m+n)}L_{m+n}^2(x). \tag{3.28}$$

Proof. We prove only (3.23), and other identities are proved similarly. According to the identity (1.2), we have

$$\begin{aligned}
&\left(\frac{a}{b}\right)^{\zeta(m+n)}\left(F_{m+n}^2(x)-q^{2n}(x)F_{m-n}^2(x)\right)\\
&=\left(\frac{a}{b}\right)^{\zeta(m+n)}\left\{\left(\left(\frac{a^{1-\zeta(m+n)}}{(ab)^{\lfloor\frac{m+n}{2}\rfloor}}\right)\frac{\sigma^{m+n}(x)-\tau^{m+n}(x)}{\sigma(x)-\tau(x)}\right)^2-q^{2n}(x)\left(\left(\frac{a^{1-\zeta(m+n)}}{(ab)^{\lfloor\frac{m-n}{2}\rfloor}}\right)\frac{\sigma^{m-n}(x)-\tau^{m-n}(x)}{\sigma(x)-\tau(x)}\right)\right\}\\
&=\frac{a^2}{(ab)^{m+n}}\left(\frac{\sigma^{2(m+n)}(x)+\tau^{2(m+n)}(x)}{(\sigma(x)-\tau(x))^2}\right)-\frac{2a^2(-q(x))^{m+n}}{(\sigma(x)-\tau(x))^2}\\
&\quad-\frac{a^2}{(ab)^{m+n}}\left(\frac{\sigma^{2m}(x)\tau^{2n}(x)+\sigma^{2n}(x)\tau^{2m}(x)}{(\sigma(x)-\tau(x))^2}\right)+\frac{2a^2(-q(x))^{m+n}}{(\sigma(x)-\tau(x))^2}\\
&=\frac{a^2}{(ab)^{m+n}}\cdot\frac{\sigma^{2m}(x)-\tau^{2m}(x)}{\sigma(x)-\tau(x)}\cdot\frac{\sigma^{2n}(x)-\tau^{2n}(x)}{\sigma(x)-\tau(x)}\\
&=F_{2m}(x)F_{2n}(x).
\end{aligned}$$

□

Theorem 3.10. Let $\mathcal{F}_m(x)$ and $\mathcal{L}_m(x)$ denote the $m\times m$ tridiagonal matrix defined by

$$\mathcal{F}_m(x)=\begin{bmatrix} ap(x) & q(x) & & & \\ -1 & bp(x) & q(x) & & \\ & -1 & ap(x) & \ddots & \\ & & \ddots & \ddots & q(x) \\ & & & -1 & a^{\zeta(m)}b^{1-\zeta(m)}p(x)\end{bmatrix},\quad m\geq 1 \tag{3.29}$$

and

$$\mathcal{L}_m(x) = \begin{bmatrix} ap(x) & q(x) & & & \\ -2 & bp(x) & q(x) & & \\ & -1 & ap(x) & \ddots & \\ & & \ddots & \ddots & q(x) \\ & & & -1 & a^{\zeta(m)}b^{1-\zeta(m)}p(x) \end{bmatrix}, \quad m \geq 1, \tag{3.30}$$

with

$$\mathcal{F}_0(x) = \begin{bmatrix} 0 \end{bmatrix}$$

and

$$\mathcal{L}_0(x) = \begin{bmatrix} 2 \end{bmatrix}.$$

Therefore,

$$det\mathcal{F}_m(x) = F_{m+1}(x)$$

and

$$det\mathcal{L}_m(x) = L_m(x).$$

Proof. We prove (3.29) and (3.30) by mathematical induction. Obviously, the identity is true when $m = 1$ and $m = 2$:

$$det\mathcal{F}_1(x) = ap(x) = F_2(x), \quad det\mathcal{F}_2(x) = abp^2(x) + q(x) = F_3(x)$$

and

$$det\mathcal{L}_1(x) = ap(x) = L_1(x), \quad det\mathcal{L}_2(x) = abp^2(x) + 2q(x) = L_2(x).$$

We assume that the identity is true when $m - 1$:

$$det\mathcal{F}_{m-1}(x) = F_m(x), \quad det\mathcal{F}_{m-2}(x) = F_{m-1}(x)$$

and

$$det\mathcal{L}_{m-1}(x) = L_{m-1}(x), \quad det\mathcal{L}_{m-2}(x) = L_{m-2}(x).$$

Next, we prove that the identity is true with m.

According to the identities (2.3) and (3.1) and mathematical induction, we have

$$\begin{aligned} det\mathcal{F}_m(x) &= a^{\zeta(m)}b^{1-\zeta(m)}p(x)\,det\mathcal{F}_{m-1}(x) + q(x)\,det\mathcal{F}_{m-2}(x) \\ &= a^{\zeta(m)}b^{1-\zeta(n)}p(x)\,F_m(x) + q(x)\,F_{m-1}(x) \\ &= F_{m+1}(x) \end{aligned}$$

and

$$\begin{aligned} det\mathcal{L}_m(x) &= a^{\zeta(m)}b^{1-\zeta(m)}p(x)\,det\mathcal{L}_{m-1}(x) + q(x)\,det\mathcal{L}_{m-2}(x) \\ &= a^{\zeta(m)}b^{1-\zeta(m)}p(x)\,L_{m-1}(x) + q(x)\,L_{m-2}(x) \\ &= L_m(x). \end{aligned}$$

This completes the proof of Theorem 3.10. □

4. Conclusions

In this paper, we extend the generalized bi-periodic Fibonacci polynomial $F_n(x)$ defined in [18] and we consider $F_n(x)$ using of matrix methods. In addition, we define the generalized bi-periodic Lucas polynomial $L_n(x)$ and obtain some identities related to $L_n(x)$. Finally, we obtain a series of identities connecting $F_n(x)$ and $L_n(x)$. An interesting idea is that perhaps we can obtain a series of identities related to generalized bi-periodic Lucas polynomials using matrix methods.

Use of AI tools declaration

The authors declare they have not use Artificial Intelligence (AI) tools in the creation of this paper.

Acknowledgments

The authors would like to thank the editors and reviewers for their helpful suggestions. All the authors have contributed equally to this work and have read and approved this final manuscript. This work is supported by the National Natural Science Foundation of China (11701448).

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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